Triangle varieties and surface decomposition of hyper-Kähler manifolds

Claire Voisin

Pour Miles Reid, avec estime et amitié

Abstract

We introduce and study the notion of "surface decomposable" variety, and discuss the possibility that any projective hyper-Kähler manifold is surface decomposable, which would produce new evidence for Beauville's weak splitting conjecture. We show that surface decomposability relates to the Beauville-Fujiki relation, a constraint on the cohomology ring of the variety, and that general varieties with $h^{2,0} \neq 0$ are not surface decomposable. We also formalize the notion of triangle variety that is useful to produce surface decomposition. We show the existence of these geometric structures on most explicitly constructed classes of projective hyper-Kähler manifolds of Picard number 1.

0 Introduction

Let X be a complex manifold of dimension $2n$ equipped with a holomorphic 2-form σ_X which is everywhere of maximal rank $2n$. Locally for the Euclidean topology on X, Darboux's theorem tells that one can write, for an adequate choice of holomorphic coordinates,

$$\sigma_X = \sum_{i=1}^{n} dz_i \wedge dz_{n+i},$$

that is, σ_X is the sum of n closed holomorphic 2-forms of rank 2.

A natural question is whether this statement can be made more global, particularly in the projective case:

Question 0.1. *Does there exist a generically finite cover $\phi : Y \to X$, such that $\phi^*\sigma_X$ is the sum of n closed holomorphic 2-forms of rank 2 on Y?*

Our goal in this paper is to study a geometric variant of this question, namely the possibility that any projective hyper-Kähler manifold is *surface decomposable* (or admits a *surface decomposition*) in the following sense:

Definition 0.2. *A smooth projective variety X of dimension $2n$ will be said to be surface decomposable if there exist a smooth variety Γ, smooth projective surfaces $S_1, \ldots, S_n$, and generically finite surjective morphisms $\phi : \Gamma \to X$, $\psi : \Gamma \to S_1 \times \ldots \times S_n$ such that for any holomorphic 2-form $\sigma \in H^0(X, \Omega_X^2)$,*

$$\phi^*\sigma = \psi^*(\sum_i \mathrm{pr}_i^*\sigma_i) \tag{1}$$

for some holomorphic 2-forms σ_i on S_i.

We will show that surface decomposability is restrictive for general projective varieties of dimension $2n \geq 4$ (see Theorem 1.6). The reason is that it essentially implies the Beauville-Fujiki formulas describing the top self-intersection on cohomology of degree 2 as the power of a quadratic form, or at least a product of quadratic forms (see Proposition 1.5).

Remark 0.3. Surface decompositions are natural in the hyper-Kähler context which is the one of this paper, but one can of course introduce in general decompositions into summands of other dimensions. For example, we can consider *curve decompositions* of any variety X of dimension n, given by the data of generically finite surjective morphisms

$$\phi : \Gamma \to X,\ \ \psi : \Gamma \to C_1 \times \ldots \times C_n, \tag{2}$$

such that for any 1-form $\alpha \in H^{1,0}(X)$,

$$\phi^*\alpha = \psi^*(\sum_i \mathrm{pr}_i^*\alpha_i) \tag{3}$$

for some forms $\alpha_i \in H^{1,0}(C_i)$. Abelian varieties are clearly curve decomposable. One can show similarly that this notion puts strong restrictions on the structure of the intersection pairing $\bigwedge^{2n} H^1(X, \mathbb{C}) \to H^{2n}(X, \mathbb{C}) = \mathbb{C}$, when $h^{1,0}(X)$ is large compared to n.

The first examples of hyper-Kähler manifolds were constructed by Beauville [4] and Fujiki [14] as punctual Hilbert schemes of $K3$ surfaces or abelian surfaces and hence were rationally dominated by products of surfaces. They were thus obviously surface decomposable. However, it follows from deformation theory that these $K3$ or abelian surfaces disappear under a general deformation to a projective hyper-Kähler manifold with Picard number 1. Indeed, the parameter space for $K3$ surfaces is too small to parameterize also general deformations with Picard number 1 of their punctual Hilbert schemes. The starting point of this paper is the observation that on many explicitly described general deformations as above, a surface decomposition still exists.

Let us make several remarks concerning Definition 0.2. First of all, the condition (1) has been asked only for holomorphic 2-forms, but by an elementary argument involving Hodge structures (see Section 1), it then follows that it is satisfied for any transcendental class $\eta \in H^2(X, \mathbb{Q})_{tr}$, the latter space being defined as the smallest Hodge substructure of $H^2(X, \mathbb{Q})$ whose complexification contains $H^{2,0}(X)$.

Next, if we allow an arbitrarily large number N of summands S_i and only ask that ϕ is surjective and ψ is generically finite, then the definition is (at least conjecturally) not restrictive since the property is satisfied by any smooth projective variety X satisfying the Lefschetz standard conjecture in degree 2 (see Proposition 1.2). Similarly, we could consider, instead of (1), the weaker condition that ϕ is surjective generically finite and

$$\sigma = \phi_*(\psi^*(\sum_i \mathrm{pr}_i^*\sigma_i)) \text{ in } H^{2,0}(X), \tag{4}$$

but, as before, this is implied by Lefschetz standard conjecture, and we even can take $N = 1$ to achieve (4), as shows the case of the $(2, 0)$-forms on the symmetric product or rather punctual Hilbert scheme $X = S^{[k]}$ of a simply connected surface S: they are all obtained starting from a $(2, 0)$-form on S by a formula like (4).

The reason why (4) is much weaker than (1) is the fact that pull-back maps are compatible with cup-products, while push-forward maps are not. More precisely, we will show (see Proposition 1.5) that the surface decomposability implies and provides a geometric explanation for Beauville-Fujiki's famous formula for the self-intersection of degree 2 cohomology on a hyper-Kähler manifold:

$$\int_X \alpha^{2n} = \lambda q(\alpha)^n,$$

where q is the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{Q})_{tr}$ (see [4]). To prove this implication, we have to assume that the Mumford-Tate group of the Hodge structure on $H^2(X, \mathbb{Q})_{tr}$ is large enough to guarantee that all the quadratic forms on $H^2(X, \mathbb{Q})_{tr}$ induced from $(\ ,\ )_{S_i}$ via the morphism of Hodge structures $\psi_i : \eta \mapsto \eta_i$ (see Section 1) are proportional, but this is automatic if X is the general member of a family of polarized hyper-Kähler manifolds.

This observation suggests that surface decomposability could be a way to approach the weak splitting property of hyper-Kähler manifolds (see Conjecture 1.7) conjectured by Beauville in [6]. It says that cohomological polynomial relations between divisors on hyper-Kähler manifolds X are satisfied on the Chow level. A weaker version asks that X has a canonical 0-cycle $o_X \in \mathrm{CH}_0(X)$ such that D^{2n} is proportional to o_X in $\mathrm{CH}_0(X)$ for any divisor D on X.

In this direction, we prove Theorem 1.8 which has the following consequence:

Theorem 0.4. *Assume that the general member X_t of a family $(X_t)_{t\in B}$ of hyper-Kähler manifolds with given Picard lattice Λ has a surface decomposition with a simply connected Γ. Then the Beauville weak splitting property holds for the divisor classes belonging to $\mathrm{NS}(X_t)^{\perp\Lambda}$ if and only if there exists a 0-cycle $o_{X_t} \in \mathrm{CH}_0(X_t)$ such that D^{2n} is proportional to o_{X_t} in $\mathrm{CH}_0(X_t)$ for any divisor $D \in \mathrm{NS}(X_t)^{\perp\Lambda}$.*

The importance of Theorem 0.4 is that it reduces the weak splitting property to checking the weak version, namely in top degree. The weakness of the result is that it applies only to divisors of class perpendicular to Λ, which means, in practice, primitive. One has to understand separately what happens with the powers h^k of the polarizing class. In all the geometric examples we have, the natural surface decomposition that we exhibit provides (1) only on primitive cohomology.

Remark 0.5. This raises the question whether hyper-Kähler manifolds could admit a surface decomposition such that formula (1) holds on the whole cohomology H^2, instead of only primitive or transcendental cohomology. This question is very interesting in the case of the Hilbert scheme $S^{[n]}$ of a $K3$ surface S, where the natural surface decomposition provides (1) only on the part of H^2 which is orthogonal to the class of the exceptional divisor over the diagonal. Combined with the theorem above, this could be a way to reproving the weak splitting conjecture for $S^{[n]}$ (proved by Maulik and Negut [22]), as the weak version is known for them by [30].

Remark 0.6. The statement above is empty for the *very general* member of the family since it has $\mathrm{NS}(X_t)^{\perp\Lambda} = 0$. The statement above is interesting for special hyper-Kähler manifolds with higher Picard rank $\rho \geq \rho_{gen} + 2$, which are parameterized by a countable union of closed algebraic subsets in the base B, which is dense in B if $\dim B \geq 2$.

The second geometric notion that will play an important role in this paper is the following.

Definition 0.7. *[29] An algebraically coisotropic subvariety of a hyper-Kähler manifold X of dimension $2n$ is a subvariety $Z \subset X$ of codimension $k \leq n$ which admits a rational fibration $\phi : Z \dashrightarrow W$, where W is smooth and $\dim W = 2n - 2k$, such that*

$$\sigma_{X|Z_{\mathrm{reg}}} = \phi_{\mathrm{reg}}^*\sigma_W, \tag{5}$$

where $\phi_{\mathrm{reg}} : Z_{\mathrm{reg}} \dashrightarrow W$ is the restriction of ϕ to the regular locus of Z and σ_W is a holomorphic 2-form on W.

This notion is to be distinguished from the notion of coisotropic subvariety, which just asks that the restriction $\sigma_{X|Z_{\mathrm{reg}}}$ has rank $2n - 2k$ at any point, or equivalently that

$$T_{Z_{\mathrm{reg}},x}^{\perp\sigma_X} \subset T_{Z_{\mathrm{reg}},x} \tag{6}$$

at any point x of Z_{reg}. Equation (6) defines then a foliation on Z_{reg} and Z is algebraically coisotropic when the leaves of this foliation are algebraic. The two notions coincide in the case $n = k$ of Lagrangian varieties. For $k = 1$, any divisor is coisotropic but smooth ample divisors are not algebraically isotropic (see [2]). It is not easy a priori to construct algebraically coisotropic divisors in a projective hyper-Kähler manifolds. Examples are given by uniruled (singular) divisors. Indeed, starting from a singular uniruled divisor

$D \subset X$, consider a desingularization $\widetilde{D} \to D$ with induced morphism $j : \widetilde{D} \to X$. Then by assumption $\widetilde{D}$ is uniruled, hence its maximal rationally connected fibration (or MRC fibration, see [18]) is nontrivial, producing, possibly after changing the birational model of $\widetilde{D}$, a morphism

$$f : \widetilde{D} \to B$$

with rationally connected fibers of positive dimension. Any holomorphic form on $\widetilde{D}$ is pulled-back via f from a holomorphic form on B. We apply this to the pull-back $j^*\sigma_X \in H^{2,0}(\widetilde{D})$ and conclude that

$$j^*\sigma_X = f^*\sigma_B \text{ in } H^{2,0}(\widetilde{D}). \tag{7}$$

Formula (7) implies that the fibers of f are tangent to the kernel of $j^*\sigma_X$. As the generic rank of $j^*\sigma_X$ is $2n-1$, $n = \dim X$, we conclude that the fibers of f are 1-dimensional (so in fact D is ruled), and D is algebraically coisotropic. Unfortunately, the uniruled divisors in a hyper-Kähler manifold X are rigid, because rational curves cannot cover X (indeed, the MRC fibration of X is trivial as shows the argument above). One open question is whether a projective hyper-Kähler manifold can always be swept out by algebraically coisotropic divisor. This is certainly true if X has a surface decomposition (see Proposition 0.11 below). In the other direction, we show in Proposition 1.10 that the existence of a 1-parameter family of algebraically coisotropic divisors for X implies a decomposition, on a generically finite cover of X, of the holomorphic 2-form of X as a sum of one rank 2 and one rank $2n-2$ holomorphic forms.

The theory of coisotropic subvarieties of higher codimension is more complicated. In the paper [29], we discussed the constraints on the cohomology classes of coisotropic subvarieties of higher codimension and asked whether the space of coisotropic classes, namely those satisfying these constraints, are generated by classes of algebraically coisotropic subvarieties. We also proposed the construction of algebraically coisotropic subvarieties as total spaces of $2n-2k$-dimensional families of constant cycles varieties (in the sense of Huybrechts [16]) of dimension k.

The third notion that will be introduced and studied in this paper is that of *triangle variety*:

Definition 0.8. *A triangle variety for X (equipped with a holomorphic 2-form σ_X) is a subvariety of $X \times X \times X$ which dominates X by the three projections, maps in a generically finite way to its image in $X \times X$ via the three projections and is Lagrangian for the holomorphic form $\sigma_1 + \sigma_2 + \sigma_3$ on X^3, where $\sigma_i = \mathrm{pr}_i^*\sigma_X$.*

The following example will be generalized in Section 2.5.

Example 0.9. Let $S \to B$ be an elliptic surface with a section. Then the graph of minus the relative sum map $S \times_B S \dashrightarrow S$, which is naturally contained in S^3, is a triangle variety for any holomorphic 2-form σ on S.

Triangle varieties seem to exist for most explicitly constructed classes of projective hyper-Kähler manifolds. In fact, the simplest example of them, namely actual triangles in the Fano variety $F_1(Y)$ of lines of a smooth cubic fourfold Y, is studied with detail in [25] by Shen and Vial, who use them to study a decomposition (Beauville splitting) of the Chow groups of $F_1(Y)$. The main geometric examples, including this one, will be presented in Section 2.

Remark 0.10. More generally, we can also introduce (and we will use) $k+1$-angle subvarieties $T_{k+1} \subset X^{k+1}$ which are Lagrangian for the 2-form $\sum_i \pm \mathrm{pr}_i^*\sigma_X$. They are easily constructed by iteration starting from a triangle variety.

The first link between triangle varieties, surface decompositions and algebraically coisotropic subvarieties is the following obvious implication:

Proposition 0.11. *If X has a surface decomposition, then it has mobile algebraically coisotropic subvarieties of any codimension $k \leq n$. If the surfaces appearing in a surface decomposition of X have triangle varieties, (for example, if they are elliptic,) then so does X.*

Proof. Indeed let $\phi : \Gamma \to X$, $\psi : \Gamma \to S_1 \times \ldots \times S_n$ be surjective generically finite maps such that

$$\phi^*\sigma_X = \psi^*(\mathrm{pr}_1^*\sigma_{S_1} + \ldots + \mathrm{pr}_n^*\sigma_{S_n}) \text{ in } H^{2,0}(\Gamma).$$

For any integer $k \leq n$, let $C_i \subset S_i$, $i = 1, \ldots, k$, be very ample curves in general position. Then $\phi(\psi^{-1}(C_1 \times \ldots C_k \times S_{k+1} \times \ldots \times S_n))$ is an algebraically coisotropic subvariety of X of codimension k. It $T_i \subset S_i^3$ are triangle varieties, then $\phi^3(T_1 \times \ldots \times T_n)$ is a triangle variety for X. $\square$

We will show in the paper how conversely triangle (or $n+1$-angle) subvarieties for X and algebraically coisotropic subvarieties of X of codimension $n-1$, where dim $= 2n$, can be used to construct surface decompositions and algebraically coisotropic varieties of X of any codimension $1 \leq k \leq n$. We prove the following result (see Theorem 3.1).

Theorem 0.12. *Let X be a projective hyper-Kähler variety of dimension $2n$. Assume X has a $n+1$-angle variety (see Remark 0.10) $T_{n+1} \subset X^{n+1}$ and an algebraically coisotropic subvariety $\tau : Z \dashrightarrow \Sigma$ of dimension $n+1$. Let $F \subset X$ be the general fiber of τ. Then if the intersection number $F^n \cdot p_{1\ldots n}(T_{n+1})$ of cycles in X^n is nonzero, X admits a surface decomposition.*

Finally we will prove (see Theorem 3.3), as an application of Theorem 0.12 or variants of it, that most hyper-Kähler manifolds that have been explicitly constructed from algebraic geometry admit surface decompositions.

1 The decomposition problem for hyper-Kähler varieties

Let X be a smooth projective manifold of dimension $2n$ (we will later on focus on the hyper-Kähler case). We wish to study the existence and consequences of a surface decomposition of the form described in the introduction (see Definition 0.2), namely the existence of smooth projective surfaces S_i, $i = 1, \ldots, n$, and an effective correspondence Γ (which can be assumed to be smooth and projective)

$$\phi : \Gamma \to X, \quad \psi : \Gamma \to S_1 \times \ldots \times S_n,$$

with ϕ and ψ dominant generically finite, such that for any $\sigma_X \in H^{2,0}(X)$,

$$\phi^*\sigma_X = \psi^*(\sum_{i=1}^{n} \mathrm{pr}_i^*\sigma_{S_i}) \text{ in } H^{2,0}(\Gamma), \tag{8}$$

for some $\sigma_{S_i} \in H^{2,0}(S_i)$, where the $\mathrm{pr}_i : S_1 \times \ldots \times S_n \to S_i$ are the various projections. Let us spell-out the proof of the following

Lemma 1.1. *Condition (8) implies more generally that, for any $\eta \in H^2(X, \mathbb{Q})_{tr}$,*

$$\phi^*\eta = \psi^*(\mathrm{pr}_1^*\eta_1 + \ldots + \mathrm{pr}_n^*\eta_n) \text{ in } H^2(\Gamma, \mathbb{Q}), \tag{9}$$

for some $\eta_i \in H^2(S_i, \mathbb{Q})_{tr}$.

Proof. Indeed, the form $\sigma_{S_i} \in H^{2,0}(S_i)$ in (8) can be reconstructed from σ_X by the action of the morphism of Hodge structures

$$\psi_i : H^2(X, \mathbb{Q})_{tr} \to H^2(S_i, \mathbb{Q}), \ \ \eta \mapsto \eta_i$$

$$\psi_i(\eta) = \frac{1}{NN_i}\mathrm{pr}_{i*}(d^{2n-2} \cup \psi_*(\phi^*\eta)), \tag{10}$$

where N is the degree of ψ and $d = \sum_i \mathrm{pr}_i^* d_i$ is the first Chern class of an ample divisor on $\prod_i S_i$ with the property that $\mathrm{pr}_{i*}(d^{2n-2}) = N_i 1_{S_i}$ in $H^0(S_i, \mathbb{Q})$ for all i. The last condition indeed guarantees that

$$\frac{1}{NN_i}\mathrm{pr}_{i*}(d^{2n-2} \cdot \psi_*\psi^*(\sum_j \mathrm{pr}_j^*\eta_j)) = \eta_i$$

for any cohomology classes η_j on S_j such that $\eta_j \cup d_j = 0$ for all j. The morphisms of Hodge structures ψ_i being defined as in (10), condition (8) then rewrites as

$$\phi^*\sigma_X = \psi^*(\sum_i \mathrm{pr}_i^*(\psi_i(\sigma_X))) \text{ in } H^{2,0}(\Gamma).$$

This equality defines a Hodge substructure of $H^2(X, \mathbb{Q})$. Hence, once it is satisfied on $H^{2,0}(X)$, it is satisfied on $H^2(X, \mathbb{Q})_{tr}$. $\square$

A variant of this definition assumes that $S_1 = \ldots = S_n$ and the correspondence Γ is symmetric with respect to the symmetric group action on S^n, but this is not essential. Another variant asks that condition (9) is satisfied for any $\eta \in H^2(X, \mathbb{Q})_{prim}$, where the subscript "prim" refers to the choice of an ample line bundle L on X, and primitive cohomology is primitive with respect to $l = c_1(L)$. If we work with very general hyper-Kähler manifolds of Picard number 1, the two notions coincide. In the hyper-Kähler case, equation (8) decomposes the smooth projective manifold X in the sense that the rank $2n$ holomorphic 2-form on X gets decomposed as the sum of n (generically) rank 2 holomorphic 2-forms $\psi^*(\mathrm{pr}_i^*\sigma_{S_i})$ on the generically finite cover Γ.

If we now relax the conditions on ϕ, ψ in Definition 0.2 and just ask that ϕ is surjective and $\phi^*\sigma_X = \psi^*(\sum_{i=1}^N \mathrm{pr}_i^*\sigma_{S_i})$ for any holomorphic 2-form on X, allowing an arbitrarily large number of summands, then a decomposition as in (8) should always exist, for any smooth projective variety X. More precisely:

Proposition 1.2. *Let X be a smooth projective variety. Assume X satisfies the Lefschetz standard conjecture for degree 2 cohomology. Then there is a generically finite cover $\phi : \Gamma \to X$, surfaces $S_1, \ldots, S_N$, and a morphism $\psi : \Gamma \to S_1 \times \ldots \times S_N$, such that any $(2,0)$-form σ on X satisfies*

$$\phi^*\sigma = \psi^*(\sum_i \mathrm{pr}_i^*\sigma_i) \text{ in } H^{2,0}(\Gamma) \tag{11}$$

for some $(2,0)$-forms σ_i on S_i.

Remark 1.3. The proof will even show that we can take $S_1 = \ldots = S_N$ and Γ symmetric.

Proof of Proposition 1.2. The Lefschetz standard conjecture for degree 2 cohomology on X provides a codimension 2-cycle Z on $X \times X$ such that, if $n = \dim X$, $Z^* : H^{2n-2}(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is the inverse of the Lefschetz isomorphism $l^{n-2} : H^2(X, \mathbb{Q}) \cong H^{2n-2}(X, \mathbb{Q})$ induced by the first Chern class l of a very ample line bundle L on X. Let $j : S \to X$ be a smooth surface which is the complete intersection of $n-2$ ample hypersurfaces in $|L|$. Then by the Lefschetz theorem on hyperplane sections, the Gysin morphism $j_* : H^2(S, \mathbb{Q}) \to H^{2n-2}(X, \mathbb{Q})$ is surjective, so that, denoting by Z_S the restriction of Z to $X \times S$, we find that

$$Z_S^* = Z^* \circ j_* : H^2(S, \mathbb{Q}) \to H^2(X, \mathbb{Q})$$

is also surjective. We can make Z_S is effective by replacing if necessary its negative components $-Z_{S,i}$ by effective residual cycles Z'_i of class $H^2 - Z_{S,i}$, where $H = \mathrm{pr}_1^*H_1 + \mathrm{pr}_2^*H_2$ is a sufficiently ample line bundle on $X \times S$. The cycle H^2 acts trivially on transcendental cohomology, so this change does not affect $Z_S^*: H^2(S,\mathbb{Q})_{tr} \to H^2(X,\mathbb{Q})_{tr}$. Because Z_S is effective, it is given by a rational map

$$\phi_{Z_S}: X \dashrightarrow S^{(N)},$$

so that

$$Z_S^*\sigma_S = \phi_{Z_S}^*\sigma_{S^{(N)}} \text{ in } H^{2,0}(X),$$

for any holomorphic 2-form σ_S on S, where $\sigma_{S^{(N)}}$ denotes the induced 2-form on $S^{(N)}$. Recall that, denoting by $\mu: S^N \to S^{(N)}$ the quotient map,

$$\mu^*\sigma_{S^{(N)}} = \sum_{i=1}^{N} \mathrm{pr}_i^*\sigma_S \text{ in } H^{2,0}(S^N). \tag{12}$$

The finite cover μ induces a finite cover $\phi: \Gamma := X \times_{S^{(N)}} S^N \to X$, $\psi: \Gamma \to S^N$ and we have a commutative diagram

$$\begin{tikzcd} \Gamma \arrow[r, "\psi"] \arrow[d, "\phi"] & S^N \arrow[d, "\mu"] \\ X \arrow[r, "\phi_{Z_S}"] & S^{(N)} \end{tikzcd} \tag{13}$$

From the commutativity of (13), we deduce the equality of holomorphic 2- forms on Γ

$$\phi^*(\phi_{Z_S}^*\sigma_{S^{(N)}}) = \psi^*(\mu^*\sigma_{S^{(N)}}). \tag{14}$$

Combining (12) and (14), we get (11) with $\sigma_X = Z_S^*\sigma_S$, $\sigma_i = \sigma_S$ for all i. $\square$

Remark 1.4. To be fully rigorous in the above proof, we should introduce desingularizations of $S^{(n)}$ and Γ to write the equalities above. This is done in [23].

1.1 Surface decomposition and cohomology ring

It is a well-known and fundamental result (see [8], [14]) that for a hyper-Kähler manifold X of dimension $2n$, there exist a quadratic form q on $H^2(X,\mathbb{Q})$ and a positive rational number λ such that for any $\eta \in H^2(X,\mathbb{Q})$

$$\int_X \eta^{2n} = \lambda q(\eta)^n. \tag{15}$$

Let us show how this property follows, at least on transcendental cohomology, from the existence of a surface decomposition for a smooth projective variety X, assuming it satisfies the following property (*). Recall first that a quadratic form q on a rational weight 2 Hodge structure, consisting of a $\mathbb{Q}$-vector space H and a Hodge decomposition of $H_{\mathbb{C}} := H \otimes \mathbb{C}$

$$H_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},\ \overline{H^{p,q}} = H^{q,p},$$

is said to satisfy the first Hodge-Riemann relations if the Hodge decomposition is orthogonal for the Hermitian pairing $h(\alpha,\beta) = q(\alpha,\overline{\beta})$ on $H_{\mathbb{C}}$ or, equivalently, $q(H^{2,0}, H^{2,0} \oplus H^{1,1}) = 0$. We will say that it satisfies the weak second Hodge-Riemann relations if $q(\alpha,\overline{\alpha}) \geq 0$ for $\alpha \in H^{2,0}$ and $q(\alpha,\overline{\alpha}) \leq 0$ for $\alpha \in H^{1,1}$. Consider the condition

(*) *There exists up to a coefficient a unique quadratic form q satisfying the first Hodge-Riemann relations on $H^2(X,\mathbb{Q})_{tr}$.*

Property (*) is well-known to be satisfied by a very general lattice polarized projective hyper-Kähler manifold. Note that we need in any case to use transcendental cohomology, namely $H^2(X,\mathbb{Q})^{\perp NS(X)}$, instead of primitive cohomology, as (*) is never satisfied on $H^2(X,\mathbb{Q})_{prim}$ if it is different from $H^2(X,\mathbb{Q})_{tr}$, that is, if it contains rational classes of type $(1,1)$. We have the following.

Proposition 1.5. *(i) If a smooth projective variety X of dimension $2n$ admits a surface decomposition as in Definition 0.2, there exist quadratic forms $q_1, \ldots, q_n$ satisfying the first and weak second Hodge-Riemann relations on $H^2(X, \mathbb{Q})_{tr}$, such that, for any $\eta \in H^2(X, \mathbb{Q})_{tr}$*

$$\int_X \eta^{2n} = q_1(\eta) \ldots q_n(\eta). \tag{16}$$

(ii) If furthermore X satisfies property (*), there exists a rational number λ and a quadratic form q satisfying the first and weak second Hodge-Riemann relations on $H^2(X, \mathbb{Q})_{tr}$, such that, for any $\eta \in H^2(X, \mathbb{Q})_{tr}$,

$$\int_X \eta^{2n} = \lambda q(\eta)^n. \tag{17}$$

Proof. We have by assumption, for any $\eta \in H^2(X, \mathbb{Q})_{tr}$, an equality

$$\phi^*\eta = \psi^*(\sum_{i=1}^n \mathrm{pr}_i^*\eta_i), \tag{18}$$

where ψ, ϕ are as in (8). For each surface S_i, we have the Poincaré pairing $(\ ,\ )_{S_i}$ on $H^2(S_i, \mathbb{Q})_{tr}$ which satisfies the first and second Hodge-Riemann relations, and, as the morphism ψ_i which maps η to η_i is a morphism of Hodge structures (see (10)), it provides an intersection form $q_i(\eta) := (\eta_i, \eta_i)_{S_i}$ on $H^2(X, \mathbb{Q})_{tr}$, which satisfies the first and weak second Hodge-Riemann relations.

Let now N, M be the respective degrees of the maps ϕ, ψ. We deduce from (18) the following equality:

$$N \int_X \eta^{2n} = M \int_{S_1 \times \ldots \times S_n} \sum_{i=1}^n (\mathrm{pr}_i^*\eta_i)^{2n} = M \frac{(2n)!}{2^n n!} (\eta_1, \eta_1)_{S_1} \ldots (\eta_n, \eta_n)_{S_n}. \tag{19}$$

Let $q_1(\eta) := (\eta_1, \eta_1)_{S_1}, \ldots, q_n(\eta) := (\eta_n, \eta_n)_{S_n}$, where the η_i's are defined by (18). Then (19) gives (16) up to a multiplicative coefficient, which proves (i).

We next assume property (*) which implies that $(\eta_i, \eta_i)_{S_i} = \mu_i q(\eta)$ for some rational numbers μ_i, since q_i satisfies the first Hodge-Riemann relations. Equation (19) then gives:

$$N \int_X \eta^{2n} = M \frac{(2n)!}{2^n n!} \mu_1 \ldots \mu_n q(\eta)^n,$$

proving (ii). ∎

Proposition 1.5 (i) now implies the following result, showing that having a surface decomposition is a restrictive condition:

Theorem 1.6. *Let S_1, S_2, S_3 be three smooth projective surfaces with $h^{1,0}(S_i) = 0$, $h^{2,0}(S_i) \neq 0$ for all i, and let $H = \mathrm{pr}_1^*H_1 + \mathrm{pr}_2^*H_2 + \mathrm{pr}_3^*H_3 \in \mathrm{Pic}\,(S_1 \times S_2 \times S_3)$ be a very ample divisor on $S_1 \times S_2 \times S_3$. Let $Y \subset S_1 \times S_2 \times S_3$ be the smooth complete intersection of two general members of $|H|$. Then Y is not surface decomposable.*

Proof. As $h^{1,0}(S_i) = 0$, we have

$$H^2(S_1 \times S_2 \times S_3, \mathbb{Q}) = H^2(S_1, \mathbb{Q}) \oplus H^2(S_2, \mathbb{Q}) \oplus H^2(S_3, \mathbb{Q})$$

and similarly for transcendental cohomology. By the Lefschetz hyperplane section theorem, we get, as $\dim Y = 4$:

$$H^2(Y, \mathbb{Q})_{tr} = H^2(S_1, \mathbb{Q})_{tr} \oplus H^2(S_2, \mathbb{Q})_{tr} \oplus H^2(S_3, \mathbb{Q})_{tr},$$

We now compute $\int_Y \alpha^4$ for $\alpha \in H^2(Y,\mathbb{Q})_{tr}$: For $\alpha = \alpha_1+\alpha_2+\alpha_3$, using $\int_{S_i}\alpha_i \cup h_i = 0$, where $h_i := c_1(H_i)$, we get

$$\int_Y \alpha^4 = \int_{S_1\times S_2\times S_3} (\mathrm{pr}_1^*\alpha_1 + \mathrm{pr}_2^*\alpha_2 + \mathrm{pr}_3^*\alpha_3)^4 (\mathrm{pr}_1^*h_1 + \mathrm{pr}_2^*h_2 + \mathrm{pr}_3^*h_3)^2$$

$$= \lambda_1 q_2(\alpha_2)q_3(\alpha_3) + \lambda_2 q_1(\alpha_1)q_3(\alpha_3) + \lambda_3 q_1(\alpha_1)q_2(\alpha_2), \tag{20}$$

where $q_i(\alpha_i) := \int_{S_i}\alpha_i^2$, and the constants λ_i are nonzero rational numbers. It is immediate to see that (20) is not of the form (16), namely the product of two quadrics in $\alpha = \alpha_1+\alpha_2+\alpha_3$. Indeed, the hypersurface in $\mathbb{P}(H^2(Y,\mathbb{C})_{tr})$ defined by (20) is irreducible, being fibered with irreducible fibers over the smooth conic in $\mathbb{P}^2_{\mathbb{C}}$ with equation $\lambda_1 y_2y_3 + \lambda_2 y_1y_3 + \lambda_3 y_1y_2 = 0$, via the rational map

$$\mathbb{P}(H^2(Y,\mathbb{C})_{tr}) \dashrightarrow \mathbb{P}^2_{\mathbb{C}},$$
$$\alpha = \alpha_1+\alpha_2+\alpha_3 \mapsto (q_1(\alpha_1), q_2(\alpha_2), q_3(\alpha_3)).$$

□

1.2 Application to Beauville's weak splitting conjecture

In the paper [7], it was observed that a projective $K3$ surface has the following property: there is a canonical 0-cycle $o_S \in \mathrm{CH}_0(S)$ of degree 1 (in fact, it can be defined as $\frac{c_2(S)}{24}$) such that for any divisor $D \in \mathrm{Pic}\,S = \mathrm{CH}^1(S) = \mathrm{NS}(S)$, one has

$$D^2 = q(D)o_S \text{ in } \mathrm{CH}_0(S), \tag{21}$$

where $q(D) = ([D],[D])_S$. One can rephrase this result by saying :

(*) *Any cohomological polynomial relation*

$$Q([D_1],\ldots,[D_k]) = 0 \text{ in } H^*(S,\mathbb{Q})$$

involving only divisor classes is already satisfied in $\mathrm{CH}(S)_{\mathbb{Q}}$.

In [6], Beauville made the following conjecture, generalizing the result above:

Conjecture 1.7. *Let X be a projective hyper-Kähler manifold. Then the cycle class map is injective on the subalgebra of $\mathrm{CH}^*(X)_{\mathbb{Q}}$ generated by divisor classes.*

This conjecture is called Beauville's weak splitting conjecture. It is equivalent to the fact that property (*), that we will call the weak splitting property, is satisfied for any divisor classes on X. Let us discuss Conjecture 1.7 in relation with the notion of surface decomposition. Let X be a projective hyper-Kähler manifold, and let $\Lambda \subset \mathrm{NS}(X)$ be a lattice polarization (which means that Λ contains an ample class). The very general Λ-polarized deformation X_t of X is the very general fiber of a family $\mathcal{X} \to B$ parameterized by a quasiprojective base B and it satisfies $H^2(X_t,\mathbb{Q})_{tr} = H^2(X_t,\mathbb{Q})^{\perp\Lambda}$. We assume that the general (or very general) Λ-polarized deformation X_t of X has a surface decomposition. Then, by standard spreading arguments involving relative Chow varieties, after passing to a generically finite cover B' of B, we have projective morphisms $\Gamma \to B'$, $\mathcal{S}_i \to B'$, with $\dim \mathcal{S}_i/B' = 2$, and morphisms over B'

$$\phi : \Gamma \to \mathcal{X},\ \psi : \Gamma \to \mathcal{S}_1 \times_{B'} \ldots \times_{B'} \mathcal{S}_n$$

inducing a surface decomposition at the general point $t \in B'$. After shrinking B', by desingularization of the general fiber, one can assume that the fibers Γ_t and $S_{i,t}$ are smooth and we get by specialization a diagram

$$\phi_t : \Gamma_t \to X_t,\ \psi_t : \Gamma_t \to S_{1,t} \times \ldots \times S_{n,t} \tag{22}$$

such that

$$\phi_t^*\sigma_{X_t} = \psi_t^*(\sum_i \mathrm{pr}_i^*\sigma_{S_{i,t}}) \text{ in } H^{2,0}(\Gamma_t) \tag{23}$$

for some $(2,0)$-forms $\sigma_{S_{i,t}}$ on $S_{i,t}$. We proved in Lemma 1.1 that the relation (23) then holds in fact for any class $\alpha \in H^2(X_t,\mathbb{Q})_{tr} = H^2(X_t,\mathbb{Q})^{\perp\Lambda}$ and that there is for each i a (locally constant) morphism of Hodge structures

$$\psi_{i,t} : H^2(X_t,\mathbb{Q})^{\perp\Lambda} \to H^2(S_{i,t},\mathbb{Q})$$

given by (10) such that

$$\phi_t^*\alpha = \psi_t^*(\sum_i \mathrm{pr}_i^*(\psi_{i,t}(\alpha))) \text{ in } H^2(\Gamma_t,\mathbb{Q}). \tag{24}$$

Let us now assume furthermore that $H^1(\Gamma_t,\mathbb{Z}) = 0$, or equivalently

$$\mathrm{NS}\,(\Gamma_t) = \mathrm{Pic}\,(\Gamma_t). \tag{25}$$

Note that, since Γ_t dominates each $S_{i,t}$, this implies the same equality for each $S_{i,t}$. In the situation described above, we have the following result.

Theorem 1.8. *For any $t \in B'$, the weak splitting property holds for divisor classes on X_t which are in $H^2(X_t,\mathbb{Q})^{\perp\Lambda}$ if and only if, for each surface $S_{i,t}$, the Beauville-Voisin relation (21) holds on $\mathrm{Im}\,\psi_{i,t}$ for an adequate 0-cycle $o_{S_{i,t}} \in \mathrm{CH}_0(S_{i,t})$.*

Proof. Using (25), we conclude that (24) holds in $\mathrm{Pic}\,(\Gamma_t)_{\mathbb{Q}}$ for $\alpha \in \mathrm{Pic}\,(X_t)^{\perp\Lambda} = \mathrm{NS}\,(X_t)^{\perp\Lambda}$ (where the point t is now special in B', being in a Noether-Lefschetz locus), and more precisely, that the morphism of Hodge structures $\psi_{i,t}$ induces for any $t \in B'$ a $\mathbb{Q}$-linear map

$$\psi_{i,t} : \mathrm{Pic}\,(X_t)_{\mathbb{Q}}^{\perp\Lambda} \to \mathrm{Pic}\,(S_{i,t})_{\mathbb{Q}}$$

such that, for any $D \in \mathrm{Pic}\,(X_t)_{\mathbb{Q}}^{\perp\Lambda}$:

$$\phi_t^*D = \psi_t^*(\sum_i \mathrm{pr}_i^*(\psi_{i,t}(D))) \text{ in } \mathrm{Pic}(\Gamma_t)_{\mathbb{Q}} = \mathrm{CH}^1(\Gamma_t)_{\mathbb{Q}}. \tag{26}$$

As in the cohomological setting which has been studied in the previous section, the important point here is the fact that the pull-back maps appearing on both sides are compatible with intersection product. Note also that they are injective since the maps ϕ_t and ψ_t are dominant. For any point $t \in B$, let $D_1,\ldots, D_k \in \mathrm{CH}^1(X)_{\mathbb{Q}}$ and let Q be a degree l homogeneous polynomial with $\mathbb{Q}$-coefficients in k variables. Then we get from (26):

$$\phi_t^*Q(D_1,\ldots,D_k) = \psi_t^*(Q(D'_1,\ldots, D'_k)) \quad \text{in } \mathrm{CH}^l(\Gamma_t)_{\mathbb{Q}}, \tag{27}$$

where $D'_j := \sum_i \mathrm{pr}_i^*(\psi_{i,t}(D_j))$. Assume that X_t satisfies the weak splitting property, at least for divisor classes $D \in \mathrm{CH}^1(X_t)^{\perp\Lambda}$. There is then a 0-cycle $o_X \in \mathrm{CH}_0(X)$ of degree 1 such that

$$D^{2n} = (\deg D^n)o_X \text{ in } \mathrm{CH}_0(X) \tag{28}$$

for any $D \in \mathrm{CH}^1(X_t)^{\perp\Lambda}$. Pulling-back to Γ_t and using (27), we have now

$$\phi_t^*(D^{2n}) = \frac{(2n)!}{2^n n!}\psi_t^*(\prod_{j=1}^n \mathrm{pr}_j^*(\psi_{j,t}(D)^2)) \text{ in } \mathrm{CH}_0(\Gamma_t). \tag{29}$$

Note that any $D \in \mathrm{CH}^1(X_t)^{\perp\Lambda}$ satisfies $q(D) \neq 0$ by the Hodge index theorem, where q is the Beauville-Bogomolov quadratic form on $H^2(X_t, \mathbb{Q})$, which can also be defined as the Lefschetz intersection pairing on $H^2(X_t, \mathbb{Q})^{\perp\Lambda}$ (see [4]). As we have $\deg D^{2n} = \lambda q([D])^n$ with $\lambda \neq 0$ by (15), we conclude that $\deg D^{2n} \neq 0$. Let

$$o_{S_{j,t}} := \mathrm{pr}_{j*}(\frac{1}{\deg \phi_t}(\psi_{t*}(\phi_t^* o_X))) \ \in \ \mathrm{CH}_0(S_{j,t})_{\mathbb{Q}}.$$

This cycle has degree 1 and we get from (29) by pushing-forward to $S_{j,t}$ via $\mathrm{pr}_j \circ \psi$ that $\psi_{j,t}(D)^2$ is proportional to $o_{S_{j,t}}$. Indeed, $(\mathrm{pr}_j \circ \psi)_*(\phi_t^*(D^{2n}))$ is a 0-cycle of degree different from 0 on $S_{j,t}$, which by (29) is proportional to both $o_{S_{j,t}}$ and $\psi_{j,t}(D)^2$. This proves the "only if" direction.

Conversely, assume each surface $S_{i,t}$ has a 0-cycle $o_{S_{i,t}}$ of degree 1 with the property that divisors D_i in $\mathrm{Im}\,\psi_{i,t} \subset \mathrm{NS}(S_{i,t})_{\mathbb{Q}} = \mathrm{Pic}\,(S_{i,t})_{\mathbb{Q}}$ satisfy $D_i^2 = (D_i, D_i)_{S_{i,t}} o_{S_{i,t}}$ in $\mathrm{CH}_0(S_{i,t})$ or equivalently that for any D_i, $D_i' \in \mathrm{Im}\,\psi_{i,t}$

$$D_i \cdot D_i' = (D_i, D_i')_{S_{i,t}} o_{S_{i,t}} \text{ in } \mathrm{CH}_0(S_{i,t}). \tag{30}$$

We now use the fact that, at the very general point of B', the Mumford-Tate group of the Hodge structure on $H^2(X_t, \mathbb{Q})^{\perp\Lambda}$ is the orthogonal group, and thus the intersection form $\psi_{i,t}^*((\ ,\ )_{S_{i,t}})$ equals $\mu_i q$ on $H^2(X_t, \mathbb{Q})^{\perp\Lambda}$, for some coefficient μ_i. It then follows from (30) that a numerical relation $q(D) = 0$ for $D \in \mathrm{Pic}\,(X_t)_{\mathbb{C}}$ produces relations

$$D_i^2 = 0 \text{ in } \mathrm{CH}_0(S_{i,t})_{\mathbb{C}}, \tag{31}$$

for any $i = 1, \ldots, n$, where $D_i := \psi_{i,t}(D)$.

By [8] (or rather, the same arguments as in [8] using the fact that the Beauville-Bogomolov pairing q remains nondegenerate on $\mathrm{NS}(X_t)_{\mathbb{Q}}$), we know that the relations in the subalgebra of $H^*(X_t, \mathbb{C})$ generated by $\mathrm{NS}\,(X_t)_{\mathbb{C}}$ are generated by the Bogomolov-Verbitsky relations

$$d^{n+1} = 0 \text{ if } q(d) = 0. \tag{32}$$

This is true as well (for the same reasons) if we restrict to the subalgebra generated by $\mathrm{NS}\,(X_t)_{\mathbb{C}}^{\perp\Lambda} = \mathrm{Pic}\,(X_t)_{\mathbb{C}}^{\perp\Lambda}$. Next, (27) provides for any $D \in \mathrm{Pic}\,(X_t)_{\mathbb{C}}^{\perp\Lambda}$

$$\phi_t^*(D^{n+1}) = \psi_t^*((\sum_{i=1}^n \mathrm{pr}_i^* D_i)^{n+1}) = \sum_i \mathrm{pr}_1^* D_1 \cdot \ldots \cdot \mathrm{pr}_i^* D_i^2 \cdot \mathrm{pr}_n^* D_n \\ + \ldots \text{ in } \mathrm{CH}(\Gamma_t)_{\mathbb{C}}, \tag{33}$$

where the remaining term "..." involves products $\mathrm{pr}_i^* D_i^2 \cdot \mathrm{pr}_j^* D_j^2$ of two squares, then three squares $\mathrm{pr}_i^* D_i^2 \cdot \mathrm{pr}_j^* D_j^2 \cdot \mathrm{pr}_k^* D_k^2$ etc... Using (31) and (33), we get $\phi_t^*(D^{n+1}) = 0$ in $\mathrm{CH}^{n+1}(\Gamma_t)_{\mathbb{C}}$, hence $D^{n+1} = 0$ in $\mathrm{CH}^{n+1}(X_t)_{\mathbb{C}}$, whenever $q(D) = 0$. In other words, the Bogomolov-Verbitsky relations (32) are satisfied in $\mathrm{CH}^{n+1}(X_t)_{\mathbb{C}}$, which concludes the proof. $\square$

We get the following corollary:

Corollary 1.9. *(Cf. Theorem 0.4) Under the same assumptions as in Theorem 1.8, the weak splitting property holds for divisor classes on X_t which are in $H^2(X_t, \mathbb{Q})^{\perp\Lambda}$ if and only if they hold in top degree, that is,*

(*) there exists a canonical 0-cycle $o_{X_t} \in \mathrm{CH}_0(X_t)$ such that for any $D \in \mathrm{NS}(X_t)^{\perp\Lambda}$, D^{2n} is proportional to o_{X_t} in $\mathrm{CH}_0(X_t)$.

Proof. The "only if" is clear. In the other direction, examining the proof of Theorem 1.8, we observe that we only used relations (29) in top degree $2n$ to conclude that, if (*) holds, defining $o_{S_{i,t}} \in \mathrm{CH}_0(S_{i,t}) := \frac{1}{\deg \phi_t} \mathrm{pr}_{i*}(\psi_{t*}(\phi_t^* o_{X_t})] \in \mathrm{CH}_0(S_{i,t})$, the zero-cycle $D_{i,t}^2$ is proportional to $o_{S_{i,t}}$ in $\mathrm{CH}_0(S_{i,t})$, for any $D_t \in \mathrm{NS}(X_t)^{\perp\Lambda}$, where $D_{i,t} := \psi_{i,t}(D_t)$. Hence by Theorem 1.8, (*) implies the weak splitting property for $\mathrm{NS}(X_t)^{\perp\Lambda}$. $\square$

1.3 Decomposition from families of algebraically coisotropic divisors

We study in this section a weaker notion of decomposition for a holomorphic 2-form into forms of smaller rank (see Question 0.1). The following is a weak converse to Proposition 0.11.

Proposition 1.10. *Let X be smooth projective variety of dimension $2n$ equiped with a generically nondegenerate holomorphic 2-form σ_X. Assume X is swept-out by (possibly singular) algebraically coisotropic divisors. Then there exists a generically finite cover $\Phi : \mathcal{D}' \to X$ such that*

$$\Phi^*\sigma_X = \eta_1 + \eta_2 \text{ in } H^{2,0}(\mathcal{D}'), \tag{34}$$

where $\operatorname{rank}\eta_1 = 2$, *and* $\operatorname{rank}\eta_2 = 2n-2$. *More precisely, η_2 is the pull-back of a holomorphic 2-form on a variety of dimension $\leq 2n-1$.*

Here by the rank we mean the generic rank of the considered forms.

Proof of Proposition 1.10. By assumption, there exists a 1-parameter family

$$\mathcal{D} \to C,\ \mathcal{D} \to X$$

of divisors $D_t \subset X$ whose characteristic foliation (on the regular locus of D_t) is algebraically integrable, that is, there exists a rational map

$$\phi_t : D_t \dashrightarrow B_t$$

with $\dim B_t = 2n-2$, such that the equality $\sigma_{X|D_t} = \phi_t^*\sigma_{B_t}$, for some holomorphic 2-form σ_{B_t} on the regular locus of B_t, holds on the regular locus of D_t. Note that, by desingularisation, we can assume D_t and B_t smooth, at least for general t. Indeed, the 2-form σ_{B_t} extends holomorphically on any smooth projective model $\widetilde{B}_t$ of B_t, because it can be constructed as

$$\tilde{\phi}_{t*}(\tilde{j}_t^*\sigma_X \wedge \omega)$$

where $\tilde{j}_t : \widetilde{D}_t \to X$ is a smooth model of D_t such that $\tilde{\phi}_t : \widetilde{D}_t \to \widetilde{B}_t$ is a morphism, and ω is a closed $(1,1)$-form on $\widetilde{D}_t$ whose integral over the fibers of $\tilde{\phi}_t$ is 1.

As usual, the data above (namely the family of varieties B_t and morphisms ϕ_t) can be put in a family, possibly after base change from the original family $\mathcal{D} \to C$ of divisors on X and birational transformations. We thus get the following diagram

$$\begin{tikzcd} \mathcal{D}' \arrow[r, "J"] \arrow[d, "\Phi"] & X \\ B & \end{tikzcd} \tag{35}$$

where all the varieties are smooth and projective, the morphism J is surjective generically finite, $\dim B = 2n-1$ and B admits a morphism $f : B \to C$ such that, considering the induced diagram of fibers over a general point $t \in C$

$$\begin{tikzcd} \mathcal{D}'_t \arrow[r, "J_t"] \arrow[d, "\Phi_t"] & X\ , \\ B_t & \end{tikzcd} \tag{36}$$

one has

$$J_t^*\sigma_X = \Phi_t^*\sigma_{B_t} \text{ in } H^{2,0}(B_t). \tag{37}$$

We deduce from this last equality that the forms σ_{B_t}, $t \in C$, form a locally constant section of the bundle $\mathcal{H}^{2,0} \subset R^2 f_* \mathbb{C} \otimes \mathcal{O}_C$ on the open set of C of regular values of f. By the global invariant cycles theorem [11], [28, 4.3.3], there exists a holomorphic 2-form $\sigma_B \in H^{2,0}(B)$ such that

$$\sigma_{B|B_t} = \sigma_{B_t}. \tag{38}$$

We conclude from (37) and (38) that the 2-form $\Phi^*\sigma_B - J^*\sigma_X$ vanishes on the divisors $\mathcal{D}'_t = \Phi^{-1}(B_t)$ which cover $\mathcal{D}'$. This form thus has rank ≤ 2 on $\mathcal{D}'$. Finally, as the rank of $\Phi^*\sigma_B - J^*\sigma_X$ is ≤ 2 and the rank of $\Phi^*\sigma_B$ is $\leq 2n-2$, while rank $J^*\sigma_X = 2n$, one concludes that rank $\Phi^*\sigma_B - J^*\sigma_X = 2$ and rank $\Phi^*\sigma_B = 2n-2$. □

This statement raises the following question.

Question 1.11. *Is any projective hyper-Kähler manifold swept out by algebraically coisotropic divisors?*

The following question was asked by G. Pacienza.

Question 1.12. *Is any projective hyper-Kähler manifold swept out by elliptic curves?*

The following proposition relates Question 1.11 and Question 1.12.

Proposition 1.13. *If a very general polarized hyper-Kähler manifold with $b_2 \geq 5$ is swept out by elliptic curves, then it is swept out by algebraically coisotropic divisors.*

Here, "very general" means that X is the very general member of a complete family of polarized hyper-Kähler manifolds.

Proof. There exists by assumption a covering family of elliptic curves

$$\phi : \mathcal{E} \to X,\ \psi : \mathcal{E} \to B$$

with ϕ surjective generically finite and $\dim B = 2n-1$. If these elliptic curves have constant moduli, after passing to a generically finite cover of B, $\mathcal{E}$ becomes birational to a product $E \times B$ and we conclude that there is an injective morphism of Hodge structures

$$H^2(X,\mathbb{Q})_{tr} \to H^1(E,\mathbb{Q}) \otimes H^1(B,\mathbb{Q}).$$

Indeed, $\phi^*\sigma_X$ is not in the image of ψ^* because $\psi^* H^{2,0}(B)$ consists of holomorphic forms of generic rank $< \dim X$, while $\phi^*\sigma_X$ has generic rank equal to $\dim X$. Hence $\phi^*\sigma_X$ has a nontrivial image in $H^1(E,\mathbb{C}) \otimes H^1(B,\mathbb{C})$. The natural morphism $H^2(X,\mathbb{Q})_{tr} \to H^1(E,\mathbb{Q}) \otimes H^1(B,\mathbb{Q})$ given by pull-back and projection to a Leray summand is thus nonzero on $H^{2,0}(X)$, hence injective on $H^2(X,\mathbb{Q})_{tr}$. When $b_2 \geq 5$ and X is very general, the existence of such injective morphism contradicts the result of [15]. Hence the elliptic curves E must have variable modulus. For a fixed $t \in \mathbb{P}^1$, consider the divisor $B_t \subset$ parameterizing elliptic curves E_b with fixed j-invariant determined by t. Over B_t, the family $\mathcal{E}_t = \psi^{-1}(B_t)$ is birational (possibly after after base change) to $E_t \times B_t$. Let

$$\phi_t : \mathcal{E}_t \to X,\ \psi_t : \mathcal{E}_t \to B_t$$

be the restricted family. The same argument as above shows that $\phi_t^*\sigma_X$ has to vanish in $H^1(E_t,\mathbb{C}) \otimes H^1(B_t,\mathbb{C})$. This is exactly saying that $\phi(\mathcal{E}_t)$ is an algebraically coisotropic divisor in X, as this implies that $\phi_t^*\sigma_X$ is pulled-back from B_t. □

It seems plausible that Question 1.12 has a negative answer while Question 1.11 has a positive answer.

2 Triangle varieties: examples

Recall from Definition 0.8 in the introduction that a triangle variety T for a hyper-Kähler manifold X of dimension $2n$ is a subvariety of $X \times X \times X$ which has dimension $3n$, maps surjectively onto the various summands and maps in a generically finite way on its image in the product of two summands, and is such that

$$(\mathrm{pr}_1^*\sigma_X + \mathrm{pr}_2^*\sigma_X + \mathrm{pr}_3^*\sigma_X)_{|T_{\mathrm{reg}}} = 0, \tag{39}$$

where σ_X is the holomorphic 2-form of X. Note that Equation (39) says that T is Lagrangian for the everywhere nondegenerate holomorphic 2-form $\mathrm{pr}_1^*\sigma_X + \mathrm{pr}_2^*\sigma_X + \mathrm{pr}_3^*\sigma_X$ on X^3. A variant of the main deformation invariance theorem of [27] says now the following:

Theorem 2.1. *Let X be hyper-Kähler and let $j : L \hookrightarrow X \times X \times X$ be a smooth triangle subvariety (hence L is Lagrangian for the 2-form $\mathrm{pr}_1^*\sigma_X + \mathrm{pr}_2^*\sigma_X + \mathrm{pr}_3^*\sigma_X$). Then for a small deformation X_t of X with constant Picard group, there is a deformation $j_t : L_t \hookrightarrow X_t \times X_t \times X_t$ of L, and L_t is a triangle variety for X_t.*

The last statement follows from the fact that, denoting $\Lambda = \mathrm{NS}(X)$, the subgroup $H^2(X_t,\mathbb{Q})^{\perp\Lambda}$ and the restriction map

$$j_t^* : H^2(X_t \times X_t \times X_t, \mathbb{Q}) \to H^2(L_t, \mathbb{Q})$$

are locally constant on the base B of deformations of X with fixed Picard number. Hence the diagonal image of $H^2(X_t,\mathbb{Q})^{\perp\Lambda}$ in $H^2(X_t \times X_t \times X_t, \mathbb{Q}) = H^2(X_t,\mathbb{Q})^3$ is annihilated by j_t^*, since it is annihilated by j^* (note here that $H^2(X,\mathbb{Q})^{\perp\Lambda} = H^2(X,\mathbb{Q})_{tr}$).

Remark 2.2. A smooth triangle subvariety $L \subset X \times X \times X$ cannot deform in products $X_t \times X_{t'} \times X_{t''}$ unless $t = t' = t''$. Indeed, the kernel H of $j^* : H^2(X,\mathbb{Q})^3_{tr} \to H^2(L,\mathbb{Q})$ is exactly the diagonal image of $H^2(X,\mathbb{Q})_{tr}$, as it follows from the fact that L maps to a subvariety of dimension $3n$ in the three products $X \times X$. If there is a deformation $L_{t,t',t''}$ of L in $X_t \times X_{t'} \times X_{t''}$, there is a Hodge substructure

$$H_{t,t',t''} \subset H^2(X_t,\mathbb{Q})^{\perp\Lambda} \oplus H^2(X_{t'},\mathbb{Q})^{\perp\Lambda} \oplus H^2(X_{t''},\mathbb{Q})^{\perp\Lambda}$$

deforming H. But then $H_{t,t',t''}$ is isomorphic by projections to the three Hodge structures $H^2(X_t,\mathbb{Q})^{\perp\Lambda}$, $H^2(X_{t'},\mathbb{Q})^{\perp\Lambda}$, $H^2(X_{t''},\mathbb{Q})^{\perp\Lambda}$. By the local Torelli theorem, we then have $t = t' = t''$.

Theorem 2.1 suggests possibly that triangle subvarieties tend to be stable under deformations with constant Picard number, but in the examples we will describe below, the triangle subvarieties are never smooth, so in fact Theorem 2.1 does not apply.

Considering the conjectures made in [6], [29], it would be very nice if the triangle varieties T satisfied a cycle-theoretic variant of (39), asking the following: for any $t = (t_1,t_2,t_3) \in T \subset X^3$

$$t_1 + t_2 + t_3 = c \text{ in } \mathrm{CH}_0(X), \tag{40}$$

for some fixed zero-cycle c of X. Formula (40) implies indeed (39) by Mumford's theorem [23]. Let us explain why it is not possible to achieve (40) starting from dimension 4.

Proposition 2.3. *Let X be a projective hyper-Kähler manifold of dimension $2n \geq 4$. Let T be a triangle subvariety of $X \times X \times X$. Then the cycle $t_1 + t_2 + t_3 \in \mathrm{CH}_0(X)$ for $t = (t_1,t_2,t_3) \in T$ is not constant along T.*

Proof. Indeed, if (40) holds, then Mumford's theorem [23] says that for any power σ_X^l, $l > 0$, of σ_X,

$$(\mathrm{pr}_1^*\sigma_X^l + \mathrm{pr}_2^*\sigma_X^l + \mathrm{pr}_3^*\sigma_X^l)_{|T_{\mathrm{reg}}} = 0 \text{ in } H^0(T_{\mathrm{reg}}, \Omega^{2l}_{T_{\mathrm{reg}}}). \tag{41}$$

We now set $l=2$. We then have the two equations

$$(\mathrm{pr}_1^*\sigma_X)_{|T_{\mathrm{reg}}} = -(\mathrm{pr}_2^*\sigma_X + \mathrm{pr}_3^*\sigma_X)_{|T_{\mathrm{reg}}} \text{ in } H^0(T_{\mathrm{reg}}, \Omega^2_{T_{\mathrm{reg}}}), \tag{42}$$
$$(\mathrm{pr}_1^*\sigma_X^2)_{|T_{\mathrm{reg}}} = -(\mathrm{pr}_2^*\sigma_X^2 + \mathrm{pr}_3^*\sigma_X^2)_{|T_{\mathrm{reg}}} \text{ in } H^0(T_{\mathrm{reg}}, \Omega^4_{T_{\mathrm{reg}}}).$$

It follows that

$$-(\mathrm{pr}_2^*\sigma_X^2 + \mathrm{pr}_3^*\sigma_X^2)_{|T_{\mathrm{reg}}} = (\mathrm{pr}_2^*\sigma_X + \mathrm{pr}_3^*\sigma_X)^2_{|T_{\mathrm{reg}}} \text{ in } H^0(T_{\mathrm{reg}}, \Omega^4_{T_{\mathrm{reg}}}). \tag{43}$$

Let us write the above equation as

$$\omega_{|T_{\mathrm{reg}}} \wedge \omega'_{|T_{\mathrm{reg}}} = 0 \text{ in } H^0(T_{\mathrm{reg}}, \Omega^4_{T_{\mathrm{reg}}}), \tag{44}$$

where

$$\omega := \mathrm{pr}_2^*\sigma_X - \frac{-1+i\sqrt{3}}{2}\mathrm{pr}_3^*\sigma_X,\ \omega' := \mathrm{pr}_2^*\sigma_X - \frac{-1-i\sqrt{3}}{2}\mathrm{pr}_3^*\sigma_X.$$

We next have the following

Lemma 2.4. *Let V be a vector space and let $\omega, \omega' \in \bigwedge^2 V^*$ such that $\omega \neq 0$, $\omega' \neq 0$ and $\omega \wedge \omega' = 0$ in $\bigwedge^4 V^*$. Then there exists a quotient $V \to V'$ with $\dim V' \leq 4$ such that both ω and ω' are pulled back from 2-forms on V'.*

Proof. This follows from the fact that for a 2-form ω of rank 6 on a 6-dimensional vector space V, the wedge product map $\omega\wedge : \bigwedge^2 V^* \to \bigwedge^4 V^*$ is an isomorphism. This fact already implies that if $\omega \wedge \omega' = 0$ in $\bigwedge^4 V^*$, with $\omega \neq 0$, $\omega' \neq 0$, the rank of ω is at most 4 and similarly for ω'. If both forms ω and ω' are of rank 2, the conclusion of the lemma holds. If $\omega = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$ is of rank 4, let $V' = \langle e_1^*, \ldots, e_4^*\rangle$. Choosing a decomposition $V^* = V'^* \oplus W^*$, we can write $\omega' = \alpha + \beta + \gamma$ with

$$\alpha \in \bigwedge^2 V'^*,\ \beta \in V'^* \otimes W^*,\ \gamma \in \bigwedge^2 W^*,$$

and we must have $\omega \wedge \beta = 0$, $\omega \wedge \gamma = 0$, which clearly implies that $\beta = 0$ and $\gamma = 0$ because ω has rank 4 so $\omega\wedge$ is injective on V'^*. Thus ω' belongs to $\bigwedge^2 V'^*$. ☐

The contradiction now comes from (43) and Lemma 2.4 which imply that either $\omega_{|T_{\mathrm{reg}}} = 0$ or $\omega'_{|T_{\mathrm{reg}}} = 0$, or both forms $\omega_{|T_{\mathrm{reg}}}$ and $\omega'_{|T_{\mathrm{reg}}}$ have rank ≤ 4 at any point t of T and more precisely, at any point $t \in T_{\mathrm{reg}}$, are pulled-back via a quotient map $T_{T,t} \to T'$, with $\dim T' \leq 4$. The form $\omega_{|T}$ cannot be 0 because this would imply that the projection of T in $X \times X$ via $(\mathrm{pr}_2, \mathrm{pr}_3)$ is Lagrangian for a form which has rank $4n$ everywhere on $X \times X$ while by assumption $\dim(\mathrm{pr}_2, \mathrm{pr}_3)(T) = 3n$. The same argument also works for ω'. We thus conclude that the last possibility should hold. In that case, the restrictions to T of $\mathrm{pr}_2^*\sigma_X$ and $\mathrm{pr}_3^*\sigma_X$ are also pulled-back via the quotient map $T_{T,t} \to T'$ hence have rank ≤ 4. The form σ_X on X is everywhere nondegenerate and the projections pr_2, pr_3 restricted to T are dominant, so we conclude that the forms $\mathrm{pr}_2^*\sigma_X$, $\mathrm{pr}_3^*\sigma_X$ restricted to T have rank equal to $\dim X$. As they are of rank ≤ 4 at a general point $t \in T$, we get a contradiction if $n \geq 3$. If $n = 2$, these forms pulled-back to T have rank 4 but they do not have the same kernel, because their respective kernels at a point $t \in T_{\mathrm{reg}}$ are the spaces $\mathrm{Ker}\,(\mathrm{pr}_{2|T})_*$ and $\mathrm{Ker}\,(\mathrm{pr}_{3|T})_*$ which are different generically on T by the assumption that $(\mathrm{pr}_2, \mathrm{pr}_3)$ is generically of maximal rank. This contradiction concludes the proof. ☐

We construct in the next subsections triangle varieties for the main “ known” classes of hyper-Kähler manifolds, for which we have an explicit projective model.

2.1 Hilbert schemes of $K3$ surfaces

Recall from [7] (see also Section 1.2) that a projective $K3$ surface S has a canonical 0-cycle o_S of degree 1 satisfying many properties, including the following : for any integer $k \geq 1$, the degree-k 0-cycle ko_S on S has a k-dimensional orbit

$$O_{ko_S} = \{z \in S^{(k)},\, z = ko_S \text{ in } \mathrm{CH}_0(S)\}$$

in $S^{(k)}$ for rational equivalence on S. An explicit example of a k-dimensional orbit component of ko_S and of a triangle variety for $S^{[n]}$ is as follows. Assume S has a very ample polarization $L \in \mathrm{Pic}\, S$ with $\deg L^2 = 2g-2$. Let $k = 2g-2$. One component of the orbit $O_{L^2} \subset S^{(2g-2)}$ of the zero-cycle L^2 is birational to the Grassmannian $G(2, H^0(S,L))$ and is made of complete intersections $H_1 \cap H_2$, with $H_1,\, H_2 \in |L|$, or rather of their supports. Note that $\dim G(2, H^0(S,L)) = 2g-2$ as we want. Assume furthermore that $2g-2 = 3n$ is divisible by 3 and consider

$$T := \{(z_1,\, z_2,\, z_3) \in (S^{[n]})^3,\; c(z_1) + c(z_2) + c(z_3) \in O_{L^2} \subset S^{(3n)}\}, \tag{45}$$

where $c : S^{[l]} \to S^{(l)}$ denotes the Hilbert-Chow morphism.

Proposition 2.5. *T is a triangle variety for $S^{[n]}$.*

Proof. The relation $(\mathrm{pr}_1^*\sigma_{S^{[n]}} + \mathrm{pr}_2^*\sigma_{S^{[n]}} + \mathrm{pr}_3^*\sigma_{S^{[n]}})_{|T_{\mathrm{reg}}}$ follows from the fact that the 0-cycle $c(z_1) + c(z_2) + c(z_3)$ is constant in $\mathrm{CH}_0(S)$ along T and from Mumford's theorem [23] because the holomorphic 2-form $\sigma_{S^{[n]}}$ is induced by the holomorphic 2-form σ_S via the incidence correspondence. The fact that the dimension of T is $3n$ follows from the fact that T is birational to a generically finite cover of O_{L^2} which has dimension $2g-2 = 3n$. It remains to see that T dominates the three summands and that it maps in a generically finite way to its images in the three products $S^{[n]} \times S^{[n]}$. The first statement follows from the fact that L is very ample with $h^0(S,L) = g+1$, where $3n = 2g-2$. This implies that for a general set $z_1 = \{x_1, \dots, x_n\}$ of n points of S, there is a reduced complete intersection Z of two members of $|L|$ containing all the x_i. Then writing Z as the union $z_1 \sqcup z_2 \sqcup z_3$ of three sets of cardinality n, we have $(z_1, z_2, z_3) \in T$.

For the second statement, we observe that for such a general reduced 0-dimensional complete intersection

$$Z = H_1 \cap H_2 = \{x_1, \dots, x_{2g-2}\},$$

with $2g-2 = 3n$, the first $2n$ points $x_1, \dots, x_{2n}$ already impose $g-1$ conditions on $|L|$, hence the space of hypersurfaces in $|L|$ containing these $2n$ points is the projective line $\langle H_1,\, H_2\rangle$. Setting

$$z_1 = \{x_1, \dots, x_n\},\; z_2 = \{x_{n+1}, \dots, x_{2n}\},\; z_3 = \{x_{2n+1}, \dots, x_{3n}\},$$

we have $(z_1, z_2, z_3) \in T$ and the fiber of the projection $p_{12} : T \to S^{[n]} \times S^{[n]}$ over (z_1, z_2) consists by definition of the single element z_3. ◻

The numerical condition $3n = 2g-2$ used for the construction above is not important, as there are variants of this construction, starting from other Lagrangian subvarieties of $S^{[3n]}$, also obtained as components of dimension $3n$ of the orbit of $3no_S$ in $S^{(3n)}$.

2.2 Fano variety of lines in a cubic fourfold

The Fano variety $F_1(Y)$ of lines in a smooth cubic fourfold Y is a hyper-Kähler fourfold (see [5]). In this case, the triangles are just triangles in a usual sense, namely the plane sections of Y which are the unions of three lines (plus an ordering of these lines). They form a 6-dimensional subvariety of $F_1(Y)^3$. Indeed, for each line $l \subset Y$, consider the $\mathbb{P}^3_l$ of planes containing l. Each of these planes cuts Y along the union of l and a conic, and when the conic is degenerate, that is along a surface in $\mathbb{P}^3_l$, the conic becomes the union of two lines, which together with l form a triangle. In this case, the fact that the family of these triangles is a Lagrangian subvariety of $F_1(Y)^3$ is a consequence of Mumford's theorem [23]. Indeed we know that, via the incidence correspondence $p : P \to F_1(Y)$, $q : P \to Y$ given by the universal family of lines in Y, one has $\sigma_{F_1(Y)} = P^*\eta_Y$ for some class $\eta_Y \in H^1(Y, \Omega^3_Y)$, and furthermore, for any triangle $([l], [l'], [l'']) \in F_1(Y)^3$

$$P_*l + P_*l' + P_*l'' = h^3 \text{ in } \mathrm{CH}_1(Y),$$

where $h = c_1(\mathcal{O}_Y(1))$.

2.3 Debarre-Voisin hyper-Kähler fourfolds

Let V_{10} be a 10-dimensional vector space and let $\lambda \in \bigwedge^3 V_{10}^*$. The associated Debarre-Voisin fourfold $F_\lambda \subset G(6, V_{10})$ is the set of 6-dimensional vector subspaces $W_6 \subset V_{10}$ such that $\lambda_{|W_6} = 0$. This is a hyper-Kähler fourfold for a general parameter λ (see [10]). In [3], Bazhov considered the subvariety $T \subset F_\lambda \times F_\lambda \times F_\lambda$ parameterizing triples $([W], [W'], [W'']) \in F_\lambda^3$ such that the three subspaces W, W' and W'' of V_{10} generate only a $V_9 \subset V_{10}$. He proved the following:

Theorem 2.6. *(i) T has dimension 6. It is birationally equivalent via the projection p_{12} to the incidence subvariety $I \subset F_\lambda \times F_\lambda$ defined as the set of couples $([W], [W'])$ such that W and W' generate only a $V_9 \subset V_{10}$.*

(ii) I dominates F_λ by the first projection.

(iii) T is a Lagrangian subvariety of $F_\lambda \times F_\lambda \times F_\lambda$.

These three facts together say that T is a triangle variety for F_λ.

2.4 Double EPW sextics

The double EPW sextics X constructed by O'Grady in [24] are quasi-étale double covers of sextic hypersurfaces in $\mathbb{P}^5$ singular along a surface discovered by Eisenbud-Pospescu-Walter [12]. We will follow here the description given by Iliev and Manivel [17], which is very convenient to study subvarieties and relations between zero-cycles of X. More precisely, the Iliev-Manivel description provides a Fano fourfold Y, such that X parameterizes 1-cycles in Y and the $(2, 0)$-form on X is induced via the incidence relation from a cohomology class of Hodge type $(3, 1)$ in Y. By Proposition 2.3, we cannot obtain enough relations (39) in $\mathrm{CH}_0(X)$ to construct using Mumford's theorem triangle varieties in X, that is, Lagrangian subvarieties of X^3. In the present case, and this was also exploited in the case of the Fano variety of lines of a cubic fourfold, it suffices to exhibit relations between the corresponding 1-cycles in $\mathrm{CH}_1(Y)$.

The Iliev-Manivel construction is as follows. Let V_5 be a 5-dimensional vector space and let $\mathbb{G} := G(2, 5) \subset \mathbb{P}^9$. Let $Y \subset \mathbb{G}$ be the generic complete intersection of a linear Plücker section $H \subset \mathbb{G}$ and a quadratic Plücker section Q of $\mathbb{G}$. The fourfold Y is Fano of index 2 with Picard number 1 and its variety of conics $\mathcal{H}_{2,0}$ is 5-dimensional. It is fibered into $\mathbb{P}^1$'s, because if $C \subset \mathbb{G}$ is a conic, there exists a hyperplane $V_4 \subset V_5$ such that $C \subset G(2, V_4)$. (Indeed, the surface in $\mathbb{P}(V_5)$ swept-out by lines parameterized by C has degree 2, hence is contained in some $\mathbb{P}(V_4) \subset \mathbb{P}(V_5)$.) Thus C is contained in the del Pezzo surface $\Sigma = H \cap Q \cap G(2, V_4)$ which has index 1 and degree 4. But then C moves in a pencil in Σ. Next Iliev and Manivel show that Y has a $(3, 1)$-form $\eta_Y \in H^{3,1}(Y)$ and considering the incidence diagram given by the universal conic

$$\begin{tikzcd} \mathcal{C} \arrow[r, "q"] \arrow[d, "p"] & Y , \\ \mathcal{H}_{0,2} & \end{tikzcd} \qquad (46)$$

they show that the $(2,0)$-form $p_*q^*\eta$ has generic rank 4 on $\mathcal{H}_{0,2}$. It follows that the base of the MRC fibration of $\mathcal{H}_{0,2}$ is 4-dimensional, with fibers given by the $\mathbb{P}^1$'s described above. Finally, it is shown in [17] that this base is birational to a general double EPW sextic X.

This construction is very convenient to exhibit Lagrangian subvarieties in X^l and, for $l=1$, this is already done in [17]. For example, the variety of conics contained in a general hyperplane section $Y' \subset Y$ is 3-dimensional and its image in X is a Lagrangian surface constructed in [17]. This follows from the fact that the class η vanishes on Y' and that the pull-back of σ_X to $\mathcal{H}_{0,2}$ is defined as $p_*(q^*\eta)$. We now explain how to use this description of X to produce a triangle variety for X.

First of all, we observe that nondegenerate rational curves of degree 4 on Y are parameterized by a 9-dimensional variety $\mathcal{H}_{0,4}$, while nondegenerate elliptic curves of degree 6 are parameterized by a 12-dimensional variety $\mathcal{H}_{1,6}$. Furthermore, there is a dominant rational map

$$\Phi : \mathcal{H}_{1,6} \dashrightarrow \mathcal{H}_{0,4} \tag{47}$$

with general fiber $\mathbb{P}^3$. This map is obtained by liaison. Indeed, a nondegenerate rational curve C of degree 4 on Y has $h^0(C, \mathcal{O}_C(1)) = 5$ and the restriction map $H^0(Y, \mathcal{O}_Y(1)) \to H^0(C, \mathcal{O}_C(1))$ is surjective, hence has a 4-dimensional kernel. As C is general, C is defined in Y by linear Plücker equations. Thus, taking three general equations $\sigma_1, \sigma_2, \sigma_3$ vanishing on C, the locus defined by these 3 equations is a curve of degree 10 that contains C and is the union of C and an elliptic curve of degree 6. Conversely, starting from a nondegenerate elliptic curve E of degree 6, we have $h^0(E, \mathcal{O}_E(1)) = 6$, and the restriction map $H^0(Y, \mathcal{O}_Y(1)) \to H^0(E, \mathcal{O}_E(1))$ is surjective, hence has a 3-dimensional kernel. The locus defined by this 3-dimensional set of linear Plücker equations is a curve of degree 10 containing E and is in fact the union of E and a residual rational curve of degree 4.

There is a 4-dimensional (or codimension 1) family $\Gamma_4 \subset \mathcal{H}_{0,2}$ of conics in Y (which must be contracted to a surface in X), which is constructed as follows. Consider the variety $Z := H \cap \mathbb{G}$ and its variety of planes $P \subset Z$. The equation defining H is a 2-form $\omega \in \bigwedge^2 V_5^*$. It is well-known that a plane in $\mathbb{G}$ corresponds to a point $x \in \mathbb{P}(V_5)$ together with a $\mathbb{P}(V_4) \subset \mathbb{P}(V_5)$ passing through x and defining the plane P of lines in $\mathbb{P}(V_4)$ passing through x. This plane is contained in Z when V_4 is contained in $x^{\perp\omega}$, which provides the desired 4-dimensional family (parameterized birationally by the choice of $x \in \mathbb{P}(V_5)$). Any such plane P determines a conic $C = P \cap Y$ in Y (or is contained in Y, but this does not happen for generic Y). This provides us with a rational 4-dimensional subvariety

$$\Gamma_4 \subset \mathcal{H}_{0,2}.$$

It is obvious that the subvariety of X we get this way is Lagrangian for σ_X, because it is dominated by the rational variety Γ_4.

We now make the following construction. Inside $\Gamma_4 \times \Gamma_4$, there is a 6-dimensional subvariety Γ_6 consisting of pairs of intersecting conics. We observe that Γ_6 maps naturally to $\mathcal{H}_{0,4}$, via the 2 to 1 map which associates to a pair of intersecting conics the rational curve of degree 4 which is the union of the two conics. This way we get a 6-dimensional variety parameterizing degree 4 rational curves in Y, and applying the residual construction explained previously, we get a 9-dimensional subvariety Γ_9^1 of $\mathcal{H}_{1,6}$.

Let now $\mathcal{T} \subset \mathcal{H}_{0,2} \times \mathcal{H}_{0,2} \times \mathcal{H}_{0,2}$ be the set of triples of conics (C_1, C_2, C_3) in Y, intersecting each other (a triangle of conics), and such that the singular elliptic curve $E = C_1 \cup C_2 \cup C_3$ is a member of the family parameterized by Γ_9^1.

Theorem 2.7. *For general Y, the image T of $\mathcal{T}$ in X^3 is a triangle variety.*

Proof. The triples (C_1, C_2, C_3) of conics in Y parameterized by $\mathcal{T}$ have the property that the singular elliptic curve $E = C_1 \cup C_2 \cup C_3 \subset Y$ is residual in Y to a rational curve of degree 4 which is the union of two conics C_4, C_5 meeting at one point, where C_4 and C_5 are cut on Y by planes in Z. All the planes contained in Z are rationally equivalent in Z, so we conclude that the elliptic curves E parameterized by Γ_9^1 are all rationally equivalent. By [17], the holomorphic 2-form on X pulls-back to a holomorphic 2-form $\widetilde{\sigma}_X$ on $\mathcal{H}_{0,2}$ which is induced from a cohomology class of type $(3,1)$ on Y via the incidence correspondence. Mumford's theorem [23] implies that $\mathrm{pr}_1^*\widetilde{\sigma}_X + \mathrm{pr}_2^*\widetilde{\sigma}_X + \mathrm{pr}_3^*\widetilde{\sigma}_X$ vanishes on $\mathcal{T}$, hence equivalently that $\mathrm{pr}_1^*\sigma_X + \mathrm{pr}_2^*\sigma_X + \mathrm{pr}_3^*\sigma_X$ vanishes on T. We leave to the reader checking the dimension count for general Y and the fact that T dominates X by the various projections and is generically finite on its image in $X \times X$ by the various projections. $\square$

Remark 2.8. The method described in the next section and the existence of a covering of X by a family of Lagrangian surfaces given in [17] can also be used to construct triangle varieties for X, see Theorem 2.11.

2.5 Lagrangian fibrations and Lagrangian coverings

Let $\phi : X \to B$ be a projective Lagrangian fibration on a hyper-Kähler manifold of dimension $2n$. Recall from Lin's paper [21] that ϕ has a Lagrangian constant cycle multisection $\widetilde{B}$. By base change from B to $\widetilde{B}$, we get (possibly after desingularization) an induced fibration $\widetilde{X} \to \widetilde{B}$ which has a section, hence is (over a dense open set of $\widetilde{B}$) a family of abelian varieties. Let $\widetilde{I} := \widetilde{X} \times_{\widetilde{B}} \widetilde{X} \subset \widetilde{X} \times \widetilde{X}$. Using the relative addition map, we get a rational map $\mu : \widetilde{I} \dashrightarrow \widetilde{X}$ and finally we define T as the image of $\widetilde{I}$ in $X \times X \times X$ under the rational map $(r \circ \mathrm{pr}_1, r \circ \mathrm{pr}_2, r \circ -\mu)$ where $r : \widetilde{X} \to X$ is the natural map and the pr_i's are the projections from $\widetilde{X} \times \widetilde{X}$ to $\widetilde{X}$, restricted to $\widetilde{I}$.

Proposition 2.9. *The variety T is a triangle variety.*

Proof. As T is the union over $t \in \widetilde{B}$ of the graphs of the sum map in the fibers X_b, it is clear that T dominates X by the three projections and maps in a generically finite way to the products $X \times X$ of any two factors (the image is $X \times_B X$ but the map is not birational because of the base change $\widetilde{B} \to B$). We want to prove that $\mathrm{pr}_1^*\sigma_X + \mathrm{pr}_2^*\sigma_X + \mathrm{pr}_3^*\sigma_X = 0$ on T_{reg}, or, equivalently

$$\mathrm{pr}_1^*\sigma_{\widetilde{X}} + \mathrm{pr}_2^*\sigma_{\widetilde{X}} = \mu^*\sigma_{\widetilde{X}} \tag{48}$$

on $\widetilde{I}$, where $\sigma_{\widetilde{X}} := r^*\sigma_X$. As $\tilde{\phi} : \widetilde{X}_{\mathrm{reg}} \to \widetilde{B}_{\mathrm{reg}}$ is a Lagrangian fibration with respect to $\sigma_{\widetilde{X}}$, we have

$$\sigma_{\widetilde{X}|\widetilde{X}_{\mathrm{reg}}} \in H^0(\widetilde{X}_{\mathrm{reg}}, F^1\Omega^2_{\widetilde{X}_{\mathrm{reg}}}), \tag{49}$$

where $F^1\Omega_{\widetilde{X}^2_{\mathrm{reg}}} := \tilde{\phi}^*\Omega_{\widetilde{B}} \wedge \Omega_{\widetilde{X}_{\mathrm{reg}}}$. Let $F^2\Omega^2_{\widetilde{X}_{\mathrm{reg}}} = \tilde{\phi}^*\Omega^2_{\widetilde{B}}$. The quotient bundle $F^1\Omega_{\widetilde{X}^2_{\mathrm{reg}}}/F^2\Omega_{\widetilde{X}^2_{\mathrm{reg}}}$ is isomorphic to $\tilde{\phi}^*\Omega_{\widetilde{B}} \otimes \Omega_{\widetilde{X}_{\mathrm{reg}}/\widetilde{B}}$. We have

$$\mathrm{pr}_1^*\sigma_{\widetilde{X}} + \mathrm{pr}_2^*\sigma_{\widetilde{X}} = \mu^*\sigma_{\widetilde{X}} \text{ in } H^0(\widetilde{I}_{\mathrm{reg}}, F^1\Omega_{\widetilde{I}^2_{\mathrm{reg}}}/F^2\Omega_{\widetilde{I}^2_{\mathrm{reg}}}) \tag{50}$$

by (49) and because on the fibers $\widetilde{X}_b$, we have $\mu^*\alpha = \mathrm{pr}_1^*\alpha + \mathrm{pr}_2^*\alpha$ for any $\alpha \in H^0(\widetilde{X_b}, \Omega_{\widetilde{X_b}})$, so that

$$\mu^* = \mathrm{pr}_1^* + \mathrm{pr}_2^* : \mu^*\Omega_{\widetilde{X}_{\mathrm{reg}}/\widetilde{B}} \to \Omega_{\widetilde{I}_{\mathrm{reg}}/\widetilde{B}}.$$

It follows from (50) that $\mathrm{pr}_1^*\sigma_{\widetilde{X}} + \mathrm{pr}_2^*\sigma_{\widetilde{X}} - \mu^*\sigma_{\widetilde{X}} \in H^0(\widetilde{I}_{\mathrm{reg}}, \tilde{\phi}^*\Omega^2_{\widetilde{B}_{\mathrm{reg}}}) \subset H^{2,0}(\widetilde{I}_{\mathrm{reg}})$, which gives an equality of 2-forms on $\widetilde{I}_{\mathrm{reg}}$

$$\mathrm{pr}_1^*\sigma_{\widetilde{X}} + \mathrm{pr}_2^*\sigma_{\widetilde{X}} - \mu^*\sigma_{\widetilde{X}} = \tilde{\phi}^*\eta \tag{51}$$

for some $\eta \in H^0(\widetilde{B}, \Omega^2_{\widetilde{B}_{\mathrm{reg}}})$. On the other hand, recall that the multisection $\widetilde{B}$ of ϕ, or 0-section $\widetilde{B}$ of $\tilde{\phi}$, was chosen to be Lagrangian for σ_X (or equivalently $\sigma_{\widetilde{X}}$). Restricting (51) to the 0-section $\widetilde{B}$, we then conclude that $\eta = 0$, which proves (48). $\square$

Let us say that a hyper-Kähler manifold X has a Lagrangian covering if there exists a diagram

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\Phi"] \arrow[d, "\pi"] & X \\ B & \end{tikzcd} \tag{52}$$

where $\mathcal{L}$ and B are smooth projective varieties, the morphism Φ is surjective and maps birationally the general fiber L_t, $t \in B$, of π to a (possibly singular) Lagrangian subvariety of X, and furthermore, the following condition holds. As $\Phi(L_t)$ is Lagrangian, one has a natural morphism of coherent sheaves

$$\lrcorner\sigma_X : N_{L_t/X} \to \Omega_{L_t}$$

which is a generic isomorphism, and induces a morphism at the level of global sections

$$H^0(L_t, N_{L_t/X}) \to H^0(L_t, \Omega_{L_t}).$$

We ask that for $t \in B$ generic, the composite map

$$T_{B,t} \to H^0(L_t, N_{L_t/X}) \stackrel{\lrcorner\sigma_X}{\to} H^0(L_t, \Omega_{L_t})$$

where the first map is the classifying map, is an isomorphism. In particular $\dim B = h^{1,0}(L_t) =: g$. This condition is satisfied by unobstructedness results for deformations of Lagrangian submanifolds (see [27]) if, for general $t \in B$, the fiber L_t is isomorphic via Φ to a smooth Lagrangian subvariety of X, and $\mathcal{L} \to B$ identifies near t to the universal family of deformations of L_t in X. For singular Lagrangian subvarieties, the deformation theory is not well understood. Note that, with the hypotheses above, the surjectivity of Φ has the following interpretation.

Lemma 2.10. *The surjectivity of Φ is equivalent to the fact that the Albanese map $\mathrm{alb}_{L_t} : L_t \to \mathrm{Alb}\, L_t$ is generically finite on its image for general t.*

Proof. The second property is equivalent to the fact that, for general $t \in B$, alb_{L_t} has a generically injective differential, or equivalently, that the evaluation map

$$\mathrm{ev} : H^0(L_t, \Omega_{L_t}) \otimes \mathcal{O}_{L_t} \to \Omega_{L_t} \tag{53}$$

is generically surjective on L_t. The surjectivity of Φ is equivalent to the fact that Φ is submersive generically along L_t for general t. As L_t imbeds generically into X, this is well-known to be equivalent to the fact that the map

$$\mathrm{ev} : T_{B,t} \otimes \mathcal{O}_{L_t} \to N_{L_t/X}, \tag{54}$$

which is the composition of the evaluation map and of the classifying map $T_{B,t} \to H^0(L_t, N_{L_t/X})$ is generically surjective. We now use the map

$$\lrcorner\sigma_X : N_{L_t/X} \to \Omega_{L_t}$$

which is a generic isomorphism on L_t, and induces a morphism at the level of global sections which composed with the classifying map makes the following diagram commutative

$$\begin{tikzcd} T_{B,t} \otimes \mathcal{O}_{L_t} \arrow[r, "\mathrm{ev}"] \arrow[d, "\lrcorner\sigma_X"] & N_{L_t/X} \arrow[d, "\lrcorner\sigma_X"] \\ H^0(L_t, \Omega_{L_t}) \otimes \mathcal{O}_{L_t} \arrow[r, "\mathrm{ev}"] & \Omega_{L_t} \end{tikzcd}, \tag{55}$$

As the first vertical map is by assumption an isomorphism, it follows that the generic surjectivity of the evaluation map (54) is equivalent to the generic surjectivity of the evaluation map (54) $\square$

We show the following variant of Proposition 2.9.

Theorem 2.11. *Let X be a projective hyper-Kähler manifold admitting a Lagrangian covering $\Phi : \mathcal{L} \to X$. Assume that the general fibers L_t have the property that the sum map $L_t \times L_t \to \operatorname{Alb} L_t$ is surjective (in particular $2n \geq g$). Assume there exists a Lagrangian subvariety $K \subset X$ such that the general fiber $\Phi(L_t)$ intersects K in a finite (nonzero) number of points. Then X admits a triangle variety.*

Note that by the same arguments as above, the assumption on K will be satisfied by taking $K = L_s$, for general s, assuming that the fibers $L_t \subset X$ are smooth Lagrangian, and a general form $\alpha \in H^0(L_t, \Omega_{L_t})$ has finitely many zeroes.

Proof of Theorem 2.11. Consider, over the open set B_{reg} of regular values of π, the Albanese fibration $\mathcal{A} \to B_{\mathrm{reg}}$ with fiber $\operatorname{Alb} L_b$ over $b \in B$. By assumption, a general variety $\Phi(L_b) \subset X$ intersects K in finitely many points, which provides a generically finite cover

$$B_K = \Phi^{-1}(K) \to B_{\mathrm{reg}}$$

parameterizing the pairs (b, k), where $k \in L_b$ is such that $\Phi(k) \in K$. Denoting by $\mathcal{L}_K$ the fibered product $\mathcal{L} \times_B B_K$, there is a natural section

$$\sigma : B_K \to \mathcal{L}_K = \mathcal{L} \times_B B_K,$$

$$(b, k) \mapsto k.$$

We denote by $\mathrm{alb}_K : \mathcal{L}_K \to \mathcal{A}_K = \mathcal{A} \times_{B_{\mathrm{reg}}} B_K$ the relative Albanese map defined by the section σ, so that

$$\mathrm{alb}_K(x) = \mathrm{alb}_{L_b}(x - \sigma(b)),$$

where $b = \pi(x) \in B_K$. Let now, for any integer $N \neq 0$, $\widetilde{T}_N \subset \mathcal{L}_K \times_{B_K} \mathcal{L}_K \times_{B_K} \mathcal{L}_K$ be defined as

$$\begin{aligned}\widetilde{T}_N := \{(x, y, z) \in \mathcal{L}_K \times_{B_K} \mathcal{L}_K \times_{B_K} \mathcal{L}_K,\ & N(\mathrm{alb}_K(x) + \mathrm{alb}_K(y) + \mathrm{alb}_K(z)) = 0 \\ & \text{in } \operatorname{Alb} L_b,\ b := \pi(x) = \pi(y) = \pi(z)\}.\end{aligned} \tag{56}$$

The variety $\mathcal{L}_K$ has a morphism $\Phi_K : \mathcal{L}_K \to X$ composed from Φ and the natural map $\mathcal{L}_K \to \mathcal{L}$. We define T_N as the Zariski closure in $X \times X \times X$ of $(\Phi_K, \Phi_K, \Phi_K)(\widetilde{T}_N^0)$, where $\widetilde{T}_N^0$ is the union of the irreducible components of $\widetilde{T}_N$ dominating B_K, and where the point $\mathrm{alb}_K(x) + \mathrm{alb}_K(y) + \mathrm{alb}_K(z)$ is of order exactly N. It remains to show that T_N has the required properties for large N. First of all, the proof given for Proposition 2.9 works as well to show

Lemma 2.12. *One has $(\mathrm{pr}_1^*\sigma_X + \mathrm{pr}_2^*\sigma_X + \mathrm{pr}_3^*\sigma_X)_{|T_{N,\mathrm{reg}}} = 0$. Equivalently, $(\mathrm{pr}_1^*(\Phi_K^*\sigma_X) + \mathrm{pr}_2^*(\Phi_K^*\sigma_X) + \mathrm{pr}_3^*(\Phi_K^*\sigma_X))_{|\widetilde{T}_{N,\mathrm{reg}}} = 0$.*

We next observe that, if X is of dimension $2n$, $\widetilde{T}_N$ has expected dimension $3n$, which is the dimension of a triangle variety. Indeed, let $g := \dim B = \dim \operatorname{Alb} L_b$. Then, as $\dim L_b = n$,

$$\dim \mathcal{L}_K \times_{B_K} \mathcal{L}_K \times_{B_K} \mathcal{L}_K = g + 3n,$$

while from (56), we see that $\widetilde{T}_N^0$ is the inverse image of the N-torsion multisection of $\mathcal{A}_K \to B_K$ via the sum morphism $\mathrm{alb}_K \circ \mathrm{pr}_1 + \mathrm{alb}_K \circ \mathrm{pr}_2 + \mathrm{alb}_K \circ \mathrm{pr}_3$, over the regular locus B_K^0 of $\mathcal{L}_K \to B_K$. Hence the expected codimension of $\widetilde{T}_N$ is g and the expected dimension of $\widetilde{T}_N$ is $3n$. The proof of the theorem concludes with

Lemma 2.13. *Under the assumptions of the theorem, $\widetilde{T}_N^0$ is actually of dimension $3n$, the projections $\widetilde{T}_N^0 \to \mathcal{L}_K$ are dominant and the projections $\widetilde{T}_N^0 \to \mathcal{L}_K \times_{B_K} \mathcal{L}_K$ are generically finite on their images.*

Proof. By assumption, the sum map $L_t \times L_t \to \mathrm{Alb}\, L_t$ is surjective for general t, while by Lemma 2.10, the Albanese map $L_t \to \mathrm{Alb}\, L_t$ is generically finite on its image. (Here the Albanese map of L_t is computed using one of the finitely many points of $L_t \cap K$, in other words, t is taken in B_K rather than B.) This implies that for general $x \in L_t$, there is a solution to the equation

$$N(\mathrm{alb}_{L_t} x + \mathrm{alb}_{L_t} y + \mathrm{alb}_{L_t} z) = 0, \tag{57}$$

with y, $z \in L_t$. This is saying that the three projections $\widetilde{T} \to \mathcal{L}_K$ are surjective. Finally, using (57), we find that the projections $\widetilde{T}_N \to \mathcal{L}_K \times_{B_K} \mathcal{L}_K$ are generically finite on their image because the Albanese map of L_t is generically finite on its image by Lemma 2.10. $\square$

It remains to see that the same properties hold for $T_N \subset X \times X \times X \times X$. This follows from the following lemma which is proved exactly as Lemma 2.10.

Lemma 2.14. *The assumptions that the sum map $L_t \times L_t \to \mathrm{Alb}\, L_t$ is surjective is equivalent to the fact that the natural map $(\Phi, \Phi) : \mathcal{L}_K \times_{B_K} \mathcal{L}_K \to X \times X$ is generically finite on its image.*

As Φ is surjective, the fact that the projections $\mathrm{pr}_i : \widetilde{T}_N^0 \to \mathcal{L}_K$ are dominant for $i = 1, 2, 3$ implies the same property for the projections $\mathrm{pr}_i : T_N \to X$. As $(\Phi, \Phi) : \mathcal{L}_K \times_{B_K} \mathcal{L}_K \to X \times X$ is generically finite on its image, the fact that the projections $\mathrm{pr}_{ij} : \widetilde{T}_N^0 \to \mathcal{L}_K \times_{B_K} \mathcal{L}_K$ are dominant for $i = 1, 2, 3$ does not necessarily imply the same property for the projections $\mathrm{pr}_{ij} : T_N \to X \times X$, but it will imply it if N is large, using the Zariski density of torsion points. $\square$

Example 2.15. *In the case of the variety of lines $X = F_1(Y)$ of a smooth cubic fourfold, we get by applying Theorem 2.11 constructions of triangle varieties for X, different from the one constructed in Section 2.2, by using its Lagrangian covering by Fano surfaces $S_H := F_1(Y_H)$, or rather their desingularizations, where $Y_H \subset Y$ is a singular (generically 1-nodal) hyperplane section $Y \cap H$ of Y, and S_H is its surface of lines. The construction depends on the choice of a Lagrangian surface $K \subset X$.*

A similar construction can be done for the double covers of EPW sextics, using again the constructions of Iliev and Manivel.

3 Construction of surface decompositions from triangle varieties

Let X be a smooth projective variety of dimension $2n$ and $\sigma_X \in H^{2,0}(X)$ a holomorphic 2-form on X. First of all, note that from a triangle variety $T \subset X \times X \times X$, we can construct for each $k \geq 3$ a subvariety T_k of X^k of dimension kn satisfying the following property: the holomorphic 2-form $\sum_{i=1}^k \epsilon_i \mathrm{pr}_i^* \sigma_X$ vanishes on T_k, with $\epsilon_i = \pm 1$. The k-angle variety T_k is defined inductively by composition in the sense of correspondences. For $k = 4$, let

$$T'_4 = \mathrm{pr}_{1245*}(\mathrm{pr}_{123}^{-1}(T) \cap \mathrm{pr}_{345}^{-1}(T)) \subset X^4, \tag{58}$$

where the projections are defined on X^5, pr_{1245} takes value in X^4 and pr_{123}, pr_{345} take value in X^3. On $\mathrm{pr}_{123}^{-1}(T)$, one has $\mathrm{pr}_1^*\sigma_X + \mathrm{pr}_2^*\sigma_X + \mathrm{pr}_3^*\sigma_X = 0$ and on $\mathrm{pr}_{345}^{-1}(T)$, one has $\mathrm{pr}_3^*\sigma_X + \mathrm{pr}_4^*\sigma_X + \mathrm{pr}_5^*\sigma_X = 0$ so that, by subtracting, one has on the regular locus of $\mathrm{pr}_{123}^{-1}(T) \cap \mathrm{pr}_{345}^{-1}(T)$, hence also on $T'_{4,reg}$:

$$\mathrm{pr}_1^*\sigma_X + \mathrm{pr}_2^*\sigma_X - \mathrm{pr}_4^*\sigma_X - \mathrm{pr}_5^*\sigma_X = 0, \tag{59}$$

where now the projections are defined on X^4 with factors indexed by $1, 2, 4, 5$. As T dominates X by the projections, the variety T'_4 so defined also dominates X by the various projections. As the fibers of the projection $T \to X$ have dimension n, T_4' has at least one component which is of dimension $\geq 4n$. We take for T_4 the union of the irreducible components of dimension $4n$ of T_4. Note that, if X is hyper-Kähler, the 2-form σ_X is everywhere nondegenerate, so T_4 does not have components of dimension $> 4n$, because we already know by (59) that the components are Lagrangian for the holomorphic symplectic form $\mathrm{pr}_1^*\sigma_X + \mathrm{pr}_2^*\sigma_X - \mathrm{pr}_4^*\sigma_X - \mathrm{pr}_5^*\sigma_X$ on X^4. The variety T_k is similarly defined inductively by composing T_{k-1} and T.

Recall that for X as above, an algebraically coisotropic subvariety $Z \subset X$ of dimension $n+1$ admits a rational map $\tau : Z \dashrightarrow \Sigma$, where Σ is a surface and, denoting $\tau_{\mathrm{reg}} := \tau_{|Z_{reg}}$,

$$\sigma_{X|Z_{\mathrm{reg}}} = \tau_{\mathrm{reg}}^*\sigma_\Sigma$$

for some holomorphic 2-form σ_Σ on Σ.

Theorem 3.1. *Let X be a projective hyper-Kähler variety of dimension $2n$. Assume X has a triangle variety $T \subset X^3$ and an algebraically coisotropic subvariety $\tau : Z \dashrightarrow \Sigma$ of dimension $n+1$. Let $F \subset X$ be the general fiber of τ. Then if the intersection of $F^n \subset X^n$ and $\mathrm{pr}_{1\ldots n}(T_{n+1}) \subset X^n$ satisfies*

$$F^n \cdot \mathrm{pr}_{1\ldots n}(T_{n+1}) \neq 0, \tag{60}$$

X admits a surface decomposition. In particular it admits mobile algebraically coisotropic subvarieties of any codimension $\leq n$.

In (60), we have $\dim F = n-1$, so $\dim F^n = n(n-1)$ and $\dim T_{n+1} = n(n+1)$, while the intersection takes place in X^n which has dimension $2n^2 = n(n-1) + n(n+1)$.

Proof. We construct $\phi : \Gamma_0 \to X$, $\psi : \Gamma_0 \to \Sigma^n$ by the formulas

$$\begin{gathered}\Gamma_0 = \mathrm{pr}_{1\ldots n}^{-1}(Z^n) \cap T_{n+1} \subset T_{n+1} \subset X^{n+1}, \\ \phi := \mathrm{pr}_{n+1} : \Gamma_0 \to X,\ \psi = \tau^n \circ \mathrm{pr}_{1\ldots n} : \Gamma_0 \to \Sigma^n.\end{gathered} \tag{61}$$

As $\Gamma_0 \subset T_{n+1}$, the form $\sum_{i=1}^{n+1} \epsilon_i \mathrm{pr}_i^*\sigma_X$ vanishes on $\Gamma_{0,reg}$, where the ϵ_i are the signs introduced in the construction of T_{n+1}. In other words, using $\phi = \mathrm{pr}_{n+1}$

$$\phi^*\sigma_X = \sum_{i=1}^n \epsilon_i' \mathrm{pr}_i^*\sigma_{X|\Gamma_{0,reg}} \tag{62}$$

in $H^0(\Omega^2_{\Gamma_{0,reg}})$, where $\epsilon_i' = \pm\epsilon_i$. We next use the fact that $\mathrm{pr}_i(\Gamma) \subset Z$ and that $\sigma_{X|Z} = \tau^*(\sigma_\Sigma)$. We then get the desired formula characterizing a surface decomposition

$$\phi^*\sigma_X = \psi^*(\sum_{i=1}^n \epsilon_i' \mathrm{pr}_i^*\sigma_\Sigma) \text{ in } H^{2,0}(\Gamma_{0,reg}). \tag{63}$$

We need to show that ϕ and ψ are dominant, and that we can assume that they are generically finite. The fact that ψ is dominant is a consequence of (60), which can be seen as saying that $\mathrm{pr}_{1\ldots n}^{-1}(Z^n) \cap T_{n+1}$ intersects nontrivially the fibers of $\tau^n : \mathrm{pr}_{1\ldots n}^{-1}(Z^n) \to \Sigma^n$. Knowing that ψ is dominant, we conclude that the form $\psi^*(\sum_{i=1}^n \epsilon_i' \mathrm{pr}_i^*\sigma_\Sigma)$ has generic rank $2n$ on $\Gamma_{0,reg}$. It thus follows from (63) that $\phi^*\sigma_X$ has generic rank $2n$ on Γ, hence that ϕ is also dominant. The last argument applies to any irreducible component Γ_0' of Γ_0 dominating Σ^n, which thus also has to dominate X. Finally, by cutting Γ_0' by hyperplane sections and reapplying the same arguments if necessary, we get a Γ which is generically finite onto both Σ^n and X, and still satisfies (63). ☐

We also have the following result, whose proof is a variant of that of Theorem 3.1, and shows how to construct new algebraically coisotropic subvarieties out of old ones, using a triangle variety:

Let X be smooth projective variety of dimension $2n$ with an everywhere nondegenerate holomorphic 2-form σ_X. Denote by $I_r \subset X$ an algebraically coisotropic subvariety of X of codimension $n-r$. Hence there exists a rational map

$$\phi_r : I_r \dashrightarrow B_r$$

to a smooth projective variety B_r of dimension $2r$, with general fiber F_r of dimension $n-r$, such that

$$\sigma_{X|I_r} = \phi_r^* \sigma_{B_r}, \tag{64}$$

for some holomorphic 2-form σ_{B_r} on B_r which is generically of maximal rank $2r = \dim B_r$.

Theorem 3.2. *Assume that X has a triangle variety T relative to σ_X and let I_r, $I_{r'}$ be two algebraically coisotropic varieties of X of respective codimensions r, r'. Assume that*

(*) the class $\mathrm{pr}_{3*}([T] \cup \mathrm{pr}_{12}^*([I_r \times I_{r'}]))$ is nonzero in $H^{2n-2r-2r'}(X, \mathbb{Q})$ (so in particular $r + r' \leq n$).

Then $\mathrm{pr}_3(T \cap \mathrm{pr}_{12}^{-1}(I_r \times I_{r'})) \subset X$ contains an algebraically coisotropic subvariety $I_{r+r'}$ of codimension $n - r - r'$.

As usual, pr_i and pr_{ij} denote the projections from $X \times X \times X$ to its factors, or products of two factors.

Proof. The variety $Y := T \cap \mathrm{pr}_{12}^{-1}(I_r \times I_{r'}) \subset X \times X \times X$ maps to $B_r \times B_{r'}$ by the map $\phi_{r+r'} := (\phi_r, \phi_{r'}) \circ \mathrm{pr}_{12|Y}$. By the definition of a triangle variety and using (64), we get that

$$\mathrm{pr}_3^* \sigma_{X|Y} = -\phi_{r+r'}^*(\mathrm{pr}_1^* \sigma_{B_r} + \mathrm{pr}_2^* \sigma_{B_{r'}}) \text{ in } H^{2,0}(Y_{\mathrm{reg}}). \tag{65}$$

It thus follows that the rank of $\mathrm{pr}_3^* \sigma_X$ restricted to Y_{reg} is nowhere greater than $2r + 2r'$. On the other hand, Condition (*) implies that $\mathrm{pr}_3(Y)$ has at least one component of dimension $\geq n+r+r'$. This component thus must have dimension exactly $n+r+r'$ and is coisotropic. This is the desired variety $I_{r+r'}$, and it is in fact algebraically coisotropic, choosing a subvariety $Y' \subset Y$ mapping to $I_{r+r'}$ in a generically finite way and using the diagram

$$\begin{tikzcd} Y' \arrow[r, "\mathrm{pr}_3"] \arrow[d, "(\phi_r, \phi_{r'}) \circ \mathrm{pr}_{12}"] & I_{r+r'} \\ B_r \times B_{r'} & \end{tikzcd}, \tag{66}$$

in which $\mathrm{pr}_3^* \sigma_{X|Y'} = \phi_r^* \sigma_{B_r} + \phi_{r'}^* \sigma_{B_{r'}}$. □

As a consequence of Theorem 3.1 (or using methods similar as above), we conclude now that many explicitly constructed projective hyper-Kähler manifolds admit a surface decomposition :

Theorem 3.3. *The following hyper-Kähler manifolds admit surface decompositions:*

1. *The Fano variety of lines $X = F_1(Y)$ of a cubic fourfold Y (see [5]).*
2. *The Debarre-Voisin hyper-Kähler fourfold (see [10]).*
3. *The double EPW sextics (see [24]).*
4. *The LLSvS hyper-Kähler 8-fold (see [20]).*
5. *The LSV compactification of the intermediate Jacobian fibration associated with a cubic fourfold (see [19]).*

Proof of cases 1 and 2. The case of the Beauville-Donagi hyper-Kähler fourfold $X = F_1(Y)$ is done as follows: recall first that $F_1(Y)$ has an ample (singular) uniruled divisor D which can be constructed using the rational self-map of degree 16

$$\phi : X \dashrightarrow X$$

constructed in [26]. This map associates to a general point $[l]$ parameterizing a line $l \subset Y$ the point $[l']$ parameterizing the line $l' \subset Y$ such that there is a unique plane $P \subset \mathbb{P}^5$ with $P \cap Y = 2l + l'$. It satisfies the property that

$$\phi^* \sigma_X = -2\sigma_X. \tag{67}$$

This map has indeterminacies when the plane P is not unique, and this happens along a surface Σ which is studied in [1]. After blowing-up Σ, the map ϕ becomes a morphism $\tilde{\phi} : \widetilde{F_1(Y)} \to F_1(Y)$ which is finite (see [1]). The image of the exceptional divisor E under $\tilde{\phi}$ is thus a uniruled divisor E' in $F_1(Y)$ which has in fact normalization isomorphic to E. We thus have a diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{\phi}_E"] \arrow[d, "\tau_E"] & E' \subset X = F_1(Y) \\ \Sigma & \end{tikzcd} , \tag{68}$$

where τ_E is the restriction of the blowing-up morphism to E, such that $\tilde{\phi}_E^* \sigma_X = \tau_E^* \sigma_\Sigma$ for some holomorphic 2-form σ_Σ on Σ.

On the other hand, we have the triangle variety $T \subset X \times X \times X$ described in Section 2.2. We thus have the ingredients needed to apply Theorem 3.1, but we have to check the condition (60). This is easy because $\mathrm{pr}_{12}(T) \subset X \times X$ has codimension 2, and the classes of the fibers F of $\tau_E : E' \to \Sigma$ must be proportional to h^3, where h is the first Chern class of an ample line bundle on X, because $\rho(X) = 1$. Hence the intersection number $(F \times F) \cdot \mathrm{pr}_{12}(T)$ is nonnegative. We claim that the intersection number is strictly positive. If the intersection number is 0, then for a general complete intersection curve $C \subset X$, the 3-fold $(C \times X) \cap \mathrm{pr}_{12}(T)$ does not map to a 3-fold by the second projection to X. This implies that $\mathrm{pr}_{12}(T)$ contains $C \times \Sigma$ for some surface $\Sigma \subset X$. Recalling that $X = F_1(Y)$ and that $\mathrm{pr}_{12}(T)$ is the set of pairs of intersecting lines in Y, we can easily contradict this conclusion, proving the claim. We thus get a surface decomposition given by

$$\Gamma = \mathrm{pr}_{12}^{-1}(E \times E) \cap T \subset X^3, \tag{69}$$
$$\psi = (\tau_E, \tau_E) \circ \mathrm{pr}_{12} : \Gamma \to \Sigma \times \Sigma,\ \phi = \mathrm{pr}_{3|\Gamma} : \Gamma \to X.$$

The proof in the case 2 works similarly. We use on the one hand the triangle variety T constructed by Bazhov (see [3] or Section 2.3), and on the other hand the existence of a uniruled divisor $\tau : D \to \Sigma$, $D \to X$ that we can exhibit either by looking at the indeterminacies of Bazhov's construction, or by applying [9], using the fact that the Debarre-Voisin fourfold has the deformation type of $K3^{[2]}$. As the very general Debarre-Voisin varieties have Picard number 1, the fibers of a uniruled divisors have as before a class proportional to h^3, where h is an ample divisor class, hence the variety

$$\mathrm{pr}_{12}^{-1}(D \times D) \subset T \subset X \times X \times X$$

dominates $\Sigma \times \Sigma$ by $(\tau, \tau) \circ \mathrm{pr}_{12}$, hence X by the projection pr_3. The rest of the argument is identical. □

Proof of cases 4 and 5. The LLSvS manifold $F_3'(Y)$ is a hyper-Kähler 8-fold contructed in [20] as a smooth hyper-Kähler model of the basis of the rationally connected fibration of the Hilbert scheme $F_3(Y)$ of degree 3 rational curves in a smooth cubic fourfold X not containing a plane. In the paper [29], we constructed a dominating rational map

$$\mu : F_1(Y) \times F_1(Y) \dashrightarrow F_3'(Y).$$

Two non intersecting lines l, l' in Y generate a $\mathbb{P}^3$ which intersects Y along a cubic surface S. The $\mathbb{P}^2$ of degree 3 rational curves corresponding to $\mu(l, l')$ is the linear system $|h+l-l'|$ on S, where $h = \mathcal{O}_S(1)$. It follows from this formula, Mumford's theorem and the fact that the holomorphic 2-forms on the considered varieties come from a $(3,1)$-class on Y by the corresponding incidence correspondences, that

$$\mu^*\sigma_{F_3'(Y)} = \mathrm{pr}_1^*\sigma_{F_1(Y)} - \mathrm{pr}_2^*\sigma_{F_1(Y)}. \tag{70}$$

Together with Case 1, this immediately gives us a surface decomposition for $F_3'(Y)$. Indeed, we have the surface decomposition $\phi : \Gamma \to F_1(Y)$, $\psi : \Gamma \to \Sigma \times \Sigma$ for $F_1(Y)$ of (69). The maps satisfy

$$\phi^*\sigma_{F_1(Y)} = \psi^*(\mathrm{pr}_1^*\sigma_\Sigma + \mathrm{pr}_2^*\sigma_\Sigma). \tag{71}$$

Taking products, we get

$$\phi' : \Gamma \times \Gamma \to F_1(Y) \times F_1(Y),\ \psi' : \Gamma \times \Gamma \to \Sigma \times \Sigma \times \Sigma \times \Sigma.$$

Composing the first map with μ and desingularizing, we get

$$\phi'' : \widetilde{\Gamma \times \Gamma} \to F_3(Y),\ \psi'' : \widetilde{\Gamma \times \Gamma} \to \Sigma \times \Sigma \times \Sigma \times \Sigma. \tag{72}$$

By (71) and (70), the morphisms in (72) satisfy

$$\phi''^*\sigma_{F_3(Y)} = \psi''^*(\mathrm{pr}_1^*\sigma_\Sigma + \mathrm{pr}_2^*\sigma_\Sigma - \mathrm{pr}_3^*\sigma_\Sigma - \mathrm{pr}_4^*\sigma_\Sigma), \tag{73}$$

which gives the desired decomposition in case 4.

We now turn to the LSV hyper-Kähler fourfold $J(Y)$. It is a 10-dimensional hyper-Kähler manifold associated to a general cubic fourfold Y. As it has a Lagrangian fibration, we will be able to use the triangle variety described in Section 2.5. Another ingredient we will use is the following:

Lemma 3.4. *There exists a codimension* 3 *algebraically coisotropic subvariety of* $J(Y)$ *which is birational to a* $\mathbb{P}^3$*-bundle over* $F_1(Y)$*.*

Proof. For each line $l \subset Y \subset \mathbb{P}^5$, there is a $\mathbb{P}^3 \subset (\mathbb{P}^5)^*$ of hyperplane sections of Y containing l. This determines a $\mathbb{P}^3$-bundle $P \to F_1(Y)$. Each of these hyperplanes H determines a hyperplane section Y_H of Y. Then $l \subset Y_H$ and the 1-cycle $3l - h^2$ is homologous to 0 on Y_H, at least assuming Y_H smooth, which allows to define a point $\Psi_{Y_H}(3l - h^2) \in J(Y_H)$ using the Abel-Jacobi map Ψ_{Y_H} of Y_H. As $J(Y)$ is fibered over $(\mathbb{P}^5)^*$ into intermediate Jacobians $J(Y_H)$, we thus constructed the desired rational map $P \dashrightarrow J(Y)$. It is not hard to see that this map is birational onto its image P' which has thus dimension 7. As the only holomorphic 2-forms on P are those coming from $F_1(Y)$, we conclude that $P' \subset J(Y)$ is algebraically coisotropic. □

Corollary 3.5. *There exists an algebraically coisotropic subvariety* Z *of* $J(Y)$ *which has codimension* 4 *(dimension* 6*). This variety dominates* $(\mathbb{P}^5)^*$*.*

Proof. We use for this the existence of the uniruled divisor $E' \subset F_1(Y)$ appearing in (68). Let now $P_{E'}$ be the inverse image of E' in P and let Z be its image in $J(Y)$. We have to prove that Z dominates $(\mathbb{P}^5)^*$. This is saying that any hyperplane section Y_H of Y contains a line residual to a special line of Y, which is implied by the fact that no hyperplane section of Y contains a 2-dimensional family of lines residual to a special line of Y. The last statement is proved in [19]. □

We now construct a surface decomposition for $J(Y)$: Let Z be as in Corollary 3.5, so Z is fibered into curves over B. We use the sum map on the fibers of the fibration $J(Y) \to B := (\mathbb{P}^5)^*$. We then get a morphism:

$$\mu_{Z,5} : Z \times_B \ldots \times_B Z \dashrightarrow J(Y),$$

$$(a_1, \ldots, a_5) \mapsto \sum_i a_i.$$

We first observe that μ_5 is dominant. This is because the fibers of $J(Y) \to B$ are generically irreducible abelian varieties and the fibers of $Z \to B$ are curves Z_H which must generate $J(Y_H)$, which is 5-dimensional. Finally, it remains to prove that the construction above provides a surface decomposition for $J(Y)$. First of all, by Proposition 2.9, for the relative sum map

$$\mu_5 : J(Y) \times_B \ldots \times_B J(Y) \dashrightarrow J(Y),$$

$$(a_1, \ldots, a_5) \mapsto \sum_i a_i,$$

one has

$$\mu_5^*(\sigma_{J(Y)}) = \sum_i \mathrm{pr}_i^* \sigma_{J(Y)}. \tag{74}$$

Next we use the rational map $f : Z \to \Sigma$ which is the composition of $Z \to E' \subset F_1(Y)$, with E' birational to E, and $\tau_E : E \to \Sigma$. We clearly have

$$f^* \sigma_\Sigma = \sigma_{J|Z}. \tag{75}$$

Next f induces a morphism

$$f_5 : Z \times_B \ldots \times_B Z \to \Sigma^5$$

and combining (74) and (75), one concludes that the diagram

$$\begin{tikzcd} Z \times_B \ldots \times_B Z \arrow[r, "\mu_5"] \arrow[d, "f_5"] & J(Y) \\ \Sigma^5 & \end{tikzcd} \tag{76}$$

provides a surface decomposition of $J(Y)$. □

References

[1] E. Amerik. A computation of invariants of a rational self-map. Ann. Fac. Sci. Toulouse Math. (6) 18 (2009), no. 3, 445-457.

[2] E. Amerik, F. Campana. Characteristic foliation on non-uniruled smooth divisors on hyperkähler manifolds. J. Lond. Math. Soc. (2) 95 (2017), no. 1, 115-127.

[3] I. Bazhov. On the variety of triangles for a hyper-Kaehler fourfold constructed by Debarre and Voisin, J. Pure Appl. Algebra 223 (2019), no. 6, 2530-2542.

[4] A. Beauville. Variétes Kählériennes dont la première classe de Chern est nulle, J. Differential Geom. 18, no. 4, 755-782 (1984).

[5] A. Beauville, R. Donagi. La variété des droites d'une hypersurface cubique de dimension 4, *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985), 703–706.

[6] A. Beauville. On the splitting of the Bloch-Beilinson filtration, in *Algebraic cycles and motives* (vol. 2), London Math. Soc. Lecture Notes 344, 38-53; Cambridge University Press (2007).

[7] A. Beauville, C. Voisin. On the Chow ring of a $K3$ surface, J. Algebraic Geom. 13 (2004), no. 3, 417–426.

[8] F. Bogomolov. On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky). Geom. Funct. Anal. 6 (1996), no. 4, 612-618.

[9] F. Charles, G. Mongardi, G. Pacienza. Families of rational curves on holomorphic symplectic varieties and applications to 0-cycles, (a corrected version of arXiv:1401.4071).

[10] O. Debarre, C. Voisin. Hyper-Kähler fourfolds and Grassmann geometry, J. reine angew. Math. 649 (2010), 63-87.

[11] P. Deligne. Théorie de Hodge II, Publ. Math. IHES 40 (1971), 5-57.

[12] D. Eisenbud, S. Popescu and C. H. Walter. Lagrangian subbundles and codimension 3 subcanonical subschemes, Duke Math. J. 107 (2001), no. 3, 427-467.

[13] L. Fu. Beauville-Voisin conjecture for generalized Kummer varieties, Int. Math. Res. Not. IMRN 2015, no. 12, 3878-3898.

[14] A. Fujiki. On the de Rham cohomology group of a compact Kähler symplectic manifold. Algebraic geometry, Sendai, 1985, 105-165, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[15] B. van Geemen, C. Voisin. On a conjecture of Matsushita, Int. Math. Res. Not. IMRN 2016, no. 10, 3111-3123.

[16] D. Huybrechts. Curves and cycles on K3 surfaces. Algebr. Geom. 1 (2014), no. 1, 69-106.

[17] A. Iliev, L. Manivel. Fano manifolds of degree ten and EPW sextics. Ann. Sci. Ec. Norm. Supér. (4) 44 (2011), no. 3, 393-426.

[18] J. Kollár, S. Miyaoka, S. Mori. Rationally connected varieties. J. Algebraic Geom. 1 (1992), no. 3, 429-448.

[19] R. Laza, G. Saccà, C. Voisin. A hyper-Kähler compactification of the intermediate Jacobian fibration associated with a cubic 4-fold. Acta Math. 218 (2017), no. 1, 55-135.

[20] C. Lehn, M. Lehn, C. Sorger, D. van Straten. Twisted cubics on cubic fourfolds. J. Reine Angew. Math. 731 (2017), 87-128.

[21] H.-Y. Lin. Lagrangian constant cycle subvarieties in Lagrangian fibrations, arXiv:1510.01437, to appear in IMRN.

[22] D. Maulik, A. Negut. Lehn's formula in Chow and Conjectures of Beauville and Voisin, arXiv:1904.05262.

[23] D. Mumford. Rational equivalence of 0-cycles on surfaces. J. Math. Kyoto Univ. 9 (1968) 195-204.

[24] K. O'Grady. Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics. Duke Math. J. 134 (2006), no. 1, 99-137.

[25] M. Shen, C. Vial. The Fourier transform for certain hyperkähler fourfolds. Mem. Amer. Math. Soc. 240 (2016), no. 1139.

[26] C. Voisin. Intrinsic pseudo-volume forms and K-correspondences, in The Fano Conference, 761-792, Univ. Torino, Turin, (2004).

[27] C. Voisin. Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes, in *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, 294-303, London Math. Soc. Lecture Note Ser., 179, Cambridge Univ. Press, Cambridge, 1992.

[28] C. Voisin. *Hodge theory and complex algebraic geometry* II. Cambridge Studies in Advanced Mathematics, 77. Cambridge University Press, Cambridge, (2003).

[29] C. Voisin. Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties. in *K3 surfaces and their moduli*, 365-399, Progr. Math., 315, Birkhäuser/Springer, 2016.

[30] C. Voisin. On the Chow ring of certain algebraic hyper-Kähler manifolds, Pure and Applied Mathematics Quarterly, Volume 4, Number 3, (Special issue in honor of Fedya Bogomolov), (2008).

Collège de France, 3 rue d'Ulm, 75005 Paris FRANCE
claire.voisin@imj-prg.fr