Green's canonical syzygy conjecture for generic curves of odd genus

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0 Introduction

For X a projective variety, L a line bundle on X, and $\mathcal{F}$ a coherent sheaf on X, denote, following [4], by $K_{p,q}(X, L, \mathcal{F})$ the cohomology at the middle of the exact sequence

$$\bigwedge^{p+1} H^0(X, L) \otimes H^0(X, \mathcal{F}((q-1)L)) \to \bigwedge^{p} H^0(X, L) \otimes H^0(X, \mathcal{F}(qL))$$

$$\to \bigwedge^{p-1} H^0(X, L) \otimes H^0(X, \mathcal{F}((q+1)L)),$$

where the maps are Koszul differentials. For $\mathcal{F} = \mathcal{O}_X$, use the notation $K_{p,q}(X, L)$. Green's conjecture on syzygies of canonical curves (see [4]) relates the Koszul cohomology groups

$$K_{p,1}(C, K_C),$$

for C a smooth projective curve, to the Clifford index of the curve :

$$Cliff(C) := Min_D\{deg\, D - 2r\},$$

where D runs through the set of divisors D on C satisfying :

$$r + 1 := h^0(D) \geq 2,\ h^1(D) \geq 2.$$

Conjecture 1 *(Green)*

$$K_{l,1}(C, K_C) = 0, \forall l \geq p \Leftrightarrow Cliff(C) > g - p - 2.$$

The direction $\Rightarrow$ is proved by Green and Lazarsfeld in the appendix to [4]. The case $p = g-2$ of the conjecture is equivalent to Noether's theorem, and the case $p = g-3$ to Petri's theorem (see [6]). The case $p = g-4$ has been proved in any genus by Schreyer [10] and by the author [13] for $g > 10$.

More recently, the conjecture has been studied in [11], [12], for generic curves of fixed gonality. Teixidor proves the following

Theorem 1 *([11]) Green's conjecture is true for generic curves of genus g and fixed gonality γ, in the range*

$$\gamma \leq \frac{g+7}{3}.$$

Note that Brill-Noether theory says that the gonality γ always satisfies the inequality

$$\gamma \leq [\frac{g+3}{2}],$$

with equality for the generic curve. We proved the following

Theorem 2 *([12]) Green's conjecture is true for generic curves of genus g and fixed gonality γ, in the range*

$$\gamma \geq \frac{g}{3},$$

except possibly for the generic curves of odd genus $g = 2k+1$, whose gonality is $k+2$.

So, for generic curves of fixed gonality, the only remaining case is that of generic curves of odd genus $g = 2k+1$. Green's conjecture together with Brill-Noether theory predicts that

$$K_{k,1}(C, K_C) = 0. \tag{0.1}$$

This is the main result proved in this paper. We give the precise statement below; it gives slightly more, since it proves the vanishing (0.1) for some explicit curves which we know to be generic in the Brill-Noether sense. Applications of this result to the gonality conjecture for generic curves of even genus can be found in [2], [1].

Note that this last case was especially challenging, first of all because, as noticed in [7], the locus of jumping syzygies, i.e. the locus where $K_{k,1}(C, K_C) \neq 0$ is of codimension 1 in $\mathcal{M}_g$ in this case, and in fact has a natural structure of determinantal hypersurface, and also because of the following important result of Hirschowitz and Ramanan :

Theorem 3 *([7]) If the Green conjecture is true for generic curves of genus $2k+1$, then the locus of jumping syzygies in $\mathcal{M}_{2k+1}$ is equal set theoretically to the $k+1$-gonal divisor, which is also the locus where the Clifford index is one less the generic Clifford index.*

Combined with the generic Green conjecture for genus $2k+1$-curves, this provides a strong evidence for conjecture 1.

Coming back to our result, the curves we consider are the following : we consider a smooth projective $K3$ surface S, such that $Pic(S)$ is isomorphic to $\mathbb{Z}^2$, and is freely generated by L and $\mathcal{O}_S(\Delta)$, where Δ is a smooth rational curve such that $deg\, L_{|\Delta} = 2$, and L is a very ample line bundle with $L^2 = 2g-2$, $g = 2k+1$. By the hyperplane section theorem (see [4]), we have

$$K_{k,1}(S,L) \cong K_{k,1}(C,L),\ \forall C \in\mid L \mid .$$

As we shall see in the next section, curves in $\mid L \mid$ have the generic Clifford index. Hence we expect from Green's conjecture that

$$K_{k,1}(C,L) = 0 = K_{k,1}(S,L).$$

Our theorem says indeed :

Theorem 4 *The $K3$ surface S being as above, we have*

$$K_{k,1}(S,L) = 0.$$

In the first section, we show how to adapt the arguments of [12], to the line bundle $L+\Delta$ on S, in order to show that

$$K_{k+1,1}(S, L+\Delta) = 0.$$

Note that the proof of [12] worked under the assumption $Pic(S) = \mathbb{Z}$, which is why a few supplementary arguments are needed.

In the second section, we show how to deduce from this the vanishing $K_{k,1}(S,L) = 0$. The last section is devoted to the proof of the crucial proposition 8 used in the proof of Theorem 4.

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1 The case of curves of even genus on a $K3$ surface with a node

Let S be a $K3$ surface, whose Picard group is freely generated by a very ample line bundle L, such that

$$L^2 = 2g-2,\ g = 2k+1,$$

and $\mathcal{O}_S(\Delta)$, where Δ is a rational curve such that

$$deg\, L_{|\Delta} = 2.$$

Let $L' = L(\Delta)$; smooth curves in $\mid L' \mid$ do not meet Δ and are of genus $2k+2 = 2(k+1)$. Contracting Δ to a node, the line bundle L' descends, and we are essentially in the situation considered in [12]. (Note however the change of notations from k to $k+1$.)

We first apply Lazarsfeld's argument in [8] to show :

Proposition 1 *Smooth curves C in $\mid L \mid$ or in $\mid L' \mid$ are generic in the Brill-Noether sense, i.e., do not have a g^r_d when the Brill-Noether number $\rho(g(C), d, r)$ is negative. In particular, their Clifford index is the generic one.*

Proof. It follows from [8] that if $C \subset S$ is a smooth curve in a linear system $\mid M \mid$, and D is a g^r_d on C with $\rho(g(C), d, r) < 0$, there exists a line bundle H on S with

$$h^0(H) \geq 2,\ h^0(M - H) \geq 2.$$

Apply this to $M = L$ or $M = L'$. Writing $H = \alpha L + \beta\Delta$, the condition $h^0(H) \geq 2$ implies that $\alpha \geq 1$. Similarly, the condition $h^0(M-H) \geq 2$ implies that $1 - \alpha \geq 1$, whether $M = L$ or $M = L'$. This is a contradiction. ∎

It is now expected from Green's conjecture 1 that

$$K_{k+1,1}(S, L') = 0,\ K_{k,1}(S, L) = 0.$$

In [12], we proved the vanishing

$$K_{k+1,1}(S, L') = 0,$$

for a line bundle L' on S with

$$L'^2 = 2g' - 2,\ g' = 2k + 2,$$

under the assumption that L' generates $Pic(S)$. Our first goal is to extend this result in our situation.

Theorem 5 *For S, $L' = L(\Delta)$ as above, we have*

$$K_{k+1,1}(S, L') = 0. \tag{1.2}$$

The proof of this theorem occupies the rest of this section. Let $C' \in\mid L' \mid$ be smooth; by Brill-Noether theory, there is a smooth g^1_{k+2}, say D, on C'. By proposition 1, both D and $K_{C'} - D$ are generated by sections. Consider the Lazarsfeld bundle

$$E = F^* = F \otimes L', \tag{1.3}$$

where F is the rank 2 vector bundle fitting in the exact sequence

$$0 \to F \to H^0(C', D) \otimes \mathcal{O}_S \to D \to 0. \tag{1.4}$$

Here the last map is the evaluation map along C'. One can show that E does not depend on the curve C', and neither on D'. The bundle E has $det\, E \cong L'$, and $h^0(E) = k + 3$. The following key point, which was used constantly throughout the proof of [12], remains true in our situation :

Proposition 2 *The determinant map*

$$d : \bigwedge^2 H^0(S, E) \to H^0(S, L')$$

does not vanish on decomposable elements.

Proof. Indeed assume s, $s' \in H^0(S,E)$ are not proportional but satisfy $d(s \wedge s') = 0$. Then s, s' generate a sub-line bundle of E, say H, which we may assume saturated, and which satisfies

$$h^0(H) \geq 2.$$

Hence there is an exact sequence

$$0 \rightarrow H \rightarrow E \rightarrow H' \rightarrow T \rightarrow 0,$$

where H' is a line bundle such that $H + H' = det\, E = L'$, and T is torsion supported on points of S. Since E is generated by sections, H' is generated by sections away from the support of T. On the other hand H' is not trivial, since $H^0(S, E^*) = 0$. So $h^0(H') \geq 2$. But this contradicts the fact we already mentioned, that we cannot write L' as the sum of two line bundles admitting at least two sections. ■

We now recall the main points of the proof of the vanishing (1.2) given in [12], in order to make clear what has to be added in our situation. We warn again the reader that the notation of [12] has been shifted (the integer k there becomes $k+1$ here).

First step. Let $S_{curv}^{[k+2]}$ be the open subset of the Hilbert scheme of S parametrizing curvilinear, degree $k+2$, 0-dimensional subschemes of S. Let

$$I_{k+2} \stackrel{\pi_{k+2}}{\rightarrow} S_{curv}^{[k+2]},\ I_{k+2} \subset S \times S_{curv}^{[k+1]}$$

be the incidence scheme. We established the following isomorphism :

$$K_{k+1,1}(S, L') \cong H^0(I_{k+2}, \pi_{k+2}^* L'_{k+2}) / \pi_{k+2}^* H^0(S_{curv}^{[k+2]}, L'_{k+2}), \qquad (1.5)$$

where the line bundle L'_{k+2} is the determinant of the vector bundle $\mathcal{E}_{L'}$ of rank $k+2$ on $S_{curv}^{[k+2]}$, defined as

$$\mathcal{E}_{L'} = R^0 \pi_{k+2*}(pr_1^* L').$$

From this we deduced the following criterion :

Lemma 1 *The vanishing $K_{k+1,1}(S, L') = 0$ holds if there exists a reduced scheme Z, and a morphism*

$$j : Z \rightarrow S_{curv}^{[k+2]}$$

such that, denoting

$$\widetilde{Z} \stackrel{\tilde{j}}{\rightarrow} I_{k+2}$$

the fibered product

$$\widetilde{Z} = Z \times_{S_{curv}^{[k+2]}} I_{k+2}$$

we have :

1. *The map*

$$\tilde{j}^* : H^0(I_{k+2}, \pi_{k+2}^* L'_{k+2}) \to H^0(\widetilde{Z}, \tilde{j}^* \pi_{k+2}^* L'_{k+2}) \tag{1.6}$$

is injective.

2. *Denoting by* $\pi : \widetilde{Z} \to Z$ *the first projection, the map*

$$\pi^* : H^0(Z, j^* L'_{k+2}) \to H^0(\widetilde{Z}, \tilde{j}^* \pi_{k+2}^* L'_{k+2}) \tag{1.7}$$

is surjective.

Second step. The construction of Z is as follows : we start with the vector bundle E of (1.3), (1.4). It has $c_2(E) = k + 2$. Denote by

$$\mathbb{P}(H^0(E))_{curv} \subset \mathbb{P}(H^0(S, E))$$

the open set parametrizing sections $\sigma \in H^0(S, E)$ whose 0-scheme z_σ is 0-dimensional and curvilinear. There is a natural morphism

$$\mathbb{P}(H^0(E))_{curv} \to S_{curv}^{[k+2]}$$

$$\sigma \mapsto z_\sigma.$$

Let $W = \mathbb{P}(H^0(E))_{curv} \times_{S_{curv}^{[k+2]}} I_{k+2}$. This is a degree $k+2$ cover of $\mathbb{P}(H^0(E))_{curv}$. It admits a natural morphism, say f to I_{k+2}. We use now the morphism

$$\tau_{k+2} : I_{k+2} \to S \times S_{curv}^{[k+1]},$$

which sends a point $(x, z), \{x\} \subset z$ of I_{k+2} to the residual scheme of x in z, which is curvilinear of length $k + 1$, since z is curvilinear. Let

$$\psi : W \to S_{curv}^{[k+1]}$$

be the composed map $\psi = \tau_{k+2} \circ f$. Finally we construct the sum map:

$$j : Z := (\widetilde{S \times W})_0 \to S_{curv}^{[k+2]},$$

$$(x, w) \mapsto x \cup \psi(w),$$

where the $\tilde{}$ stands here to mean "blowup along the incidence subscheme in order to make the scheme structure on the union $x \cup \psi(w)$ well defined", and the subscript 0 means, "taking an open set in order to make sure that this scheme structure is curvilinear".

Third step . The injectivity of the map (1.6) in Lemma 1 is easily reduced to the injectivity of the restriction map

$$\psi^* : H^0(S_{curv}^{[k+1]}, L'_{k+1}) \cong \bigwedge^{k+1} H^0(S, L') \to H^0(W, \psi^* L'_{k+1}).$$

Now if $\beta : W \to \mathbb{P}(H^0(E))_{curv}$ is the natural surjective map, we showed that

$$\psi^* L'_{k+1} \cong \beta^* \mathcal{O}_{\mathbb{P}(H^0(E))_{curv}}(k+1)$$

and that the map above is the composition of β^* and of an isomorphism

$$\bigwedge^{k+1} H^0(S, L') \cong H^0(\mathbb{P}(H^0(E))_{curv}, \mathcal{O}(k+1)) \cong S^{k+1} H^0(S, E)^*.$$

The construction of this isomorphism uses only the proposition 2 which remains true in our situation. Hence this step works as in [12].

Fourth step. In [12], we reduced easily the proof of the surjectivity of the map (1.7) in Lemma 1, to the proof of the following : let

$$\widetilde{W} = W \times_{S^{[k+1]}} I_{k+1},$$

and denote by $\gamma : \widetilde{W} \to W$ the natural map.

Proposition 3 *The map*

$$\gamma^* : H^0(W, \psi^* L_{k+1}) \to H^0(\widetilde{W}, \gamma^* \psi^* L_{k+1}) \tag{1.8}$$

is surjective.

Using the fact that

$$\psi^* L_{k+1} = r^* \mathcal{O}_{\mathbb{P}(H^0(E))_{curv}}(k+1),$$

where

$$r = \beta \circ \gamma : \widetilde{W} \to \mathbb{P}(H^0(E))_{curv},$$

this proposition is a consequence of the following :

Proposition 4 *The map*

$$r^* : H^0(\mathcal{O}_{\mathbb{P}(H^0(E))_{curv}}(k+1)) = S^{k+1} H^0(S, E)^* \to H^0(\widetilde{W}, r^* \mathcal{O}_{\mathbb{P}(H^0(E))_{curv}}(k+1)) \tag{1.9}$$

is surjective.

This is in the proof of this proposition that we shall see a difference between the case considered in [12] and the present case. Indeed, let us introduce as in [12], the codimension 4 subscheme

$$W' = \{(z, \sigma) \in \widetilde{S \times S} \times \mathbb{P}(H^0(S, E)),\ \sigma_{|z} = 0\}, \tag{1.10}$$

where $\widetilde{S \times S}$ is the blowup of $S \times S$ along its diagonal, hence parametrizes ordered length 2 subschemes of $S \times S$.

In [12], we used the fact that $\widetilde{W}$ can be seen as a large (i.e. the complementary set has codimension ≥ 2) Zariski open set in W', and the fact (which is inaccurately not mentioned explicitly) that W' is normal (in fact it is smooth for $k+1>3$, see below) to conclude that

$$H^0(\widetilde{W}, r^*\mathcal{O}_{\mathbb{P}(H^0(E))_{curv}}(k+1)) = H^0(W', pr_2^*\mathcal{O}_{\mathbb{P}(H^0(S,E))}(k+1)).$$

Here we cannot do that because it is not true anymore that W' is normal, nor that $\widetilde{W}$ is large in W'. In fact W' is not irreducible. Indeed, consider the rational curve $\Delta \subset S$. The exact sequence (1.4) together with the fact that $L'_{|\Delta}$ is trivial, shows that $E_{|\Delta}$ is trivial and that the restriction map

$$H^0(S,E) \to H^0(\Delta, E_{|\Delta})$$

is surjective, the right hand side being of rank 2. So $H^0(S, E(-\Delta))$ is of codimension 2 in $H^0(S,E)$, so that W' has one component isomorphic to $\Delta \times \Delta \times \mathbb{P}(H^0(S, E(-\Delta))$.

However, what remains true in our situation is the following

Lemma 2 *Assume $k+1>3$ (Theorem 5 is already known for $k+1 \leq 3$). Let*

$$U := \mathbb{P}(H^0(S,E)) - \mathbb{P}(H^0(S,E(-\Delta))).$$

Then $W'_U := W' \cap (\widetilde{S\times S} \times U)$ is smooth and $\widetilde{W}$ is a large open set in it.

Proof. If a section of E vanishes at a point of Δ, then it vanishes along Δ, hence its 0-locus is not 0-dimensional. So $\widetilde{W} \subset W'$. The proof that it is a large open set is easy. To prove that W'_U is smooth, it suffices to show the following :

For any $z \in (S-\Delta)^{[2]}$, the restriction map

$$H^0(S,E) \to H^0(E_{|z}) \tag{1.11}$$

is surjective.

Choose a smooth curve $C' \in \mid L' \mid$ containing z. It exists because z does not meet Δ. There is an exact sequence

$$0 \to D \to E_{|C'} \to K_{C'} - D \to 0,$$

where D is a divisor of degree $k+2$ on C', with $h^0(D)=2$. Furthermore the map $H^0(S,E) \to H^0(C', E_{|C'})$ is surjective and there is an exact sequence

$$0 \to H^0(C', D) \to H^0(C', E_{|C'}) \to H^0(C', K_{C'} - D) \to 0.$$

Now, by proposition 1, the curve C' is generic in the Brill-Noether sense. Hence, since $k+1>3$, it does not possesses a g^2_{k+4}. Hence the map

$$H^0(C', K_{C'} - D) \to H^0((K_{C'} - D)_{|z})$$

is surjective. So the map (1.11) has at least rank 3 since $\mid D \mid$ has no base point so that the restriction map $H^0(C', D) \to H^0(D_{|z})$ has at least rank 1, and our statement will be proved if we can furthermore choose C' and D so that the restriction map

$$H^0(C', D) \to H^0(D_{|z})$$

is injective. Take now two sections s, s' of E such that $d(s \wedge s')$ vanishes on z, but s, s' have independent restrictions in $H^0(E_{|z})$. It is easily shown to exist once we know that the restriction map (1.11) has rank at least 3. Let C' be defined by $d(s \wedge s')$. The sections s, s' generate a subline bundle D of $E_{|C'}$ as above and the two sections of D restrict injectively to z. ∎

Last step. Lemma 2 shows that we have an isomorphism

$$H^0(\widetilde{W}, r^*\mathcal{O}_{\mathbb{P}(H^0(E))_{curv}}(k+1)) \cong H^0(W'_U, pr_2^*\mathcal{O}_U(k+1)),$$

so that proposition 4 reduces to

Proposition 5 *The map*

$$pr_2^* : H^0(U, \mathcal{O}_U(k+1)) \cong S^{k+1}H^0(S,E)^* \to H^0(W'_U, pr_2^*(\mathcal{O}_U(k+1)))$$

is an isomorphism.

Proof. The proof works as in [12] ; we note that W'_U is the zero locus of a section $\tilde{\sigma}$ of a certain rank 4 vector bundle $pr_1^*\tilde{E}_2 \otimes pr_2^*(\mathcal{O}_U(1))$ on $\widetilde{S \times S} \times U$. We use then the corresponding Koszul resolution of $\mathcal{I}_{W'_U}$ to conclude that

$$H^1(\widetilde{S \times S} \times U, \mathcal{I}_{W'_U} \otimes pr_2^*(\mathcal{O}_U(k+1))) = 0. \tag{1.12}$$

There is one difference with the case considered in [12]: namely, the spectral sequence which converges to $H^*(\widetilde{S \times S} \times U, \mathcal{I}_{W'_U} \otimes pr_2^*\mathcal{O}_U(k+1))$, has degree 1 terms

$$E_1^{i,1-i} = H^i(\widetilde{S \times S} \times U, pr_1^* \bigwedge^i \tilde{E}_2^* \otimes pr_2^*(\mathcal{O}_U(k+1-i))),\ i \geq 1.$$

Of course we have

$$H^0(U, \mathcal{O}_U(k+1)) = S^{k+1}H^0(S,E)^*.$$

But unlike the case considered in [12], where we worked over the whole $\mathbb{P}(H^0(S,E))$, there might be some terms

$$H^{i-1}(\widetilde{S \times S}, \bigwedge^i \tilde{E}_2^*) \otimes H^1(U, \mathcal{O}_U(k+1))$$

contributing to the term $E_1^{i,1-i}$ above. It turns out that this is not the case, thanks to the following lemma :

Lemma 3 *We have*

$$H^{i-1}(\widetilde{S\times S},\bigwedge^i\tilde{E}_2^*)=0,$$

for any $i\geq 1$.

Proof. We refer to [12], Proposition 6, for more details and similar computations.

First of all, the vanishing $h^0(\widetilde{S\times S},\tilde{E}_2^*)=0$ follows from the fact that the dual vector bundle $\tilde{E}_2$ admits for space of global sections the space $H^0(E)$, which generates it generically, and that all of these sections vanish somewhere.

Next, $\bigwedge^4\tilde{E}_2^*\cong det\,\tilde{E}_2^*=(-L)\boxtimes(-L)(2D)$ where D is the exceptional divisor of $\widetilde{S\times S}$, and we denote abusively by $(-L)\boxtimes(-L)$ the pull-back of the line bundle $(-L)\boxtimes(-L)$ on $S\times S$ to $\widetilde{S\times S}$ via the blowing-down map. Since the canonical divisor of $\widetilde{S\times S}$ is equal to D, we get by Serre duality :

$$H^3(\widetilde{S\times S},\bigwedge^4\tilde{E}_2^*)=H^1(\widetilde{S\times S},L\boxtimes L(-D))^*,$$

and the space on the right is 0 because the multiplication map $S^2H^0(S,L)\rightarrow H^0(S,2L)$ is surjective.

To compute $H^2(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*)$, we use the isomorphism

$$\bigwedge^3\tilde{E}_2^*\cong det\,\tilde{E}_2^*\otimes\tilde{E}_2,$$

and the exact sequence

$$0\rightarrow\tilde{E}_2\rightarrow pr_1^*E\oplus pr_2^*E\rightarrow\tau^*E\rightarrow 0,\qquad(1.13)$$

where $\tau:D\rightarrow S$ is the restriction of the blowing-down map to D, and the pr_i are the projections to S composed with the blowing-down map. It follows then from the associated long exact sequence that we only have to prove the vanishings

$$H^1(D,\tau^*E(-2L))(2D_{|D}))=0,\qquad(1.14)$$

$$H^2(\widetilde{S\times S},pr_i^*E((-L)\boxtimes(-L))(2D))=0.\qquad(1.15)$$

(1.14) comes from the fact that $R^1\tau_*(2D_{|D})\cong\mathcal{O}_S$ and from $H^0(S,E(-2L))=0$. (1.15) is deduced from Serre's duality and from

$$H^2(\widetilde{S\times S},pr_i^*E^*(L\boxtimes L)(-D))=0.$$

This last property is itself deduced from $H^2(S\times S,pr_i^*E^*(L\boxtimes L)=0$ and from

$$H^1(D,pr_i^*E^*(L\boxtimes L)_{|D})=H^1(D,\tau^*(E(2L)))=0.$$

It remains only to prove the vanishing $H^1(\widetilde{S\times S},\bigwedge^2\tilde{E}_2^*)=0$. We use for this the fact, which follows from the dualization of the exact sequence (1.13) cf [12], that $\bigwedge^2\tilde{E}_2^*$ has a filtration whose successive quotients are

$$\bigwedge^2(pr_1^*E^*\oplus pr_2^*E^*),\ (pr_1^*E^*\oplus pr_2^*E^*)\otimes\tau^*E^*\otimes\mathcal{O}_D(D),\ \bigwedge^2\tau^*E^*\otimes\mathcal{O}_D(2D).$$

It is immediate to prove that each term has $H^1=0$.

■

Once we have these vanishings, the spectral sequence converging to $H^*(\widetilde{S\times S}\times U,\mathcal{I}_{W'_U}\otimes pr_2^*\mathcal{O}_U(k+1))$ has the same shape in degree 1 as in [12], and then the proof of the vanishing (1.12) works as in [12]. This concludes the proof of proposition 5, hence of theorem 5.

2 Proof of Theorem 4

We start recalling the duality theorem of [4], which we state here only in the case of surfaces :

Theorem 6 *(Green) Let X be a smooth projective surface, M be a line bundle on X which is generated by sections, and $\mathcal{F}$ be a coherent sheaf on X satisfying the condition*

$$H^1(X,\mathcal{F}(sM))=0,\ \forall s\in\mathbb{Z}.$$

Then there is for all p, q a duality isomorphism (which is canonical up to a multiplicative coefficient):

$$K_{p,q}(X,M,\mathcal{F})\cong K_{r-2-p,3-q}(X,M,\mathcal{F}^*\otimes K_X)^*,$$

where $r+1=h^0(X,M)$.

We consider now the case where X is the $K3$ surface S of the previous section, M is either L or L', and $\mathcal{F}$ is trivial. Then in the first case, $r+1=g+1=2k+2$, and in the second case $r'+1=g'+1=2k+3$. So the duality theorem above gives, using the fact that $K_S\cong\mathcal{O}_S$:

$$K_{k,1}(S,L)^*\cong K_{k-1,2}(S,L), \tag{2.16}$$
$$K_{k+1,1}(S,L')^*\cong K_{k-1,2}(S,L'). \tag{2.17}$$

Theorem 5 now says that $K_{k+1,1}(S,L')=0$ or equivalently by (2.17)

$$K_{k-1,2}(S,L')=0. \tag{2.18}$$

Next, recall that we want to prove that $K_{k,1}(S,L)=0$, and by (2.16), this is equivalent to

$$K_{k-1,2}(S,L)=0. \tag{2.19}$$

Recalling that $L'=L+\Delta$, and choosing a $\sigma\in H^0(S,L')$ such that σ generates $H^0(S,L')/H^0(S,L)$, (equivalently, σ nowhere vanishes along Δ), we have now the following :

Proposition 6 *The space $K_{k-1,2}(S,L)$ is generated as follows : consider the Koszul differential*

$$\delta' : \bigwedge^{k-1} H^0(S,L)\otimes H^0(S,L-\Delta)\to \bigwedge^{k-2} H^0(S,L)\otimes H^0(S,2L-\Delta). \quad (2.20)$$

For any $\alpha\in Ker\,\delta'$, multiplication on the right by $\sigma\in H^0(S,L+\Delta)$ provides an element $\alpha\cdot\sigma$ which is in

$$Ker\,\delta : \bigwedge^{k-1} H^0(S,L)\otimes H^0(S,2L)\to \bigwedge^{k-2} H^0(S,L)\otimes H^0(S,3L),$$

where δ is also the Koszul differential, however acting on a different space. The classes of these elements $\alpha\cdot\sigma$ generate $K_{k-1,2}(S,L)$.

Proof. Let $\beta\in Ker\,\delta\subset\bigwedge^{k-1}H^0(S,L)\otimes H^0(S,2L)$. Since we know that $K_{k-1,2}(S,L')=0$ by (2.18), we can write

$$\beta=\delta\gamma$$

for some $\gamma\in\bigwedge^k H^0(S,L')\otimes H^0(S,L')$. Since $H^0(S,L')=H^0(S,L)\oplus<\sigma>$, we can now decompose γ as

$$\gamma=\gamma_1+\sigma\wedge\gamma_2+\gamma_3\otimes\sigma+\sigma\wedge\gamma_4\otimes\sigma,$$

where

$$\gamma_1\in\bigwedge^k H^0(S,L)\otimes H^0(S,L),\quad \gamma_2\in\bigwedge^{k-1} H^0(S,L)\otimes H^0(S,L),$$

$$\gamma_3\in\bigwedge^k H^0(S,L),\quad \gamma_4\in\bigwedge^{k-1} H^0(S,L).$$

The fact that $\delta\gamma=\beta$ belongs to $\bigwedge^{k-1}H^0(S,L)\otimes H^0(S,2L)$ implies that $\gamma_4=0$ since γ_4 identifies to the image of $\delta\gamma$ in $\bigwedge^{k-1}H^0(S,L')\otimes H^0(2L'_{|\Delta})$. Next, since we consider β only modulo

$$Im\,\delta : \bigwedge^{k} H^0(S,L)\otimes H^0(S,L)\to \bigwedge^{k-1} H^0(S,L)\otimes H^0(S,2L),$$

we may assume, modifying β by an exact element, that $\gamma_1=0$. Finally, we note that γ is defined up to δ-closed and in particular up to δ-exact elements. Using the relation

$$\gamma_3\otimes\sigma=-\delta(\sigma\wedge\gamma_3)-\sigma\wedge\delta\gamma_3,$$

we conclude that modifying γ we may also assume that $\gamma_3=0$.

In conclusion, $K_{k-1,2}(S,L)$ is generated by classes of δ-closed elements β such that in $\bigwedge^{k-1}H^0(S,L')\otimes H^0(S,2L')$, we have

$$\beta=\delta(\sigma\wedge\gamma),$$

for some $\gamma \in \bigwedge^{k-1} H^0(S,L) \otimes H^0(S,L)$. Now we observe that the condition

$$\delta(\sigma \wedge \gamma) \in \bigwedge^{k-1} H^0(S,L) \otimes H^0(S,2L)$$

implies that $\delta\gamma = 0$. Hence

$$\delta(\sigma \wedge \gamma) = -\gamma \cdot \sigma.$$

The condition that

$$\gamma \cdot \sigma \in \bigwedge^{k-1} H^0(S,L) \otimes H^0(S,2L),$$

with $\sigma \in H^0(S,L+\Delta)$, $\sigma_{|\Delta} \neq 0$, implies now that $\gamma \in \bigwedge^{k-1} H^0(S,L) \otimes H^0(S,L-\Delta)$. Hence $\gamma \in Ker\,\delta'$ and the proposition is proved. ■

Our next task is to compute the dimension of the space

$$K := Ker\,(\delta' : \bigwedge^{k-1} H^0(S,L) \otimes H^0(S,L-\Delta) \to \bigwedge^{k-2} H^0(S,L) \otimes H^0(S,2L-\Delta)).$$

Notice that the Koszul complex of $(S,L,L-\Delta)$ equipped with the Koszul differential δ' has the following shape :

$$0 \to \bigwedge^{k-1} H^0(S,L) \otimes H^0(S,L-\Delta) \stackrel{\delta'}{\to} \bigwedge^{k-2} H^0(S,L) \otimes H^0(S,2L-\Delta) \to \quad (2.21)$$
$$\ldots \to \bigwedge^{k-i} H^0(S,L) \otimes H^0(S,iL-\Delta) \to \ldots.$$

So K is the first cohomology group of this complex, while the next ones are the $K_{k-i,i-1}(S,L,L-\Delta)$ for $i \geq 2$. We have now

Lemma 4 *The Koszul cohomology groups $K_{k-i,i-1}(S,L,L-\Delta)$ vanish for $i \geq 2$.*

Proof. We observe that the triple $(S,L,L-\Delta)$ satisfies the assumptions of the duality theorem 6. Hence, using $K_S \cong \mathcal{O}_S$, and $h^0(L) = 2k+2$, we conclude that $K_{k-i,i-1}(S,L,L-\Delta)$ is dual to $K_{k-1+i,3-i+1}(S,L,-L+\Delta)$.

If $i = 2$, the last group is the cohomology at the middle of the sequence

$$\bigwedge^{k+2} H^0(S,L) \otimes H^0(S,\Delta) \stackrel{\delta_1}{\to} \bigwedge^{k+1} H^0(S,L) \otimes H^0(S,L+\Delta) \stackrel{\delta_2}{\to} \bigwedge^{k} H^0(S,L) \otimes H^0(S,2L+\Delta).$$

Now we use the equality $L' = L + \Delta$ and Theorem 5 to conclude that if $\beta \in Ker\,\delta_2$, then we have

$$\beta = \delta\gamma,$$

for some $\gamma \in \bigwedge^{k+2} H^0(S,L')$. As in the previous proof, we now write

$$\gamma = \gamma_1 + \sigma \wedge \gamma_2,$$

with

$$\gamma_1 \in \bigwedge^{k+2} H^0(S,L),\ \gamma_2 \in \bigwedge^{k+1} H^0(S,L).$$

The fact that

$$\delta\gamma = \beta \in \bigwedge^{k+1} H^0(S,L) \otimes H^0(S,L+\Delta)$$

implies immediately that $\gamma_2 = 0$. Hence in fact, we have in $\bigwedge^{k+1} H^0(S,L') \otimes H^0(S,L')$ the equality

$$\beta = \delta\gamma,\ \gamma \in \bigwedge^{k+2} H^0(S,L).$$

Using the fact that the inclusion $H^0(S,L) \subset H^0(S,L+\Delta)$ is the multiplication by the unique section of $H^0(S,\Delta)$, it is obvious that this is equivalent to $\beta \in Im\,\delta_1$. So the claim is proved in this case.

Next assume that $i = 3$. Then $K_{k-1+i,3-i+1}(S,L,-L+\Delta)$ is the cohomology in the middle of the sequence

$$\bigwedge^{k+3} H^0(S,L)\otimes H^0(S,-L+\Delta) \stackrel{\delta_1}{\to} \bigwedge^{k+2} H^0(S,L)\otimes H^0(S,\Delta) \stackrel{\delta_2}{\to} \bigwedge^{k+1} H^0(S,L)\otimes H^0(S,L+\Delta).$$

But since $H^0(S,\Delta)$ is of dimension 1, it is easy to see that $Ker\,\delta_2 = 0$. So this case is also proved.

Finally, if $i \geq 4$, $K_{k-1+i,3-i+1}(S,L,-L+\Delta)$ is 0 because it is the cohomology at the middle of a complex with vanishing middle term, since $H^0(S,sL+\Delta) = \{0\}$ for $s < 0$.

■

Corollary 1 *The dimension of K is equal to the binomial coefficient* $\binom{2k+1}{k-1}$.

Proof. K is the degree 0 cohomology group of the complex (2.21) whose all next cohomology groups vanish. Hence the dimension of K is equal to the Euler characteristic of this complex. Since the terms of the complex are $\bigwedge^{k-i} H^0(S,L) \otimes H^0(S,iL-\Delta)$ put in degree $i-1$, for $i \geq 1$, and since

$$h^0(S,L) = 2k+2,\ h^0(S,iL-\Delta) = 1+2ki^2-2i,$$

we are reduced to proving the following identity:

$$\binom{2k+1}{k-1} = \sum_{i\geq 1}(-1)^{i-1}\binom{2k+2}{k-i}(1+2ki^2-2i).$$

The proof is left to the reader. ■

Recall now the vector bundle E from (1.3). Our strategy to conclude the proof of Theorem 4, or equivalently the vanishing $K_{k-1,2}(S,L)=0$, will be to construct a map

$$\phi : S^{k-1}H^0(S,E) \to K = Ker\,\delta'$$

and to prove first of all that it is an isomorphism and secondly that all the elements in $Im\,\phi$ are annihilated by the map $\cdot\sigma$ of Proposition 6. The vanishing $K_{k-1,2}(S,L)=0$ will then be a consequence of Proposition 6.

Construction of ϕ. Recall that $E_{|\Delta} \cong \mathcal{O}_\Delta^2$, and that the restriction map

$$H^0(S,E) \to H^0(\Delta, E_{|\Delta})$$

is surjective. Since $H^0(S,E)$ is of dimension $k+3$, $H^0(S,E(-\Delta))$ is of dimension $k+1$. Consider the determinant map

$$d : \bigwedge^2 H^0(S,E) \to H^0(S,L').$$

Note that for $v \in \bigwedge^2 H^0(S,E(-\Delta))$ we have $d(v) \in H^0(S,L(-\Delta))$ and for $v \in H^0(S,E) \wedge H^0(S,E(-\Delta))$, we have $d(v) \in H^0(S,L)$. Let

$$w_1, \ldots, w_{k+1}$$

be a basis of $H^0(S,E(-\Delta))$. The map ϕ is defined by the following formula

$$\phi(\tau^{k-1}) = \sum_{i<j} (-1)^{i+j} d(\tau\wedge w_1)\wedge \ldots \hat{i} \ldots \hat{j} \ldots \wedge d(\tau \wedge w_{k+1}) \otimes d(w_i \wedge w_j) \quad (2.22)$$

By the remarks above, we have

$$\phi(\tau^{k-1}) \in \bigwedge^{k-1} H^0(S,L)\otimes H^0(S,L(-\Delta)) \subset \bigwedge^{k-1} H^0(S,L')\otimes H^0(S,L').$$

We prove now:

Lemma 5 *The image of* ϕ *is contained in* $Ker\,\delta'$*, where* δ' *is the Koszul differential of (2.20).*

Proof. Observe that we have the following quadratic equations for S, imbedded in projective space via $\mid L' \mid$, (these equations are in fact quadratic equations defining the Grassmannian of codimension 2 subspaces of $H^0(E)$, in which S lies naturally): consider the natural map

$$\psi : \bigwedge^3 H^0(S,E)\otimes H^0(S,E) \to \bigwedge^2 H^0(S,E) \otimes \bigwedge^2 H^0(S,E)$$

$$\stackrel{d\otimes d}{\to} H^0(S,L')\otimes H^0(S,L') \to S^2H^0(S,L').$$

Here the first map sends $v_1 \wedge v_2 \wedge v_3 \otimes \gamma$ to

$$v_2 \wedge v_3 \otimes v_1 \wedge \gamma - v_1 \wedge v_3 \otimes v_2 \wedge \gamma + v_1 \wedge v_2 \otimes v_3 \wedge \gamma.$$

We claim that the image of ψ is contained in the ideal of S. The reason is simply that the map ψ commutes with evaluation at $x \in S$ and that since $rank\, E = 2$, we have $\bigwedge^3 E_x = 0$.

So we conclude that we have the following equalities :

$$\begin{aligned} d(v_2 \wedge v_3) \cdot d(v_1 \wedge \gamma) - d(v_1 \wedge v_3) \cdot d(v_2 \wedge \gamma) \\ + d(v_1 \wedge v_2) \cdot d(v_3 \wedge \gamma) = 0 \quad \text{in} \quad H^0(S, 2L'). \end{aligned} \tag{2.23}$$

We now compute :

$$\delta'(\phi(\tau^{k-1})) = \sum_{k<i<j} (-1)^{i+j+k} d(\tau \wedge w_1) \wedge \ldots \hat{k}\, \hat{i}\, \hat{j} \ldots \wedge d(\tau \wedge w_{k+1}) \otimes d(\tau \wedge w_k) \cdot d(w_i \wedge w_j)$$

$$- \sum_{i<k<j} (-1)^{i+j+k} d(\tau \wedge w_1) \wedge \ldots \hat{i}\, \hat{k}\, \hat{j} \ldots \wedge d(\tau \wedge w_{k+1}) \otimes d(\tau \wedge w_k) \cdot d(w_i \wedge w_j)$$

$$+ \sum_{i<j<k} (-1)^{i+j+k} d(\tau \wedge w_1) \wedge \ldots \hat{i}\, \hat{j}\, \hat{k} \ldots \wedge d(\tau \wedge w_{k+1}) \otimes d(\tau \wedge w_k) \cdot d(w_i \wedge w_j).$$

This is also equal to

$$\sum_{i<j<k} (-1)^{i+j+k} d(\tau \wedge w_1) \wedge \ldots \hat{i} \ldots \hat{j} \ldots \hat{k} \ldots \wedge d(\tau \wedge w_{k+1})$$

$$\otimes (d(\tau \wedge w_k) \cdot d(w_i \wedge w_j) + d(\tau \wedge w_i) \cdot d(w_j \wedge w_k) - d(\tau \wedge w_j) \cdot d(w_i \wedge w_k)).$$

Hence by (2.23), we find that

$$\delta'(\phi(\tau^{k-1})) = 0 \text{ in } \bigwedge^{k-2} H^0(S, L) \otimes H^0(S, 2L - \Delta).$$

■

Remark 1 The map ϕ is strongly related to the construction due to Green and Lazarsfeld (see [4], Appendix) of non trivial syzygies in $K_{r_1+r_2-1,1}(X, L_1 \otimes L_2)$, where for $i = 1, 2$, L_i are line bundles on X with $r_i + 1 = h^0(X, L_i)$. The precise relation is obtained by taking $X = C \in \mid L \mid$, L_1 a line bundle of degree $k+2$ on C with $h^0(L_1) = 2$, $h^0(L_1 - \Delta_{|C}) = 1$, and $L_2 = K_C - L_1$. One has to use for that the relation (given by sequences like (1.4)) between the Lazarsfeld vector bundle E and linear systems on the curve C, or more precisely $C \cup \Delta$.

We shall prove the following :

Proposition 7 *The map*

$$\phi : S^{k-1} H^0(S, E) \to K$$

is an isomorphism.

Proof. By corollary 1, both spaces have the same dimension, since $rank\, H^0(S,E) = k+3$. The fact that ϕ is an isomorphism reduces then to the following :

Proposition 8 *The map ϕ is injective.*

We postpone the proof of Proposition 8 to the next section. ■

Assuming Proposition 8, the proof of the vanishing

$$K_{k-1,2}(S,L) = 0$$

is then a consequence of Proposition 7, Proposition 6 and of the following :

Proposition 9 *For $\beta \in Im\,\phi \subset K$, we have*

$$\beta \cdot \sigma = 0 \text{ in } K_{k-1,2}(S,L).$$

Proof. Let $\beta = \phi(\tau^{k-1})$. We may assume first that $\tau \notin H^0(S, E(-\Delta))$, and then that $\sigma = d(\tau \wedge w)$, for some $w \in H^0(S,E)$, because the result depends only on the class of σ modulo $H^0(S,L)$, and the map

$$H^0(S,E) \to H^0(S, L+\Delta)/H^0(S,L),$$

$$v \mapsto d(\tau \wedge v) \ mod\ H^0(S,L)$$

is surjective. Next recall the formula (2.22)

$$\phi(\tau^{k-1}) = \sum_{i<j} (-1)^{i+j} d(\tau \wedge w_1) \wedge \ldots \hat{i} \ldots \hat{j} \ldots \wedge d(\tau \wedge w_{k+1}) \otimes d(w_i \wedge w_j).$$

Using equations (2.23) applied to $v_1 = w_i$, $v_2 = w_j$, $v_3 = \tau$, $\gamma = w$, we get now

$$d(w_i \wedge w_j) \cdot d(\tau \wedge w) = d(\tau \wedge w_j) \cdot d(w_i \wedge w) - d(\tau \wedge w_i) \cdot d(w_j \wedge w).$$

Hence

$$\phi(\tau^{k-1}) \cdot d(\tau \wedge w) = \sum_{i<j} (-1)^{i+j} d(\tau \wedge w_1) \wedge \ldots \hat{i} \ldots \hat{j} \ldots \wedge d(\tau \wedge w_{k+1})$$

$$\otimes(-d(\tau \wedge w_i) \cdot d(w_j \wedge w) + d(\tau \wedge w_j) \cdot d(w_i \wedge w)).$$

Now the expression on the right is equal to $\delta\beta'$, with

$$\beta' = \sum_i (-1)^i d(\tau \wedge w_1) \wedge \ldots \hat{i} \ldots \wedge d(\tau \wedge w_{k+1}) \otimes d(w_i \wedge w),$$

and since $w_i \in H^0(S, E(-\Delta))$ we have

$$d(\tau \wedge w_i) \in H^0(S,L),\ d(w_i \wedge w) \in H^0(S,L)$$

so that $\beta' \in \bigwedge^k H^0(S,L) \otimes H^0(S,L)$. So $\beta \cdot \sigma = 0$ in $K_{k-1,2}(S,L)$. ■

3 Proof of Proposition 8

Let us first recall the statement: we have the determinant map

$$d : \bigwedge^2 H^0(E) \to H^0(S, L'),$$

which has the property that it does not vanish on non-zero decomposable elements. Here the rank of $H^0(E)$ is $k+3$ and the rank of $H^0(E(-\Delta))$ is $k+1$. We defined the map

$$\phi : S^{k-1}H^0(E) \to \bigwedge^{k-1} H^0(L) \otimes H^0(L(-\Delta)),$$

explicitly by the formula (cf (2.22))

$$\phi(\tau^{k-1}) = \sum_{i<j} (-1)^{i+j} d(\tau \wedge w_1) \wedge \ldots \hat{i} \ldots \hat{j} \ldots \wedge d(\tau \wedge w_{k+1}) \otimes d(w_i \wedge w_j),$$

where the w_l's form a basis of $H^0(E(-\Delta))$. Proposition 8 states that this map is injective.

We give an ad hoc, presumably not optimal, proof of this, relying on the particular geometry of the determinant map d. We believe that it is in fact true for any d satisfying the condition that d does not vanish on decomposable elements.

We assume in the following that $k \geq 2$.

In our situation, let $x \in S$ be a generic point. Consider the composition ϕ_x of ϕ with the evaluation at x:

$$\phi_x : S^{k-1}H^0(E) \to \bigwedge^{k-1} H^0(S, L) \otimes H^0(S, L(-\Delta)_{|x}).$$

Choose the basis $w_1, \ldots w_{k+1}$ in such a way that $w_1, \ldots, w_{k-1}$ form a basis of $H^0(S, E(-\Delta) \otimes \mathcal{I}_x)$. Then the $d(w_i \wedge w_j)$ vanish at x if i or j is non greater than $k-1$, while $d(w_k \wedge w_{k+1})$ does not vanish in $H^0(S, L(-\Delta)_{|x})$. Identifying this last space with $\mathbb{C}$, it follows that ϕ_x has the following form up to a coefficient:

$$\phi_x(\tau^{k-1}) = d(\tau \wedge w_1) \wedge \ldots \wedge d(\tau \wedge w_{k-1}). \qquad (3.24)$$

First step. We first use formula (3.24) to express the map ϕ, or rather its transpose, as the map induced in cohomology by the top exterior power of a vector bundle map over an adequate variety. That will allow us later on to use the Koszul resolution of such top exterior powers.

Denote by $\mathcal{V}$ the vector bundle on the $K3$ surface, which is defined by the exact sequence

$$0 \to \mathcal{V} \to H^0(E(-\Delta)) \otimes \mathcal{O}_S \to E(-\Delta) \to 0.$$

So the fiber $\mathcal{V}_x$ at $x \in S$ is the space generated by the $w_1, \dots, w_{k-1}$ introduced above.

On $Y := \mathbb{P}(H^0(E)) \times S$, there is a natural map

$$h : pr_2^* \mathcal{V} \otimes pr_1^* \mathcal{O}_{\mathbb{P}(H^0(E))}(-1) \to H^0(S, L) \otimes \mathcal{O}_Y,$$

which at the point (τ, x) is the map

$$d(\tau \wedge \ ) : H^0(E(-\Delta) \otimes \mathcal{I}_x) \to H^0(L).$$

This map is injective when $\tau \notin H^0(E(-\Delta) \otimes \mathcal{I}_x)$, and has for kernel $< \tau >$ otherwise. Since we want to study the map induced in cohomology by the top exterior power of h, we first want to make h into a morphism which is everywhere injective. This is done as follows (we refer to diagram (3.31) for the notations) : Let

$$\mathbb{P}(\mathcal{V}) =: Z = \{(\tau, x) \in Y, \, \tau \in H^0(E(-\Delta) \otimes \mathcal{I}_x)\}.$$

So Z is the locus of points (τ, x) where $h_{\tau,x}$ is not injective. Denote by $f : \widetilde{Y} \to Y$ the blow-up of Y along Z. For simplicity, denote by $\mathbb{P}$ the space $\mathbb{P}(H^0(E))$ and by p the map $pr_1 \circ f : \widetilde{Y} \to \mathbb{P}$. Let also $q := pr_2 \circ f : \widetilde{Y} \to S$. The map h extends to a map

$$\tilde{h} : \mathcal{G} \to H^0(S, L) \otimes \mathcal{O}_{\widetilde{Y}},$$

which is now injective everywhere, where $\mathcal{G}$ is obtained from $q^* \mathcal{V} \otimes p^* \mathcal{O}_{\mathbb{P}}(-1)$ by an elementary transform along the exceptional divisor D of f. Namely $\mathcal{G}$ fits in an exact sequence

$$0 \to q^* \mathcal{V} \otimes p^* \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{G} \to \mathcal{H}_D \to 0,$$

where $\mathcal{H}_D$ is a line bundle supported on D, and the restriction of the first map to D has exactly for kernel the kernel of the map $h_{|D}$, that is the sub-line bundle

$$p^* \mathcal{O}_{\mathbb{P}}(-2)_{|D} \subset q^* \mathcal{V} \otimes p^* \mathcal{O}_{\mathbb{P}}(-1)_{|D}.$$

Note that

$$\bigwedge^{k-1} \mathcal{G} = det \, \mathcal{G} = p^* \mathcal{O}_{\mathbb{P}}(-k+1) \otimes q^*(L^{-1}(\Delta))(D). \tag{3.25}$$

Let

$$\tilde{h}_{k-1} : det \, \mathcal{G} \to \bigwedge^{k-1} H^0(L) \otimes \mathcal{O}_{\widetilde{Y}}$$

be the map which is the $k-1$-th exterior power of $\tilde{h}$, and let

$$h'_{k-1} : \bigwedge^{k-1} H^0(L)^* \otimes q^*(L^{-1}(\Delta))(D) \to \bigwedge^{k-1} \mathcal{G}^* \otimes q^*(L^{-1}(\Delta))(D)$$

be the transpose of $\tilde{h}_{k-1}$ twisted by $q^*(L^{-1}(\Delta))(D)$. We first claim that the transpose of the map ϕ identifies to the map $h^2(h'_{k-1})$:

$$\begin{array}{ccc} H^2(\widetilde{Y}, \bigwedge^{k-1} H^0(L)^* \otimes q^*(L^{-1}(\Delta))(D)) & \rightarrow & H^2(\widetilde{Y}, \bigwedge^{k-1} \mathcal{G}^* \otimes q^*(L^{-1}(\Delta))(D)) \\ \| & & \| \\ \bigwedge^{k-1} H^0(L)^* \otimes H^0(L(-\Delta)) & \stackrel{^t\phi}{\rightarrow} & H^2(\widetilde{Y}, p^*\mathcal{O}(k-1)) = S^{k-1}H^0(E)^* \end{array} \quad . \tag{3.26}$$

Indeed, by (3.25), we have

$$R^{k+2}q_*(K_{\widetilde{Y}} \otimes p^*\mathcal{O}_{\mathbb{P}}(-k+1)) = R^{k+2}q_*(K_{\widetilde{Y}} \otimes det\, \mathcal{G} \otimes q^*(L(-\Delta))(-D)),$$

and by Serre duality and $K_S = \mathcal{O}_S$, the left hand side identifies to $S^{k-1}H^0(E)\otimes \mathcal{O}_S$. Now, formula (3.24) says that the map induced (up to a twist) by $\tilde{h}_{k-1}$:

$$S^{k-1}H^0(E) \otimes \mathcal{O}_S \cong R^{k+2}q_*(K_{\widetilde{Y}} \otimes det\, \mathcal{G} \otimes q^*(L(-\Delta))(-D))$$

$$\rightarrow R^{k+2}q_*(K_{\widetilde{Y}} \otimes \bigwedge^{k-1} H^0(L) \otimes q^*(L(-\Delta))(-D)) = \bigwedge^{k-1} H^0(L) \otimes L(-\Delta)$$

is exactly the map ϕ followed with evaluation. Taking global sections on S, we conclude that ϕ is the map induced by $\tilde{h}_{k-1}$ (up to a twist):

$$H^{k+2}(\widetilde{Y}, K_{\widetilde{Y}} \otimes det\, \mathcal{G} \otimes q^*(L(-\Delta))(-D)) \rightarrow H^{k+2}(\widetilde{Y}, K_{\widetilde{Y}} \otimes \bigwedge^{k-1} H^0(L) \otimes q^*(L(-\Delta))(-D)),$$

and applying Serre duality gives the result.

So the content of Proposition 8 is the surjectivity of the map $h^2(h'_{k-1})$.

Second step. We shall now analyse the spectral sequence associated to the Koszul resolution of $Ker\, h'_{k-1}$: associated to the surjective map ${}^t\tilde{h}$, there is a resolution

$$0 \rightarrow \bigwedge^{2k+2} H^0(L)^* \otimes S^{k+3}\mathcal{G} \rightarrow \ldots \rightarrow \bigwedge^{k} H^0(L)^* \otimes \mathcal{G} \tag{3.27}$$
$$\rightarrow \bigwedge^{k-1} H^0(L)^* \otimes \mathcal{O}_{\widetilde{Y}} \rightarrow \bigwedge^{k-1} \mathcal{G}^* \rightarrow 0.$$

We claim now that the surjectivity of the map (3.26) follows from the following lemmas.

Lemma 6 *For $1 \leq l \leq k$, or $l = k+2$, we have*

$$H^{l+2}(\widetilde{Y}, S^l\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D)) = 0.$$

Lemma 7 *The sequence*

$$\bigwedge^{2k+1} H^0(L)^* \otimes H^{k+3}(\widetilde{Y}, S^{k+2}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D)) \tag{3.28}$$
$$\rightarrow \bigwedge^{2k} H^0(L)^* \otimes H^{k+3}(\widetilde{Y}, S^{k+1}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D))$$
$$\rightarrow \bigwedge^{2k-1} H^0(L)^* \otimes H^{k+3}(\widetilde{Y}, S^{k}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D))$$

induced by the complex (3.27) is exact at the middle.

Indeed, these two lemmas together imply that the vector bundle $Ker\,h'_{k-1}$ satisfies $H^3(\widetilde{Y}, Ker\,h'_{k-1}) = 0$, which implies the surjectivity of $h^2(h'_{k-1})$: In fact, twisting the complex (3.27) by $q^*(L^{-1}(\Delta))(D)$, we get a resolution of $Ker\,h'_{k-1}$ as follows :

$$0 \to \bigwedge^{2k+2} H^0(L)^* \otimes S^{k+3}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D) \to \ldots \to \bigwedge^{k} H^0(L)^* \otimes \mathcal{G} \otimes q^*(L^{-1}(\Delta))(D)$$

$$\to Ker\,h'_{k-1} \to 0.$$

The associated spectral sequence abutting to the hypercohomology of the complex

$$0 \to \bigwedge^{2k+2} H^0(L)^* \otimes S^{k+3}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D) \to \ldots \to \bigwedge^{k} H^0(L)^* \otimes \mathcal{G} \otimes q^*(L^{-1}(\Delta))(D) \to 0,$$

where we put the last term on the right in degree 0, has

$$E_1^{p,q} \cong \bigwedge^{k-p} H^0(L)^* \otimes H^q(\widetilde{Y}, S^{1-p}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D)).$$

Now Lemma 6 says that the $E_1^{p,q}$ for $p + q = 3$, $q \geq 3$ are 0 unless $q = k + 3$. For $q = k + 3$, we have $E_1^{-k,k+3} \neq 0$ but Lemma 7 says that $E_2^{-k,k+3} = 0$. Hence this complex has $\mathbb{H}^3 = 0$ and thus $H^3(\widetilde{Y}, Ker\,h'_{k-1}) = 0$. ∎

Third step. We start now proving Lemmas 6 and 7. We shall use for this another geometric definition of the vector bundle $\mathcal{G}$. We refer to diagram (3.31) for the notations.

Proof of Lemma 6. Let $\mathcal{F}$ be the quotient bundle $H^0(S, E) \otimes \mathcal{O}_S/\mathcal{V}$. There is the relative projection

$$\chi : \widetilde{Y} \to \mathbb{P}(\mathcal{F}),$$

which makes $\widetilde{Y}$ isomorphic to $\mathbb{P}(\mathcal{H})$, where $\mathcal{H}$ is a vector bundle on $\mathbb{P}(\mathcal{F})$ which fits in the exact sequence

$$O \to \pi^*\mathcal{V} \to \mathcal{H} \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1) \to 0, \tag{3.29}$$

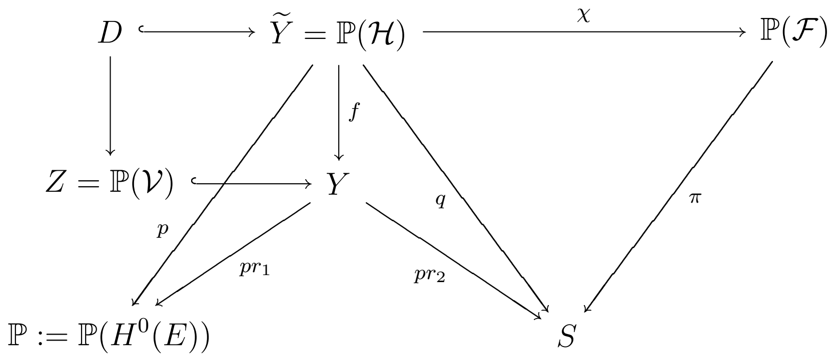
where $\pi : \mathbb{P}(\mathcal{F}) \to S$ is the structural map. We observe now that $\mathcal{G}$ is naturally isomorphic to the twisted relative tangent bundle $T_\chi \otimes p^*\mathcal{O}_{\mathbb{P}}(-2)$. To see this, we consider the relative Euler sequence

$$0 \to p^*\mathcal{O}_{\mathbb{P}}(-2) \to \chi^*\mathcal{H} \otimes p^*\mathcal{O}_{\mathbb{P}}(-1) \to T_\chi(-2) \to 0. \tag{3.30}$$

It induces a map

$$q^*\mathcal{V} \otimes p^*\mathcal{O}_{\mathbb{P}}(-1) \to T_\chi,$$

and it follows from the Euler sequence that this map is injective away from D and has $p^*\mathcal{O}_{\mathbb{P}}(-2)_{|D}$ as kernel along D. Hence T_χ is deduced from $q^*\mathcal{V} \otimes p^*\mathcal{O}_{\mathbb{P}}(-1)$ by the same elementary transform as $\mathcal{G}$.



(3.31)

The relative Euler sequence (3.30) describes $\mathcal{G}$ by the exact sequence :

$$0 \to p^*\mathcal{O}_{\mathbb{P}}(-2) \to p^*\mathcal{O}_{\mathbb{P}}(-1) \otimes \chi^*\mathcal{H} \to \mathcal{G} \to 0. \tag{3.32}$$

Taking the l-th symmetric power, we get the exact sequence :

$$0 \to p^*\mathcal{O}_{\mathbb{P}}(-l-1) \otimes \chi^* S^{l-1}\mathcal{H} \to p^*\mathcal{O}_{\mathbb{P}}(-l) \otimes \chi^* S^l\mathcal{H} \to S^l\mathcal{G} \to 0. \tag{3.33}$$

Assume first that $1 \leq l \leq k-1$. By the exact sequence (3.33), the vanishing

$$H^{l+2}(\widetilde{Y}, S^l\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D)) = 0$$

is implied by the vanishings :

$$\begin{aligned} H^{l+2}(\widetilde{Y}, p^*\mathcal{O}_{\mathbb{P}}(-l) \otimes \chi^* S^l\mathcal{H} \otimes q^*(L^{-1}(\Delta))(D)) = 0, \\ H^{l+3}(\widetilde{Y}, p^*\mathcal{O}_{\mathbb{P}}(-l-1) \otimes \chi^* S^{l-1}\mathcal{H} \otimes q^*(L^{-1}(\Delta))(D)) = 0. \end{aligned} \tag{3.34}$$

Now, if $k-1 \geq l \geq 2$, the line bundles $p^*\mathcal{O}_{\mathbb{P}}(-l)(D)$, $p^*\mathcal{O}_{\mathbb{P}}(-l-1)(D)$, have trivial cohomology along the fibers of χ, which are $\mathbb{P}^{k-1}$'s, on which D restricts to $\mathcal{O}(1)$. Hence the vanishings (3.34) are proved in this case. The case $l = 1$ is also easy.

If $l = k$, the argument above gives an inclusion

$$H^{k+2}(\widetilde{Y}, S^k\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D))$$

$$\hookrightarrow H^{k+3}(\widetilde{Y}, p^*\mathcal{O}_{\mathbb{P}}(-k-1) \otimes \chi^* S^{k-1}\mathcal{H} \otimes q^*(L^{-1}(\Delta))(D)).$$

By Serre duality, this dualizes as

$$H^1(\widetilde{Y}, p^*\mathcal{O}_{\mathbb{P}}(-2) \otimes \chi^* S^{k-1}\mathcal{H}^* \otimes q^*(L(-\Delta))(2D)).$$

Since $p^*\mathcal{O}_{\mathbb{P}}(-2)(2D) = \chi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-2)$ and $S^{k-1}\mathcal{H}^* = R^0\chi_*(p^*\mathcal{O}_{\mathbb{P}}(k-1))$, the last space is equal to

$$H^1(\widetilde{Y}, p^*\mathcal{O}_{\mathbb{P}}(k-3)(2D) \otimes q^*(L-(\Delta))),$$

which is 0 because the map $f = (p, q)$ is the blow-down map, and $2D$ has trivial cohomology along the fibers of f. It follows that $H^1(\widetilde{Y}, p^*\mathcal{O}_{\mathbb{P}}(k-3)(2D) \otimes q^*(L - (\Delta))) = H^1(Y, pr_1^*\mathcal{O}_{\mathbb{P}}(k-3) \otimes pr_2^*(L - (\Delta))) = 0$.

Next, if $l = k+2$, we use the inclusion $p^*\mathcal{O}_{\mathbb{P}}(-1) \otimes q^*\mathcal{V} \subset \mathcal{G}$, which is an isomorphism away from D, to get a surjective map

$$H^{k+4}(\widetilde{Y}, p^*\mathcal{O}_{\mathbb{P}}(-k-2) \otimes q^*S^{k+2}\mathcal{V} \otimes q^*(L^{-1}(\Delta))(D))$$

$$\to H^{k+4}(\widetilde{Y}, S^{k+2}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D)).$$

The left hand side is zero because D has trivial cohomology along the fibers of (p, q), so that

$$H^{k+4}(\widetilde{Y}, p^*\mathcal{O}_{\mathbb{P}}(-k-2) \otimes q^*S^{k+2}\mathcal{V} \otimes q^*(L^{-1}(\Delta))(D))$$

$$= H^{k+4}(Y, pr_1^*\mathcal{O}_{\mathbb{P}}(-k-2) \otimes pr_2^*(S^{k+2}\mathcal{V} \otimes L^{-1}(\Delta))) = 0$$

(recall that $Y = \mathbb{P}(H^0(E)) \times S$, with $rk\, H^0(E) = k+3$). Hence we conclude that $H^{k+4}(\widetilde{Y}, S^{k+2}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D)) = 0$. ■

In order to prove Lemma 7, we will need the following :

Lemma 8 *1. The space*

$$H^{k+3}(\widetilde{Y}, S^{k+1}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D)) \tag{3.35}$$

is canonically isomorphic to $S^{k-2}H^0(E) \otimes H^2(S, \mathcal{V} \otimes L^{-1}(\Delta))$.

2. *The space* $H^{k+3}(\widetilde{Y}, S^{k+2}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D))$ *is canonically isomorphic to* $S^{k-1}H^0(E) \otimes H^2(S, S^2\mathcal{V} \otimes L^{-1}(\Delta))$.

3. *The space* $H^{k+3}(\widetilde{Y}, S^{k}\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D))$ *is canonically isomorphic to* $S^{k-3}H^0(E) \otimes H^2(S, L^{-1}(\Delta))$.

Proof. We use the exact sequence

$$0 \to \chi^*S^{l-1}\mathcal{H} \otimes p^*\mathcal{O}_{\mathbb{P}}(-l-1) \to \chi^*S^{l}\mathcal{H} \otimes p^*\mathcal{O}_{\mathbb{P}}(-l) \to S^l\mathcal{G} \to 0.$$

It implies by the associated long exact sequence that the space

$$H^{k+3}(\widetilde{Y}, S^l\mathcal{G} \otimes q^*(L^{-1}(\Delta))(D))$$

is isomorphic to

$$Ker\,(H^{k+4}(\widetilde{Y}, \chi^*S^{l-1}\mathcal{H} \otimes p^*\mathcal{O}_{\mathbb{P}}(-l-1) \otimes q^*(L^{-1}(\Delta))(D))$$

$$\to H^{k+4}(\widetilde{Y}, \chi^*S^{l}\mathcal{H} \otimes p^*\mathcal{O}_{\mathbb{P}}(-l) \otimes q^*(L^{-1}(\Delta))(D))).$$

Recalling that

$$p^*\mathcal{O}_{\mathbb{P}}(1) = \chi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)(D),$$

we can rewrite this as

$$Ker\,(H^{k+4}(\widetilde{Y},\chi^*S^{l-1}\mathcal{H}\otimes\chi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)\otimes p^*\mathcal{O}_{\mathbb{P}}(-l)\otimes q^*(L^{-1}(\Delta))) \quad (3.36)$$
$$\to H^{k+4}(\widetilde{Y},\chi^*S^{l}\mathcal{H}\otimes\chi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)\otimes p^*\mathcal{O}_{\mathbb{P}}(-l+1)\otimes q^*(L^{-1}(\Delta)))).$$

Now, we use the formula

$$K_{\widetilde{Y}/\mathbb{P}(\mathcal{F})}=p^*\mathcal{O}_{\mathbb{P}}(-k)\otimes\chi^*det\,\mathcal{H}^*,$$

that is, by the exact sequence (3.29), which gives $det\,\mathcal{H}^*=\pi^*(L(-\Delta))\otimes\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$,

$$K_{\widetilde{Y}/\mathbb{P}(\mathcal{F})}=p^*\mathcal{O}_{\mathbb{P}}(-l)\otimes\chi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})(1)}\otimes q^*(L(-\Delta)).$$

It follows then by Leray spectral sequence and relative Serre duality, that (3.36) is also equal to:

$$Ker\,(H^5(\mathbb{P}(\mathcal{F}),S^{l-1}\mathcal{H}\otimes S^{l-k}\mathcal{H}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-2)\otimes\pi^*(L^{-2}(2\Delta)))$$
$$\to H^5(\mathbb{P}(\mathcal{F}),S^{l}\mathcal{H}\otimes S^{l-k-1}\mathcal{H}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-2)\otimes\pi^*(L^{-2}(2\Delta)))),$$

where we make the convention that negative symmetric powers are 0, and where the map is induced by the natural map

$$S^{l-1}\mathcal{H}\otimes S^{l-k}\mathcal{H}\to S^{l}\mathcal{H}\otimes S^{l-k-1}\mathcal{H}.$$

We now apply again relative Serre duality and Leray spectral sequence to conclude that this is also equal to :

$$Ker\,(H^{k+4}(\widetilde{Y},p^*\mathcal{O}_{\mathbb{P}}(-k-l)\otimes\chi^*S^{l-k}\mathcal{H}\otimes q^*(L^{-1}(\Delta))(D))$$
$$\to H^{k+4}(\widetilde{Y},p^*\mathcal{O}_{\mathbb{P}}(-k-l-1)\otimes\chi^*S^{l-k-1}\mathcal{H}\otimes q^*(L^{-1}(\Delta))(D))).$$

We now distinguish according to the value of l.

- *Case* $l=k$. We proved that in this case we have

$$H^{k+3}(\widetilde{Y},S^k\mathcal{G}\otimes q^*(L^{-1}(\Delta))(D))\cong H^{k+4}(\widetilde{Y},p^*\mathcal{O}_{\mathbb{P}}(-2k)\otimes q^*(L^{-1}(\Delta))(D)).$$

Since $K_{\widetilde{Y}}=p^*\mathcal{O}_{\mathbb{P}}(-k-3)(3D)$, this is Serre dual to

$$H^0(\widetilde{Y},p^*\mathcal{O}_{\mathbb{P}}(k-3)\otimes q^*(L(-\Delta))(2D))=S^{k-3}H^0(E)^*\otimes H^0(S,L(-\Delta)).$$

Applying Serre's duality on S gives then 3.

- *Case* $l=k+1$. In this case, we have

$$H^{k+3}(\widetilde{Y},S^{k+1}\mathcal{G}\otimes q^*(L^{-1}(\Delta))(D))\cong \quad (3.37)$$
$$Ker\,(H^{k+4}(\widetilde{Y},p^*\mathcal{O}_{\mathbb{P}}(-2k-1)\otimes\chi^*\mathcal{H}\otimes q^*(L^{-1}(\Delta))(D))$$
$$\to H^{k+4}(\widetilde{Y},p^*\mathcal{O}_{\mathbb{P}}(-2k-2)\otimes q^*(L^{-1}(\Delta))(D))).$$

The right hand side is computed as before : by Serre duality on $\widetilde{Y}$, we get

$$H^{k+4}(\widetilde{Y}, p^*\mathcal{O}_{\mathbb{P}}(-2k-2)\otimes q^*(L^{-1}(\Delta))(D)) \cong S^{k-1}H^0(E)\otimes H^2(S, L^{-1}(\Delta)) \tag{3.38}$$

To compute the left hand side, we use the exact sequence (3.29):

$$O \to \pi^*\mathcal{V} \to \mathcal{H} \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1) \to 0.$$

Pulling back to $\widetilde{Y}$ and tensoring with $p^*\mathcal{O}_{\mathbb{P}}(-2k-1)\otimes q^*(L^{-1}(\Delta))(D)$, it provides the exact sequence:

$$\begin{aligned} O &\to q^*\mathcal{V}\otimes p^*\mathcal{O}_{\mathbb{P}}(-2k-1)\otimes q^*(L^{-1}(\Delta))(D) \\ &\to \chi^*\mathcal{H}\otimes p^*\mathcal{O}_{\mathbb{P}}(-2k-1)\otimes q^*(L^{-1}(\Delta))(D) \\ &\to \chi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)\otimes p^*\mathcal{O}_{\mathbb{P}}(-2k-1)\otimes q^*(L^{-1}(\Delta))(D) \to 0. \end{aligned} \tag{3.39}$$

Note that, as (3.29) is (non-canonically) split along the fibers of π, the sequence (3.39) is (non-canonically) split along the fibers of q. Hence there is an induced exact sequence on S:

$$\begin{aligned} O &\to \mathcal{V}\otimes L^{-1}(\Delta)\otimes R^{k+2}q_*(p^*\mathcal{O}_{\mathbb{P}}(-2k-1)(D)) \\ &\to L^{-1}(\Delta)\otimes R^{k+2}q_*(\chi^*\mathcal{H}\otimes p^*\mathcal{O}_{\mathbb{P}}(-2k-1)(D)) \\ &\to L^{-1}(\Delta)\otimes R^{k+2}q_*(\chi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)\otimes p^*\mathcal{O}_{\mathbb{P}}(-2k-1)(D)) \to 0. \end{aligned} \tag{3.40}$$

Using the equality

$$K_{\widetilde{Y}} = K_{\widetilde{Y}/S} = p^*\mathcal{O}_{\mathbb{P}}(-k-3)(3D)$$

and relative Serre duality, we get :

$$\begin{aligned} R^{k+2}q_*(p^*\mathcal{O}_{\mathbb{P}}(-2k-1)(D)) &\cong R^0q_*(p^*\mathcal{O}_{\mathbb{P}}(k-2)(2D))^* \\ &= S^{k-2}H^0(E)\otimes\mathcal{O}_S. \end{aligned}$$

Similarly, we get

$$\begin{aligned} &R^{k+2}q_*(p^*\mathcal{O}_{\mathbb{P}}(-2k-1)\otimes\chi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)(D)) \\ &= R^{k+2}q_*(p^*\mathcal{O}_{\mathbb{P}}(-2k-2)(2D)) \cong R^0q_*(p^*\mathcal{O}_{\mathbb{P}}(k-1)(D))^* \\ &= S^{k-1}H^0(E)\otimes\mathcal{O}_S. \end{aligned}$$

The exact sequence (3.40) thus rewrites as :

$$\begin{aligned} 0 \to \mathcal{V}\otimes L^{-1}(\Delta)\otimes S^{k-2}H^0(E) &\to R^{k+2}q_*(\chi^*\mathcal{H}\otimes p^*\mathcal{O}_{\mathbb{P}}(-2k-1)(D))) \\ &\to L^{-1}(\Delta)\otimes S^{k-1}H^0(E)\to 0. \end{aligned}$$

This last exact sequence is now canonically split because

$$H^0(S,\mathcal{V}) = H^1(S,\mathcal{V}) = 0.$$

It follows that we have a canonical isomorphism :

$$H^{k+4}(\widetilde{Y},\chi^*\mathcal{H}\otimes p^*\mathcal{O}_{\mathbb{P}}(-2k-1)(D)) = H^2(S,R^{k+2}q_*(\chi^*\mathcal{H}\otimes p^*\mathcal{O}_{\mathbb{P}}(-2k-1)(D)))$$
$$= H^2(S,\mathcal{V}\otimes L^{-1}(\Delta))\otimes S^{k-2}H^0(E)\oplus H^2(S,L^{-1}(\Delta))\otimes S^{k-1}H^0(E) \quad (3.41)$$

In conclusion, using (3.37), (3.38) and (3.41), we have found a canonical identification :

$$H^{k+3}(\widetilde{Y},S^{k+1}\mathcal{G}\otimes q^*(L^{-1}(\Delta))(D)) \cong$$
$$Ker\,(S^{k-2}H^0(E)\otimes H^2(S,\mathcal{V}\otimes L^{-1}(\Delta))\oplus S^{k-1}H^0(E)\otimes H^2(S,L^{-1}(\Delta))$$
$$\to S^{k-1}H^0(E)\otimes H^2(S,L^{-1}(\Delta))).$$

One checks that the second component of the map is the identity, which gives a canonical identification :

$$H^{k+3}(\widetilde{Y},S^{k+1}\mathcal{G}\otimes q^*(L^{-1}(\Delta))(D)) \cong S^{k-2}H^0(E)\otimes H^2(S,\mathcal{V}\otimes L^{-1}(\Delta)),$$

proving 1.

The isomorphism 2 is proved in the same way. ■

Let us now compute the spaces $H^2(S,\mathcal{V}\otimes L^{-1}(\Delta))$ and $H^2(S,S^2\mathcal{V}\otimes L^{-1}(\Delta))$.

We first observe that the exact sequence

$$0\to\mathcal{V}\to H^0(E(-\Delta))\otimes\mathcal{O}_S\to E(-\Delta)\to 0,$$

and Serre duality give an isomorphism

$$H^2(S,\mathcal{V}\otimes L^{-1}(\Delta))\cong Ker\,c, \quad (3.42)$$

where c is the contraction map

$$H^0(E(-\Delta))\otimes H^0(S,L(-\Delta))^*\to H^0(E(-\Delta))^*$$

induced by the determinant map

$$d_0:\bigwedge^2 H^0(E(-\Delta))\to H^0(S,L(-\Delta)).$$

Similarly the induced exact sequence

$$0\to S^2\mathcal{V}\to S^2H^0(E(-\Delta))\otimes\mathcal{O}_S\to H^0(E(-\Delta))\otimes E(-\Delta)\to\bigwedge^2 E(-\Delta)\to 0$$

gives a surjective map :

$$H^2(S,S^2\mathcal{V}\otimes L^{-1}(\Delta))\twoheadrightarrow Ker\,c', \quad (3.43)$$

where the contraction map

$$c':S^2H^0(E(-\Delta))\otimes H^0(S,L(-\Delta))^*\to H^0(E(-\Delta))\otimes H^0(E(-\Delta))^*,$$

is also induced by d_0.

Using Lemma 8, we rewrite now the sequence (3.28) as follows: We first identify

$$\bigwedge^{2k+1} H^0(L)^*, \bigwedge^{2k} H^0(L)^*, \bigwedge^{2k-1} H^0(L)^*$$

respectively to

$$H^0(L), \bigwedge^2 H^0(L), \bigwedge^3 H^0(L).$$

Then via the isomorphisms given in Lemma 8, and (3.42), (3.43) above, our sequence (3.28) becomes, after replacing the first term by its quotient given in (3.43), through which the first map factors :

$$H^0(L) \otimes S^{k-1}H^0(E) \otimes Ker\, c' \to \bigwedge^2 H^0(L) \otimes S^{k-2}H^0(E) \otimes Ker\, c \quad (3.44)$$
$$\to \bigwedge^3 H^0(L) \otimes S^{k-3}H^0(E) \otimes H^0(S, L(-\Delta))^*.$$

It is immediate to check that the maps of the complex are induced by the determinant map

$$d : H^0(E) \otimes H^0(E(-\Delta)) \to H^0(L)$$

and by the natural maps, for $(i, j) = (k-1, 2)$, $(k-2, 1)$, $(k-3, 0)$:

$$S^iH^0(E) \otimes S^jH^0(E(-\Delta)) \to S^{i-1}H^0(E) \otimes S^{j-1}H^0(E(-\Delta)) \otimes H^0(E) \otimes H^0(E(-\Delta)).$$

Fourth step. We now want to prove Lemma 7, which we have just proved to be equivalent to exactness at the middle of the sequence (3.44). We do not need at this point the $K3$ surface anymore. We shall use now only the map d and do geometry on the Grassmannian of subspaces of $H^0(E)$. We believe that this step is the only essential one in the proof of proposition 8.

Denote by G the Grassmannian of rank 2 subspaces of $H^0(E)$, and let $\mathcal{L}$ be the Plücker line bundle on G, $\mathcal{E}$ the tautological rank 2 quotient bundle on G. Let $\widetilde{G'}$ be the desingularization of the hypersurface $G' \subset G$ parametrizing the $V \subset H^0(E)$ meeting $H^0(E(-\Delta))$, defined as

$$\widetilde{G'} = \{(v, V) \in \mathbb{P}(H^0(E(-\Delta))) \times G,\, v \in V\}.$$

We shall also denote by $\mathcal{L}$, $\mathcal{E}$ the pull-backs of $\mathcal{L}$, $\mathcal{E}$ to $\widetilde{G'}$ by the second projection. Let

$$g : \widetilde{G'} \to \mathbb{P}H^0(E(-\Delta))$$

be the first projection, and denote by H the line bundle $g^*\mathcal{O}_{\mathbb{P}H^0(E(-\Delta))}(1)$.

Next, the map

$$d_0 : \bigwedge^2 H^0(E(-\Delta)) \to H^0(L(-\Delta))$$

allows to define

$$I \subset \mathbb{P}H^0(L(-\Delta))^* \times \mathbb{P}H^0(E(-\Delta)),$$

$$I = \{(\sigma, v),\ \sigma(d_0(v \wedge \cdot)) = 0 \text{ in } H^0(E(-\Delta))^*\}.$$

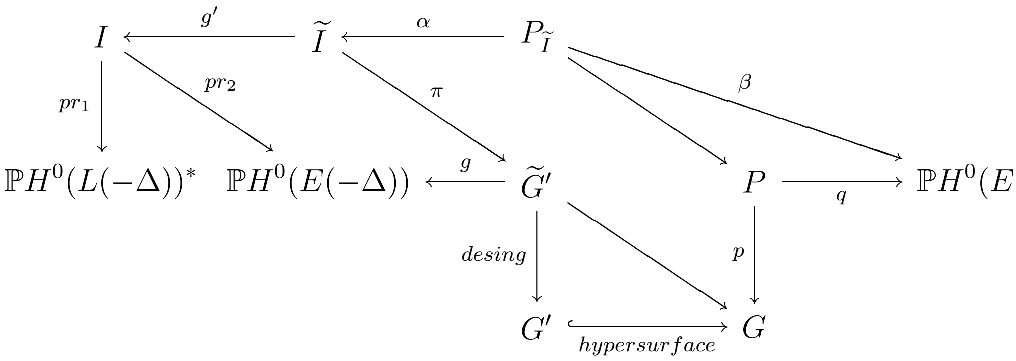
Let $\widetilde{I} \xrightarrow{\pi} \widetilde{G}'$ be the fibered product $I \times_{\mathbb{P}H^0(E(-\Delta))} \widetilde{G}'$. We shall denote by $\mathcal{K}$ the line bundle

$$pr_1^* \mathcal{O}_{\mathbb{P}H^0(L(-\Delta))^*}(1)$$

on $\widetilde{I}$, and by $\mathcal{L}'$, $\mathcal{E}'$, H' the pull-backs to $\widetilde{I}$ of the corresponding objects on $\widetilde{G}'$, that is

$$\mathcal{L}' = \pi^* \mathcal{L}, \quad \mathcal{E}' = \pi^* \mathcal{E}, \quad H' = \pi^* H.$$

The notations are summarized in the following diagram:



(3.45)

We shall use the following Lemma:

Lemma 9 *For any positive integers* p, s, $t \geq s$, $p \geq s$ *we have*

$$H^0(\widetilde{I}, S^p \mathcal{E}' \otimes \mathcal{L}'^{-s} \otimes H'^t \otimes \mathcal{K}) \cong H^0(\widetilde{I}, S^{p-s} \mathcal{E}' \otimes H'^{t-s} \otimes \mathcal{K}),$$

$$H^0(\widetilde{I}, S^{p-s} \mathcal{E}' \otimes H'^{t-s} \otimes \mathcal{K}) \cong S^{p-s} H^0(E)^* \otimes [S^{t-s} H^0(E(-\Delta)) \otimes H^0(L(-\Delta))^*]_0^*.$$

Here, the term $[S^{t-s} H^0(E(-\Delta)) \otimes H^0(L(-\Delta))^*]_0$ has the following meaning: the map d_0 provides a contraction map

$$H^0(E(-\Delta)) \otimes H^0(L(-\Delta))^* \to H^0(E(-\Delta))^*$$

and more generally a contraction map

$$c_i : S^i H^0(E(-\Delta)) \otimes H^0(L(-\Delta))^* \to S^{i-1} H^0(E(-\Delta)) \otimes H^0(E(-\Delta))^*.$$

Then we denote

$$[S^i H^0(E(-\Delta)) \otimes H^0(L(-\Delta))^*]_0 := Ker\, c_i.$$

Note that with the previous notations, we have $c' = c_2$, $c = c_1$.

Proof of Lemma 9. These facts are proved using the exact sequence on $\widetilde{G}'$:

$$0 \to \mathcal{L} \otimes H^{-1} \to \mathcal{E} \to H \to 0,$$

or, pulling-back via π:

$$0 \to \mathcal{L}' \otimes H'^{-1} \to \mathcal{E}' \to H' \to 0$$

on $\widetilde{I}$. Taking symmetric powers gives exact sequences

$$0 \to S^{l-1}\mathcal{E}' \otimes \mathcal{L}' \otimes H'^{-1} \to S^l\mathcal{E}' \to H'^l \to 0 \tag{3.46}$$

on $\widetilde{I}$. Take $l = p$ and tensor (3.46) with $\mathcal{L}'^{-s} \otimes H'^t \otimes \mathcal{K}$. Observing that if $s > 0$, $H^0(\widetilde{I}, \mathcal{L}'^{-s} \otimes H'^{p+t} \otimes \mathcal{K}) = 0$ because the restriction of this line bundle to the fibers of g' is $\mathcal{O}(-s)$ on a projective space of dimension > 0, we conclude that

$$H^0(\widetilde{I}, S^p\mathcal{E}' \otimes \mathcal{L}'^{-s} \otimes H'^t \otimes \mathcal{K}) \cong H^0(\widetilde{I}, S^{p-1}\mathcal{E}' \otimes \mathcal{L}'^{-s+1} \otimes H'^{t-1} \otimes \mathcal{K}),$$

which proves the first equality by iteration.

The second equality follows from the following observation : denoting by $P_{\widetilde{I}} \stackrel{(\alpha,\beta)}{\to} \widetilde{I} \times \mathbb{P}(H^0(E))$ the pull-back to $\widetilde{I}$ of the tautological $\mathbb{P}^1$-bundle P on G, (see diagram (3.45),) there is a natural map

$$P_{\widetilde{I}} \stackrel{(g'\circ\alpha,\beta)}{\longrightarrow} I \times \mathbb{P}(H^0(E))$$

which is immediately seen to be birational. Furthermore, we have $S^l\mathcal{E}' \cong R^0\alpha_*(\beta^*\mathcal{O}(l))$ on $\widetilde{I}$, so that

$$\begin{aligned} H^0(\widetilde{I}, H'^i \otimes S^j\mathcal{E}' \otimes \mathcal{K}) &= H^0(P_{\widetilde{I}}, \alpha^*(H'^i \otimes \mathcal{K}) \otimes \beta^*\mathcal{O}(j)) \\ &= H^0(P_{\widetilde{I}}, (g' \circ \alpha)^*(pr_2^*\mathcal{O}(i) \otimes \mathcal{K})) \otimes \beta^*\mathcal{O}(j)). \end{aligned}$$

Because the map $(g' \circ \alpha, \beta)$ is birational, this is also equal to

$$H^0(I \times \mathbb{P}(H^0(E)), pr_2^*\mathcal{O}(i) \otimes \mathcal{K} \boxtimes \mathcal{O}(j)) = H^0(I, pr_2^*\mathcal{O}(i) \otimes \mathcal{K}) \otimes S^jH^0(E)^*.$$

To conclude, it thus suffices to show the following equality:

$$H^0(I, pr_2^*\mathcal{O}(i) \otimes \mathcal{K}) = [S^iH^0(E(-\Delta)) \otimes H^0(L(-\Delta))^*]_0^*.$$

This last fact follows from the fact that by definition of I and $\mathcal{K}$, the vector bundle

$$\mathcal{Q} := R^0pr_{2*}\mathcal{K}$$

on $\mathbb{P}H^0(E(-\Delta))$ fits into the exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \otimes H^0(E(-\Delta)) \stackrel{d_0}{\to} H^0(L(-\Delta)) \otimes \mathcal{O} \to \mathcal{Q} \to 0. \tag{3.47}$$

It follows from this exact sequence, using the fact that $rk\, H^0(E(-\Delta)) = k+1 \geq 3$ and vanishing on $\mathbb{P}H^0(E(-\Delta))$, that

$$\begin{aligned} H^0(I, pr_2^*\mathcal{O}(i) \otimes \mathcal{K}) &= H^0(\mathbb{P}H^0(E(-\Delta)), \mathcal{O}(i) \otimes R^0pr_{2*}\mathcal{K}) \\ &= H^0(\mathbb{P}H^0(E(-\Delta)), \mathcal{O}(i) \otimes \mathcal{Q}) \end{aligned}$$

is equal to the cokernel of the map induced by d_0

$$S^{i-1}H^0(E(-\Delta))^* \otimes H^0(E(-\Delta)) \to S^iH^0(E(-\Delta))^* \otimes H^0(L(-\Delta)),$$

that is to $[S^iH^0(E(-\Delta)) \otimes H^0(L(-\Delta))^*]_0^*$.

■

This Lemma will provide in particular canonical identifications:

$$H^0(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-1}) = S^{k-3}H^0(E)^* \otimes H^0(L(-\Delta)), \tag{3.48}$$

$$\begin{aligned} H^0(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H') = \\ S^{k-2}H^0(E)^* \otimes [H^0(E(-\Delta)) \otimes H^0(L(-\Delta))^*]_0^* \\ = S^{k-2}H^0(E)^* \otimes (Ker\, c)^*, \end{aligned} \tag{3.49}$$

an inclusion

$$H^0(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}') \subset H^0(\widetilde{I}, S^{k-1}\mathcal{E}' \otimes \mathcal{K} \otimes H'^2) \tag{3.50}$$

and an identification:

$$\begin{aligned} H^0(\widetilde{I}, S^{k-1}\mathcal{E}' \otimes \mathcal{K} \otimes H'^2) = \\ S^{k-1}H^0(E)^* \otimes [S^2H^0(E(-\Delta)) \otimes H^0(L(-\Delta))^*]_0^* \\ = S^{k-1}H^0(E)^* \otimes (Ker\, c')^*. \end{aligned} \tag{3.51}$$

They are used as follows: The determinant map $d : H^0(E) \otimes H^0(E(-\Delta)) \to H^0(L)$ provides dually a linear system

$$H^0(L)^* =: W \subset H^0(G', \mathcal{L}) \subset H^0(\widetilde{I}, \mathcal{L}')$$

which has no base-point by Proposition 2. This provides an exact Koszul complex on $\widetilde{I}$:

$$0 \to \bigwedge^{2k+2} W \otimes \mathcal{L}'^{-2k-2} \to \ldots \to W \otimes \mathcal{L}'^{-1} \to \mathcal{O}_{\widetilde{I}} \to 0. \tag{3.52}$$

We twist by $S^{k-2}\mathcal{E}' \otimes \mathcal{L}'^{\otimes 2} \otimes H' \otimes \mathcal{K}$ and take global sections. The relevant piece of this complex of global sections is:

$$\begin{aligned} \ldots \to \bigwedge^3 W \otimes H^0(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-1}) \to \\ \bigwedge^2 W \otimes H^0(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H') \to W \otimes H^0(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}') \to \ldots \end{aligned} \tag{3.53}$$

Using the inclusion (3.50), this sequence has the same cohomology at the middle as the sequence:

$$\begin{aligned} \ldots \to \bigwedge^3 W \otimes H^0(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-1}) \to \\ \bigwedge^2 W \otimes H^0(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H') \to W \otimes H^0(\widetilde{I}, S^{k-1}\mathcal{E}' \otimes \mathcal{K} \otimes H'^2) \to \ldots \end{aligned} \tag{3.54}$$

Finally, using the identifications (3.48), (3.49), (3.51) above, we see that the three terms of this last sequence are canonically dual to the three terms of the sequence (3.44). We leave to the reader to verify that this last sequence is indeed dual to (3.44). Hence, the exactness at the middle of (3.44) is equivalent to the exactness at the middle of the Koszul sequence (3.53), and we claim that this is implied by the following statement :

Lemma 10 *For $1 \leq i \leq 2k-1$ we have the vanishing*

$$H^i(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-i-1}) = 0. \tag{3.55}$$

Indeed, the Koszul complex (3.52) twisted by $S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^2$ reads

$$0 \to \bigwedge^{2k+2} W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-2k} \to \ldots \to \bigwedge^{2} W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H'$$
$$\to W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}' \to \ldots.$$

In particular, this provides by truncation a resolution

$$0 \to \bigwedge^{2k+2} W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-2k} \to \ldots \to \bigwedge^{4} W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-2} \to \mathcal{M} \to 0,$$

of the sheaf $\mathcal{M}$ on $\widetilde{I}$ which fits in the exact sequence:

$$0 \to \mathcal{M} \to \bigwedge^{3} W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-1} \tag{3.56}$$
$$\to \bigwedge^{2} W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \to W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}' \ldots.$$

Now the exactness at the middle of the sequence (3.53) is implied by the vanishing $H^1(\widetilde{I}, \mathcal{M}) = 0$. By the above resolution, this space is isomorphic to the hypercohomology group

$$\mathbb{H}^1(\widetilde{I}, 0 \to \bigwedge^{2k+2} W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-2k} \to \ldots \tag{3.57}$$
$$\to \bigwedge^{4} W \otimes S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-2} \to 0),$$

where the last term on the right is put in degree 0. The terms $E_1^{p,q}$ of the spectral sequence associated to the naive filtration of this complex are equal to

$$\bigwedge^{4-p} W \otimes H^q(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{p-2})$$

in degree $p+q$, that is, for $p+q=1$, to

$$\bigwedge^{q+3} W \otimes H^q(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-q-1}),\ q \geq 1.$$

Hence the vanishings (3.55) say that the terms $E_1^{p,q}$, $p+q=1$ of this spectral sequence are 0 for $1 \leq q \leq 2k-1$ and they are also obviously 0 for $q > 2k-1$, since then $\bigwedge^{q+3} W = 0$. Thus (3.57) vanishes and so does $H^1(\widetilde{I}, \mathcal{M})$. ∎

Proof of Lemma 10. Consider the Cartesian diagram (see diagram (3.45):

$$\begin{array}{ccccc} & & \widetilde{I} & \xrightarrow{\pi} & \widetilde{G}' \\ & & \downarrow & & \downarrow g \\ \mathbb{P}H^0(L(-\Delta))^* & \xleftarrow[pr_1]{} & I & \xrightarrow[pr_2]{} & \mathbb{P}H^0(E(-\Delta)) \end{array} \tag{3.58}$$

We have

$$R^0\pi_*\mathcal{K} = g^*(R^0 pr_{2*}(pr_1^*\mathcal{O}(1))) = g^*\mathcal{Q},$$

where the bundle $\mathcal{Q}$ on $\mathbb{P}H^0(E(-\Delta))$ admits the resolution (see (3.47)):

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \otimes H^0(E(-\Delta)) \xrightarrow{d_0} H^0(L(-\Delta)) \otimes \mathcal{O} \to \mathcal{Q} \to 0.$$

Since $\mathcal{E}'$, $\mathcal{L}'$, H' are pull-backs via π of the corresponding objects on $\widetilde{G}'$, we have:

$$H^i(\widetilde{I}, S^{k-2}\mathcal{E}' \otimes \mathcal{K} \otimes H' \otimes \mathcal{L}'^{-i-1}) = H^i(\widetilde{G}', S^{k-2}\mathcal{E} \otimes H \otimes \mathcal{L}^{-i-1} \otimes g^*\mathcal{Q}).$$

The bundle $g^*\mathcal{Q}$ admits the resolution:

$$0 \to H^{-2} \to H^{-1} \otimes H^0(E(-\Delta)) \xrightarrow{d_0} H^0(L(-\Delta)) \otimes \mathcal{O}_{\widetilde{G}'} \to g^*\mathcal{Q} \to 0,$$

and it follows that the desired vanishing

$$H^i(\widetilde{G}', S^{k-2}\mathcal{E} \otimes H \otimes \mathcal{L}^{-i-1} \otimes g^*\mathcal{Q}) = 0, \; 1 \le i \le 2k-1$$

is a consequence of the following:

$$\begin{aligned} H^i(\widetilde{G}', S^{k-2}\mathcal{E} \otimes H \otimes \mathcal{L}^{-i-1}) = 0, \; 1 \le i \le 2k-1, \\ H^{i+1}(\widetilde{G}', S^{k-2}\mathcal{E} \otimes \mathcal{L}^{-i-1}) = 0, \; 1 \le i \le 2k-1, \\ H^{i+2}(\widetilde{G}', S^{k-2}\mathcal{E} \otimes H^{-1} \otimes \mathcal{L}^{-i-1}) = 0, \; 1 \le i \le 2k-2, \end{aligned} \tag{3.59}$$

and of the following fact:

Lemma 11 *The map*

$$H^{i+2}(\widetilde{G}', S^{k-2}\mathcal{E} \otimes H^{-1} \otimes \mathcal{L}^{-i-1}) \to H^0(E(-\Delta)) \otimes H^{i+2}(\widetilde{G}', S^{k-2}\mathcal{E} \otimes \mathcal{L}^{-i-1})$$

induced by the natural inclusion

$$H^{-1} \subset H^0(E(-\Delta)) \otimes \mathcal{O}_{\widetilde{G}'}$$

is injective for $i = 2k-1$.

Let us first prove (3.59): We use the fact that we can see $\widetilde{G}'$ as the complete intersection of two members of $\mid H \mid$ on the tautological $\mathbb{P}^1$-bundle P on G. Indeed, let $P \subset G \times \mathbb{P}H^0(E)$ be the tautological subbundle, and denote by $p : P \to G$ the first projection, $q : P \to \mathbb{P}H^0(E)$ the second projection (see diagram (3.45)). Denote by H the line bundle $q^*\mathcal{O}(1)$ on P, and by $\mathcal{E}$, $\mathcal{L}$ the pull-backs via p of the corresponding bundles on G. Then by definition, $\widetilde{G}'$ identifies to $q^{-1}(\mathbb{P}H^0(E(-\Delta)))$, and the bundles H, $\mathcal{E}$, $\mathcal{L}$ are the restrictions to $\widetilde{G}'$ of the corresponding objects on P.

Hence there is a Koszul resolution of $\mathcal{O}_{\widetilde{G}'}$ which has the form:

$$0 \to \bigwedge^2 R \otimes H^{-2} \to R \otimes H^{-1} \to \mathcal{O}_P \to \mathcal{O}_{\widetilde{G}'} \to 0,$$

where R is a rank 2 vector space.

Using this resolution, we see that the vanishing statements (3.59) are a consequence of the following ones:

1. $H^i(P, S^{k-2}\mathcal{E} \otimes H \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i \le 2k-1,$
2. $H^{i+1}(P, S^{k-2}\mathcal{E} \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i \le 2k-1,$
3. $H^{i+2}(P, S^{k-2}\mathcal{E} \otimes H^{-1} \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i \le 2k-1,$
4. $H^{i+3}(P, S^{k-2}\mathcal{E} \otimes H^{-2} \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i \le 2k-1,$
5. $H^{i+4}(P, S^{k-2}\mathcal{E} \otimes H^{-3} \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i \le 2k-2.$

Recall now the following statement proven in the Appendix of [12]:

Proposition 10 *For $q > 0$, $q' \ge 0$, $H^p(G, \mathcal{L}^{-q} \otimes S^{q'}\mathcal{E}) = 0$ if $p \ne k+1$, $2k+2$.*
For $p = k+1$, $H^p(G, \mathcal{L}^{-q} \otimes S^{q'}\mathcal{E}) = 0$ if $-q + q' + 1 < 0$ or $q \le k+1$.

(Note the shift of notation from k there to $k+1$ here, which is due to the fact that we are now working with a space $H^0(E)$ of rank $k+3$ instead of $k+2$.)

The vanishing 2 follows directly from this Proposition. The vanishing 3 follows from the fact that H^{-1} has trivial cohomology along the fibers of $p : P \to G$. For the vanishing 1, we use the exact sequence on P:

$$0 \to \mathcal{L} \otimes H^{-1} \to \mathcal{E} \to H \to 0.$$

It provides the exact sequence:

$$0 \to \mathcal{L} \otimes H^{-1} \otimes S^{k-3}\mathcal{E} \to S^{k-2}\mathcal{E} \to H^{k-2} \to 0.$$

Hence we see that 1 is implied by the vanishings:

$$H^i(P, S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-i}) = 0,\ 1 \le i \le 2k-1,$$

$$H^i(P, H^{k-1} \otimes \mathcal{L}^{-i-1}) = H^i(G, S^{k-1}\mathcal{E} \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i \le 2k-1,$$

and they are both consequences of Proposition 10.

For the vanishing 4, one notes that $K_{P/G}$ is equal to $H^{-2}\otimes\mathcal{L}$. Hence we have

$$H^{i+3}(P,S^{k-2}\mathcal{E}\otimes\mathcal{L}^{-i-1}\otimes H^{-2})=H^{i+2}(G,S^{k-2}\mathcal{E}\otimes\mathcal{L}^{-i-1}\otimes R^1p_*H^{-2})$$
$$=H^{i+2}(G,S^{k-2}\mathcal{E}\otimes\mathcal{L}^{-i-2}).$$

This vanishes for $1\leq i\leq 2k-1$ by Proposition 10.

To conclude, 5 is proved as follows: We have as above:

$$H^{i+4}(P,S^{k-2}\mathcal{E}\otimes\mathcal{L}^{-i-1}\otimes H^{-3})=H^{i+3}(G,S^{k-2}\mathcal{E}\otimes\mathcal{L}^{-i-1}\otimes R^1p_*H^{-3})$$
$$=H^{i+3}(G,S^{k-2}\mathcal{E}\otimes\mathcal{L}^{-i-2}\otimes\mathcal{E}^*),$$

using relative Serre duality and $R^0p_*H=\mathcal{E}$ on G. Since $\mathcal{E}^*=\mathcal{E}\otimes\mathcal{L}^{-1}$, the last term is equal to

$$H^{i+3}(G,S^{k-2}\mathcal{E}\otimes\mathcal{E}\otimes\mathcal{L}^{-i-3}).$$

By the exact sequence

$$0\to\mathcal{L}\otimes S^{k-3}\mathcal{E}\to S^{k-2}\mathcal{E}\otimes\mathcal{E}\to S^{k-1}\mathcal{E}\to 0,$$

we see that the vanishing $H^{i+3}(G,S^{k-2}\mathcal{E}\otimes\mathcal{E}\otimes\mathcal{L}^{-i-3})=0$ for $1\leq i\leq 2k-2$ is a consequence of the vanishings

$$H^{i+3}(G,S^{k-3}\mathcal{E}\otimes\mathcal{L}^{-i-2})=0,$$
$$H^{i+3}(G,S^{k-1}\mathcal{E}\otimes\mathcal{L}^{-i-3})=0,$$

for $1\leq i\leq 2k-2$, which both follow from Proposition 10.

This concludes the proof of (3.59) and the proof of Lemma 10 will then be concluded with the proof of Lemma 11. ∎

Proof of Lemma 11.

Since the map

$$H^{2k+1}(\widetilde{G}',S^{k-2}\mathcal{E}\otimes H^{-1}\otimes\mathcal{L}^{-2k})\to H^0(E(-\Delta))\otimes H^{2k+1}(\widetilde{G}',S^{k-2}\mathcal{E}\otimes\mathcal{L}^{-2k})$$

is induced by the inclusion

$$H^{-1}\subset H^0(E(-\Delta))\otimes\mathcal{O}_{\widetilde{G}'},$$

which is dual to the evaluation map, where $H^0(E(-\Delta))^*$ is identified to $H^0(\widetilde{G}',H)$, we see that its dual is equal to the multiplication map:

$$H^0(E(-\Delta))^*\otimes H^0(\widetilde{G}',K_{\widetilde{G}'}\otimes S^{k-2}\mathcal{E}^*\otimes\mathcal{L}^{2k})\to H^0(\widetilde{G}',K_{\widetilde{G}'}\otimes H\otimes S^{k-2}\mathcal{E}^*\otimes\mathcal{L}^{2k}).$$

The canonical bundle of $\widetilde{G}'$ is equal to $\mathcal{L}^{-k-2}$, because $\widetilde{G}'$ is the complete intersection of two members of $\mid H\mid$ in P and $K_P=\mathcal{L}^{-k-2}\otimes H^{-2}$.

Thus we have to prove that the multiplication map

$$H^0(E(-\Delta))^* \otimes H^0(\widetilde{G}', S^{k-2}\mathcal{E}^* \otimes \mathcal{L}^{k-2}) \to H^0(\widetilde{G}', H \otimes S^{k-2}\mathcal{E}^* \otimes \mathcal{L}^{k-2})$$

is surjective.

Since $\mathcal{E}^* \otimes \mathcal{L} \cong \mathcal{E}$, this is equivalent to the surjectivity of the multiplication map

$$H^0(E(-\Delta))^* \otimes H^0(\widetilde{G}', S^{k-2}\mathcal{E}) \to H^0(\widetilde{G}', H \otimes S^{k-2}\mathcal{E}).$$

This follows from the surjectivity of the multiplication map

$$H^0(P, H) \otimes H^0(P, S^{k-2}\mathcal{E}) \to H^0(P, H \otimes S^{k-2}\mathcal{E}),$$

and of the restriction map

$$H^0(P, H \otimes S^{k-2}\mathcal{E}) \to H^0(\widetilde{G}', H \otimes S^{k-2}\mathcal{E}).$$

∎

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