On the stable birational type of Fano threefolds

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Abstract

We show that, given a very general cubic threefold X, any smooth cubic threefold stably birational to X must be isomorphic to X. The method applies as well to other Fano 3-folds, like the sextic double solid, and the quartic threefold. We also discuss the higher-dimensional case.

1 Introduction

Recall that two smooth projective varieties X and X' are stably birational if $X \times \mathbb{P}^r$ is birational to $X' \times \mathbb{P}^{r'}$, for some integers r, r' such that $\dim X + r = \dim X' + r'$. A variety X is stably rational if it is stably birational to a point. The following beautiful result was proved recently by de Gaay Fortman, Engel and Schreieder:

Theorem 1.1. *[11] A very general smooth cubic threefold is not stably rational.*

Note that it had been proved long ago by Clemens and Griffiths [6] that a smooth cubic threefold is never rational. It has been proved in [25] (using AI) that some smooth cubic threefolds are stably rational. However, it is proved in [14] (using AI) that $X \times \mathbb{P}^1$ is not rational for any smooth cubic threefold X. One difficulty in the study of rationality of $X \times \mathbb{P}^r$ with $r \geq 1$, hence of the stable rationality problem, is that the Clemens-Griffiths method can be *a priori* applied only in dimension 3, because it uses intermediate Jacobians $J^3(X)$ equipped with their unimodular polarization, and the latter exists only in dimension 3. The paper [14] uses the completely new tool invented in [16], which is adapted to treat rationality in dimension 4.

In the paper [23], Shinder initiated the study of a different and interesting problem, namely, given a family of smooth projective algebraic varieties $\mathcal{X} \to B$, whose very general fiber $\mathcal{X}_b$ is not stably rational, what can be said of the set of pairs b, $b' \in B$ such that $\mathcal{X}_b$ is stably birational to $\mathcal{X}_{b'}$? The main result proved in [23] is the following:

Theorem 1.2. *[23] Assume $\mathcal{X} \to B$ is the family of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$. Then if one fiber $\mathcal{X}_{b_0}$ is not stably rational, two very general fibers $\mathcal{X}_b$, $\mathcal{X}_{b'}$ are not stably birational.*

In other words, there is a variation of the stable birational type of the fibers, once one fiber is not stably rational. The assumption that at least one fiber is not stably rational had been established by Schreieder in [20] in the range $d \geq \log_2 n + 2$, $n \geq 3$. The very recent paper [15] by Hotchkiss (using AI) establishes this result for hypersurfaces of degree at least 5. In the case of cubic threefolds or quartic threefolds, or quartic or sextic double solids, this is proved respectively in [11], [9], [26], [4]. More precisely, these papers establish the non-existence of a decomposition of the diagonal for the very general considered hypersurface, which implies that it is not stably rational (see Section 2 for more explanations). In all these cases, the assumption of Theorem 1.2 is satisfied, and we conclude that if X, X' are very general smooth cubic threefolds, or smooth quartic threefolds, or smooth quartic or sextic double solids, then X is not stably birational to X'.

In the appendix to [23], we gave a different proof of Theorem 1.2, which works assuming the stronger statement that the very general fiber does not have a Chow decomposition of

the diagonal, and more importantly that this can be proved by applying the specialization method, exhibiting a nontrivial stable birational invariant on the desingularized special fiber, which disappears on the general fiber, (typically, a Brauer class or a unramified cohomology class of positive degree). A similar statement, assuming only that the very general fiber does not have a Chow decomposition of the diagonal, and working in nonzero characteristic, is proved by Schreieder in [21].

In this paper, we are going to work under the even stronger assumption that the very general member of the family does not have a *cohomological* decomposition of the diagonal. The non-existence of a cohomological decomposition of the diagonal is proved in [26] for the very general quartic double solid, and, following the same proof and using [18], one can prove the same statement for the quartic threefold, as it specializes to a double cover of a quadric threefold ramified along a quartic section. Following the same method, it is established by [4] for the sextic double solids. It is also proved in [11], by a completely different argument, for the very general cubic threefold.

Our main result is the following statement.

Theorem 1.3. *(a) If X is a very general cubic threefold, any smooth cubic threefold X' which is stably birational to X is in fact isomorphic to X.*

(b) The same statement holds for quartic threefolds and sextic double solids.

Remark 1.4. If one replaces "stably birational" by "birational", the conclusion follows from the Clemens-Griffiths method [6] combined with the generic Torelli theorem which is known for all these Fano threefolds. Indeed, if X is birational to X', then one must have by Clemens and Griffiths an isomorphism of principally polarized abelian varieties

$$J(X) \oplus \bigoplus_i J(C_i) \cong J(X') \oplus \bigoplus_j J(C'_j).$$

As $J(X)$ is an irreducible ppav which is not isomorphic to the Jacobian of curves (which are also irreducible), one then gets that $J(X) \cong J(X')$ as ppav, and the (generic) Torelli theorem implies that $X \cong X'$.

The proof of Theorem 1.3 is obtained by exploiting further the main result of [23, Appendix] (presented here as Theorem 3.1) in order to prove that, in the situation of Theorem 1.3, the intermediate Jacobians of X and X' must be isogenous. The conclusion of the proof relies then on a stronger form of the Torelli theorem (see Section 4) involving isogeny classes (instead of isomorphism classes) of intermediate Jacobians.

Another application of our arguments compares the stable birational types of different Fano threefolds. For example, we prove the following

Theorem 1.5. *(Cf. Corollary 3.6(b)) If X is a very general cubic threefold, quartic threefold, quartic or sextic double solid, and X' is a smooth rationally connected variety stably birational to X, then $b_3(X') \geq b_3(X)$.*

In particular:

(a) Let X be a very general cubic threefold. Then X is not stably birational to any rationally connected projective variety X' with $b_3(X') < 10$. Hence it is not stably birational to any smooth complete intersection Fano variety of dimension ≥ 4.

(b) Let X be a very general quartic double solid. Then X is not stably birational to any rationally connected threefold with $b_3(X') < 20$, and neither to any smooth complete intersection Fano variety of dimension ≥ 4.

The paper is organized as follows. In Section 2, we discuss the various notions of decompositions of the diagonal, the available criteria and non-existence results. Section 3 is devoted to recalling the statement and proof of Theorem 3.1 from [23, Appendix], expressing stable birational equivalence in terms of decompositions of the diagonal. We establish in that section the following result, from which the other theorems follow directly.

Proposition 1.6. *(See Proposition 3.5.) Let X, X' be two stably birational rationally connected smooth projective varieties, with $\dim X = 3$. Assume that $\mathrm{Tors}(H^3(X,\mathbb{Z})) = 0$ and that X does not have a cohomological decomposition of the diagonal. Then there exists a nonzero morphism $\gamma : J^3(X) \to J^3(X')$ of abelian varieties.*

This morphism γ has to be an isogeny when $b_3(X) = b_3(X')$ and $J^3(X)$ is simple, which is the first step in the proof of Theorem 1.3.

In Section 4, we recall the statement and proof of the Torelli theorem for isogenies between intermediate Jacobians of complete intersections Fano threefolds, obtained as a consequence of Donagi's generic Torelli theorem [12]. This allows us to deduce, from the existence of a nontrivial isogeny between $J^3(X)$ and $J^3(X')$, that X is isomorphic to X', concluding the proof of Theorem 1.3.

In Section 5, we extend conditionally our results to dimension 4 and higher. Although the scheme of proof is exactly the same, starting from dimension 5, we need some extra conditions like the integral Hodge conjecture to be satisfied by the very general member of the family in order to extend the results, see Remark 5.3). Furthermore, the non-existence of a *cohomological* (as opposed to Chow-theoretic) decomposition of the diagonal for the very general Fano hypersurface does not seem to have been established in the ranges considered in [20] or [15]. We discuss this last point in Section 2. The higher dimensional results are thus conditional (see Theorem 5.1).

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2 Decompositions of the diagonal

We review in this section various notions of decomposition of the diagonal, and known non-existence results.

Definition 2.1. *(i) A smooth projective variety X over an algebraically closed field K has a Chow decomposition of the diagonal if, choosing a point $x \in X(K)$ and denoting by $\Delta_X \subset X \times X$ the diagonal of X, one can write*

$$\Delta_X = X \times x + \Gamma \text{ in } \mathrm{CH}^n(X \times X),$$

where the cycle Γ is supported on $D \times X$ for some proper closed algebraic subset $D \subset X$.

(ii) Assuming $K = \mathbb{C}$ (as in the rest of the paper), X has a cohomological decomposition of the diagonal if, choosing a point $x \in X(\mathbb{C})$ and denoting by $\Delta_X \subset X \times X$ the diagonal of X, one has an equality of cycle classes

$$[\Delta_X] = [X \times x] + [\Gamma] \text{ in } H^{2n}(X \times X, \mathbb{Z}),$$

where the cycle Γ is supported on $D \times X$ for some proper closed algebraic subset $D \subset X$.

Obviously, having a Chow decomposition of the diagonal implies having a cohomological decomposition of the diagonal, by taking Betti cycles classes. Having a Chow decomposition of the diagonal is equivalent to having trivial universal CH_0 (see [2]), which means that for any overfield $L \supseteq K$, $\mathrm{CH}_0(X_L)$ is cyclic, generated by x_L. As the CH_0 group is a stable birational invariant, one gets that stably rational varieties admit a Chow decomposition of the diagonal, and thus, when $K = \mathbb{C}$, admit a cohomological decomposition of the diagonal.

Obstructions to the existence of a cohomological decomposition of the diagonal which are possibly nontrivial for some rationally connected smooth projective complex varieties, are described in [1], [8], [27], [29]. This is summarized in the following

Proposition 2.2. *If a smooth complex projective variety X has a cohomological decomposition of the diagonal, then*

a) The topological Brauer group $\mathrm{Tors}(H^3(X,\mathbb{Z}))$ *vanishes. (This condition is not satisfied by the desingularization of a certain nodal quartic double solid, as proved in [1].)*

b) Integral Hodge classes of degree 4 *on X are algebraic. (This condition is reinterpreted in [8] as the vanishing of the degree* 3 *unramified cohomology of X with value in $\mathbb{Q}/\mathbb{Z}$. It is proved in [7] to be nontrivial for certain unirational* 6*-folds and in [20] for some unirational fourfolds.)*

c) X has a universal codimension 2 *cycle parameterized by the intermediate Jacobian $J^3(X)$ (this condition is not satisfied by the desingularization of some nodal quartic double solids, as proved in [26]).*

d) If furthermore $\dim X = 3$*, the minimal class $\frac{\theta^{g-1}}{(g-1)!}$, $g := \frac{b_3(X)}{2}$, of the intermediate Jacobian $J^3(X)$ is algebraic. (This last implication is proved in [27]. This property is not satisfied by the very general cubic threefold, as proved in [11].)*

In contrast, there are many more obstructions to the existence of a Chow decomposition of the diagonal.

Proposition 2.3. *If a smooth projective variety X over a field K has a Chow decomposition of the diagonal, then*

a) (See [24]) One has $H^0(X,\Omega^l_{X/K}) = 0$ for any $l > 0$. (This condition is automatically satisfied if X is rationally connected in characteristic 0*, but not in nonzero characteristic, see [17, Proposition 5.10].)*

b) (See [8]) If $K = \mathbb{C}$, all unramified cohomology groups $H^i_{\mathrm{nr}}(X,A)$ vanish for $i > 0$ and any abelian group A.

Remark 2.4. In b), torsion coefficients A are the interesting ones. Indeed, it is proved in [8] that unramified cohomology with $\mathbb{Z}$-coefficients vanishes in positive degree on any smooth projective complex rationally connected variety.

Remark 2.5. The first two interesting unramified cohomology groups appearing in b), namely $H^2_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z})$ and $H^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z})$ are in fact (for a rationally connected variety), the groups appearing respectively in Proposition 2.2 a) and b).

Finally the original form of the specialization method, as appearing in [26], provides the following result.

Theorem 2.6. *[26] Let $\mathcal{X} \to B$ be a flat family of projective complex varieties of dimension > 1 over a smooth connected curve. Assume the general fiber $\mathcal{X}_b$ is smooth and one fiber $\mathcal{X}_0$ has ordinary quadratic singularities.*

a) Assume furthermore that a (equivalently, any) desingularization $\widetilde{X}_0$ of $\mathcal{X}_0$ does not have a cohomological decomposition of the diagonal. Then the very general fiber $\mathcal{X}_b$ does not have a cohomological decomposition of the diagonal.

b) Assume furthermore that a (equivalently, any) desingularization $\widetilde{X}_0$ of $\mathcal{X}_0$ does not have a Chow decomposition of the diagonal. Then the very general fiber $\mathcal{X}_b$ does not have a Chow decomposition of the diagonal.

In a), the non-existence of a cohomological decomposition of the diagonal for $\widetilde{X}_0$ can be studied using Proposition 2.2. In the case treated in [26], the varieties are quartic double solids, the special fiber is the Artin-Mumford double solid, and its desingularization does not admit a cohomological decomposition of the diagonal by Proposition 2.2 a) because it has a nontrivial topological Brauer group. Thus the very general quartic double solid does not have a cohomological decomposition of the diagonal. The same argument was applied by Beauville in [4] to the sextic double solid. For the quartic threefold, it suffices, in order to apply Theorem 2.6, to find a nodal degeneration X_0 of the quartic threefold such that a desingularization $\widetilde{X}_0$ of X_0 has a nontrivial topological Brauer class. This is easy using the Mori specialization (see [17, Section V.5]) of X to a general double cover of a quadric

$Q \subset \mathbb{P}^4$ ramified along a quartic section. One can then use [18] which precisely constructs an Artin-Mumford type nodal degeneration for such double covers.

The non-existence of a cohomological decomposition of the diagonal for the very general cubic threefold is proved in [11]. Hence the very general Fano threefolds in Theorems 1.3 and 1.5 do not have a cohomological decomposition of the diagonal.

Similarly, for b), the non-existence of a Chow decomposition of the diagonal for $\widetilde{X}_0$ can be studied using Proposition 2.3. Theorem 2.6 b) has been generalized in [9], which allows in particular to work with specialization to nonzero characteristic, and in [24], where differential forms on the desingularized special fiber in nonzero characteristic are used as an obstruction to Chow decomposition of the diagonal on its desingularization, as in Proposition 2.3 a).

These methods have been also improved by Schreieder in [22], with applications in a much broader numerical range in [20], and also more recently by Hotchkiss (using AI) in [15]. However, it is not clear that any of these generalizations allows to disprove the existence of a cohomological decomposition of the diagonal on the general fiber. Schreieder's improved criterion uses the existence of a nontrivial unramified cohomology class on the desingularization of the special fiber. Unfortunately, he does not work with nodal degenerations, so even if the degree of that class is not greater than 3, it is not clear that one can deduce from the non-existence of a cohomological decomposition on the desingularization of the special fiber, the non-existence of a cohomological decomposition on the very general fiber. In [20], nonzero unramified cohomology classes of any given degree are constructed on some unirational conic bundles. In particular, Schreieder constructs unirational fourfolds which do not satisfy the integral Hodge conjecture for degree 4 Hodge classes. This is an obstruction to a cohomological decomposition of the diagonal. It is not clear however that this can be combined with the specialization method to show that the very general quartic or quintic fourfold, or the very general quartic, sextic or octic double covers of $\mathbb{P}^4$ do not admit a cohomological decomposition of the diagonal.

3 Stably birational varieties

Let X, X' be smooth projective varieties of the same dimension. We proved in [23, Appendix] the following result, that will be essential for the proof of Theorem 1.3.

Theorem 3.1. *If X and X' are stably birational of the same dimension n, there exist correspondences*

$$\Gamma \in \mathrm{CH}^n(X \times X'),\, \Gamma' \in \mathrm{CH}^n(X' \times X)$$

with the following properties

$$\Gamma' \circ \Gamma = \Delta_X + W \text{ in } \mathrm{CH}^n(X \times X),\ \Gamma \circ \Gamma' = \Delta_{X'} + W' \text{ in } \mathrm{CH}^n(X' \times X'), \tag{1}$$

where the cycle W is supported on $D \times X$ for some proper closed algebraic subset $D \subset X$, and the cycle W' is supported on $D' \times X'$ for some proper closed algebraic subset $D' \subset X'$.

We include the proof for completeness, and because the reference [23, Appendix] is not so easily available.

Proof of Theorem 3.1. Let X, X' be smooth projective of the same dimension n, and let

$$\Phi : X \times \mathbb{P}^r \dashrightarrow X' \times \mathbb{P}^r \tag{2}$$

be a birational map. Let $\Phi' := \Phi^{-1} : X' \times \mathbb{P}^r \dashrightarrow X \times \mathbb{P}^r$ be its inverse. Let

$$\Gamma_\Phi \subset X \times \mathbb{P}^r \times X' \times \mathbb{P}^r,\ \Gamma_{\Phi'} \subset X' \times \mathbb{P}^r \times X \times \mathbb{P}^r$$

be the graphs of Φ, resp. Φ'. We use the same letter to denote the classes of Γ_Φ, $\Gamma_{\Phi'}$ in $\mathrm{CH}^{n+r}(X \times \mathbb{P}^r \times X' \times \mathbb{P}^r)$. As Φ, Φ' are birational and inverse to each other, we have

$$\Gamma_{\Phi'} \circ \Gamma_\Phi = \Delta_{X \times \mathbb{P}^r} + W_0 \text{ in } \mathrm{CH}^{n+r}(X \times \mathbb{P}^r \times X \times \mathbb{P}^r), \tag{3}$$

where the cycle W_0 is supported on $D_0 \times X \times \mathbb{P}^r$ for some proper closed algebraic subset D_0 of $X \times \mathbb{P}^r$. (Precisely, we can take for D_0 a divisor such that $\Phi_{|(X\times\mathbb{P}^r)\setminus D_0}$ is a morphism and an isomorphism onto its image in $X' \times \mathbb{P}^r$.) Let h be the positive generator of the Chow ring of $\mathbb{P}^r$, and let

$$h_2 := \mathrm{pr}_2^* h \in \mathrm{CH}^1(X \times \mathbb{P}^r \times X' \times \mathbb{P}^r),\ h'_4 := \mathrm{pr}_4^* h \in \mathrm{CH}^1(X \times \mathbb{P}^r \times X' \times \mathbb{P}^r),$$

where pr_i, $i = 2, 4$ denotes the projection of $X \times \mathbb{P}^r \times X' \times \mathbb{P}^r$ onto its i-th factor. (Here we use the notation h'_4 to insist on the fact that it comes from the copy of $\mathbb{P}^r$ that is associated with X' in (2).) We use a similar notation for $X' \times \mathbb{P}^r \times X \times \mathbb{P}^r$ where the graph of Φ' lies.

We now write

$$\Gamma_\Phi = \sum_{i,j} h_2^i {h'_4}^j \mathrm{pr}^*_{X\times X'} \Gamma_{ij} \text{ in } \mathrm{CH}^{n+r}(X \times \mathbb{P}^r \times X' \times \mathbb{P}^r), \tag{4}$$

where $0 \le i \le r$, $0 \le j \le r$, and similarly

$$\Gamma_{\Phi'} = \sum_{i,j} {h'_2}^i h_4^j \mathrm{pr}^*_{X'\times X} \Gamma'_{ij} \text{ in } \mathrm{CH}^{n+r}(X' \times \mathbb{P}^r \times X \times \mathbb{P}^r). \tag{5}$$

Lemma 3.2. *We have*

$$\Gamma_{\Phi'} \circ \Gamma_\Phi = \sum_{i,k,l} h_2^i h_4^l \mathrm{pr}^*_{X\times X}(\Gamma'_{kl} \circ \Gamma_{i,r-k}) \text{ in } \mathrm{CH}^{n+r}(X \times \mathbb{P}^r \times X \times \mathbb{P}^r). \tag{6}$$

Proof. To compute the composition, we have to work on

$$X \times \mathbb{P}^r \times X' \times \mathbb{P}^r \times X \times \mathbb{P}^r.$$

Following the definition of the composition of correspondences, we need to pull-back the cycles Γ_Φ, $\Gamma_{\Phi'}$ from the first four summands $X \times \mathbb{P}^r \times X' \times \mathbb{P}^r$ and the last four summands $X' \times \mathbb{P}^r \times X \times \mathbb{P}^r$ respectively, take their intersection, and push-forward the result via the projection onto $X \times \mathbb{P}^r \times X \times \mathbb{P}^r$ which forgets the middle term $X' \times \mathbb{P}^r$. Using the developments (4), (5), and applying the same notational convention as above, one sees that one has to push-forward to $X \times \mathbb{P}^r \times X \times \mathbb{P}^r$ the cycle

$$\sum_{ijkl} \mathrm{pr}^*_{X\times X'} \Gamma_{ij} \cdot \mathrm{pr}^*_{X'\times X} \Gamma'_{kl} h_2^i {h'_4}^{j+k} h_6^l.$$

on $X \times \mathbb{P}^r \times X' \times \mathbb{P}^r \times X \times \mathbb{P}^r$. It it clear that only the terms with $j + k = r$ will survive, which gives (6). □

Coming back to our proof, we use the fact that

$$\mathrm{pr}_{X\times X*}(\Delta_{X\times\mathbb{P}^r} \cdot h_2^r) = \Delta_X \text{ in } \mathrm{CH}^n(X \times X), \tag{7}$$

where the projections are now those defined on $X \times \mathbb{P}^r \times X \times \mathbb{P}^r$. Equation (3), combined with (7), then gives

$$\begin{aligned} \Delta_X &= \mathrm{pr}_{X\times X*}(\Delta_{X\times\mathbb{P}^r} \cdot h_2^r) \\ &= \mathrm{pr}_{X\times X*}((\Gamma_{\Phi'} \circ \Gamma_\Phi) \cdot h_2^r) - \mathrm{pr}_{X\times X*}(W_0 \cdot h_2^r) \text{ in } \mathrm{CH}^n(X \times X), \end{aligned} \tag{8}$$

where we observe that the cycle $W_1 := \mathrm{pr}_{X\times X*}(W_0 \cdot h_2^r)$ is supported on $D_1 \times X$ for some proper closed algebraic subset D_1 of X.

Formula (6) now gives us

$$\mathrm{pr}_{X\times X*}((\Gamma_{\Phi'} \circ \Gamma_\Phi) \cdot h_2^r) = \sum_k \Gamma'_{kr} \circ \Gamma_{0,r-k}. \tag{9}$$

Finally, we observe that $\dim \Gamma_{ij} = n - r + i + j$. It follows that $\dim \Gamma_{0,r-k} = n - k$, hence that

$$\dim \Gamma_{0,r-k} < n \text{ if } k > 0.$$

Thus, for $k > 0$, the composition $\Gamma'_{kr} \circ \Gamma_{0,r-k}$ is supported on $Z_k \times X$, for some proper closed algebraic subset $Z_k \subset X$. Thus (9) can be written as

$$\mathrm{pr}_{X\times X*}((\Gamma_{\Phi'} \circ \Gamma_\Phi) \cdot h_2^r) = \Gamma'_{0r} \circ \Gamma_{0r} + W_2 \text{ in } \mathrm{CH}^n(X \times X), \tag{10}$$

where W_2 is supported on $D_2 \times X$, for some proper closed algebraic subset $D_2 \subset X$ that contains Z_k for all $k > 0$.

Combining (10) and (8) proves the first formula in (1), with $D = D_1 \cup D_2$. The second formula, which will not be used here, is obtained by exchanging the roles of X and X'. □

Letting the correspondences Γ_Φ, $\Gamma_{\Phi'}$, W, W' act on $H^3(X, \mathbb{Z})$ or $H^3(X', \mathbb{Z})$ (via morphisms of Hodge structures), we get

Corollary 3.3. *Under the same assumptions on X, X', there exist morphisms*

$$\gamma := \Gamma_{\Phi*} : J^3(X) \to J^3(X'),\ w := W_* : J^3(X) \to J^3(X),$$
$$\gamma' := \Gamma_{\Phi'*} : J^3(X') \to J^3(X),\ w' := W'_* : J^3(X') \to J^3(X'),$$

such that

$$\gamma' \circ \gamma = \mathrm{Id}_{J^3(X)} + w : J^3(X) \to J^3(X) \tag{11}$$

and the endomorphism $w \in \mathrm{End}(J^3(X))$ is given by the action of a cycle $W \in \mathrm{CH}^n(X \times X)$ supported on $D \times X$.

Remark 3.4. The intermediate Jacobian $J^3(X)$ exists as an abelian variety for any smooth projective rationally connected variety X, not only in dimension 3. Dimension 3 is used only if we want to introduce a principal polarization on it, but we will not need it.

We now prove the main result of this section.

Proposition 3.5. *Let X be a smooth projective rationally connected threefold over $\mathbb{C}$ and let X' be a smooth projective variety stably birational to X. Assume that $\mathrm{Tors}(H^3(X, \mathbb{Z})) = 0$ and that X does not have a cohomological decomposition of the diagonal. Then there exists a nonzero morphism $\gamma : J^3(X) \to J^3(X')$ of abelian varieties.*

Proof. Note that, in this statement, X' does not have *a priori* the same dimension as X, but we can replace X by $X \times \mathbb{P}^s$ or X' by $X' \times \mathbb{P}^s$ to achieve the equality of dimensions, so that we can apply Theorem 3.1 and Corollary 3.3. This gives us a morphism $\gamma : J^3(X) \to J^3(X')$ and we just have to prove that $\gamma \neq 0$. This is where the assumption that $\dim X = 3$ will be used. If $\gamma = 0$, equation (11) becomes

$$\mathrm{Id}_{J^3(X)} = -w : J^3(X) \to J^3(X), \tag{12}$$

where the endomorphism $w \in \mathrm{End}(J^3(X))$ is given by the action of a cycle $W \in \mathrm{CH}^3(X \times X)$ supported on $D \times X$. It follows from (12) and the fact that $H^3(X, \mathbb{Z})$ is torsion-free, that the cycle $\Delta_X + W \in \mathrm{CH}^3(X \times X)$ has the property that its cycle class $[\Delta_X] - [W] \in H^6(X \times X, \mathbb{Z})$ has its $(3, 3)$-Künneth component equal to zero in $\mathrm{End}(H^3(X, \mathbb{Z}))$. Using the fact that $H^1(X, \mathbb{Z}) = 0$ and $H^*(X, \mathbb{Z})$ is torsion-free, we deduce by Künneth decomposition that

$$[\Delta_X] + [W] \in H^{2*}(X, \mathbb{Z}) \otimes H^{2*}(X, \mathbb{Z}). \tag{13}$$

As $H^{2*}(X, \mathbb{Z})$ is algebraic by [30], we can write (13) as

$$[\Delta_X] + [W] = [Z] \text{ in } H^6(X \times X, \mathbb{Z}), \tag{14}$$

where the cycle $Z \in \mathrm{CH}^3(X \times X)$ is decomposable, that is, of the form $Z = \sum_i \mathrm{pr}_1^* Z_i \cdot \mathrm{pr}_2^* Z_i'$ for some cycles Z_i, $Z_i' \in \mathrm{CH}^*(X)$. Finally, writing $Z = \alpha\, X \times \mathrm{pt} + \sum_{\mathrm{codim}\, Z_i > 0} \mathrm{pr}_1^* Z_i \cdot \mathrm{pr}_2^* Z_i'$, formula (14) and the fact that W is supported on $D \times X$ easily imply $\alpha = 1$, which gives a cohomological decomposition of the diagonal for X, contradicting our assumption. □

Corollary 3.6. *(a) Let X be a very general cubic theefold, and let X' be a cubic threefold which is stably birational to X. Then there exists an isogeny $J^3(X) \to J^3(X')$.*

The same statement holds for the very general quartic threefold or quartic or sextic double solid.

(b) (Cf. Theorem 1.5.) Let X be a very general cubic threefold, quartic threefold or quartic or sextic double solid. If X' is any smooth projective variety which is stably birational to X, then $b_3(X') \geq b_3(X)$.

Proof. (a) It is proved in [11] that a very general cubic threefold X does not admit a cohomological decomposition of the diagonal. Furthermore, there is no torsion in $H^3(X, \mathbb{Z})$. Proposition 3.5 thus provides a nonzero morphism $\gamma : J^3(X) \to J^3(X')$ of abelian varieties. As $J^3(X)$ is simple because X is very general, hence has Mumford-Tate group equal to the symplectic group of $H^3(X, \mathbb{Q})$, and $\dim J^3(X') = \dim J^3(X)$, we get that γ is an isogeny.

The other cases have been discussed in Section 2. The conclusion is that the very general Fano threefolds in the considered families do not have a cohomological decomposition of the diagonal. The other properties used above also hold, because they are ample hypersurfaces in (weighted) projective space.

We finally prove (b). Assuming X is stably birational to X', we get as above a nonzero morphism

$$\gamma = \Gamma_* : J^3(X) \to J^3(X'),$$

which has to be injective since the intermediate Jacobian of X is as before a simple abelian variety. It follows that $b_3(X') \geq b_3(X)$. □

4 Applications of the variational Torelli theorem for hypersurfaces

We complete in this section the proof of Theorem 1.3. We first recall for completeness the statement and the proof of the following generic Torelli theorem for cubic threefolds, quartic threefolds and sextic double solids up to isogeny. The proof relies on the Carlson-Griffiths generic Torelli theorem for cubic hypersurfaces of dimension $3m$ (see [5] and in the quartic case, on the Donagi generic Torelli theorem for hypersurfaces (see [10], [12], [13]). The similar result for curves of genus ≥ 5 appears in [3], and we follow here the shorter alternative argument mentioned at the end of that paper, and which is due to us. All the proofs below are due to Carlson-Griffiths [5] and Donagi and collaborators (see [10], [12]) except for the observation that these results apply as well to prove a generic Torelli for rational Hodge structures, which is crucial here.

Theorem 4.1. *(a) (Cf. [5].) Let X be a very general cubic threefold, and let X' be a smooth cubic threefold. Then, if $J(X)$ is isogenous to $J(X')$, X is isomorphic to X'.*

(b) (Cf. [12].) The same statement holds for quartic threefolds, and sextic double solids.

Before explaining the proof, we complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Let X be a very general cubic threefold and let X' be a smooth cubic threefold which is stably birational to X. By Proposition 3.6, there exists a nontrivial isogeny $J^3(X) \to J^3(X')$. By Theorem 4.1, one has $X \cong X'$.

Statement b) of Theorem 1.3 is proved similarly, using statements b) for both Proposition 3.6 and Theorem 4.1. □

Remark 4.2. The quartic double solids are not listed here, due to the quartic case in Donagi's list of possible exceptions, see [12].

Proof of Theorem 4.1. As X is very general, $J^3(X)$ has only one polarization θ_X (up to a coefficient). It follows that the isogeny

$$\gamma : J^3(X) \to J^3(X')$$

satisfies $\gamma^*\theta_{X'} = \mu\theta_X$ for some nonzero rational number μ. Equivalently, the isomorphism of Hodge structures

$$\gamma^* : H^3(X', \mathbb{Q}) \cong H^1(J^3(X'), \mathbb{Q}) \to H^1(J^3(X), \mathbb{Q}) \cong H^3(X, \mathbb{Q})$$

preserves the intersection pairings, up to a multiplicative coefficient μ. The period map for cubic threefolds, which associates to a smooth cubic threefold the isomorphism class of its intermediate Jacobian equipped with its canonical polarization (or equivalently, its polarized Hodge structure on $H^3(X, \mathbb{Z})$), is an immersion. As X is very general and $J^3(X')$ is isogenous to $J^3(X)$, it follows that X' is also very general and, using the local period map, the isogeny γ induces, by considering small deformations of X and by applying a Baire countability argument, a holomorphic diffeomorphism

$$\psi : U \cong U'$$

between Euclidean open sets in the moduli space of smooth cubic threefolds, where $0 \in U$ parameterizes X and $0' \in U'$ parameterizes X'. Furthermore, this diffeomorphism is constructed in such a way that, for any $y \in U$, there is an isogeny (preserving the polarizations up to the coefficient μ)

$$\gamma_y : J^3(\mathcal{X}_y) \to J^3(\mathcal{X}'_{\psi(y)}),$$

where $\mathcal{X} \to U$, $\mathcal{X}' \to U'$ are the local universal (or Kuranishi) families over U and U' respectively. This family of isogenies translates into an isomorphism of rational Hodge structures

$$\gamma_H : H^3(X_y, \mathbb{Q}) \cong H^3(X_{\psi(y)}, \mathbb{Q})$$

and more generally, an isomorphism of variations of Hodge structures which is compatible with the polarizations (up to the coefficient μ). Differentiating this isomorphism gives an isomorphism of polarized infinitesimal variations of Hodge structures:

$$\begin{array}{ccc} T_{U,0} & \to & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \\ \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \gamma_H} \\ T_{U',0'} & \to & \mathrm{Hom}(H^{2,1}(X'), H^{1,2}(X')), \end{array} \tag{15}$$

where the vertical isomorphism γ_H on the right is the map induced by γ_H via the Hodge decomposition on both spaces $H^3(X_0, \mathbb{C})$ and $H^3(X_{0'}, \mathbb{C})$ and the duality given by the polarization. Using the Jacobian description due to Carlson and Griffiths [5], it is proved in [5] that the data on the top line determine X (and the data on the bottom line determine X' as well), hence we get $X \cong X'$. More precisely, when X is a smooth cubic 3-fold, the dual of the morphism

$$T_{U,0} \to \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X))$$

above (with value in the space of symmetric linear maps), identifies, thanks to the Carlson-Griffiths Jacobian formalism [5], to the map induced by multiplication

$$m_f : S^1 \otimes S^1 \to R_f^2,$$

where $S^1 = H^0(X, \mathcal{O}_X(1))$, f is the defining equation of X and $R_f^2 = S^2/J_f^2$, J_f^2 being the Jacobian ideal of f in degree 2. The kernel of m_f equals, (after symmetrization) the Jacobian

ideal $J_f^2 \subset S^2$. The end of the proof of a) is the Mather-Yau reconstruction theorem, proved in [19] and improved in [12], saying that the isomorphism class under $\mathrm{Gl}(S^1)$ of a smooth hypersurface of degree d defined by a homogeneous polynomial equation $f \in S^d := \mathrm{Sym}^d S^1$ is determined by its Jacobian ideal $J_f^{d-1} \subset S^{d-1}$. This proves (a).

Carlson-Griffiths' theorem above recovering a cubic threefold from its infinitesimal variation of Hodge structures applies as well to quartic threefolds, with one important extra ingredient which is provided in [12]. If X is a smooth quartic threefold defined by an equation $f = 0$, the morphism

$$T_{U,0} \to \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X))$$

given by the infinitesimal variation of Hodge structure of X, identifies by Carlson-Griffiths theory to the multiplication map

$$m_f : R_f^4 \to \mathrm{Hom}(R_f^3, R_f^7), \tag{16}$$

where f is the defining equation of X and $R_f^3 = S^3/J_f^3$, $R_f^4 = S^4/J_f^4$, J_f^l being the Jacobian ideal of f in degree l. The extra step in Donagi's method to reconstruct X from the data (16) is the symmetrizer lemma (see [13] or [28, Proposition 6.21]). The symmetrizer lemma indeed allows to reconstruct from (16) the multiplication map

$$m_f : R_f^1 \to \mathrm{Hom}(R_f^3, R_f^4). \tag{17}$$

Precisely, one recovers $R_f^1 \subset \mathrm{Hom}(R_f^3, R_f^4)$ as the set of linear maps $\phi : R_f^3 \to R_f^4$ such that, for any A, $B \in R_f^3$,

$$m_f(\phi(A))(B) = m_f(\phi(B))(A) \text{ in } R_f^7.$$

Having reconstructed canonically (17) by the symmetrizer lemma, recovering the polynomial structure $S^4 \cong \mathrm{Sym}^4 S^1$ and the Jacobian ideal $J_f^4 \subset S^4$ works as in the previous case.

The similar statement for sextic double solids is proved along the same lines, as is mentioned in the introduction of [12], because they can be represented as hypersurfaces in a weighted projective space, so that Carlson-Griffiths theory applies. In that case, the infinitesimal variation of Hodge structures is given by the multiplication map

$$m_f : R_f^6 \to \mathrm{Hom}(R_f^2, R_f^8). \tag{18}$$

The symmetrizer lemma allows as before to recover canonically from the data (18) the multiplication map

$$m'_f : R_f^4 \to \mathrm{Hom}(R_f^2, R_f^6). \tag{19}$$

Applying again the symmetrizer lemma, we recover canonically the multiplication map

$$m'_f : R_f^2 \to \mathrm{Hom}(R_f^2, R_f^4). \tag{20}$$

The multiplication map (20) identifies with the multiplication map

$$m_2 : S^2 \to \mathrm{Hom}(S^2, S^4). \tag{21}$$

The data of m_2 provides the isomorphism $S^2 \cong \mathrm{Sym}^2 S^1$, that is, the polynomial structure of $S^2 \cong R_f^2$, because the kernel of the associated map $\mathrm{Sym}^2 S^2 \to S^4$ is the ideal of the second Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}(S^2)$. The data of (18) and (19) then determine $J_f^6 \subset \mathrm{Sym}^6 S^1$, hence, by Mather-Yau theorem, the isomorphism class of f. □

5 Higher dimension

Although we used the intermediate Jacobians in the 3-dimensional case, following the Clemens-Griffiths tradition, most of the arguments (assuming we know that the general hypersurfaces do not have a cohomological decomposition of the diagonal) work as well for fourfolds, working with Hodge structure on primitive cohomology, which is an equivalent datum. We spell-out in this section the case of dimension 4. In higher dimension, the statement will be also conditional to the integral Hodge conjecture, so we state separately the result in dimension 4.

Theorem 5.1. *a) Assume that the very general hypersurface X of degree $d \geq 3$ in $\mathbb{P}^5$ does not admit a cohomological decomposition of the diagonal. Then, if X' is any smooth projective rationally connected variety which is stably birational to X, one has $b_4(X') \geq b_4(X)$.*

b) Under the same assumption, if $d = 5$, any hypersurface X' of degree d in $\mathbb{P}^5$ that is stably birational to X is isomorphic to X.

c) Statements a) and b) hold for quartic, sextic and octic double covers of $\mathbb{P}^4$.

Remark 5.2. In the case of degree 3 (cubic fourfolds), the Torelli theorem with rational coefficients is clearly wrong, so the proof of b) given below does not apply. In the case of quartic fourfolds, which is one of the potential exceptions in [12], the variational Torelli theorem we need in order to prove the Torelli theorem with rational coefficients does not seem to be known.

Proof of Theorem 5.1. Let X be as above and assume that X' is stably birational to X. We first use Theorem 3.1, that provides correspondences

$$\Gamma \in \mathrm{CH}^4(X \times X'),\, \Gamma' \in \mathrm{CH}^4(X' \times X)$$

such that

$$\Gamma' \circ \Gamma = \Delta_X + W \text{ in } \mathrm{CH}^4(X \times X), \tag{22}$$

where W is supported on $D \times X$ for some proper c:losed algebraic subset $D \subset X$.

We now denote by $\gamma := [\Gamma]_* : H^4(X, \mathbb{Z})_{\mathrm{prim}} \to H^4(X', \mathbb{Z})$, where $H^4(X, \mathbb{Z})_{\mathrm{prim}}$ is the primitive degree 4 cohomology of X, (which agrees by the generalized Noether-Lefschetz theorem with the transcendental part $H^4(X, \mathbb{Z})_{\mathrm{tr}}$ of $H^4(X, \mathbb{Z})$ because X is very general, and X is not a quadric), and $\gamma' := [\Gamma']_* : H^4(X', \mathbb{Z}) \to H^4(X, \mathbb{Z})$. Equation (22) gives

$$\gamma' \circ \gamma = \mathrm{Id}_{H^4(X,\mathbb{Z})_{\mathrm{prim}}} + w, \tag{23}$$

where $w := [W]_* : H^4(X, \mathbb{Z})_{\mathrm{prim}} \to H^4(X', \mathbb{Z})$.

As X is very general, the Hodge structure on $H^4(X, \mathbb{Z})_{\mathrm{prim}}$ is simple. It follows that $\gamma = 0$ or γ is injective. In the situation of a), if γ is injective, one concludes that

$$b_4(X') \geq b_4(X')_{\mathrm{tr}} + 1 \geq b_4(X)_{\mathrm{tr}} + 1 = b_4(X),$$

which proves a). In case b), if γ is injective, it has to induce a isomorphism of rational Hodge structures

$$H^4(X, \mathbb{Q})_{\mathrm{prim}} \cong H^4(X', \mathbb{Q})_{\mathrm{prim}}.$$

We then conclude by the same arguments as in the previous section, using the Donagi variational Torelli theorem [12], that X is isomorphic to X'. This proves b).

We thus only have to prove in both cases that $\gamma \neq 0$. Assume by contradiction that $\gamma = 0$. Then (23) becomes

$$0 = \mathrm{Id}_{H^4(X,\mathbb{Z})_{\mathrm{prim}}} + w, \tag{24}$$

and $w := [W]_* : H^4(X,\mathbb{Z})_{\rm prim} \to H^4(X',\mathbb{Z})$, where $W \in {\rm CH}^4(X\times X)$ is supported on $D\times X$ for some proper closed algebraic subset D of X. Another way to write (24) is to say that the cycle class $\alpha := [\Delta_X] + [W]$ has the property that

$$\alpha_{*|H^4(X,\mathbb{Z}_{\rm prim}} = 0,$$

from which one deduces, using the fact that $H^*(X,\mathbb{Z})$ is torsion free, that the Künneth decomposition of α writes

$$\begin{aligned} \alpha \in H^0(X,\mathbb{Z})\otimes H^8(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z})\otimes H^6(X,\mathbb{Z}) \\ \oplus \mathbb{Z}\, h^2\otimes h^2 \oplus H^6(X,\mathbb{Z})\otimes H^2(X,\mathbb{Z}\otimes H^8(X,\mathbb{Z})\otimes H^0(X,\mathbb{Z}), \end{aligned} \tag{25}$$

where $h = c_1(\mathcal{O}_X(1)) \in H^2(X,\mathbb{Z})$, so that $(h^2)^{\rm perp} = H^4(X,\mathbb{Z})_{\rm prim}$. All the cohomology groups $H^{2i}(X,\mathbb{Z})$, $i\neq 2$ are algebraic (the only nontrivial case being the case of $H^6(X,\mathbb{Z})$ which is generated by the class of a line). Furthermore, the first term in (25) has to be $[X\times{\rm pt}]$ since α acts as Id on $[{\rm pt}\times X]$. It thus follows from (25) that $[\Delta_X] + [W] - [X\times{\rm pt}]$ is the class of a cycle supported over $D'\times X$, hence that X admits a cohomological decomposition of the diagonal.

The proof of c) is identical (the Donagi variational Torelli theorem holds in all three cases). □

Remark 5.3. Starting from dimension $n\geq 5$, this proof will not work as it is, because we will need the integral cohomology classes $\frac{h^k}{d}$ to be algebraic for $k\geq m+1$, where $n=2m$ or $n=2m+1$, and this is not known for general Fano hypersurfaces, that is, in the range $d\leq n+1$. This statement is known when X is a cubic of dimension 5, because it contains a plane, but it is unknown (to our knowledge) for a general quartic or quintic of dimension 5.

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