

Unboundedness of zero-cycles on higher dimensional Fano manifolds

Claire Voisin*

Abstract

We show that, unlike del Pezzo surfaces, higher dimensional Fano manifolds do not satisfy in general boundedness properties for their CH_0 group of 0-cycles. For example, for quartic threefolds having a point of odd degree, there is no “Coray type” upperbound on the minimal odd degrees of points. Also, the CH_0 -group of Fano hypersurfaces can be “unbounded” (a notion which is related to infinite dimensionality in the sense of Mumford), meaning that there is no integer N such that 0-cycles of degree at least N are effective.

1 Introduction

In this paper, a variety defined over a field K is reduced and geometrically irreducible, that is, irreducible over \overline{K} . In particular, the Fano varieties, including del Pezzo surfaces, that we will consider, are smooth projective, geometrically irreducible, with ample anticanonical bundle. For a projective variety X over a field K , and for a finite extension $K \subset L$, the degree of a L -point x of X is defined to be the degree of the field extension $K \subset L$. This is also the degree of x seen as a 0-cycle of X over K . It is expected by a conjecture of Cassels and Swinnerton-Dyer that any smooth cubic hypersurface with a point of degree coprime to 3 has a K -point. In [5] and [22], the following results on points on del Pezzo surfaces of degrees 2 and 3 are proved, improving a classical result of Coray [7].

Theorem 1.1. (i) (Colliot-Thélène [5]) *Let S be a del Pezzo surface of degree 2 over a field K of characteristic 0. Then if S has a 0-cycle of degree 1 (equivalently, if S has a point of odd degree), S has a closed point of degree 1, 3 or 7.*

(ii) (Voisin [22]) *Let S be a smooth cubic surface defined over a field K of characteristic 0. If S has a 0-cycle of degree 1 (equivalently, if S has a point of degree coprime to 3), S has a closed point of degree 1 or 4.*

Coray had established a weaker version of statement (ii), allowing also points of degree 10. Concerning statement (i), Kollár and Mella noticed in [14] that there are del Pezzo surfaces of degree 2 with a point of degree 3 but no K -point (see [5, Remarque 4.3] for a complete description of their argument).

Coming back to the case of cubic hypersurfaces, the following weak version of Cassels-Swinnerton-Dyer conjecture asking for the existence of higher dimensional Coray-type bounds seems to be open.

Question 1.2. *Does there exist a number $N(n)$ such that any smooth cubic hypersurface X of dimension n over a field K of characteristic 0 having a 0-cycle of degree 1 has a point of degree at most $N(n)$ coprime to 3?*

Colliot-Thélène proved more generally in [5] that Coray type bounds as in Theorem 1.1 exist for any class of del Pezzo surfaces. It is thus natural to ask Question 1.2 for higher dimensional Fano varieties, as suggested in [5]. One natural extension of Question 1.2 is the following

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Question 1.3. *Does there exist a number $N(n)$ such that any smooth Fano degree d hypersurface $X \subset \mathbb{P}^{n+1}$ over a field K of characteristic 0 having a 0-cycle of degree 1 has closed points of degree $d_i \leq N(n)$ that are globally coprime?*

A weaker form of this question is whether such a bound N exists depending on X defined over K but working for any X_L , for all field extensions $K \subset L$. Our first main result in this paper is a negative answer to (the weak form of) Question 1.3.

Theorem 1.4. *(i) There exist a field K of characteristic 0 and a quartic hypersurface $X \subset \mathbb{P}^4$ defined over K , with the following property. For any odd integer N , there exists an overfield $L \supseteq K$ such that X_L has a closed point of degree N and no closed point of odd degree strictly smaller than N .*

(ii) The same statement holds for quartic hypersurfaces in \mathbb{P}^5 and quartic or sextic double solids instead of quartic hypersurfaces of dimension 3.

Remark 1.5. For the generic hypersurface $X_{\text{gen}} \subset \mathbb{P}^{n+1}$ of degree d , denoting by $h_X := c_1(\mathcal{O}_{X_{\text{gen}}}(1))^n \in \text{CH}_0(X_{\text{gen}})$, one has $\deg h_X = d$ and $\text{CH}_0(X_{\text{gen}}) = \mathbb{Z}h_X$ (see Lemma 3.4). Our hypersurface will be the hypersurface X_{gen} , which is defined over the function field K of $\mathbb{P}(H^0(\mathbb{P}_{\mathbb{C}}^4, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^4}(4)))$. Here \mathbb{C} could be replaced by any field of characteristic 0. The field L (depending on N) will be obtained by adding universally a point of degree N . These fields have big transcendence degrees over \mathbb{Q} .

Theorem 1.4 says in particular that a Coray type bound does not exist for Fano 3-folds in general. Statement (ii) is more striking in view of Theorem 1.1(i), since quartic double solids are Fano threefolds of index 2 and their hyperplane sections are del Pezzo surfaces of degree 2. Their geometry is very close to that of cubic threefolds, for which Question 1.2 is still open.

We will also study in this paper a slightly different notion of (un)boundedness for zero-cycles, that has been investigated first by Mumford in [16] in the geometric setting and later by Colliot-Thélène in [5] over non-algebraically closed fields. Consider a smooth projective variety X defined over a field K of characteristic 0. Let us consider the following

Definition 1.6. *We will say that X has bounded CH_0 -group if there exists a number N (possibly depending on X and K), such that for any field $L \supseteq K$, any 0-cycle $z \in \text{CH}_0(X_L)$ of degree $\deg z \geq N$ is effective (that is, rationally equivalent to an effective 0-cycle).*

We will say that X has unbounded CH_0 -group if an integer N as above does not exist. We note that smooth projective curves X have bounded CH_0 -group, as follows from the Riemann-Roch theorem. It follows that a variety X has bounded CH_0 -group if $\text{CH}_0(X)$ is strongly representable, in the sense that there exists a curve $C \xrightarrow{j} X$ such that for any field $L \supseteq K$, the morphism

$$j_* : \text{CH}_0(C_L) \rightarrow \text{CH}_0(X_L)$$

is surjective.

One should note that for varieties defined over the complex numbers, the notion of (un)boundedness introduced above is close to, but different from, the notion of (in)finite dimensionality used by Mumford in [16]. Indeed, Mumford only considers $\text{CH}_0(X)$, not $\text{CH}_0(X_L)$ for any field L containing \mathbb{C} . For a smooth projective variety X over the complex numbers, it is in fact proved by Mumford [16] and Roitman [17], [18], that Mumford's finite dimensionality is equivalent to the representability of $\text{CH}_0(X)$ in the weaker sense that there exists a curve $C \xrightarrow{j} X$ such that the morphism $j_* : \text{CH}_0(C) \rightarrow \text{CH}_0(X)$ is surjective. This is also equivalent to the fact that the Albanese map induces an isomorphism between the group $\text{CH}_0(X)_0$ of zero-cycles of degree 0 on X and the (group of complex points of the) Albanese variety $\text{Alb}(X)$.

For varieties over \mathbb{C} , the two notions (boundedness as above and finite dimensionality in the Mumford sense) coincide if one considers Chow groups with rational coefficients, thanks to the Bloch-Srinivas argument [1].

Inspired by the work of Colliot-Thélène [5], we proved in [22] the following

Theorem 1.7. *Let S be a smooth del Pezzo surface of degree $d_S \leq 3$, where $d_S := \deg c_1(K_S)^2$, defined over a field of characteristic 0. Then any 0-cycle $z \in \text{CH}_0(S)$ of degree $d = \deg z \geq 21$ is effective, hence $\text{CH}_0(S)$ is bounded.*

Colliot-Thélène establishes Theorem 1.7 for rational surfaces over a field K of characteristic 0 having a K -point. Applying the same method as in [22], we will extend in Section 4 Theorem 1.7 to del Pezzo surfaces of degree d_S , with $4 \leq d_S \leq 9$.

Theorem 1.8. *There exists an integer N with the following property: for any smooth del Pezzo surface S defined over a field K of characteristic 0, any 0-cycle $z \in \text{CH}_0(S)$ of degree $d = \deg z \geq N$ is effective. In particular, $\text{CH}_0(S)$ is bounded.*

Theorem 1.8 proves boundedness of the CH_0 group of del Pezzo surfaces with a number N which is even independent of S . The integer N can easily be made explicit as in Theorem 1.7 by examining the numerics case by case, since there are finitely many families of del Pezzo surfaces. In the case of smooth projective geometrically integral curves X , the number N can be taken to be the genus of X , hence depends only on the deformation class of X .

Remark 1.9. As Colliot-Thélène mentioned to me, for del Pezzo surfaces of degree ≥ 5 , it is possible to give another proof of the fact that their CH_0 group is bounded, building on many beautiful known results on the arithmetic of del Pezzo surfaces of degree ≥ 5 (see [21], [2], [12], [4]). It is likely that one can even prove that they are uniformly bounded as above with a better bound than the one obtained here.

When X is a Fano variety over a field K of characteristic 0, X is geometrically rationally connected, so we have $\text{CH}_0(X_{\bar{K}}) = \mathbb{Z}$, because any two points $x, y \in X(\bar{K})$ belong to a rational curve in X defined over \bar{K} (see [11]). The Chow group $\text{CH}_0(X)_0$ of 0-cycles of degree 0 is thus of torsion, and more precisely, there exists an integer M (depending on X) such that for any field $L \supseteq K$, $\text{CH}_0(X_L)_0$ is of M -torsion. This follows from a decomposition of the diagonal argument as in [1], or directly using the rational connectedness, that provides over a finite extension $K' \supseteq K$ of degree d , a dominant rational map

$$\phi : Y \times \mathbb{P}^1 \dashrightarrow X$$

mapping $Y \times \{0\}$ to a given point $x \in X(K')$, and such that the restriction of ϕ to $Y \times \{\infty\}$ is dominant of degree N . Then by a trace argument, for any field extension $L \supseteq K$, any 0-cycle of degree 0 on X_L is annihilated by $M := Nd$.

At the opposite, when S is a surface over \mathbb{C} with $p_g(S) \neq 0$, or more generally any smooth projective variety X over \mathbb{C} with $H^{l,0}(X) \neq 0$ for some $l \geq 2$, Mumford's celebrated theorem [16], later generalized by Rojzman [17], is the following

Theorem 1.10. ([16], [17]) *Let X be a smooth projective variety over \mathbb{C} such that*

$$H^0(X, \Omega_{X/\mathbb{C}}^l) \neq 0 \text{ for some } l \geq 2.$$

Then, choosing a point $x \in X(\mathbb{C})$, for any $N \geq 0$, the natural map

$$X^{(N)}(\mathbb{C}) \rightarrow \text{CH}_0(X)_0/\text{Torsion},$$

$$z \mapsto z - Nx$$

is not surjective.

In the theorem above, $X^{(N)}$ denotes the N -th symmetric product of X . The conclusion says that for any integer $N \geq 0$, there exists a cycle $z_0 \in \text{CH}_0(X_{\mathbb{C}})_0$ such that the 0-cycle $z_0 + Nx \in \text{CH}_0(X_{\mathbb{C}})$, which is of degree N , is not effective modulo torsion. This shows that X has unbounded CH_0 -group in the sense of Definition 1.6, but also that it has infinite dimensional CH_0 -group in the Mumford sense.

In the case where X is a smooth Fano variety defined over a field K of characteristic 0, we know that $H^0(X, \Omega_{X/K}^l) = 0$ for any $l > 0$, and furthermore we know as explained above that $\text{CH}_0(X)_0$ is universally of torsion, so a priori we cannot directly use Mumford's strategy. Our second main result is

Theorem 1.11. *Assume d is even, $n \geq 3$ and $d \geq 2\lceil \frac{n+2}{3} \rceil$. Then the very general degree d hypersurface $X \subset \mathbb{P}^{n+1}$ defined over \mathbb{C} has unbounded CH_0 -group in the sense of Definition 1.6.*

There exist smooth Fano degree d hypersurfaces $X \subset \mathbb{P}^{n+1}$ defined over $\overline{\mathbb{Q}}$, with unbounded CH_0 -group.

The same results hold for double covers of \mathbb{P}^n , $n \geq 3$, ramified along a smooth hypersurface of degree $2d \geq n + 1$.

The proof of Theorem 1.11 will be obtained by combining the arguments of Mumford [16] and Totaro [20]. We will start by reviewing Mumford's argument in Section 2 and we will present it in the following form, inspired by Bloch-Srinivas' paper [1], which works as well in nonzero characteristic. For any smooth projective variety X of dimension n over a field K , we denote by $X_0^{(N)}$ the Zariski open set of $X_0^{(N)}$ parameterizing unordered sets of N distinct points in X and by $\Gamma_N \subset X_0^{(N)} \times X$ the universal subscheme.

Theorem 1.12. *Assume there exists an effective cycle $Z \in \text{CH}^n(X_0^{(N)} \times X)$ with the property that the degree of Z over $X_0^{(N)}$ is $N' < N$, and a 0-cycle $z_0 \in \text{CH}_0(X)$ of degree $N - N'$, such that, over the generic point η of $X_0^{(N)}$, one has*

$$\Gamma_{N,\eta} = Z_\eta + z_{0,\eta} \text{ in } \text{CH}_0(X_\eta), \quad (1)$$

where $X_\eta := X \times \text{Spec } K(X_0^{(N)})$ and so on. Then one has $H^0(X, \Omega_{X/K}^l) = 0$, assuming we are in one of the following situations.

1. One has $l \geq 2$ and $\text{char } K = 0$.
2. One has $l = 2$ and $\text{char } K$ is arbitrary.
3. One has $l = 3$, $\text{char } K = 2$ and the dimension of X is ≤ 5 .
4. One has $l \geq 2$, $\text{char } K$ is arbitrary and $N \geq \frac{N'(n+1)}{l}$.

Remark 1.13. The case 1 of characteristic 0 is due to Mumford [16] and Roitman [17].

Case 3 may seem special but it will be needed for the proof of Theorem 1.4. The reason to limit ourselves to the dimensions 3, 4 and 5 in case 3 is the fact that, for these dimensions, the tensor rank of a l -form on X is lower-semicontinuous (see [13]), so the tensor rank at any point is not greater than the tensor rank at a generic point, a property that is not satisfied when $3 \leq l \leq n - 3$, see Remark 2.7 for more detail.

Case 4 is the one used for the proof of theorem 1.11.

As a corollary of Theorem 1.12, 4 we get the following analogue of Mumford's theorem 1.10.

Corollary 1.14. *Let X be a smooth projective variety over any field K . If $\text{CH}_0(X)$ is bounded, then $h^0(X, \Omega_{X/K}^l) = 0$ for $l \geq 2$.*

Indeed, if $\text{CH}_0(X)$ is bounded, there exists a number N' such that for any surfield $L \supseteq K$, any 0-cycle $z \in \text{CH}_0(X_L)$ of degree at least N' is effective. Choose $z_0 \in \text{CH}_0(X)$ of degree $d > 0$, and let $N := N' + ld$. Let $L := K(X_0^{(N)})$. The 0-cycle $\Gamma_{N,\eta} - lz_{0,\eta} \in \text{CH}_0(X_L)$ is of degree N' , hence effective, thus producing an effective cycle $Z' \in \text{CH}^n(X_0^{(N)} \times X)$ as above, of degree N' over $X_0^{(N)}$, and satisfying (1). As N is arbitrarily large and N' is fixed, we can apply Theorem 1.12, 4.

The paper is organized as follows. In Section 2, we will recall Mumford's argument and prove Theorem 1.12. In Section 3, we will give the proof of Theorems 1.11 and 1.4. In Section 4, we explain the method of proof of Theorem 1.8, following [22].

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2 A review and extension of Mumford's results on infinite dimensionality of CH_0

Our goal in this section is to prove Theorem 1.12. Our plan is to adapt Mumford's argument in [16] (proving over the complex numbers infinite dimensionality of CH_0 in presence of differential forms of degree > 1) to prove over any field unboundedness of CH_0 under the same assumptions, but we will meet difficulties due to inseparability that prevent to directly apply Proposition 2.2 below. We first review Mumford's theorem, adapted following [20] to work over any field. Let X be a smooth projective variety of dimension n over a field K , and let $X^{(N)}$ be the N -th symmetric product of X . It is singular but we will work with the open set $X_0^{(N)}$ parameterizing N distinct unordered points, which is smooth. Consider the universal subscheme

$$\Gamma_N \subset X_0^{(N)} \times X,$$

which is flat and proper over $X_0^{(N)}$. We denote by

$$\text{pr}_{X_0^{(N)}} : X_0^{(N)} \times X \rightarrow X_0^{(N)}, \quad \text{pr}_X : X_0^{(N)} \times X \rightarrow X$$

the two projections.

As X is smooth projective of dimension n , there is for any l a push-forward map

$$\text{pr}_{X_0^{(N)*}} : H^n(X_0^{(N)} \times X, \Omega_{X_0^{(N)} \times X/K}^{n+l}) \rightarrow H^0(X_0^{(N)}, \Omega_{X_0^{(N)}/K}^l)$$

and thus, for any class $\gamma \in H^n(X_0^{(N)} \times X, \Omega_{X_0^{(N)} \times X/K}^n)$, an action

$$\begin{aligned} \gamma^* : H^0(X, \Omega_{X/K}^l) &\rightarrow H^0(X_0^{(N)}, \Omega_{X_0^{(N)}/K}^l), \\ \gamma^*(\alpha) &= \text{pr}_{X_0^{(N)*}}(\gamma \wedge \text{pr}_X^* \alpha). \end{aligned} \tag{2}$$

In particular, for $\gamma = [\Gamma_N]$, where $[\Gamma_N] \in H^n(X_0^{(N)} \times X, \Omega_{X_0^{(N)} \times X/K}^n)$ is the Gros cycle class used in [20], and which depends only on the class of Γ_N in $\text{CH}^n(X_0^{(N)} \times X)$, we get for any $\alpha \in H^0(X, \Omega_{X/K}^l)$ a l -form

$$[\Gamma_N]^* \alpha \in H^0(X_0^{(N)}, \Omega_{X_0^{(N)}/K}^l).$$

Remark 2.1. The construction above gives more generally, for any l , for any smooth variety Y over K and any cycle $\Gamma \in \text{CH}^n(Y \times X)$, a morphism

$$[\Gamma]^* : H^0(X, \Omega_{X/K}^l) \rightarrow H^0(Y, \Omega_{Y/K}^l).$$

In practice, we will compute $[\Gamma]^* \omega$ as a trace, at least at the generic point of Y .

Let $q : X^N \rightarrow X^{(N)}$ be the natural quotient map, and let $X_0^N := q^{-1}(X_0^{(N)}) \subset X^N$. We observe next that

$$(q, \text{Id}_X)^* \Gamma_N = \sum_i \Gamma_i \text{ in } \text{CH}^n(X_0^N \times X),$$

where $\Gamma_i \subset X_0^N \times X$ is the graph of the i -th projection $\text{pr}_i : X_0^N \rightarrow X$. It follows that

$$(q, \text{Id}_X)^* [\Gamma_N] = \sum_i [\Gamma_i] \text{ in } H^n(X_0^N \times X, \Omega_{X_0^N \times X/K}^n). \tag{3}$$

The contravariant functoriality of the constructions above gives

$$q^* [\Gamma_N]^* \alpha = \sum_i [\Gamma_i]^* \alpha = \sum_i \text{pr}_i^* \alpha \text{ in } H^0(X_0^N, \Omega_{X_0^N/K}^l) \tag{4}$$

for any $\alpha \in H^0(X, \Omega_{X/K}^l)$. We now make the following assumption

(*) *there exist an effective cycle*

$$Z \in \text{CH}^n(X^{(N)} \times X)$$

of relative degree N' over $X^{(N)}$ and a Zariski open set $U \subset X_0^{(N)}$ such that
(a) *the corresponding morphism*

$$\phi_Z : U \rightarrow X^{(N')}$$

is well-defined and takes value in $X_0^{(N')}$ and,

(b) *for some $z_0 \in \text{CH}_0(X)$ of degree $N - N'$,*

$$\Gamma_{N|U \times X} = U \times z_0 + \Gamma_{N'} \circ \phi_Z \text{ in } \text{CH}^n(U \times X). \quad (5)$$

The cycle class

$$[\Gamma_N] \in H^n(X_0^{(N)} \times X, \Omega_{X_0^{(N)} \times X/K}^n)$$

then satisfies

$$[\Gamma_N]_{|U \times X} = [U \times z_0] + (\phi_Z, \text{Id}_X)^* [\Gamma_{N'}] \text{ in } H^n(U \times X, \Omega_{U \times X}^n). \quad (6)$$

From (6), we deduce that, under the assumption (5)

$$([\Gamma_N]^* \alpha)_{|U} = \phi_Z^*([\Gamma_{N'}]^* \alpha) \text{ in } H^0(U, \Omega_{U/K}^l) \quad (7)$$

for any $l > 0$ and $\alpha \in H^0(X, \Omega_{X/K}^l)$, because $[U \times z_0]^*$ acts as zero on l -forms with $l > 0$. Let $\widehat{U} \subset X^N$ be the inverse image of U via the natural quotient map $q : X_0^N \rightarrow X_0^{(N)}$ and let $\widehat{\phi}_Z := \phi_Z \circ q : \widehat{U} \rightarrow X_0^{(N')}$.

Equation (7) gives after pull-back to \widehat{U} , using (4)

$$\sum_{i=1}^N \text{pr}_i^* \alpha = \widehat{\phi}_Z^*([\Gamma_{N'}]^* \alpha) \text{ in } H^0(\widehat{U}, \Omega_{\widehat{U}/K}^l) \quad (8)$$

for any $l > 0$ and $\alpha \in H^0(X, \Omega_{X/K}^l)$.

We now prove

Proposition 2.2. *(A version of Mumford-Roitman theorem [16], [17].) If X has dimension n and satisfies assumption (*), then $H^0(X, \Omega_{X/K}^l) = 0$ for $l \geq 2$.*

Proof of Proposition 2.2. We will use the naive notion of rank $\text{rk}(\omega)$ of a l -form $\omega \in \bigwedge^l V^*$, for any finite dimensional vector space V . Namely $\text{rk}(\omega)$ is defined as the rank of the linear map

$$\begin{aligned} \lrcorner \omega : V &\rightarrow \bigwedge^{l-1} V^*, \\ v &\mapsto v \lrcorner \omega. \end{aligned} \quad (9)$$

We have the following rather obvious

Lemma 2.3. (i) Let $\phi : V \rightarrow W$ be a linear map, and let $\omega \in \bigwedge^l W^*$. Then the rank of $\phi^*\omega$ is non greater than the rank of ω .

(ii) Let V be a vector space of the form $V = V_1 \oplus \dots \oplus V_N$ and let $\omega_i \in \bigwedge^l V_i^*$. Then, if $l \geq 2$, the rank of

$$\omega = \sum_i \text{pr}_i^* \omega_i \quad (10)$$

is given by

$$\text{rk}(\omega) = \sum_i \text{rk}(\omega_i). \quad (11)$$

(iii) The rank of a l -form $\omega \in \bigwedge^l V^*$ is lower-semicontinuous on $\bigwedge^l V^*$ equipped with the Zariski topology.

(iv) The rank of a form is subadditive, that is : $\text{rk}(\sum_i \omega_i) \leq \sum_i \text{rk}(\omega_i)$.

Proof. (i) follows immediately from the definition of the rank, and from the fact that

$$\lrcorner \phi^* \omega = \phi^* \circ \lrcorner \omega \circ \phi : V \rightarrow \bigwedge^{l-1} V^*.$$

For (ii), a form ω as in (10) has rank $\text{rk}(\omega) \leq \sum_i \text{rk}(\omega_i)$ because the map $\lrcorner \omega$ can be written as

$$\lrcorner \omega = \sum_i \text{pr}_i^* \circ \lrcorner \omega_i \circ \text{pr}_i. \quad (12)$$

To see that $\text{rk}(\omega)$ is actually $\sum_i \text{rk}(\omega_i)$ when $l \geq 2$, let V_ω be the kernel of $\lrcorner \omega$. We observe from (12) that a vector

$$v = v_1 + \dots + v_N \in V = V_1 \oplus \dots \oplus V_N$$

belongs to V_ω if and only if

$$v_i \lrcorner \omega_i = 0 \text{ for all } i. \quad (13)$$

Note that this is at this point that we use the condition $l \geq 2$, which implies that each $v_i \lrcorner \omega_i$ has degree $l-1 > 0$, so that the spaces $\text{pr}_i^*(\bigwedge^{l-1} V_i^*)$ are in direct sum in $\bigwedge^{l-1} V^*$. Equality (11) immediately follows.

(iii) The lower semi-continuity of $\text{rk}(\omega)$ immediately follows from the lower semi-continuity of the rank of a linear map or matrix, since it is defined as the rank of linear map $\lrcorner \omega$ of (9).

(iv) follows from (i) and (ii) when $l \geq 2$, and is trivial when $l = 1$. \square

Remark 2.4. Lemma 2.3(ii) is completely wrong if $l = 1$, as any 1-form has rank 1. This observation is at the core of Mumford's theorem.

We now conclude the proof of Proposition 2.2. We argue by contradiction and assume that there exists a nonzero $\alpha \in H^0(X, \Omega_{X/K}^l)$, with $l \geq 2$. We observe that thanks to Lemma 2.3(iii), a nonzero l -form α on X has a generic rank $\text{rk}_{\text{gen}}(\alpha)$, and that at any point $x \in X$, the rank of α is $\leq \text{rk}_{\text{gen}}(\alpha)$. Formula (4) for N' and Lemma 2.3(i) and (ii), imply that the rank of $\phi_Z^*([\Gamma_{N'}]^* \alpha)$ at the generic point of $X_0^{(N)}$ is $\leq N' \text{rk}_{\text{gen}}(\alpha)$, while, as $l \geq 2$, formula (4) and Lemma 2.3(ii) give that

$$\text{rk}_{\text{gen}}([\Gamma_N]^* \alpha) = N \text{rk}_{\text{gen}}(\alpha), \quad (14)$$

which contradicts (7). \square

For the proof of Theorem 1.12 in the cases (ii) and (iii) of nonzero characteristic, we will not use however the naive rank of a form as above, but its tensor rank. Let V be a vector space and let $\omega \in \bigwedge^l V^*$. Following [13], we define the tensor rank $\text{trk}(\omega)$ of any nonzero $\omega \in \bigwedge^l V$ as the minimal integer r such that

$$\omega = \omega_1 + \dots + \omega_r,$$

where each ω_i is decomposable.

We have the following variant of Lemma 2.3.

Lemma 2.5. (i) Let $\phi : V \rightarrow W$ be a linear map, and let $\omega \in \bigwedge^l W^*$. Then the tensor rank of $\phi^*\omega$ is non greater than the tensor rank of ω .

(ii) Let V be a vector space of the form $V = V_1 \oplus \dots \oplus V_N$ and let $\omega_i \in \bigwedge^l V_i^*$. Assume that $\dim V_i = l$ or $\dim V_i = l + 1$. Then if $l \geq 2$, the tensor rank of

$$\omega = \sum_i \text{pr}_i^* \omega_i \tag{15}$$

is equal to N , assuming $\omega_i \neq 0$ for all i .

(iii) In the situation of (ii), assume that either $l = 2$ or $\dim V_i = l + 2$, so that ω_i has an even rank $2r_i$ (hence tensor rank r_i). Then if $l \geq 2$, the tensor rank of $\omega = \sum_i \text{pr}_i^* \omega_i$ is equal to $\sum_i r_i$.

(iv) The tensor rank of a l -form $\alpha \in \bigwedge^l V^*$ on a vector space V of dimension n is lower-semicontinuous on $\bigwedge^l V^*$ equipped with the Zariski topology, assuming $l \leq 2$ or $l \geq n - 2$.

(v) The tensor rank of a l -form is subadditive : $\text{trk}(\sum_i \omega_i) \leq \sum_i \text{trk}(\omega_i)$.

Proof. (i) follows immediately from the definition of the tensor rank since the pull-back of a decomposable form is decomposable.

For (ii), as any l -form on a vector space of dimension l or $l + 1$ is decomposable, we get by definition of the tensor rank that a form ω as in (15) has tensor rank $\leq N$. Suppose it has tensor rank $< N$. Then it can be written as

$$\omega = \omega'_1 + \dots + \omega'_{N'},$$

with $N' < N$ and ω'_i decomposable. It follows that there is a vector subspace $V_\omega \subset V$ of dimension $\geq \dim V - N'l$ consisting of vectors $v \in V$ such that $v \lrcorner \omega = 0$, namely the space annihilated by all $\xi_{sr} \in V^*$, where $\omega'_s = \xi_{s1} \wedge \dots \wedge \xi_{sl}$. However, for ω given by (15), a vector

$$v = v_1 + \dots + v_N \in V = V_1 \oplus \dots \oplus V_N$$

belongs to V_ω if and only if

$$v_i \lrcorner \omega_i = 0 \text{ for all } i. \tag{16}$$

(Note that, as before, this is at this point that we use the condition $l \geq 2$, which implies that each $v_i \lrcorner \omega_i$ has degree $l - 1 > 0$.) As all ω_i are nonzero, condition (16) defines a space of codimension Nl , which provides a contradiction.

(iii) is proved similarly as (ii), except that the computation of the tensor rank is different. We use here the fact that when $\dim V_i = l + 2$, $\bigwedge^l V_i^* \cong \bigwedge^2 V_i^*$ so the tensor rank of ω_i is computed as half the rank as ω_i seen as a 2-form on V_i^* .

(iv) When $n = l$ or $n = l + 1$, the tensor rank of any $\omega \in \bigwedge^l V^*$ is 1 if $\omega \neq 0$ and 0 otherwise. When $n = l + 2$, it was already explained that the tensor rank is half the rank of the corresponding element of $\bigwedge^2 V$, and similarly when $l \leq 2$, hence in all these cases, the rank is lower-semicontinuous.

(v) trivially follows from the definition of the tensor rank. \square

Remark 2.6. As before, Lemma 2.5(ii) is wrong if $l = 1$, as any nonzero 1-form has tensor rank 1.

Remark 2.7. Lemma 2.5(iv) is wrong if $3 \leq l \leq n - 3$. For example, being of tensor rank 2 means belonging to the proper secant variety of the Grassmannian $\text{Grass}(l, n)$. But the Zariski closure of the proper secant variety of the Grassmannian contains its tangential variety, where we get elements of the form

$$\alpha_1 \wedge \dots \wedge \alpha_l + \sum_i \alpha_1 \wedge \dots \wedge \hat{\alpha}_i \wedge \dots \wedge \alpha_l \wedge \beta_i,$$

which are of tensor rank l if n is large enough and β_i are chosen in a general way.

Proof of theorem 1.12, 1 to 3. Proposition 2.2 proves almost directly Theorem 1.12 in characteristic 0, that is 1. Indeed, under the assumptions of Theorem 1.12, we have an effective cycle Z of relative degree N' satisfying (1) over the generic point of $X_0^{(N)}$, hence over a dense Zariski open set. In characteristic 0, such a cycle Z automatically provides a rational map ϕ to the symmetric product $X^{(N')}$. To avoid generically the singularities of $X^{(N')}$, one should in fact decompose the effective cycle Z as a sum $\sum_i Z'_i$, where each Z'_i is irreducible and has degree N'_i over $X^{(N)}$, and $\sum_i N'_i = N'$. Then generically the rational map $\phi_i : X^{(N)} \dashrightarrow X^{(N'_i)}$ induced by Z'_i takes value in $X_0^{(N'_i)}$ and (1) can be rewritten as

$$Z_{|U \times X} = \sum_i \Gamma_{N'_i}' \circ \phi_i \text{ in } \text{CH}^n(U \times X) \quad (17)$$

for a dense Zariski open set U of $X_0^{(N)}$. The relation (17) then implies the following variant of (7)

$$([\Gamma_N]^* \alpha)_{|U} = \sum_i \phi_i^*([\Gamma_{N'_i}]^* \alpha) \text{ in } H^0(U, \Omega_{U/K}^l) \quad (18)$$

for any $l > 0$ and $\alpha \in H^0(X, \Omega_{X/K}^l)$.

We then get a contradiction if $N > N'$ and $l \geq 2$ by the same rank argument as in the proof of Proposition 2.2, using the subadditivity and lower continuity of the rank (Lemma 2.3 (iii), (iv)). Indeed, if r is the generic rank of α , then the generic rank of the left hand side in (18) is Nr , while the generic rank of the right hand side is $\leq N'r$.

In nonzero characteristic, nonseparability phenomena prevent, even at the generic point, to construct ϕ from Z and to write the factorization (17), but still we can deduce from (1) the equality of traces

$$[\Gamma_N]^* \alpha = [Z]^* \alpha \text{ in } H^0(X_0^{(N)}, \Omega_{X_0^{(N)}/K}^l), \quad (19)$$

for any $\alpha \in H^0(X, \Omega_{X/K}^l)$, with $l > 0$. We now restrict equality (19) to the generic point $\eta = \text{Spec } K(X_0^{(N)})$ of $X_0^{(N)}$, which allows to compute the pull-backs of forms in (19) as traces, and argue by comparing the generic tensor ranks of both sides in (19).

The effective cycle $Z \in \text{CH}_0(X_\eta)$ is a sum of points $Z_i \in X(L_i)$ where each L_i is a finite extension of $K(X^{(N)})$. Such an extension decomposes as

$$K(X^{(N)}) \subset L_{i,\text{sep}} \subset L_i$$

where the first extension is separable and the second one is purely inseparable.

We now use the following

Lemma 2.8. *Let $L_1 \hookrightarrow L_2$ be a degree d purely inseparable extension of fields containing a given field K of characteristic p . For any l , denote by $f^* : \Omega_{L_1/K}^l \rightarrow \Omega_{L_2/K}^l$ the natural L_1 -linear map. Let $\omega \in \bigwedge^l \Omega_{L_2/K}$ be a l -form.*

- (i) *If $l = 2$, and $0 \neq \text{char } K$ is arbitrary, $\text{Tr}_{L_2/L_1}(\omega)$ has tensor rank ≤ 1 .*
- (ii) *If $l = 3$ and $\text{char } K = 2$,*

$$\text{trk}(\text{Tr}_{L_2/L_1}(\omega)) \leq d \text{trk}(\omega). \quad (20)$$

Proof. We can assume that $d = p$ and $L_2 = L_1[x]/(x^p - a)$, for some $a \in L_1$. It suffices to prove that for any decomposable l -form $\omega \in \Omega_{L_2/K}^l$, hence of tensor rank 1, the tensor rank of $\text{Tr}_{L_2/L_1}(\omega)$ is ≤ 1 if $l = 2$ (case (i)), and is ≤ 2 if $l = 3$ and $\text{char } K = 2$.

The trace map on forms of degree $l \geq 1$ on $\bigwedge^l \Omega_{L_2/K}$ is explicitly described in [19, 0ADY, Lemma 2.2] as follows:

$$\begin{aligned} \text{Tr}(f^*\eta_1 \wedge \dots \wedge f^*\eta_{l-1} \wedge x^i dx) &= 0 \text{ if } 0 \leq i \leq p-2 \\ \text{Tr}(f^*\eta_1 \wedge \dots \wedge f^*\eta_{l-1} \wedge x^{p-1} dx) &= \eta_1 \wedge \dots \wedge \eta_{l-1} \wedge da \\ \text{Tr}(\omega) &= 0 \text{ if } \omega \in \text{Im}(\Omega_{L_1/K}^l \otimes L_2 \rightarrow \Omega_{L_2/K}^l). \end{aligned} \quad (21)$$

From (21), we conclude that any trace $\text{Tr}_{L_2/L_1}(\alpha_1 \wedge \dots \wedge \alpha_l)$ can be written as $da \wedge \beta$ for some $\beta \in \Omega_{L_1/K}$. Hence if $l = 2$, it has tensor rank ≤ 1 , proving (i).

We next assume $p = 2$ and $l = 3$. Let $\alpha, \beta, \gamma \in \Omega_{L_2/K}$. We can assume after changing the basis of the 3-dimensional L_1 -vector space $\langle \alpha, \beta, \gamma \rangle$ that

$$\alpha = f^*a_1 + xf^*a_2 + xdx, \beta = f^*b_1 + xf^*b_2 + dx, \gamma = f^*c_1 + xf^*c_2,$$

where $a_i, b_j, c_l \in \Omega_{L_1/K}$. We then get

$$\alpha \wedge \beta \wedge \gamma = (xdx \wedge f^*b_1 + adx \wedge f^*b_2 + f^*a_1 \wedge dx + xf^*a_2 \wedge dx) \wedge (f^*c_1 + xf^*c_2) + \zeta, \quad (22)$$

where dx does not appear in ζ . Applying the rules (21), we get

$$\begin{aligned} &\text{Tr}_{L_2/L_1}(\alpha \wedge \beta \wedge \gamma) \\ &= \text{Tr}((xdx \wedge f^*b_1 + adx \wedge f^*b_2 + f^*a_1 \wedge dx + xf^*a_2 \wedge dx) \wedge (f^*c_1 + xf^*c_2)) \\ &= da \wedge b_1 \wedge c_1 + ada \wedge b_2 \wedge c_2 + da \wedge a_1 \wedge c_2 + da \wedge a_2 \wedge c_1 \\ &= da \wedge (b_1 \wedge c_1 + ab_2 \wedge c_2 + a_1 \wedge c_2 + a_2 \wedge c_1). \end{aligned} \quad (23)$$

The 2-form $\eta := b_1 \wedge c_1 + ab_2 \wedge c_2 + a_1 \wedge c_2 + a_2 \wedge c_1$ can be written as

$$\eta = c_1 \wedge (b_1 + a_2) + c_2 \wedge (ab_2 + a_1), \quad (24)$$

hence it has tensor rank ≤ 2 , which proves (ii). \square

Remark 2.9. Statement (ii) in Lemma 2.8 is wrong if we use the rank instead of the tensor rank (see Example 2.12 below). This is the reason why we use the tensor rank in the proof of Theorem 1.12, while the rank has better properties (see Remark 2.7).

Remark 2.10. Statement (ii) in Lemma 2.8 is wrong if $l \geq 4$ (see Example 2.13 below). This is the reason for the condition $l \leq 3$ in Theorem 1.12, 3.

We now conclude the proof of Theorem 1.12 in cases 2 and 3 as follows. We write over the generic point of $X_0^{(N)}$

$$Z = \sum_i Z_i,$$

where $Z_i \in X(L_i)$, $L_i := K(Z_i)$ is a field extension of $K(X_0^{(N)})$ of degree N'_i , and

$$N' = \sum_i N'_i. \quad (25)$$

The field L_i is a purely inseparable extension $L_i \supseteq L_{i,\text{sep}}$ of a separable extension

$$K(X_0^{(N)}) \subset L_{i,\text{sep}}.$$

Let $f_i : Z_i \rightarrow X$ be the natural morphism over K . For a l -form α on X , the l -form $[Z]^*\alpha$, taken at the generic point of $X_0^{(k)}$, can be written as

$$[Z]^*\alpha = \sum_i \mathrm{Tr}_{L_i/K(X_0^{(N)})} f_i^*(\alpha) = \sum_i \mathrm{Tr}_{L_{i,\mathrm{sep}}/K(X_0^{(N)})} (\mathrm{Tr}_{L_i/L_{i,\mathrm{sep}}} f_i^*(\alpha)). \quad (26)$$

Using our assumptions that either $l = 2$ or $l \geq \dim X - 2$, we get the inequality

$$\mathrm{trk}_{\mathrm{gen}}(f_i^*(\alpha)) \leq \mathrm{trk}_{\mathrm{gen}}(\alpha), \quad (27)$$

thanks to the semi-continuity statement (iv) in Lemma 2.5. In (27), the generic tensor rank $\mathrm{trk}_{\mathrm{gen}}(\omega)$ of a form ω on a variety is the tensor rank of ω at its generic point.

We now use our assumptions 2 and 3 in Theorem 1.12. In both cases, by Lemma 2.8, we get the inequalities

$$\mathrm{trk}_{\mathrm{gen}}(\mathrm{Tr}_{L_i/L_{i,\mathrm{sep}}}(f_i^*\alpha)) \leq \deg(L_i/L_{i,\mathrm{sep}})\mathrm{trk}_{\mathrm{gen}}(f_i^*\alpha). \quad (28)$$

Finally, the separable trace $\mathrm{Tr}_{L_{i,\mathrm{sep}}/K(X_0^{(N)})}$ works as in Corollary 2.2. More precisely, the following statement is almost obvious.

Lemma 2.11. *Let L_1 be a field over K and $L_1 \subset L_2$ be a finite separable field extension of degree k' . Then the trace*

$$\mathrm{Tr}_{L_2/L_1} : \Omega_{L_2/K}^l \rightarrow \Omega_{L_1/K}^l$$

has the property that for any $\omega \in \Omega_{L_2/K}^l$,

$$\mathrm{trk}(\mathrm{Tr}_{L_2/L_1}(\omega)) \leq k' \mathrm{trk}(\omega).$$

We now get from (28) and Lemma 2.11

$$\begin{aligned} \mathrm{trk}(\mathrm{Tr}_{L_i/K(X_0^{(N)})}(f_i^*\alpha)) &= \mathrm{trk}(\mathrm{Tr}_{L_{i,\mathrm{sep}}/K(X_0^{(N)})}(\mathrm{Tr}_{L_i/L_{i,\mathrm{sep}}}(f_i^*\alpha))) \\ &\leq \deg(L_{i,\mathrm{sep}}/K(X_0^{(N)})) \deg(L_i/L_{i,\mathrm{sep}}) \mathrm{trk}_{\mathrm{gen}}(f_i^*\alpha) \\ &= \deg(L_i/K(X_0^{(N)})) \mathrm{trk}_{\mathrm{gen}}(f_i^*\alpha) = N'_i \mathrm{trk}_{\mathrm{gen}}(f_i^*\alpha) \leq N'_i \mathrm{trk}_{\mathrm{gen}}(\alpha). \end{aligned} \quad (29)$$

By (25) and subadditivity of the tensor rank (see Lemma 2.5(v)), we finally get under our assumptions

$$\mathrm{trk}_{\mathrm{gen}}([Z]^*\alpha) \leq N' \mathrm{trk}_{\mathrm{gen}}(\alpha). \quad (30)$$

However, as already used previously, Lemma 2.5(ii) and (iii) implies

$$\mathrm{trk}_{\mathrm{gen}}([\Gamma_N]^*\alpha) = N \mathrm{trk}_{\mathrm{gen}}(\alpha),$$

because $l \geq 2$. Inequality (30) and formula (19) thus imply that $\alpha = 0$ since $N' < N$. \square

Let us give two examples concerning the rank or tensor rank of an inseparable trace. We first give an example with $l = 3$, where inequality (20) is not satisfied if we replace the tensor rank by the rank.

Example 2.12. We consider as above a field extension $L_1 \subset L_2$, of the form $x^2 = a$, in characteristic 2. Let $K \subset L_1$ be a subfield and let

$$V \subset \Omega_{L_2/K}$$

be a 6-dimensional vector space. An element in $\bigwedge^3 V \subset \Omega_{L_2/L_1}^3$ thus has rank ≤ 6 . We write now elements of V as

$$\alpha = a_\alpha + x b_\alpha + c_\alpha dx + d_\alpha x dx, \quad (31)$$

where $a_\alpha, b_\alpha \in \Omega_{L_1/K}, c_\alpha, d_\alpha \in L_1$. (Here we omit the notation f^* and identify $\Omega_{L_1/k}$ and its image in $\Omega_{L_2/k}$.) We assume that da, a_α, b_α generate a vector subspace of dimension 13 of $\Omega_{L_1/K}$. We claim that for adequately chosen $\omega \in \bigwedge^3 V$, one has

$$\mathrm{Tr}_{L_2/L_1}(\omega) = da \wedge \eta,$$

where $\eta \in \bigwedge^2(\Omega_{L_1/K}/da)$ has rank 12. Then the rank of $\mathrm{Tr}_{L_2/L_1}(\omega)$ is 13, while the rank of ω is 6. So we do not have the inequality

$$\mathrm{rk}(\mathrm{Tr}_{L_2/L_1}(\omega)) \leq 2\mathrm{rk}(\omega).$$

We now prove the claim. Let $W_{12} \subset \Omega_{L_1/K}$ be the 12-dimensional vector space generated by the a_α, b_α introduced above. The L_1 -vector space W_{12} contains a 10-dimensional vector subspace W_{10} , namely the space of those forms α for which the scalars $c_\alpha, d_\alpha \in L_1$ of (31) vanish. From the computation (23), (24), we get that for $\alpha, \beta, \gamma \in V_6$ with $\gamma \in W_{10} \subset W_{12} = V_6$, one has

$$\mathrm{Tr}_{L_2/L_1}(\alpha \wedge \beta \wedge \gamma) = da \wedge \eta. \quad (32)$$

where the 2-form η is given by

$$\eta = c_\alpha \wedge (a_\beta + b_\alpha) + d_\alpha \wedge (ab_\beta + a_\alpha). \quad (33)$$

It is clear that by taking L_1 -linear combinations of forms η as in (33), one gets arbitrary elements in $W_{10} \wedge W_{12} \subset \bigwedge^2 \Omega_{L_1/K}$, hence forms of rank 12, even modulo da , which proves the claim.

Next we give an example of computation of traces of l -forms, with $l \geq 4$, where inequality (20) is not satisfied.

Example 2.13. With the same notation as in Example 2.12, let

$$\omega = \alpha_1 \wedge \dots \wedge \alpha_4.$$

This is an element of tensor rank 1 in $\bigwedge^4 \Omega_{L_2/K}$. Then we claim that, in general, the tensor rank of $\mathrm{Tr}_{L_2/L_1}(\omega)$ is at least 3, so that we do not have the desired inequality $\mathrm{trk}(\mathrm{Tr}_{L_2/L_1}(\omega)) \leq 2\mathrm{trk}(\omega)$. To prove the claim, we can assume up to changing the L_1 basis of $\langle \alpha_i \rangle_{L_1}$ that

$$\alpha_1 = a_1 + xb_1 + dx, \alpha_2 = a_2 + xb_2 + xdx, \alpha_3 = a_3 + xb_3, \alpha_4 = a_4 + xb_4,$$

where $a_i := a_{\alpha_i}, b_i := b_{\alpha_i} \in \Omega_{L_1/k}$. We thus get

$$\begin{aligned} \mathrm{Tr}_{L_2/L_1}(\omega) &= \mathrm{Tr}_{L_2/L_1}(dx \wedge (a_2 + xb_2) \wedge (a_3 + xb_3) \wedge (a_4 + xb_4)) \\ &\quad + \mathrm{Tr}_{L_2/L_1}(xdx \wedge (a_1 + xb_1) \wedge (a_3 + xb_3) \wedge (a_4 + xb_4)) \\ &= da \wedge (b_2 \wedge a_3 \wedge a_4 + ab_2 \wedge b_3 \wedge b_4 + a_2 \wedge a_3 \wedge b_4 + a_2 \wedge b_3 \wedge a_4) \\ &\quad + da \wedge (a_1 \wedge a_3 \wedge a_4 + a(b_1 \wedge b_3 \wedge a_4 + b_1 \wedge b_4 \wedge a_3 + a_1 \wedge b_3 \wedge b_4)). \end{aligned}$$

It remains to see that the tensor rank of the 3-form

$$\begin{aligned} \eta &:= b_2 \wedge a_3 \wedge a_4 + ab_2 \wedge b_3 \wedge b_4 + a_2 \wedge a_3 \wedge b_4 + a_2 \wedge b_3 \wedge a_4 \\ &\quad + a_1 \wedge a_3 \wedge a_4 + a(b_1 \wedge b_3 \wedge a_4 + b_1 \wedge b_4 \wedge a_3 + a_1 \wedge b_3 \wedge b_4) \in \bigwedge^3(\Omega_{L_1/K}/da) \end{aligned}$$

is at least 3, when the forms a_i, b_j are globally independent modulo da . Indeed, this implies that the tensor rank of

$$da \wedge \eta = \mathrm{Tr}_{L_2/L_1}(\omega)$$

is at least 3. We have

$$\begin{aligned}\eta &= a_3 \wedge a_4 \wedge (b_2 + a_1) + ab_3 \wedge b_4 \wedge (b_2 + a_1) + a_3 \wedge b_4 \wedge (a_2 + ab_1) + b_3 \wedge a_4 \wedge (a_2 + aa_1), \\ &= (a_3 \wedge a_4 + ab_3 \wedge b_4) \wedge (b_2 + a_1) + a_3 \wedge b_4 \wedge (a_2 + ab_1) + b_3 \wedge a_4 \wedge (a_2 + aa_1).\end{aligned}$$

We conclude that the rank of η modulo da is at least 7. It follows that its tensor rank modulo da is at least 3 when the forms a_i, b_j are globally independent modulo da .

We finally complete the proof of Theorem 1.12 by treating the case 4.

Proof of Theorem 1.12,4. We will use the following result concerning the rank of inseparable traces.

Lemma 2.14. *Let L_1 be a field containing a given field K of characteristic p , and let $L_2 = L_1[x]/(x^p - a)$, for some $a \in L_1$. Let $\omega \in \bigwedge^l \Omega_{L_2/K}$ be a l -form. We have*

$$\text{rk}(\text{Tr}_{L_2/L_1}(\omega)) \leq p \text{rk}(\omega) + 1. \quad (34)$$

Proof. As before, for any l , we denote by $f^* : \Omega_{L_1/K}^l \rightarrow \Omega_{L_2/K}^l$ the natural L_1 -linear map. We assume that $\text{rk}(\omega) \leq r$. This means that there exists a L_2 -vector subspace $W \subset \Omega_{L_2/K}$ of dimension $\leq r$ such that

$$\omega \in \bigwedge^l W \subset \bigwedge^l \Omega_{L_2/K}.$$

Let $\alpha_i, i = 1, \dots, r$ be a basis of W over L_2 . We write

$$\alpha_i = \sum_{j=0}^{p-1} x^j f^* a_{ij} + \sum_{j=0}^{p-1} b_{ij} x^j dx,$$

where $a_{ij} \in \Omega_{L_1/K}, b_{ij} \in L_1$. Let $V \subset \Omega_{L_1/K}$ be the vector subspace generated by the a_{ij} , and let $V' \subset \Omega_{L_1/K}$ be the vector subspace generated by the a_{ij} and da . Formula (21) shows that for any $\omega \in \bigwedge^l W$,

$$\text{Tr}_{L_2/L_1}(\omega) \in \bigwedge^l V',$$

which proves (34) since $\dim V' \leq pr + 1$. \square

By writing any purely inseparable extension as a sequence of extensions as above, we get

Corollary 2.15. *Let L_1 be a field containing a given field K of characteristic p , and let $L_1 \subset L_2$ be a finite purely inseparable extension of degree p^s . Then for any $\omega \in \bigwedge^l \Omega_{L_2/K}$, we have*

$$\text{rk}(\text{Tr}_{L_2/L_1}(\omega)) \leq p^s \text{rk}(\omega) + \frac{p^s - 1}{p - 1} < p^s (\text{rk}(\omega) + 1). \quad (35)$$

Next we have the following obvious fact on separable traces.

Lemma 2.16. *Let L_1 be a field over K and $L_1 \subset L_2$ be a finite separable field extension of degree k' . Then for any $\omega \in \bigwedge^l \Omega_{L_2/K}$,*

$$\text{rk}(\text{Tr}_{L_2/L_1}(\omega)) \leq k' \text{rk}(\omega). \quad (36)$$

Writing any field extension $L_1 \subset L_2$ as a separable extension followed by a purely inseparable extension, we get by combining inequality (35) and (36).

Corollary 2.17. *Let L_1 be a field containing a given field K , and let $L_1 \subset L_2$ be a finite extension of degree d . Then for any $\omega \in \bigwedge^l \Omega_{L_2/K}$, we have*

$$\text{rk}(\text{Tr}_{L_2/L_1}(\omega)) < d(\text{rk}(\omega) + 1). \quad (37)$$

Corollary 2.18. *For any smooth variety Y defined over a field K , any smooth projective variety X over K and any effective cycle $Z \subset Y \times X$ of relative degree N' over Y , for any algebraic form $\omega \in H^0(X, \Omega_{X/K}^l)$, we have*

$$\mathrm{rk}_{\mathrm{gen}}([Z]^*(\omega)) < N'(\mathrm{rk}_{\mathrm{gen}}(\omega) + 1). \quad (38)$$

Proof. We decompose Z as a sum of irreducible closed algebraic subsets Z_i of $Y \times X$, and restrict over the generic point of Y . Then Z_i provides a field extension $K(Y) \subset L_i$ of degree N'_i , with $\sum_i N'_i = N'$. We apply inequality (37) to each of these extensions and get that

$$\mathrm{rk}_{\mathrm{gen}}([Z]^*(\omega)) \leq \sum_i \mathrm{rk}_{\mathrm{gen}}([Z_i]^*(\omega)) < \sum_i N'_i(\mathrm{rk}_{\mathrm{gen}}(\omega) + 1). \quad (39)$$

The first inequality in (39) is by subadditivity in Lemma 2.3(iv), and for the second inequality, we use Lemma 2.3(i), lower-semicontinuity (Lemma 2.3(i)(iii)) and (37). As $\sum_i N'_i = N'$, we get (38). \square

We now fix the dimension n of X and observe that by definition, for any l -form ω on X ,

$$\mathrm{rk}_{\mathrm{gen}}(\omega) \leq n.$$

We thus get from (39) the inequality

$$\mathrm{rk}_{\mathrm{gen}}([Z]^*(\omega)) < N'(n + 1), \quad (40)$$

which is valid for any cycle Z as above. However, letting $Y := X_0^{(N)}$, we know (see (14)) that for any $\omega \in H^0(X, \Omega_{X/K}^l)$, with $l \geq 2$ we have

$$\mathrm{rk}_{\mathrm{gen}}(\Gamma_N^*(\omega)) = N \mathrm{rk}_{\mathrm{gen}}(\omega). \quad (41)$$

We conclude arguing as in the previous proofs. As $\mathrm{rk}_{\mathrm{gen}}(\omega) \geq l$ if $\omega \neq 0$, equality (19) and (41), (40) imply that $\omega = 0$ for $l \geq 2$, once $N \geq \frac{N'(n+1)}{l}$. \square

3 Unboundedness of zero-cycles on Fano hypersurfaces of dimension ≥ 3

We prove in this section Theorem 1.11. The method is by specialization and uses the Totaro arguments in [20], combined with theorem 1.12. Totaro's paper built on the Colliot-Thélène-Pirutka specialization arguments [6], generalizing the degeneration method in [23], used in all three cases to disprove the CH_0 -universal triviality (hence the stable rationality) of the very general (or geometric generic) fiber. However Totaro used a completely different criterion to prove that the special fiber in his specialization does not have a universally trivial CH_0 -group. While Colliot-Thélène and Pirutka constructed a specialization with nontrivial Brauer class on the special fiber, Totaro uses the same specialization as Kollár in [10]. Mori first observed in [15] that hypersurfaces of even degree $2d$ specialize to double covers of hypersurfaces of degree d ramified along a general member of $|\mathcal{O}(2d)|$. Kollár specializes these cyclic double covers to characteristic 2, where they become inseparable and mildly singular. He then proves that starting from dimension 3, they have desingularizations with a nonzero algebraic forms on the special fiber. This differential form is the desired obstruction to stable rationality used in [20]. We summarize here the results of [10], [9] and [20], that will quickly lead us to the proof of Theorem 1.11.

Let us recall the following definition from [6].

Definition 3.1. *A projective morphism*

$$\phi : Y \rightarrow X$$

between varieties defined over a field K is universally CH_0 -trivial if for any field $L \supseteq K$, the induced morphism

$$\phi_* : \text{CH}_0(Y_L) \rightarrow \text{CH}_0(X_L)$$

is an isomorphism.

The hypersurfaces $X \subset \mathbb{P}^{n+1}$ we are going to consider are the same as in [20]. They were first introduced by Kollár [10], but Totaro works in a slightly broader range of degrees.

Theorem 3.2. (Cf. [10], [9, Section V.5], [20].) *Assume d is even, $n \geq 3$ and $d \geq 2\lceil \frac{n+2}{3} \rceil$. Then there exists a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ defined over a number field K_0 , with a specialization \overline{X} defined over a field \mathbb{F} of characteristic 2 (a finite extension of \mathbb{F}_2) satisfying the following properties*

1) *The special fiber \overline{X} admits a desingularization $\tau : \widetilde{X} \rightarrow \overline{X}$ and the morphism τ is universally CH_0 -trivial.*

2) *The smooth projective variety \widetilde{X} admits a nonzero algebraic form*

$$0 \neq \omega \in H^0(\widetilde{X}, \Omega_{\widetilde{X}/\mathbb{F}}^{n-1}). \quad (42)$$

By specialization, we mean as usual that there is a smooth projective morphism $\mathcal{X} \rightarrow \text{Spec } R$, where R is a local ring (in our case, a localization of a ring of integers), whose special fiber is isomorphic to \overline{X} , and generic fiber is isomorphic to X .

Remark 3.3. The same result holds for cyclic double covers of \mathbb{P}^n , $n \geq 3$, ramified along a hypersurface of degree $2d$ with $2d \geq n + 1$. Condition (2) above is proved in [9, Proposition 5.10].

Proof of Theorem 1.11. Note that the conditions on n and d are the same in Theorem 1.11 and Theorem 3.2. Let $\mathcal{X}^0 \rightarrow B^0$ be the universal smooth hypersurface. By standard arguments playing on the countability of the relative Chow varieties of $(\mathcal{X}^0)(N/B) \times_{B^0} \mathcal{X}^0$ (see [23]), the set of smooth complex hypersurfaces of degree d and dimension n with bounded CH_0 group is a countable union of closed algebraic subsets in the parameter space B^0 , so in order to prove that the very general one has unbounded CH_0 group, it suffices to prove that there exists one smooth hypersurface of degree d and dimension n with unbounded CH_0 group.

Let X be as in Theorem 3.2. We prove that $\text{CH}_0(X)$ is not bounded. Assume by contradiction that for some integer N' , and for any overfield $L \supseteq K_0$, any 0-cycle $z \in \text{CH}_0(X_L)$ of degree $\geq N'$ is effective. For any integer $l \geq 0$, we take for L the field $K_0(X^{(N)})$, $N := N' + dl$. Then X_L has a natural point z_L of degree N given by the generic fiber of the universal subscheme

$$\Gamma_N \subset X_0^{(N)} \times X.$$

Recall the notation $h_X := c_1(\mathcal{O}_X(1))^n \in \text{CH}_0(X)$. We have $\deg h_X = d$, so $\deg(z_L - lh_X) = N'$, and our hypothesis implies that $z_L - lh_X$ is effective of degree N' . This implies that there exists an effective codimension n cycle

$$T \subset X_0^{(N)} \times X$$

of degree N' over $X_0^{(N)}$, such that for some dense Zariski open set $U \subset X_0^{(N)}$

$$T|_{U \times X} = \Gamma_N|_{U \times X} - lU \times h_X \text{ in } \text{CH}^n(U \times X). \quad (43)$$

We now specialize these cycles to \overline{X} and get a specialized cycle

$$\overline{T} \subset \overline{X}_0^{(N+d)} \times \overline{X}$$

and a dense Zariski open set $\bar{U} \subset \bar{X}_0^{(N+d)}$ satisfying

$$\bar{T}|_{\bar{U} \times \bar{X}} = \Gamma_{N+d|\bar{U} \times \bar{X}} - l\bar{U} \times h_{\bar{X}} \text{ in } \text{CH}^n(\bar{U} \times \bar{X}). \quad (44)$$

Thanks to the CH_0 universal triviality of the desingularization map τ , stated in Theorem 3.2(1), a lift $\tilde{\bar{T}} \subset \tilde{\bar{X}}_0^{(N+d)} \times \tilde{\bar{X}}$ of the cycle \bar{T} exists (at least over a Zariski open set $U' \subset \tilde{\bar{X}}_0^{(N+d)}$) and satisfies (maybe after shrinking U')

$$\tilde{\bar{T}}|_{U' \times \tilde{\bar{X}}} = \Gamma_{N+d|U' \times \tilde{\bar{X}}} - U' \times h_{\tilde{\bar{X}}} \text{ in } \text{CH}^n(U' \times \tilde{\bar{X}}). \quad (45)$$

As $N = N' + dl$ is arbitrarily large while N' is fixed, and $\tilde{\bar{X}}$ has a nonzero $n - 1$ -form by Theorem 3.2(2), we get a contradiction with Theorem 1.12, 4, since $n \geq 3$. \square

3.1 Proof of Theorem 1.4

We focus in this section on the case of quartic threefolds and prove Theorem 1.4. The case of quartic or sextic double solids works exactly in the same way, using Remarks 3.3 and 3.5, 3.8 below. The case of quartic fourfolds also works well with a small extra work.

In order to prove Theorem 1.4, we will need the following two extra ingredients. Over any field k , let $B = \mathbb{P}(H^0(\mathbb{P}_k^{n+1}, \mathcal{O}_{\mathbb{P}_k^{n+1}}(d)))$ and let

$$\mathcal{X} \subset B \times \mathbb{P}_k^{n+1} \quad (46)$$

be the universal hypersurface, with morphism $\pi = \text{pr}_1 : \mathcal{X} \rightarrow B$ and generic fiber X_η over $K := k(B)$.

Lemma 3.4. *One has $\text{CH}_0(X_\eta) = \mathbb{Z}h_{X_\eta}$ and the image of $\text{CH}^n(X_\eta \times X_\eta)$ in*

$$\varinjlim_{U \subset_{\text{open}} X_\eta} \text{CH}^n(U \times X_\eta)$$

*is generated by Δ_{X_η} and $\text{pr}_2^*h_X$.*

Proof. This statement appears in several places (see for example [24]). We give a proof for completeness. The variety \mathcal{X} admits the morphism

$$f := \text{pr}_2 : \mathcal{X} \rightarrow \mathbb{P}^{n+1}.$$

The fiber of f over $x \in \mathbb{P}^{n+1}$ is the projective space $B_x = \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{I}_x(d)))$. Hence \mathcal{X} is a projective bundle over \mathbb{P}^{n+1} and it follows that $\text{CH}^*(\mathcal{X})$ is generated as a ring by $h_1 := \text{pr}_1^*c_1(\mathcal{O}_B(1))$ and $h_2 := f^*c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$. The restriction map

$$\text{CH}(\mathcal{X}) \rightarrow \text{CH}(X_\eta)$$

is surjective and annihilates h_1^i , $i > 0$. It follows that $\text{CH}(X_\eta)$ is generated as a ring by the restriction of f^*h_2 , which proves the result since $h_{X_\eta} = (h_2^2)|_{X_\eta}$.

For the second statement, we use a similar argument except that we consider now

$$\mathcal{Y} := \mathcal{X} \times_B \mathcal{X}. \quad (47)$$

We have a surjection given by restriction over the generic point of B

$$\text{CH}(\mathcal{Y}) \rightarrow \text{CH}(X_\eta \times X_\eta).$$

As above, we consider now the morphism

$$g := (f_1, f_2) : \mathcal{Y} \rightarrow \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}.$$

Above a couple $(x, y) \in \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$ with $x \neq y$, the fiber of g is a projective subspace $B_{x,y} \subset B$ of codimension 2. (Above a point $(x, x) \in \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$, the fiber of g is the projective subspace $B_x \subset B$ of codimension 1, so g is a projective bundle only over $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \setminus \Delta_{\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}}$.) Let $\Delta_{\mathcal{X}} \subset \mathcal{Y}$ be the diagonal. We have

$$\Delta_{\mathcal{X}} = g^{-1}(\Delta_{\mathbb{P}^{n+1}}).$$

Denoting $\pi_2 : \mathcal{Y} \rightarrow B$ the natural map, we see from the above analysis that $\text{CH}(\mathcal{Y} \setminus \Delta_{\mathcal{X}})$ is generated as a ring by $\alpha = \pi_2^* c_1(\mathcal{O}_B(1))$ and $\beta = f_1^* c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$, $\gamma = f_2^* c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$. It follows from the localization exact sequence that $\text{CH}^n(\mathcal{Y})$ is generated as an abelian group by the class of $\Delta_{\mathcal{X}}$ and the monomials of degree n in α, β, γ . When we restrict to $X_\eta \times X_\eta$, the powers α^i vanish for $i > 0$. Similarly, when we restrict to $U \times X_\eta$, where $U \subset X_\eta$ is a sufficiently small Zariski open subset, all powers β^i vanish for $i > 0$. Hence we conclude that for such an open set U , $\text{CH}^n(U \times X_\eta)$ is generated as an abelian group by γ^n and Δ_{X_η} . \square

Remark 3.5. The same statement holds for the generic quartic, resp. sextic double solid, and is proved similarly, as it can be defined as a hypersurface in the total space of $\mathcal{O}(2)$, resp. $\mathcal{O}(3)$, on \mathbb{P}^3 .

Let $Y = \widetilde{X}$ be the desingularization of a Kollár-Mori specialization of a quartic threefold over a field \mathbb{F} of characteristic 2 considered in Theorem 3.2. Recall that we denote by $Y_0^{(N)}$ the open set of the symmetric product $Y^{(N)}$ that parameterizes sets of N distinct points in Y .

Lemma 3.6. *For any $N \geq 1$, one has an isomorphism*

$$H^0(Y_0^{(N)}, \Omega_{Y_0^{(N)}/\mathbb{F}}^2) \cong H^0(Y, \Omega_{Y/\mathbb{F}}^2)$$

given by the trace.

Proof. The pull-back to $Y_0^{(N)}$ of 2-forms on $Y_0^{(N)}$ is injective and maps $H^0(Y_0^{(N)}, \Omega_{Y_0^{(N)}/\mathbb{F}}^2)$ to the space of invariant 2-forms under the symmetric group action. We now prove the following

Claim 3.7. *The threefold Y being as above, we have $H^0(Y, \Omega_{Y/\mathbb{F}}) = 0$.*

Proof. The variety $Y = \widetilde{X}$ is a desingularization of the variety \overline{X} , which is a cyclic double cover $r : \overline{X} \rightarrow Q$ of a quadric $Q \subset \mathbb{P}^4$ ramified along a smooth surface $S \subset Q$ which is a member of the linear system $|\mathcal{O}_Q(4)|$ defined by a section $s \in H^0(Q, \mathcal{O}_Q(4))$. The variety \overline{X} has isolated singular points over the vanishing locus $V(s)$ of the section $ds \in H^0(Q, \Omega_{Q/\mathbb{F}}(4))$ constructed in [9, Definition-Lemma V.5.4]. It clearly suffices to show that

$$H^0(\overline{X}_0, \Omega_{\overline{X}_0/\mathbb{F}}) = 0,$$

where \overline{X}_0 is the smooth locus of \overline{X} . We now restrict the exact sequence established in [9, Lemma V.5.5] to \overline{X}_0 and get

$$0 \rightarrow r^* \mathcal{F} \rightarrow \Omega_{\overline{X}_0/\mathbb{F}} \rightarrow r^* \mathcal{O}_{Q_0}(-2) \rightarrow 0, \quad (48)$$

Where $Q_0 := Q \setminus V(s)$ and \mathcal{F} is the rank 2 vector bundle on Q_0 defined as

$$\mathcal{F} := \text{Coker}(ds : \mathcal{O}_{Q_0}(-4) \rightarrow \Omega_{Q_0/\mathbb{F}}). \quad (49)$$

We have $H^0(\overline{X}_0, r^* \mathcal{O}_{Q_0}(-2)) = 0$, so by (48) we only have to prove that $H^0(\overline{X}_0, r^* \mathcal{F}) = 0$, which is clearly equivalent to $H^0(Q_0, \mathcal{F}) = 0$. By Definition (49) of \mathcal{F} , we have an exact sequence on Q_0

$$0 \rightarrow \mathcal{O}_{Q_0}(-4) \rightarrow \Omega_{Q_0/\mathbb{F}} \rightarrow \mathcal{F} \rightarrow 0. \quad (50)$$

As $Q \setminus Q_0$ has codimension 3 in Q , we have $H^1(Q_0, \mathcal{O}_{Q_0}(-4)) = H^1(Q, \mathcal{O}_Q(-4)) = 0$. Thus $H^0(Q_0, \mathcal{F}) = 0$ follows from $H^0(Q_0, \Omega_{Q_0/\mathbb{F}}) = 0$. \square

Remark 3.8. The same statement holds for the quartic or sextic double solid, replacing in the proof above the quadric Q by \mathbb{P}^3 .

Claim 3.7 implies that

$$H^0(Y_0^N, \Omega_{Y_0^N/\mathbb{F}}^2) = \bigoplus_i \mathrm{pr}_i^* H^0(Y, \Omega_{Y/\mathbb{F}}^2),$$

from which Lemma 3.6 follows by taking invariants under the symmetric group \mathfrak{S}_N . \square

Proof of Theorem 1.4. The field K and the quartic hypersurface X are defined as follows. Working over any field of characteristic 0, for example \mathbb{Q} (or \mathbb{C}), let as before $B = \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4)))$ and $\mathcal{X} \rightarrow B$ be the universal quartic hypersurface. The quartic hypersurface we consider is the generic fiber X_η of the fibration $\mathcal{X} \rightarrow B$. Let $\mathcal{X}^{(N/B)}$ be the relative N -th symmetric product. The field L is defined as

$$L := \mathbb{Q}(\mathcal{X}^{(N/B)}). \quad (51)$$

The quartic X_L is the generic fiber of the morphism

$$\mathrm{pr}_1 : \mathcal{X}^{(N/B)} \times_B \mathcal{X} \rightarrow \mathcal{X}^{(N/B)}.$$

The universal cycle

$$\mathcal{Z} \subset \mathcal{X}^{(N/B)} \times_B \mathcal{X}$$

provides a closed point of X_L , which is of degree N . Theorem 1.4 follows now from Proposition 3.9 proved below. \square

Proposition 3.9. *Assume that $N \geq 5$. Then the quartic threefold X_L has no closed point of odd degree $M < N$.*

Remark 3.10. The assumption $N \geq 5$ is not a serious one. It will be used only to simplify a detail of proof.

Proof of Proposition 3.9. We argue by contradiction and assume that there is an effective 0-cycle $z_M \in \mathrm{CH}_0(X_L)$ of odd degree $M < N$. Let $\mathcal{Z}_M \in \mathrm{CH}^3(\mathcal{X}^{(N/B)} \times_B \mathcal{X})$ be any effective cycle whose restriction over the generic point of $\mathcal{X}^{(N/B)}$ is z_M . Our first goal is to compare, at least modulo 2, the two 0-cycles

$$\Gamma_N, z_M \in \mathrm{CH}_0(X_L),$$

and get a relation as in (1). We are not, in fact, able to do this because we do not know how to compute $\mathrm{CH}_0(X_L)$, but we argue as follows.

Consider the generic quartic threefold X_η over the field $K := \mathbb{Q}(B)$. The symmetric product $X_\eta^{(N)}$ is singular along the complement $X_\eta^{(N)} \setminus (X_\eta^{(N)})_0$ and a fortiori very singular along the small diagonal

$$\delta(X_\eta) \subset X_\eta^{(N)},$$

where δ is the diagonal inclusion. Let us establish the following

Claim 3.11. *There exists a rational map*

$$\sigma : X_\eta \dashrightarrow X_\eta^{(N)}$$

with the following properties:

1. *The image of σ is not contained in the singular locus $X_\eta^{(N/B)} \setminus (X_\eta^{(N/B)})_0$ (so generically the image is contained in $(X_\eta^{(N/B)})_0$).*

2. We have

$$\Gamma_N \circ \sigma = N\Delta_{X_\eta} \text{ in } \varinjlim_{\substack{U \subset X_\eta \\ \text{open}}} \text{CH}^3(U \times X_\eta). \quad (52)$$

Proof. Let $G_\eta \rightarrow X_\eta$ be the fibration in Grassmannians of lines in \mathbb{P}^3 , whose fiber over $x \in X_\eta$ is the family of planes $P \subset \mathbb{P}^4$ passing through x . We can easily construct a rational section $\tau_1 : X_\eta \dashrightarrow G_\eta$. Over G_η , we have a fibration $C_\eta \rightarrow G_\eta$ into quartic plane curves, with fiber $C = P \cap X_\eta$ over the point $[P] \in G_\eta$. This last fibration admits a section $\tau : G_\eta \rightarrow C_\eta$ given by the point x . Composing with τ_1 , we get a section $\tau_2 : X_\eta \dashrightarrow C_\eta$. Furthermore the variety C_η admits a natural morphism $p : C_\eta \rightarrow X_\eta$, which on each fibre $C = P \cap X_\eta$ is the inclusion of C in X_η . By construction, the composition $p \circ \tau_2$ is the identity of X_η . The section τ_2 provides a divisor D in the variety $\tau_1^*(C_\eta) \rightarrow X_\eta$, which is fibered over X_η in genus 3 curves. As $N \geq 5$, linear systems of divisors of degree N on genus 3 curves have no base-points, hence we conclude that, over a Zariski open set of X_η , ND is linearly equivalent to a divisor D' which is reduced. The divisor D' gives a rational section of $(\tau_1^*(C_\eta))^{(N/X_\eta)} \rightarrow X_\eta$, which, composed with the rational map $p^{(N)}$, gives us the desired rational map σ . \square

With the notations of Claim 3.11, we now have

Corollary 3.12. *For some integers α, β, γ with α and β odd, the two correspondences Γ_N and z_N between $X_\eta^{(N)}$ and X_η satisfy the following comparison.*

$$\alpha\Gamma_N \circ \sigma = \beta z_M \circ \sigma + \gamma X_\eta \times h_{X_\eta} + \Gamma' \text{ in } \text{CH}^3(X_\eta \times X_\eta), \quad (53)$$

where Γ' is a codimension 3 cycle of $X_\eta \times X_\eta$ which is supported on $W \times X_\eta$ for some proper closed algebraic subset $W \subset X_\eta$.

Proof. Indeed, we apply Lemma 3.4 that tells us that both cycles $\Gamma_N \circ \sigma$ and $z_M \circ \sigma$ can be written, after restriction to $\varinjlim_{\substack{U \subset X_\eta \\ \text{open}}} \text{CH}^3(U \times X_\eta)$, as combinations with integral coefficients

of the cycles Δ_{X_η} and $X_\eta \times h_{X_\eta}$. Moreover, the cycle $\Gamma_N \circ \sigma$ has degree N over X_η (via the first projection), the cycle $\Gamma_M \circ \phi \circ \sigma$ has degree M over X_η , the diagonal Δ_{X_η} has degree 1 over X_η and the cycle $X_\eta \times h_{X_\eta}$ has degree 4 over X_η . Writing

$$\Gamma_N \circ \sigma = a\Delta_{X_\eta} + bX_\eta \times h_{X_\eta} \text{ in } \varinjlim_{\substack{U \subset X_\eta \\ \text{open}}} \text{CH}^3(U \times X_\eta), \quad (54)$$

$$z_M \circ \sigma = c\Delta_{X_\eta} + dX_\eta \times h_{X_\eta} \text{ in } \varinjlim_{\substack{U \subset X_\eta \\ \text{open}}} \text{CH}^3(U \times X_\eta),$$

we conclude, using the fact that both M and N are odd and by comparing the degrees in (54), that a is odd and c is odd, from which (53) follows by the localization exact sequence, since by (54) we can take $\alpha = b, \beta = a$. \square

We now conclude the proof of Proposition 3.9. The constructions above specialize from the generic fiber X_η to the special fiber \overline{X} of Theorem 3.2. The cycle \mathcal{Z}_M specializes to a cycle

$$\overline{\mathcal{Z}}_M \in \text{CH}^3(\overline{X}_0^{(N)} \times \overline{X}),$$

which is effective by [5, Lemme 2.10], and has degree M over $\overline{X}_0^{(N)}$. Looking at the proof of Claim 3.11, the rational map σ specializes well. Using the desingularization $\tau : \widetilde{\overline{X}} \rightarrow \overline{X}$, we lift the cycle $\overline{\mathcal{Z}}_M$ to an effective cycle $\widetilde{\overline{\mathcal{Z}}}_M$, and we lift similarly the specialized rational map $\overline{\sigma}$. Using the fact that τ is CH_0 -universally trivial (see Theorem 3.2), we conclude that formula (53) still holds for the specialized cycles, giving

$$\alpha\Gamma_N \circ \bar{\sigma} = \beta\tilde{Z}_M \circ \bar{\sigma} + \gamma\tilde{X} \times h_{\tilde{X}} \text{ in } \varinjlim_{U \subset \tilde{X}} \text{CH}^3(U \times \tilde{X}). \quad (55)$$

As in Totaro's argument and as was already used in the proof of Theorem 1.12, we deduce from (55) the similar equality for the actions of the considered cycles on 2-forms $\omega \in H^0(\tilde{X}, \Omega_{\tilde{X}/\mathbb{F}}^2)$. Here the field \mathbb{F} has characteristic 2, and using the fact that the numbers α and β are odd, we finally conclude that, for any $\omega \in H^0(\tilde{X}, \Omega_{\tilde{X}/\mathbb{F}}^2)$,

$$\bar{\sigma}^*([\Gamma_N]^*(\omega)) = \bar{\sigma}^*([\tilde{Z}_M]^*(\omega)) \text{ in } H^0(\tilde{X}, \Omega_{\tilde{X}/\mathbb{F}}^2). \quad (56)$$

We use now Lemma 3.6 that tells us, using the fact that N is odd and we are in characteristic 2, that $\bar{\sigma}^*$ induces an isomorphism between $H^0(\tilde{X}_0^{(N)}, \Omega_{\tilde{X}_0^{(N)}/\mathbb{F}}^2)$ and $H^0(\tilde{X}, \Omega_{\tilde{X}/\mathbb{F}}^2)$. It follows then from (56) that, for any $\omega \in H^0(\tilde{X}, \Omega_{\tilde{X}/\mathbb{F}}^2)$,

$$[\Gamma_N]^*(\omega) = [\tilde{Z}_M]^*(\omega) \text{ in } H^0(\tilde{X}_0^{(N)}, \Omega_{\tilde{X}_0^{(N)}/\mathbb{F}}^2). \quad (57)$$

as \tilde{Z}_M is effective of degree $M < N$ over $\tilde{X}_0^{(N)}$, and \tilde{X} has a nonzero 2-form, we finally get a contradiction by the rank argument used to prove Theorem 1.12, 2.

Using Theorem 1.12, 3, we can prove the same result for quartic fourfolds by a Totaro-Kollár specialization to a \bar{X} such that a desingularization \tilde{X} has a nonzero 3-form. The only extra proof needed is the analogue of Lemma 3.6, now with $l = 3$. This is left to the reader. \square

4 The case of del Pezzo surfaces

We give here for completeness the proof of Theorem 1.8 (which is established in [5], [22], only when $d_S \leq 3$). We will only describe the strategy of the proof, which follows closely [22], to which we refer for more details. In practice, one should be a little careful in checking case by case a certain technical statement (see Remark 4.2 below). Let $\mathcal{O}_S(1) := K_S^{-1}$ be the anticanonical line bundle of S and let $h_S := c_1(K_S)^2 \in \text{CH}_0(S)$. This is a 0-cycle on S of degree d_S . One has by Hirzebruch-Riemann-Roch and Kodaira vanishing

$$h^0(S, \mathcal{O}_S(l)) = 1 + \frac{(l^2 + l)d_S}{2}. \quad (58)$$

In order to prove Theorem 1.8, it suffices to prove the following

Proposition 4.1. *There exists an integer N_1 with the following property. For any smooth del Pezzo surface S over a field K of characteristic 0, and any effective 0-cycle $z \in \text{CH}_0(S)$ of degree $d > N_1$, there is an effective 0-cycle $z' := \pm z + \gamma h_S$ which is of degree strictly smaller than d .*

Let us first show how Proposition 4.1 implies Theorem 1.8. This argument already appears in [22]. From proposition 4.1, we deduce by induction on $d = \deg z$ that any effective 0-cycle on S can be written modulo rational equivalence as

$$z = \pm z' + \gamma h_S \text{ in } \text{CH}_0(S), \quad (59)$$

where $\gamma \in \mathbb{Z}$ and z' is an effective 0-cycle of degree $\leq N_1$. Next by boundedness of del Pezzo surfaces, there exists an integer $N \geq N_1$ such that for any effective zero-cycle z' on

S of degree $\leq N_1$, the 0-cycle $z'' = \gamma' h_S - z'$ is both of degree $\leq N$ and effective, for some number $\gamma' \in \mathbb{Z}$. We then conclude from (59) that any effective 0-cycle on S can be written modulo rational equivalence as

$$z = z'' + \gamma'' h_S \text{ in } \text{CH}_0(S), \quad (60)$$

where z'' is effective of degree $\leq N$. Finally (60) implies Theorem 1.8 with the same integer N . Indeed, any 0-cycle z can be written as

$$z = z_1 - z_2 \text{ in } \text{CH}_0(S) \quad (61)$$

for some effective 0-cycles z_1 and z_2 . Given z_2 , for some large enough positive integer α , $\alpha h_S - z_2$ is effective. Indeed, we choose a smooth projective curve $C \subset X$ in some linear system $| -lK_S |$ supporting z_2 . The curve C also supports a multiple βh_S of h_S , and by Riemann-Roch on C , we get that for β' large enough, the 0-cycle $\beta' w - z_2$ is effective on C , where w is any 0-cycle on C of class βh_S . It follows that (61) becomes

$$z = z_{\text{eff}} + \delta h_S \text{ in } \text{CH}_0(S) \quad (62)$$

for some effective 0-cycle z_{eff} on S . We can then apply (60) to z_{eff} and conclude that any 0-cycle z can be written as

$$z = z'' + \gamma'' h_S \text{ in } \text{CH}_0(S),$$

where z'' is effective of degree $\leq N$. If $\deg z \geq N$, then $\gamma'' \geq 0$ and z is effective, proving Theorem 1.8.

Proof of Proposition 4.1. As we are over a field K of characteristic 0, an effective 0-cycle of degree d on S is also a K -point $z \in S^{(d)}(K)$. As proved in [5, Lemme 2.10] and used repeatedly in [22], the Fulton specialization of cycles preserves effectivity. It thus suffices to prove Proposition 4.1 when $z \in S^{(d)}$ is generic (defined over the function field $L := K(S^{(d)})$), as the result for any z then follows by specialization. The proof is by induction on the degree $d = \deg z$. Let l be the unique integer such that

$$h^0(S, \mathcal{O}_S(l)) < \deg z \leq h^0(S, \mathcal{O}_S(l+1)). \quad (63)$$

Assume first that (*) $\deg z \leq h^0(S, \mathcal{O}_S(l+1)) - 2$.

Then $(l+1)^2 h_S - z$ is effective by [22, Lemma 3.6]. As $\deg (l+1)^2 h_S = (l+1)^2 d_S$, it follows that we can assume, arguing by induction on degree $d = \deg z$ and replacing z by $(l+1)^2 h_S - z$ if necessary, that

$$\deg z \leq d_S \frac{(l+1)^2}{2}. \quad (64)$$

The generic effective 0-cycle z defines as well a reduced subscheme $Z \subset S_L$ of length d . Using the left strict inequality in (63), we get that $H^1(S_L, \mathcal{I}_Z(l)) \neq 0$, hence by Serre duality on S_L , using $K_{S_L} = \mathcal{O}_{S_L}(-1)$,

$$\text{Ext}^1(\mathcal{I}_Z(l+1), \mathcal{O}_{S_L}) \neq 0.$$

A nonzero extension class $e \in \text{Ext}^1(\mathcal{I}_{S_L}(l+1), \mathcal{O}_{S_L})$ provides a rank 2 coherent sheaf E on S_L , which fits in an extension

$$0 \rightarrow \mathcal{O}_{S_L} \rightarrow E \rightarrow \mathcal{I}_Z(l+1) \rightarrow 0. \quad (65)$$

Using the genericity of z , the coherent sheaf E is in fact a vector bundle for a general choice of extension e . Furthermore it has a section s vanishing exactly along Z , which is given by the map $\mathcal{O}_{S_L} \rightarrow E$ in (65). It follows that

$$z = c_2(E) \text{ in } \text{CH}_0(S_L).$$

Furthermore E satisfies by (65)

$$h^0(S_L, E) = 1 + h^0(S_L, \mathcal{I}_Z(l+1)). \quad (66)$$

Using (64) and (58), we get from (66) that

$$h^0(S, E) \geq 2 + d_S \frac{(l+1)}{2}. \quad (67)$$

We then conclude using the genericity of Z that, if $\frac{(l+1)}{2} \geq 2$, that is, if $l \geq 3$, there exists a section s' of E vanishing along a subscheme $Z_0 \subset \bar{S}$ of class h_S . If the vanishing locus of s' has dimension 0, then the zero-cycle $V(s') - h_S$ is effective and rationally equivalent to $c_2(E) - h_S = z - h_S$. As the subschemes Z_0 are not generic, it is however not clear that there exists a section s' vanishing along Z_0 and with 0-dimensional vanishing locus. We overcome this by seeing Z_0 as the specialization of a generic 0-dimensional subscheme $Z_{0,\text{gen}} \subset S_L$ of length d_S . We prove then the existence of a section s' vanishing along $Z_{0,\text{gen}}$ and with 0-dimensional vanishing locus. Thus $z - Z_{0,\text{gen}}$ is effective, and by specialization $z - h_S$ is effective.

In conclusion, assuming (*), the induction step for Proposition 4.1 is proved if $l \geq 3$, hence if $d > h^0(S, \mathcal{O}_S(3))$. As $h^0(S, \mathcal{O}_S(3)) \leq 55$, the induction step works for $d \geq 56$, assuming (*).

In the remaining cases where

$$d = \deg z = h^0(S, \mathcal{O}_S(l+1)) - 1 \text{ or } \deg z = h^0(S, \mathcal{O}_S(l+1)), \quad (68)$$

we do the following trick. Choosing a reduced subscheme $Z_0 \subset S_L$ of class $z_0 = h_S$ and not intersecting Z , we consider the 0-dimensional subscheme $Z' = Z \cup Z_0 \subset S_L$ which is reduced and effective of degree

$$d + d_S > h^0(S_L, \mathcal{O}_{S_L}(l+1)) \quad (69)$$

(we can assume here $d_S \geq 4$ by Theorem 1.7). Using inequality (69), we now apply the same strategy of constructing a vector bundle E which fits in an extension

$$0 \rightarrow \mathcal{O}_{S_L} \rightarrow E \rightarrow \mathcal{I}_{Z'}(l+2) \rightarrow 0,$$

and has as above a section s vanishing exactly along $Z' = Z \cup Z_0$, where Z_0 represents the class h_S . One has

$$\begin{aligned} h^0(S_L, E) &= 1 + h^0(S_L, \mathcal{I}_{Z'}(l+2)) \geq 1 + h^0(S_L, \mathcal{O}_{S_L}(l+2)) - h^0(S_L, \mathcal{O}_{S_L}(l+1)) - d_S \\ &= (l+1)d_S + 1. \end{aligned}$$

It follows that, once $d_S(l+1) \geq 4d_S$, that is,

$$l+1 \geq 4, \quad (70)$$

the vector bundle E has a section s' vanishing along $Z'_0 \cup Z''_0$, where Z'_0, Z''_0 are subschemes in general position representing the class h_S . In this construction, the subscheme $Z' = Z \cup Z_0$, or rather the corresponding element $z' \in S_L^{(d+d_S)}$, is not generic in $S_L^{(d+d_S)}$, and similarly the subschemes $Z'''_0 = Z'_0 \cup Z''_0$, or rather the corresponding elements $z'''_0 = z'_0 + z''_0 \in S_L(2d_S)$, are not generic, but we can replace them by the generic points $z'_{\text{gen}} \in S_L^{(d+d_S)}$, resp. $Z'''_{0,\text{gen}} \in S_L^{(2d_S)}$ which specialize to z' , resp. z'''_0 . One checks that, when the zero-dimensional subschemes above are generic, the generic section s' of E vanishing along $Z'''_{0,\text{gen}}$ has a zero-dimensional vanishing locus, hence $z'_{\text{gen}} - z'''_{0,\text{gen}}$ is effective. It follows by specialization, applying again [5, Lemme 2.10], that

$$z' - z'_0 - z''_0 = z + z_0 - z'_0 - z''_0 = z - h_S \in \text{CH}_0(S_L)$$

is effective.

The inequality (70), together with (68) shows that in the situation (68), the induction step of Proposition 4.1 works once $l \geq 3$. When $l \leq 2$ and (68) holds, we have

$$d \leq h^0(S, \mathcal{O}_S(3)) \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(9)) = 55.$$

Proposition 4.1 is thus proved with $N_1 = 56$. □

Remark 4.2. In the sketch of proof above, what needs an extra-checking is the proof that, when the 0-cycles z (or z') and z_0 (or z_0''') are generic, one can choose the section s' as above to have a zero-dimensional vanishing locus. This proof is done carefully in the case of cubic surfaces in [22].

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