VARIETY OF POWER SUMS AND DIVISORS IN THE MODULI SPACE OF CUBIC FOURFOLDS

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Abstract. We show that a cubic fourfold F that is apolar to a Veronese surface has the property that its variety of power sums $VSP(F, 10)$ is singular along a $K3$ surface of genus 20 which is the variety of power sums of a sextic curve. This relates constructions of Mukai and Iliev and Ranestad. We also prove that these cubics form a divisor in the moduli space of cubic fourfolds and that this divisor is not a Noether-Lefschetz divisor. We use this result to prove that there is no nontrivial Hodge correspondence between a very general cubic and its VSP.

1. Introduction

For a hypersurface $F \subset \mathbf{P}^n = \mathbf{P}(V^*)$ defined by a homogeneous polynomial $f \in S^d V$ of degree d in $n + 1$ variables, we define the variety of sums of powers as the Zariski closure

$$(1)\ VSP(F, s) = \overline{\{\{[l_1], \ldots, [l_s]\} \in \mathrm{Hilb}_s(\check{\mathbf{P}}^n) \mid \exists \lambda_i \in \mathbf{C} : f = \lambda_1 l_1^d + \ldots + \lambda_s l_s^d\}},$$

in the Hilbert scheme $\mathrm{Hilb}_s(\check{\mathbf{P}}^n)$, of the set of power sums presenting f (see [15]). The minimal s such that $VSP(F, s)$ is nonempty is called the **rank** of F. We will study these power sums using apolarity. Concretely, we can see the defining equation f as the equation of a hyperplane H_f in the dual space $S^d V^*$, and more generally, we get for each $k \leq d$ a subspace $I_f^k := [H_f : \mathrm{Sym}^{d-k} V^*] \subset S^k V^*$.

Definition **1.1.** *We say that a subscheme $Z \subset \check{\mathbf{P}}^n$ is apolar to f (or to $F = V(f)$) if $I_Z \subset I_f$, or, equivalently, $I_Z^d \subset I_f^d = H_f$. We use the term symmetrically, and also say that f is apolar to Z if $I_Z^d \subset I_f^d = H_f$.*

The relation between apolarity and power sums is given by the following duality lemma (see [11]):

Lemma **1.2.** *Let $l_1, \ldots, l_s \in V$ be linear forms. Then $f = \lambda_1 l_1^d + \ldots + \lambda_s l_s^d$ for some $\lambda_i \in \mathbf{C}^*$ if and only if $Z = \{[l_1], \ldots, [l_s]\} \subset \mathbf{P}(V)$ is apolar to $F = V(f)$.*

In the case of cubic hypersurfaces F in $\mathbf{P}^5$, the generic rank is 10 and the variety of 10-power sums of F is 4-dimensional for general F. In the paper [11], Iliev and the first author exhibited cubic fourfolds $F_{IR}(S)$ associated to $K3$ surfaces S of degree 14 obtained as the transverse intersection $\mathbf{G}(2, 6) \cap \mathbf{P}_S$ of the Grassmannian $\mathbf{G}(2, 6)$ with a codimension 6 linear space $\mathbf{P}_S$ of $\mathbf{P}(\bigwedge^2 V_6) = \mathbf{P}^{14}$ (see Section 2 for the precise construction). On the other hand Beauville and Donagi, in [1], associate to such a $K3$ surface S the Pfaffian cubic $F_{BD}(S)$ which is the intersection of the Pfaffian cubic in $\mathbf{P}(\bigwedge^2 V_6^*)$ with the $\mathbf{P}^5 \subset \mathbf{P}(\bigwedge^2 V_6^*)$ orthogonal to $\mathbf{P}_S$. The following result is proved in [11].

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THEOREM **1.3.** *For general S as above, the variety $VSP(F_{IR}(S), 10)$ is isomorphic to the family of secant lines to S, i.e. to* $\mathrm{Hilb}_2(S)$.

Combining this result with those of Beauville and Donagi [1], we conclude that $VSP(F_{IR}(S), 10)$ is isomorphic to the Fano variety of lines in the Pfaffian cubic fourfold $F_{BD}(S)$. Theorem 1.3 also says that $VSP(F_{IR}(S), 10)$ is a smooth hyperkähler fourfold. A deformation argument ([11, proof of Theorem 3.17]), may therefore be applied to prove that the variety $VSP(F, 10)$ for a general cubic fourfold is a smooth hyperkähler fourfold.

Recall from [1] that the Hodge structure on $H^4(F, \mathbf{Q})$, for F a smooth cubic fourfold, is up to a shift isomorphic to the Hodge structure on H^2 of its variety of lines, the isomorphism being induced by the incidence correspondence. The construction of Iliev and Ranestad provides for general F a second hyperkähler fourfold $VSP(F, 10)$ associated to F. A natural question is whether there is also an isomorphism of Hodge structures of bidegree $(-1, -1)$ between $H^4(F, \mathbf{Q})$ and $H^2(VSP(F, 10), \mathbf{Q})$. Note that Theorem 1.3 above combined with the results of Beauville and Donagi does not imply this statement even for the particular cubic fourfolds of the type $F_{IR}(S)$, because the Hodge structures on degree 4 cohomology of the cubics $F_{IR}(S)$ and $F_{BD}(S)$ could be unrelated. Another way of stating our question is whether the two hyperkähler fourfolds associated to F, namely its variety of lines and $VSP(F, 10)$, are "isogenous" in the Hodge theoretic sense.

We prove in this paper that such a Hodge correspondence does not exist for general F.

THEOREM **1.4.** *For a very general cubic fourfold F, there is no nontrivial morphism of Hodge structure*

$$\alpha : H^4(F, \mathbf{Q})_{prim} \to H^2(VSP(F, 10), \mathbf{Q}).$$

In particular, there is no correspondence $\Gamma \in CH^3(F \times VSP(F, 10))$, such that $[\Gamma]_* : H^4(F, \mathbf{Q})_{prim} \to H^2(VSP(F, 10), \mathbf{Q})$ is non zero.

This theorem cannot be proved locally (in the usual topology), because the two variations of Hodge structures have the same shape and we have no description of the periods of $VSP(F, 10)$: it is even not clear how its holomorphic 2-form is constructed. In fact, by the general theory of the period map, there exists locally near a general point of the moduli space of cubic fourfolds and up to a local change of holomorphic coordinates, an isomorphism between the complex variations of Hodge structure on $H^4(F, \mathbf{C})_{prim}$ and $H^2(VSP(F, 10), \mathbf{C})_{prim}$. Indeed, by the work of Beauville and Donagi, we know that the variation of Hodge structure on $H^4(F, \mathbf{C})_{prim}$ is isomorphic (with a shift of degree) to the variation of Hodge structure on H^2_{prim} of the corresponding family of varieties of lines, hence in particular this is (up to a shift of degree) a complete variation of polarized Hodge structures of weight 2 with Hodge numbers $h^{2,0} = 1$, $h^{1,1}_{prim} = 20$. The same is true for the variation of Hodge structure on $H^2(VSP(F, 10), \mathbf{C})_{prim}$ once one knows that the family of VSP's is locally universal at the general point, which is equivalent to saying that the deformations of $VSP(F, 10)$ induced by the deformations of F have 20 parameters, this last fact being easy to prove. Hence both complex variations of Hodge structures are given (locally near a general point in the usual topology) by an open holomorphic embedding into a quadric in $\mathbf{P}^{21}$, and thus they are locally isomorphic since a quadric is a homogeneous space.

Notice that if we consider plane sextic curves instead of cubic fourfolds, then we are faced with an analogous situation, namely we can associate naturally to a plane sextic curve C two $K3$ surfaces, the first one being the double cover of $\mathbf{P}^2$ ramified along C, and the other one being the variety of power sums $VSP(C, 10)$, which has been proved by Mukai [14] to be a smooth $K3$ surface for general C (see also [7]).

Theorem 1.4 will be obtained as a consequence of the following construction which relates the Mukai construction for plane sextic curves to the Iliev-Ranestad construction for cubic fourfolds. This involves the introduction of the closed algebraic subset of the moduli space of the cubic F parameterizing cubic fourfolds apolar to a Veronese surface. This subset, which we will prove to be a divisor D_{V-ap}, will now be introduced in more detail.

Let W be a 3-dimensional vector space, and $V := S^2W$, which is a 6-dimensional vector space. There is a natural map

$$s : S^6W \to S^3V$$

which is dual to the multiplication map

$$m : S^3(S^2W^*) \to S^6W^*.$$

If $a \in W$, we have

$$s(a^6) = (a^2)^3. \tag{2}$$

The map s associates to a plane sextic curve C with equation $g \in S^6W$ a four dimensional cubic F with equation $f = s(g) \in S^3V$. Note that we recover g from f using the multiplication morphism $m' : S^3V \to S^6W$. Indeed we have

$$m'(f) = g, \tag{3}$$

as an immediate consequence of (2).

LEMMA **1.5.** *The cubic polynomials in the image of s are exactly those which are apolar to the Veronese surface $\Sigma \subset \mathbf{P}(S^2W)$.*

Proof. Indeed, by definition of apolarity, a cubic hypersurface defined by an equation $f \in S^3V$ is apolar to the Veronese surface if and only if the hyperplane $H_f \subset S^3V^*$ determined by f contains the ideal $I_\Sigma(3)$. Equivalently, $\langle f, k\rangle = 0$, for $k \in I_\Sigma(3)$. But as we have $g = s(f)$, (3) tells that

$$\langle f, k\rangle = \langle g, m(k)\rangle.$$

By definition of the Veronese embedding, the map $m : S^3V^* \to S^6W^*$ is nothing but the restriction map to Σ, so that $m(k) = 0$ and $\langle f, k\rangle = 0$ for $k \in I_\Sigma(3)$. This proves the statement in one direction, and the converse is proved in the same way. $\square$

It follows that the $K3$ surface $VSP(C, 10)$ embeds naturally in $VSP(F, 10)$ and we will prove in Section 5:

THEOREM **1.6.** *The variety $VSP(F, 10)$ is singular along $VSP(C, 10)$. For a general choice of C, the variety $VSP(F, 10)$ is smooth away from the $K3$ surface $VSP(C, 10)$ and has nondegenerate quadratic singularities along $VSP(C, 10)$.*

Our strategy for the proof of Theorem 1.4 is the following. We will first prove that D_{V-ap} is a divisor, and that the divisor D_{V-ap} is not a Noether-Lefschetz divisor in the moduli space $\mathcal{M}$ of cubic fourfolds (Proposition 4.15), which means that for a general cubic parameterized by this divisor, there is no nonzero Hodge class in $H^4(F, \mathbf{Q})_{prim}$. Secondly, using Theorem 1.6, we will prove that D_{V-ap} is a Noether-Lefschetz divisor for the family $\mathcal{VSP}(F, 10)$ of varieties of power sums parameterized by a Zariski open set of $\mathcal{M}$, which has to be interpreted in the sense that the generic Picard rank of the extension along D_{V-ap} of the variation of Hodge structure on the degree 2 cohomology of $VSP(F,10)$ is at least 2.

Both proofs involve a careful analysis of the variety of power sums $VSP(F,10)$ with results that we believe may have independent interest. Indeed, the set theoretic definition given in (1) of $VSP(F,s)$ as a closure in the Hilbert scheme does not give a priori any information on its schematic structure. We obtain in Section 3 the following results in the case of $VSP(F,10)$ for cubic fourfolds. Let $U\subset \mathrm{Hilb}_{10}(\mathbb{P}^5)$ be the open set of zero-dimensional subschemes imposing independent conditions to cubics. There is vector bundle E of rank 46 on U, with fiber $I_Z(3)$ over the point $[Z]\in \mathrm{Hilb}_{10}(\mathbb{P}^5)$.

THEOREM **1.7.** *(i) (cf. Proposition 3.1) For a general choice of F in the complement of explicit divisors in the moduli space of cubic fourfolds, the variety of power sums $VSP(F,10)$ is contained in U and is the zero locus of a section of the vector bundle E^* on U.*

(ii) (cf. Proposition 3.23) For a general cubic fourfold F, the variety $VSP(F,10)$ does not intersect the singular locus of $\mathrm{Hilb}_{10}(\mathbb{P}^5)$.

(iii) (cf. Proposition 4.11 and Corollary 4.14) These results remain true for a general cubic fourfold apolar to a Veronese surface.

In order to prove these results, we were led to introduce new divisors in the moduli space of cubic fourfolds, that is divisors in $\mathbf{P}^{55}=\mathbf{P}(H^0(\mathbf{P}^5,\mathcal{O}_{\mathbf{P}^5}(3)))$ invariant under the action of $PGl(6)$, along which properties stated above fail. Many $PGl(6)$-invariant divisors were already known: the discriminant hypersurface parameterizing singular cubic fourfolds and the infinite sequence of divisors of smooth cubic fourfolds containing a smooth surface which is not homologous to a complete intersection, introduced by Brendan Hassett [10]. The latter sequence includes the Beauville-Donagi hypersurface parameterizing Pfaffian cubics. These are all Noether-Lefschetz divisors. Concerning the new divisors D_{rk3}, D_{copl} and D_{V-ap} we introduce in this paper (see Section 2), we prove that D_{V-ap} is not a Noether-Lefschetz divisor, and it is presumably the case that neither D_{rk3} nor D_{copl} are Noether-Lefschetz divisors. We do not know whether the Iliev-Ranestad divisor D_{IR} parameterizing the Iliev-Ranestad cubics is a Noether-Lefschetz divisor. As a consequence of Theorem 1.3, the Picard rank of the variety $VSP(F,10)$ jumps to 2 along this divisor. Therefore proving that D_{IR} is not a Noether-Lefschetz divisor could have been another approach to Theorem 1.4.

1.1. **Notation.** We give the numerical information of the minimal free resolution of a graded $S=\mathbf{C}[x_0,\dots,x_r]$-module

$$0\leftarrow M\leftarrow F_0\leftarrow F_1\leftarrow\ldots\leftarrow F_n\leftarrow 0$$

with $F_i=\bigoplus_{j\in\mathbf{Z}}\beta_{ij}S(-j)$ in *Macaulay2* notation [13], i. e. in the form

$$\begin{array}{ccccc}
\beta_{00} & \beta_{11} & \beta_{22} & \cdots & \beta_{n,n}\\
\beta_{01} & \beta_{12} & \beta_{23} & \cdots & \beta_{n,n+1}\\
\vdots & \vdots & \vdots & \cdots & \vdots\\
\beta_{0m} & \beta_{1,m+1} & \beta_{2,m+2} & \cdots & \beta_{n,n+m}.
\end{array}$$

The β_{0j} counts the number of linearly independent generators of M of degree $j+1$, while the β_{ij}, for $i>0$ counts the homogeneous sets of linearly independent syzygies of order i.

2. Some divisors in the moduli space of cubic fourfolds

We introduce in this section two $PGl(V)$-invariant divisors D_{rk3} and D_{copl} in the open set $\mathbf{P}(S^3V)_{reg}$ of the projective space $\mathbf{P}(S^3V)$ parameterizing smooth cubic fourfolds. We also recall the definition of the Iliev-Ranestad divisor D_{IR}. These divisors are crucial in the proof that the set D_{V-ap} considered in the introduction is also a divisor (Corollary 4.10 in Section 4).

The divisor D_{rk3}. This is the set of cubic forms $[f] \in \mathbb{P}^{55}_{reg}$ such that f has a partial derivative of rank ≤ 3.

Lemma **2.1.** *The set of cubic forms $[f] \in \mathbf{P}(S^3V)_{reg}$ such that f has a partial derivative of rank ≤ 3 is an irreducible divisor in $\mathbb{P}^{55}_{reg}$.*

Proof. If $[f] \in D_{rk3}$, there exist a point $p \in \mathbf{P}(V^*)$ and a plane $\mathbf{P}(W) \subset \mathbf{P}(V^*)$ such that

$$\frac{\partial^2 f}{\partial p \partial w} = 0, \ \forall w \in W. \tag{4}$$

Consider the case where p does not belong to $\mathbf{P}(W)$ and let us compute how many conditions on f are imposed by (4) for fixed p, W. We may choose coordinates X_i, $i = 0, \ldots, 5$, such that W is defined by $X_i = 0$, $i = 3, 4, 5$ and p is defined by equations $X_i = 0$, $i = 0, \ldots, 4$. Then f has to satisfy the conditions

$$\frac{\partial^2 f}{\partial X_5 \partial X_i} = 0, \ \text{for any}\, i \in \{0,\, 1,\, 2\}.$$

Equivalently, we have

$$\frac{\partial^3 f}{\partial X_5 \partial X_i \partial X_j} = 0, \ \text{for any}\, i \in \{0,\, 1,\, 2\} \text{ and any } j \in \{0, ..., 5\}. \tag{5}$$

The number of coefficients of f annihilated by these conditions is 15. As the pair (p, W) has 14 parameters, we conclude that the f satisfying these equations for some (p, W) fill-out at most a hypersurface. On the other hand, the map

$$\mathbf{P}(S^3V)_{reg} \to G(6, S^2V);\ [f] \mapsto \langle \frac{\partial f}{\partial X_0}, ..., \frac{\partial f}{\partial X_5} \rangle$$

is generically injective; for general f, the apolar ideal is generated by the quadrics orthogonal to the partials of f, and according to Macaulays theorem, the apolar ideal defines f up to scalar. The rank 3 locus in $\mathbf{P}(S^2V)$ has codimension 6, so the 6-dimensional subspaces of S^2V that intersect the rank 3 locus form a hypersurface section in $G(6, S^2V)$. Therefore the cubic forms that have a partial of rank 3 form at least a divisor in $\mathbf{P}(S^3V)_{reg}$, i.e. they form exactly a divisor. It is irreducible, because it is dominated by a projective bundle over the parameter space for (p, W). Denote this hypersurface by D_{rk3}.

To complete the argument we consider the degenerate situation where $p \in \mathbf{P}(W)$. It may be seen as a limit of the above case: We may choose coordinates X_i, $i = 0, \ldots, 5$, such that W is defined by $X_i = 0$, $i = 3, 4, 5$ and p_t is defined by equations $X_i = 0$, $i = 1, \ldots, 4$ and $X_5 = tX_0$. Thus $p_0 \in \mathbf{P}(W)$. For any t, we consider the cubic forms f that satisfy the conditions

$$\frac{\partial^2 f}{\partial X_0 \partial X_i} - t \frac{\partial^2 f}{\partial X_5 \partial X_i} = 0,\ i = 0,\, 1,\, 2.$$

Equivalently, we have

$$\frac{\partial^3 f}{\partial X_0 \partial X_i \partial X_j} - t\frac{\partial^3 f}{\partial X_5 \partial X_i \partial X_j} = 0,\ i = 0,\ 1,\ 2\ \forall j. \tag{6}$$

These are 15 linearly independent conditions on the coefficients of f for any value of t. In particular, any cubic form f_0 satisfying the conditions with $t = 0$ is a limit of forms f that satisfy the conditions for $t \neq 0$ as t tends to 0. So also in the degenerate situation, the forms lie in the irreducible hypersurface D_{rk3}.

□

Note the following other characterization of D_{rk3}:

LEMMA **2.2.** *A cubic form belongs to D_{rk3} if it has a net (a 3-dimensional vector space) of partial derivatives which are all singular in a given point p.*

Proof. The fact that f has a net of partial derivatives which are singular in a point p is equivalent to the vanishing $\partial_p(\partial_{w_i} f) = 0$ for three independent vectors w_i. This holds if and only if $\partial_{w_i}(\partial_p f) = 0$ for $i = 1, 2, 3$, which in turn is equivalent to the fact that the partial derivative $\partial_p f$ has rank ≤ 3. □

The divisor D_{copl}. The subset $D_{copl} \subset \mathbf{P}(S^3V)_{reg}$ is the Zariski closure of the set of forms f which can be written as

$$f = \sum_{1=1}^{10} a_i^3, \tag{7}$$

such that four of the linear forms $a_i \in V$ are coplanar.

LEMMA **2.3.** *D_{copl} is an irreducible divisor in $\mathbf{P}(S^3V)_{reg}$.*

Proof. The set D_{copl} is irreducible, since it is dominated by the irreducible algebraic set parameterizing the 10 linear forms, four of which are coplanar. If we count dimensions, we find that this last algebraic set has dimension 56. However, we observe that a cubic form g in 3 variables has a two dimensional variety of power sums $VSP(E, 4)$, where $E = V(g)$. If $f = \sum_{1=1}^{i=10} a_i^3$, where $a_1, \ldots, a_4$ are coplanar, we have

$$f = g(b_1, b_2, b_3) + \sum_{i=5}^{i=10} a_i^3, \tag{8}$$

where the a_i's for $i \leq 4$ are linear combinations of the b_i's. As there is a 2-parameter family of ways of writing g as a sum of four powers of linear forms in the b_i's, we conclude that there is a 2-parameter family of ways of writing f as in (7). This proves that D_{copl} has codimension at least 1. To show that it actually is a divisor, we exhibit an affine subfamily of D_{copl} of codimension one in the space of cubic forms. In fact if we let

$$b_1 = x_0 + b_0', b_2 = x_1 + b_2', b_3 = x_2 + b_3'$$

and

$$a_5 = x_0 - x_1 + x_3 + x_4 + a_5', a_6 = x_1 + x_2 - x_3 - x_4 - x_5 + a_6',$$
$$a_7 = x_2 + x_3 - x_4 + x_5 + a_7', a_8 = x_3 + a_8', a_9 = x_4 + a_9', a_{10} = x_5 + a_{10}',$$

with $b_1', .., b_3', a_5', ..., a_{10}' \subset V$, then

$$f = g(b_1, b_2, b_3) + \sum_{i=5}^{i=10} a_i^3$$

belongs to D_{copl} for every 9-tuple of linear forms $b'_1, .., b'_3, a'_5, ..., a'_{10}$. The summands in f that are linear in the b'_i and a'_j span the tangent space to this family at the origin, where $b'_1 = ... = a'_{10} = 0$. This space may thus be shown, with *Macaulay2* [13], to have dimension 54. Therefore the family is a divisor. □

The divisor D_{IR}. This is the divisor constructed by Iliev and Ranestad in [11]. It parameterizes the cubic fourfolds $F_{IR}(S)$ mentioned in the introduction, associated to $K3$ surfaces S which are complete intersections of the Grassmannian $G(2,6) \subset \mathbf{P}^{14}$ with a $\mathbf{P}^8_S$. More precisely, these cubic fourfolds are defined as follows: Dual to $\mathbf{P}^8_S$, we get a $\mathbf{P}^5_S \subset \check{\mathbf{P}^{14}}$. The dual projective space $\check{\mathbf{P}^{14}}$ contains the Grassmannian of lines $\check{G}(2,6)$ and for generic choice of $\mathbf{P}^5_S$, the intersection $\mathbf{P}^5_S \cap \check{G}(2,6)$ is empty. It is then proved in [11] that the ideal of cubic forms on $\check{\mathbf{P}^{14}}$ vanishing on $\check{G}(2,6)$ restricts to a hyperplane in $H^0(\mathbf{P}^5_S, \mathcal{O}_{\mathbf{P}^5_S}(3))$. This hyperplane in turn determines a cubic fourfold in $\check{\mathbf{P}^5_S}$.

For later use in the paper, we recall and extend a characterization from [11] of apolar length 10 subschemes to cubic forms $[f] \in D_{IR}$ in terms of quartic surface scrolls, i.e. rational normal surface scrolls in $\mathbf{P}^5$.

LEMMA **2.4.** *Let f be a cubic form of rank 10, such that $[f] \in D_{IR}$. Then the general subscheme of length 10 apolar to f is the intersection of two quartic surface scrolls. In particular f is apolar to a quartic surface scroll.*

Conversely, if f is a cubic form of rank 10 apolar to a quartic surface scroll, then $[f] \in D_{IR}$.

Proof. The first part is shown in [11]: Let $S = \mathbf{G}(2,6) \cap \mathbf{P}^8_S$ be the $K3$-surface section associated to $F = V(f)$, i.e. $F = F_{IR}(S)$ in the notation of loc. cit. Then S parameterizes quartic surface scrolls apolar to f, and the two scrolls corresponding to a pair of points on S intersect in a length 10 subscheme apolar to f (Lemma 2.9 and the proof of Theorem 3.7 loc.cit.).

For the second part, if f is apolar to a quartic surface scroll, then by dimension count, f has a 2-dimensional family of length 10 apolar subschemes on this scroll. The general such subscheme Z has a Gale transform in $\mathbf{P}^3$ contained in a smooth quadric surface [8, Corollary 3.3]. Furthermore, the two rulings in the quadric surface correspond to two quartic surface scrolls that contain Z, see [8, Example 3.4], where an analogous case is explained. Therefore f is apolar to a 2-dimensional family of quartic surface scrolls. Now, the family of quartic surface scrolls in $\mathbf{P}^5$ is irreducible of dimension 29, and each scroll is apolar to a 27-dimensional space of cubic forms, so there is an irreducible 54-dimensional family of cubic forms apolar to some quartic surface scroll. This family must coincide with the divisor D_{IR} since it contains it. □

3. APOLARITY AND SYZYGIES

In this section we first show that for a general cubic fourfold F, the variety $VSP(F, 10)$ is defined as the zero locus, inside the Hilbert scheme, of a section of a vector bundle. In fact the variety $VSP(F, 10)$ is then entirely contained in the set $U \subset \mathrm{Hilb}_{10}(\mathbb{P}^5)$ of zero-dimensional subschemes imposing independent conditions on cubics (Proposition 3.1), and Z is apolar to F for every $[Z] \in VSP(F, 10)$.

In the second part of this section we define the cactus rank of a cubic form (Definition 3.11). For a general form the cactus rank coincides with the rank. We give a criterion (Lemma 3.16) for a cubic form f to have cactus rank 10 in terms of a syzygy variety of its apolar ideal I_f. When a cubic fourfold F has cactus rank 10, then the union of the apolar subschemes of length 10 forms a hypersurface $V_{10}(F)$ in $\mathbb{P}^5$. We will show (Lemma 3.20) that $V_{10}(F)$ is a syzygy variety of I_f, and analyze its singular locus. At the end of this section we show (Proposition 3.23) that $VSP(F,10)$ does not meet the singular locus of $\mathrm{Hilb}_{10}(\mathbf{P}(V))$ for a general F.

The results of this section that are used later, are formulated in two lemmas and two propositions. Lemmas 3.16 and 3.20 will be used in Section 4 to prove that a general $[f] \in D_{V-ap}$ is apolar to finitely many Veronese surfaces, from which we will deduce that D_{V-ap} is a divisor. Propositions 3.1 and 3.23 are applied in Section 4 to show that for a general $[f] \in D_{V-ap}$, the length 10 subscheme Z is apolar to f for every $[Z] \in VSP(F,10)$ and is a smooth point in $\mathrm{Hilb}_{10}(\mathbf{P}(V))$.

3.1. **Apolar subschemes of length 10.**

PROPOSITION **3.1.** *Let F be a cubic fourfold defined by a generic form $f \in \mathrm{Sym}^3 V$. Then any length 10 subscheme $[Z] \in VSP(F,10)$ imposes independent conditions to cubics, i.e. $h^1(\mathcal{I}_Z(3)) = 0$, and is apolar to f, that is $I_Z(3) \subset H_f$.*

Furthermore, if there is a codimension 1 component of the set of smooth cubic fourfolds not satisfying this conclusion, it must be one of the two divisors D_{rk3} and D_{copl} introduced in the previous section.

Note that the second statement follows from the first using Lemma 1.2 and the fact that the condition $I_Z(3) \subset H_f$ is a closed condition on the open set $U \subset \mathrm{Hilb}_{10}(\mathbb{P}^5)$ of zero-dimensional subschemes imposing independent conditions to cubics.

The proof of Proposition 3.1 is postponed to later in this section. The proposition will be crucial in the study of the schematic structure of $VSP(F,10)$, for f satisfying the above conditions. Indeed, it implies:

COROLLARY **3.2.** *For a general cubic fourfold F, the variety $VSP(F,10)$ is an irreducible component of the zero locus, inside U, of a section σ_f of the vector bundle $\mathcal{E}$ on U of rank 46 with fiber $I_Z(3)^*$.*

Proof. Indeed, let σ_f be the section of $\mathcal{E}$ given by $Z \mapsto f^*_{|I_Z(3)}$, where f^* denotes the linear form on $\mathrm{Sym}^3 V^*$ corresponding to f. Then σ_f has to vanish on $VSP(F,10)$ by Proposition 3.1 and Lemma 1.2. As $VSP(F,10)$ is irreducible for general F in D_{IR} (cf. [11]), it is irreducible for general F, and its intersection with the open set of $\mathrm{Hilb}_{10}(\mathbb{P}^5)$ parameterizing sets of 10 distinct points coincides with the zero locus of σ_f in this open set by Lemma 1.2, we conclude that $VSP(F,10)$ is an irreducible component of the zero locus of σ_f. □

We shall show (Lemma 3.13) that for a general F, the variety $VSP(F,10)$ contains all subschemes of length 10 that are apolar to F.

The proof of Proposition 3.1 will need a few preparatory lemmas.

For a cubic form $f \in S^3V$ such that $F = V(f)$ is not a cone, let $P(f) \subset \mathbf{P}(S^2V)$ be the space of partial derivatives of f and $Q_f = P(f)^\perp \subset S^2V^*$. Then $P(f)$ is 6-dimensional and hence $\dim Q_f = 15$. Note that $Q_f = [H_f : V^*]$, where $H_f \subset S^3V^*$ is the hyperplane defined by f^*; indeed we may identify the space of partials $P(f)$ with the image $V^*(f) \subset S^2V$, so if $q \in S^2V^*$, then $q \cdot V^*(f) = 0$ if and only if $q(P(f)) = 0$.

Consider now a subscheme $Z \subset \mathbb{P}^5$ of length 10. Since Z imposes at most 10 conditions on quadrics, the space $I_Z(2)$ of quadrics in the ideal has dimension at least 11, with equality for an open set of schemes Z. Likewise, the ideal is generated in degree 2, for an open set of length 10 schemes Z: If Z is the intersection of a rational normal quintic curve and a quadric, then $I_Z(2)$ has dimension 11 and generate the ideal I_Z. Therefore this is the case also for a general Z.

Thus, in particular, if F is a general cubic fourfold and $[Z] \in VSP(F, 10)$ is general, then $I_Z(2)$ has dimension 11 and generate the ideal I_Z. Furthermore, by Lemma 1.2, $I_Z(2) \subset Q_f$. It follows that the rank of the evaluation map

$$Q_f \to H^0(\mathcal{O}_Z(2))$$

is at most 4 for a general $[Z] \in VSP(F, 10)$, and by semicontinuity of the rank, the same remains true for any $[Z] \in VSP(F, 10)$. Therefore

LEMMA **3.3.** *Let $f \in S^3V$ be a cubic form such that $F = V(f)$ is not a cone, and let $[Z] \in VSP(F, 10)$, then* $\dim I_Z(2) \cap Q_f \geq 11$.

The linear system of quadrics Q_f gives a rational map

$$q_f : \mathbf{P}(V) \dashrightarrow \mathbf{P}(Q_f^*),$$

defined as the composition of the Veronese map $\mathbf{P}(V) \to \mathbf{P}(S^2V)$ and the projection from the subspace $P(f) \subset \mathbf{P}(S^2V)$.

The following lemma is an immediate consequence of this description.

LEMMA **3.4.** (1) *q_f is a morphism if and only if f has no partials of rank ≤ 1.*
(2) *q_f is an embedding if and only if f has no partials of rank ≤ 2.*
(3) *q_f is an embedding and the image $X_f := q_f(\mathbf{P}(V))$ contains no subscheme of length 3 contained in a line if and only if f has no partial derivative of rank ≤ 3, i.e. $f \notin D_{rk3}$.*

This lemma allows us to find possible schemes Z such that $\dim I_Z(2) \cap Q_f \geq 11$.

LEMMA **3.5.** *Let f be a cubic form with no partial derivative of rank ≤ 3, let $X_f := q_f(\mathbf{P}(V))$ be the image by the map q_f and let $P \subset \mathbf{P}(Q_f^*)$ be a $\mathbf{P}^3$. If $X_P := P \cap X_f$ contains a curve, then X_P is the image by q_f of a line and a residual finite subscheme.*

In particular, if $F = V(f)$, $[Z] \in VSP(F, 10)$ and $Z_f = q_f(Z)$, then the linear span of Z_f is a $\mathbf{P}^2$ or a $\mathbf{P}^3$, and if $I_Z(2) \cap Q_f$ is contained in the ideal of curve, this curve is a line.

Proof. Indeed, by Lemma 3.4 (3), q_f is an embedding and the image X_f has no trisecant line. Since it is a linear projection of the second Veronese embedding, every curve in the image has even degree. Consider now a 3-space $P \subset \mathbf{P}(Q_f^*)$ and the intersection $X_P = P \cap X_f$. Since every surface in P contains a line or has a trisecant line, X_P cannot contain a surface. Furthermore, any curve $C \subset P$ of even degree and no trisecant line is either a conic or a complete intersection of two quadric surfaces. But the latter is not the second Veronese embedding of a curve, so C must be a conic. In particular, if X_P contains a curve, X_P is the union of a conic and a residual finite subscheme.

If $[Z] \in VSP(F, 10)$, then $\dim I_Z(2) \cap Q_f \geq 11$ by Lemma 3.3, so the span $\langle Z_f \rangle$ is at most a $\mathbf{P}^3$. On the other hand, Z_f must span at least a plane, since X_f has no trisecant line, so that $3 \geq \dim \langle Z_f \rangle \geq 2$. The linear span $\langle Z_f \rangle$ intersect X_f in the zero locus of $I_Z(2) \cap Q_f$, so the last claim in the lemma now follows from the first. □

Notice that the span $\langle Z_f \rangle$, whether Z is apolar to f or not, has dimension 2 (resp. 3) if and only if $I_Z(2) \cap Q_f$ has dimension 12 (resp. 11).

LEMMA **3.6.** *Let $V = \mathbf{C}^6$, and let $f \in \mathrm{Sym}^3 V$ be a cubic form with no partial derivative of rank ≤ 3. Let $Z \subset \mathbf{P}(V)$ be a subscheme of length 10, and assume that $I_Z(3)$ has codimension at most 9 in $\mathrm{Sym}^3 V^*$.*

(1) *If* $\dim I_Z(2) \cap Q_f = 12$, *then* Z *is contained in a line.*
(2) *If* $\dim I_Z(2) \cap Q_f = 11$, *then* Z *contains a subscheme of length at least* 5 *in a line.*

Proof. Let $Z \subset \mathbf{P}(V)$ be a subscheme of length 10 and assume that $I_Z(3)$ has codimension at most 9 in $\mathrm{Sym}^3 V^*$. Notice first that $\dim I_Z(2) \geq 12$. In fact, the subscheme Z does not impose independent conditions on cubics, i.e. $h^1(\mathcal{I}_Z(3)) > 0$. The multiplication by a general linear form h defines an exact sequence of sheaves

$$0 \to \mathcal{I}_Z(2) \to \mathcal{I}_Z(3) \to \mathcal{O}_H(3) \to 0,$$

where $H = \{h = 0\}$. Since $h^1(\mathcal{O}_H(3)) = 0$, $h^1(\mathcal{I}_Z(3)) > 0$ implies that $h^1(\mathcal{I}_Z(2)) > 0$, and hence that $\dim I_Z(2) \geq 12$.

Now, assume furthermore that $\dim I_Z(2) \cap Q_f \geq 11$. Let $\Gamma \subset \mathbf{P}(V)$ be the zero locus of the space of quadrics $I_Z(2) \cap Q_f$. Then, $q_f(\Gamma)$ is contained in a $\mathbf{P}^3$, so by Lemma 3.5, Γ is either a line and a residual finite subscheme, or Γ is finite.

Assume first that Γ is finite. Then Z spans at least a $\mathbf{P}^4$ in $\mathbf{P}(V)$, since any finite intersection of quadrics in a $\mathbf{P}^3$ has length at most 8. Let Z_0 be a maximal length subscheme of Z that spans a $\mathbf{P}^3$ in $\mathbf{P}(V)$. The length of Z_0 is then at most 8, and at least 4 since it spans $\mathbf{P}^3$. The residual scheme $Z_1 = Z \setminus Z_0$ therefore has length at least 2 and at most 6. Let $H = \{h = 0\}$ be a general hyperplane that contains Z_0. Then multiplication by h defines a sequence of sheaves of ideals

$$0 \to \mathcal{I}_{Z_1}(2) \to \mathcal{I}_Z(3) \to \mathcal{I}_{H,Z_0}(3) \to 0,$$

which is exact. Since $h^1(\mathcal{I}_Z(3)) > 0$, either $h^1(\mathcal{I}_{H,Z_0}(3)) > 0$ or $h^1(\mathcal{I}_{Z_1}(2)) > 0$. The scheme Z_1 has length at most 6 with no subscheme of length 3 contained in a line, and by the maximality of Z_0, it has at most a subscheme of length 5 in a $\mathbf{P}^3$. Then $h^1(\mathcal{I}_{Z_1}(2)) = 0$. In fact, we may find a subscheme $Z_2 \subset Z_1$ of length 2, and a corresponding residual subscheme $Z_{1,2}$ to Z_2 in Z_1, such that the span of Z_2 is not contained in the span of $Z_{1,2}$. Furthermore the length of $Z_{1,2}$ is one more than the dimension of its span, in particular $h^1(\mathcal{I}_{H,Z_{1,2}}(2)) = 0$. If then h is a general linear form vanishing on $Z_{1,2}$ and not on Z_2, the multiplication by h defines, as above, an exact sequence of sheaves

$$0 \to \mathcal{I}_{Z_2}(1) \to \mathcal{I}_{Z_1}(2) \to \mathcal{I}_{H,Z_{1,2}}(2) \to 0.$$

Since, $h^1(\mathcal{I}_{Z_2}(1)) = h^1(\mathcal{I}_{H,Z_{1,2}}(2)) = 0$, we infer $h^1(\mathcal{I}_{Z_1}(2)) = 0$.

We may therefore assume $h^1(\mathcal{I}_{H,Z_0}(3)) > 0$. If $P = \langle Z_0 \rangle$, then, by further restriction, also $h^1(\mathcal{I}_{P,Z_0}(3)) > 0$. If Z_0 has length 4, it is a minimal scheme that spans P and $h^1(\mathcal{I}_{Z_0}(3)) = 0$, so we may assume that Z_0 has length at least 5. Let Y be a finite complete intersection of quadrics in P that contain Z_0. Then Y has length 8, and the residual scheme Y_0 to Z_0 in Y has length at most 3. Multiplication by a general quadric $q \in I_{Z_0}(2)$ defines exact sequences of sheaves of ideals

$$0 \to \mathcal{I}_{Y_0}(1) \to \mathcal{I}_Y(3) \to \mathcal{I}_{Q,Z_0}(3) \to 0,$$

and

$$0 \to \mathcal{O}_P(1) \to \mathcal{I}_{P,Z_0}(3) \to \mathcal{I}_{Q,Z_0}(3) \to 0,$$

where $Q = \{q = 0\}$. Now, $h^2(\mathcal{I}_{Y_0}(1)) = h^1(\mathcal{O}_P(1)) = 0$ and $h^1(\mathcal{I}_Y(3)) = 0$, so we deduce that $h^1(\mathcal{I}_{Q,Z_0}(3)) = h^1(\mathcal{I}_{P,Z_0}(3)) = 0$, a contradiction.

Therefore Γ contains a line Δ. Let $Z_\Delta = Z \cap \Delta$. The line Δ is mapped to a conic $q_f(\Delta)$, so if $Z_f = q_f(Z)$ spans only a plane, the image $q_f(\Gamma)$ has a subscheme of length 3 in a line, unless Z is entirely contained in Δ, i.e. $Z_\Delta = Z$. Therefore we may assume that $Z_f = q_f(Z)$ spans a $\mathbf{P}^3$, and it remains to show that in this case the intersection $Z_\Delta = Z \cap \Delta$ has length at least 5.

So we assume, for contradiction, that Z_Δ has length at most 4 and let $Z_0 \subset Z$ be a maximal length subscheme that contains Z_Δ and spans a $\mathbf{P}^3$ in $\mathbf{P}(V)$. Then Z_0 has length at least 4 and the quadrics in $I_{Z_0}(2) \cap Q_f$ define the line Δ and a finite subscheme in $\mathbf{P}^3$. The maximal length of a finite subscheme residual to a line in the intersection of quadrics in $\mathbf{P}^3$ is 4, so the scheme residual to Z_Δ in Z_0 has length at most 4. In particular, the residual scheme $Z_1 = Z \setminus Z_0$ has length at least 2 and at most 6.

Let $H = \{h = 0\}$ be a general hyperplane that contains Z_0. Then multiplication by h defines a sequence of sheaves of ideals

$$0 \to \mathcal{I}_{Z_1}(2) \to \mathcal{I}_Z(3) \to \mathcal{I}_{H,Z_0}(3) \to 0,$$

which is exact. Since $h^1(\mathcal{I}_Z(3)) > 0$, either $h^1(\mathcal{I}_{H,Z_0}(3)) > 0$ or $h^1(\mathcal{I}_{Z_1}(2)) > 0$. The scheme Z_1 has length at most 6, it has at most a subscheme of length 5 in a $\mathbf{P}^3$ and contains no subscheme of length 3 contained in a line, so as above, $h^1(\mathcal{I}_{Z_1}(2)) = 0$. Therefore $h^1(\mathcal{I}_{H,Z_0}(3)) > 0$.

Let Z_2 be the scheme residual to Z_Δ in Z_0. Then Z_2 has length at most 4 and no subscheme of length 3 in a line, so $h^1(\mathcal{I}_{Z_2}(2)) = 0$. Let P be a general plane through Δ and consider the natural exact sequence

$$0 \to \mathcal{I}_{Z_2}(2) \to \mathcal{I}_{Z_0}(3) \to \mathcal{I}_{P,Z_\Delta}(3) \to 0.$$

Since the subscheme $Z_\Delta \subset \Delta$ has length at most 4, $h^1(\mathcal{I}_{P,Z_\Delta}(3)) = 0$. Therefore, also $h^1(\mathcal{I}_{Z_0}(3)) = 0$, contradicting the above.

We conclude that Z_Δ must have length at least 5.

$\square$

Proof of Proposition 3.1. Let $[Z] \in VSP(F, 10)$. We assume, for contradiction, that Z does not impose independent conditions on cubics. Assuming f is regular and has no partial derivative of rank ≤ 3, we already proved that $12 \geq \dim I_Z(2) \cap Q_f \geq 11$. By Lemma 3.6, we conclude in both cases that there is a line Δ such that $I_Z(2) \subset I_\Delta(2)$, so that

$$I_Z(2) \cap Q_f \subset I_\Delta(2) \cap Q_f.$$

Note also that, under the same assumptions on f, the image $q_f(\Delta)$ is a conic curve in a plane that does not have any residual intersection with $X_f = q_f(\mathbf{P}(V)$. Thus $\dim I_\Delta(2) \cap Q_f = 12$ and the zero locus of $Q_{f,\Delta} := I_\Delta(2) \cap Q_f$ is Δ.

Since $[Z] \in VSP(F, 10)$, there exists a flat family of subschemes

$$(Z_t)_{t \in B},\ Z_t \subset \mathbb{P}^5,\ \text{length}\, Z_t = 10,$$

where B is a smooth curve, such that $Z_0 = Z$ for some point $0 \in B$ and for general $t \in B$, Z_t is apolar to f and imposes 10 independent conditions to quadrics. The subspace $J_t := I_{Z_t}(2) \subset Q_f$ is thus of codimension 4. Let $J \subset Q_f \cap I_Z(2)$ be the specialization of J_t at $t = 0$. Then $\dim J = 11$ and $J \subset Q_{f,\Delta} = I_\Delta(2) \cap Q_f$ so that J is a hyperplane in $Q_{f,\Delta}$.

On the other hand, note that by semicontinuity of the rank, we have for any $k \geq 0$

$$\operatorname{codim}(S^k V^* \cdot J \subset S^{k+2} V^*) \geq \operatorname{codim}(S^k V^* \cdot J_t \subset S^{k+2} V^*)$$

$$\geq \operatorname{codim}(I_{Z_t}(k+2) \subset S^{k+2} V^*) = 10.$$

The contradiction that concludes the proof of Proposition 3.1 is derived from the following statement:

LEMMA **3.7.** *Assume f is generic. Then for any line $\Delta \subset \mathbb{P}^5$, and for any hyperplane $J \subset Q_{f,\Delta} := I_\Delta(2) \cap Q_f$, we have*

$$\mathrm{codim}\,(S^3V^* \cdot J \subset S^5V^*) \leq 9.$$

Furthermore, the locus of smooth cubic fourfolds not satisfying this condition has codimension > 1 away from the union of D_{rk3} and D_{copl}.

□

Proof of Lemma 3.7. The proof is in two parts. First of all, let us deal with the case where the zero locus of $J \subset I_\Delta(2) \subset S^2V^*$ has a finite subscheme of length at most 3 residual to Δ . In this case, we have the following:

Sublemma 3.8. *Assume f is regular and has no partial derivative of rank ≤ 3. Let $J \subset I_\Delta(2) \cap Q_f$ be as above, with zero locus $\Gamma \supseteq \Delta$. Assume the scheme γ residual to Δ in Γ is finite of length at most 3. Then*

$$S^3V^* \cdot J = I_\Gamma(5). \tag{9}$$

In particular, $\mathrm{codim}\,(S^3V^* \cdot J \subset S^5V^*) \leq 9$.

Proof. Let $\tau_0 : X_0 \to \mathbf{P}^5$ be the blow-up of $\mathbf{P}^5$ along Δ. Then J provides a space J' of sections of $L_0 := \tau_0^*(\mathcal{O}_{\mathbb{P}^5}(2))(-E_\Delta)$ on X_0, where E_Δ is the exceptional divisor of τ_0 and J' has its zero locus γ' supported over γ. We claim that γ' is zero-dimensional of length at most 3 and maps isomorphically to γ. To see the last point, we are only concerned with components of γ with support on Δ. Recall that since f has no partial derivative of rank ≤ 3, the line Δ is the zero locus of $Q_{f,\Delta}$, i.e. $Q_{f,\Delta}$ generates $\mathcal{I}_\Delta(2)$ at any point of Δ. Let $E_{\Delta,x}$ be the fiber over $x \in \Delta$ in E_Δ. Then $E_{\Delta,x} \cong \mathbf{P}^3$ and $L_0|_{E_{\Delta,x}} \cong \mathcal{O}_{\mathbf{P}^3}(1)$. The restriction of the sections J' generates at least a hyperplane of sections in this line bundle, so their zero locus on $E_{\Delta,x}$ is at most a point. Via the blowup map τ_0, the subscheme γ' is therefore isomorphic to the subscheme γ, and hence finite of length at most 3.

Furthermore, we have

$$H^0(X_0, \tau_0^*(\mathcal{O}_{\mathbb{P}^5}(2))(-E_\Delta) \otimes \mathcal{I}_{\gamma'}) = H^0(X_0, L_0 \otimes \mathcal{I}_{\gamma'}) \cong H^0(\mathbf{P}^5, \mathcal{I}_\Gamma(2)),$$
$$H^0(X_0, \tau_0^*\mathcal{O}(5)(-E_\Delta) \otimes \mathcal{I}_{\gamma'}) \cong H^0(\mathbf{P}^5, \mathcal{I}_\Gamma(5)).$$

It follows from the last equality that (9) is equivalent to the fact that

$$J' \cdot H^0(X_0, \tau_0^*\mathcal{O}(2)(-E_\Delta) \otimes \mathcal{I}_{\gamma'})) = H^0(X_0, \tau_0^*\mathcal{O}(5)(-E_\Delta) \otimes \mathcal{I}_{\gamma'})).$$

Assume first that γ' is curvilinear. It follows that by successively blowing-up at most three points x_1, x_2, x_3 starting from $x_1 \in X_0$, we get a variety

$$\tau : X \to \mathbf{P}^5,\ \tau_1 : X \to X_0,$$

with three exceptional divisors E_i corresponding to the x_i's and one exceptional divisor $\tau_1^*E_\Delta$ over E_Δ. The E_i are the pullbacks to X of the exceptional divisor of the blow up at x_i such that the pull-backs J'' of the J' gives rise to a base-point free linear system of sections of

$$L := \tau^*\mathcal{O}(2)(-\tau_1^*E_\Delta - \sum_i E_i) \tag{10}$$

on X. Furthermore, we have

$$J'' \subset H^0(X, L) \cong H^0(\mathbf{P}^5, \mathcal{I}_\Gamma(2)),$$

$$H^0(X,\tau^*\mathcal{O}(5)(-\tau_1^*E_\Delta-\sum_i E_i))\cong H^0(\mathbf{P}^5,\mathcal{I}_\Gamma(5)).$$

We are thus reduced to prove that the base-point free linear system

$$J''\subset H^0(X,\tau^*\mathcal{O}(2)(-\tau_1^*E_\Delta-\sum_i E_i))$$

generates $H^0(X,\tau^*\mathcal{O}(5)(-\tau_1^*E_\Delta-\sum_i E_i))$. This is done by a Koszul resolution argument. The Koszul resolution of the surjective evaluation map

$$J''\otimes\mathcal{O}_X(-L)\to\mathcal{O}_X,$$

gives us an exact complex with terms $\bigwedge^i J''\otimes\mathcal{O}_X(-iL)$, $0\le i\le 5$. We twist this complex by

$$L':=\tau^*\mathcal{O}(5)(-\tau_1^*E_\Delta-\sum_i E_i) \tag{11}$$

and the result then follows from the vanishing

$$H^i(X,(-i-1)L+L')=0,\ i=1,\dots,5. \tag{12}$$

For $i=5$, we have by (10), (11)

$$-6L+L'=\tau^*\mathcal{O}(-7)(5\sum_i E_i+5\tau_1^*E_\Delta),$$

while

$$K_X=\tau^*\mathcal{O}(-6)(4\sum_i E_i+3\tau_1^*E_\Delta).$$

Thus

$$H^5(X,-6L+L')=H^0(X,\tau^*\mathcal{O}(1)(-\sum_i E_i-2\tau_1^*E_\Delta))^*,$$

and the right hand side is 0.

For $i=4$, we have similarly

$$-5L+L'=\tau^*\mathcal{O}(-5)(4\sum_i E_i+4\tau_1^*E_\Delta),$$

hence

$$H^4(X,-5L+L')=H^1(X,\tau^*\mathcal{O}(-1)(-\tau_1^*E_\Delta))^*,$$

and the right hand side is 0 since it is equal to $H^1(\mathbf{P}^5,\mathcal{I}_\Delta(-1))$.

For $i=3$, we have

$$-4L+L'=\tau^*\mathcal{O}(-3)(3\sum_i E_i+3\tau_1^*E_\Delta),$$

hence

$$H^3(X,-4L+L')=H^2(X,\tau^*\mathcal{O}(-3)(\sum_i E_i))^*.$$

Consider the strict transform Y on X of a general cubic fourfold whose pullback to X_0 contains γ. Then $\tau^*\mathcal{O}(-3)(\sum_i E_i)$ is the ideal sheaf of Y. On the other hand Y is regular so $H^1(Y,\mathcal{O}_Y)=0$, and hence $H^2(X,\tau^*\mathcal{O}(-3)(\sum_i E_i))=0$.

For $i=2$, we claim that

$$H^2(X,-3L+L')=H^2(X,\tau^*\mathcal{O}(-1)(2\sum_i E_i+2\tau_1^*E_\Delta)=0.$$

Consider the strict transform Y on X of a general hyperplane through Δ whose pullback to X_0 contains γ. Then Y is smooth and the multiplication by the form defining Y fits in the exact sequence of sheaves

$$0 \to \tau^*\mathcal{O}(-1)(2\sum_i E_i + 2\tau_1^* E_\Delta) \to \mathcal{O}_X(\sum_i E_i + \tau_1^* E_\Delta) \to \mathcal{O}_Y(\sum_i E_i + \tau_1^* E_\Delta) \to 0.$$

But neither of the two invertible sheaves of exceptional divisors on the right have non-vanishing higher cohomology, so the claim follows.

For $i = 1$, we get

$$H^1(X, -2L + L') = H^1(X, \tau^*\mathcal{O}(1)(\sum_i E_i + \tau_1^* E_\Delta) = 0,$$

since $H^1(X, \tau^*\mathcal{O}(1)) = H^1(X, E_1) = H^1(X, E_2) = H^1(X, E_3) = H^1(X, \tau_1^* E_\Delta) = 0$.

When γ is not curvilinear, and thus consists of one point (necessarily disjoint from Δ) with noncurvilinear schematic structure of length 3, the argument is simpler: Such a scheme γ is the first order neighborhood of a point in a plane. The image $q_f(\Gamma) = q_f(\gamma) \cup q_f(\Delta)$ spans a $\mathbf{P}^3$ and has by assumption no subscheme of length three contained in a line. But $q_f(\Delta)$ is a conic curve, while $q_f(\gamma)$ spans a plane that intersects this conic. Therefore there are lines that intersect the conic and $q_f(\gamma)$ in a subscheme of length 2, a contradiction. □

To conclude the proof of Lemma 3.7, we now show

Sublemma 3.9. *Assume f is regular with no partial derivative of rank ≤ 3. Then, for any line $\Delta \subset \mathbf{P}(V^*)$, and for any hyperplane $J \subset I_\Delta(2) \cap Q_f$, the zero locus Γ of J satisfies the condition that the subscheme residual to Δ in Γ is of finite length ≤ 3. The subset of $\mathbf{P}^{55}_{reg} \setminus D_{rk3}$ parameterizing smooth cubic forms with no partial derivative of rank ≤ 3 which do not satisfy this conclusion has only one codimension 1 irreducible component which is the divisor D_{copl} introduced in Section 2.*

Proof. Note first that the scheme Γ imposes at most 4 conditions to Q_f, since $J \subset Q_f \cap I_\Gamma(2)$ has codimension 4 in Q_f. As Γ contains a line Δ and f is generic, the previous discussion concerning the possible intersections of X_f with a $\mathbf{P}^3$ applies, so in particular the residual subscheme of Δ in Γ is finite. If it has length ≥ 4, we can replace Γ by a subscheme Γ' which is the union of Δ and a residual scheme γ' of finite length 4. And we may assume $q_f(\Gamma')$ spans a $\mathbf{P}^3$. Note that Γ', like Γ, has the property that its intersection with a plane consists either in the union of the line Δ and one residual point, or of a scheme of finite length ≤ 4. Furthermore, the residual scheme γ' is not contained in another line Δ', since otherwise the union of these two lines would be contained in Γ. It follows that Γ' imposes the maximal number of conditions to the quadrics, namely 7. Hence

$$\dim\left(I_{\Gamma'}(2)\right) = 14, \tag{13}$$

and $J \subset I_{\Gamma'}(2)$ has dimension 11. Since $q_f(\Gamma')$ spans a $\mathbf{P}^3$, the intersection $Q_f \cap I_{\Gamma'}(2)$ has dimension 11, so it equals J. Now, $Q_f = P_f^\perp$, so one concludes that

$$\dim\left(P(f) \cap I_{\Gamma'}(2)^\perp\right) = 3, \tag{14}$$

where we recall that $P(f)$ is the space of partial derivatives of f. The proof of Sublemma 3.9 is done by a dimension count, using (14). We note that as we assumed that f has no partial derivative of rank ≤ 3, it has no net of partial derivatives singular at a given point by Lemma 2.2. Thus, if f satisfies (14), the space $I_{\Gamma'}(2)^\perp$ is not contained in the space of quadrics singular at a given point, so that Γ' must span $\mathbf{P}(V)$. This is equivalent to the vanishing $H^1(\mathcal{I}_{\Gamma'}(1)) = 0$, which we assume from now on. Equation (14) determines a 3-dimensional subspace $W \subset V^*$, by $W(f) = P(f) \cap I_{\Gamma'}(2)^\perp$. Given W and Γ', we define $J^{\Gamma',W} \subset S^3V$ to be the linear space of cubic forms

$$J^{\Gamma',W} := \{f \in S^3V | W(f) \subset I_{\Gamma'}(2)^\perp = Q_{\Gamma'}\} = (W \cdot I_{\Gamma'}(2))^\perp$$

where

$$W(f) = \{\partial_u f,\, u \in W\}.$$

The space $J^{\Gamma',W}$ contains the space $J_{\Gamma'} := I_{\Gamma'}(3)^\perp$ (which is generated by the cone over the third Veronese embedding of Γ') and the space $S^3W^\perp$.

Consider the subscheme

$$\Gamma'_W := \mathbf{P}(W^\perp) \cap \Gamma' \subset \mathbf{P}(V).$$

and assume first that $\Gamma'_W = \emptyset$. In this case, we claim that

$$J^{\Gamma',W} = S^3W^\perp \oplus J_{\Gamma'}, \tag{15}$$

so that $\dim J^{\Gamma',W} = 18$. Assuming the claim, we now observe that elements

$$f \in J^{\Gamma',W} = S^3W^\perp \oplus J_{\Gamma'}$$

fill-in, when the pair (Γ', W) deforms, staying in general position, the divisor D_{copl} of Section 2. Indeed, the general Γ' is the disjoint union of a line $\Delta = \mathbf{P}(U)$ and 4 points $x_1, \ldots, x_4$. Then $J_{\Gamma'} = S^3U + \langle x_1^3, \ldots, x_4^3\rangle$ and thus $f \in S^3W^\perp \oplus J_{\Gamma'}$ belongs to $S^3W^\perp + S^3U + \langle x_1^3, \ldots, x_4^3\rangle$. The component of f lying in $S^3W^\perp$ is the sum of 4 cubes of coplanar linear forms, and the component of f lying in S^3U is the sum of 2 cubes. Thus f is the sum of 10 cubes of linear forms, 4 of which are coplanar.

In order to prove formula (15), we dualize it and note that it is equivalent to the equality

$$W \cdot I_{\Gamma'}(2) = (W \cdot S^2V^*) \cap I_{\Gamma'}(3). \tag{16}$$

The right hand side is equal to $I_{\Gamma' \cup \mathbf{P}(W^\perp)}(3)$. As $\Gamma' \cap \mathbf{P}(W^\perp) = \emptyset$, the Koszul resolution of the ideal sheaf $\mathcal{I}_{\mathbf{P}(W^\perp)}$ remains exact after tensoring by $\mathcal{I}_{\Gamma'}$, which gives the following resolution of $\mathcal{I}_{\Gamma' \cup \mathbf{P}(W^\perp)}$:

$$0 \to \bigwedge^3 W \otimes \mathcal{I}_{\Gamma'}(-3) \to \bigwedge^2 W \otimes \mathcal{I}_{\Gamma'}(-2) \to W \otimes \mathcal{I}_{\Gamma'}(-1) \to \mathcal{I}_{\Gamma' \cup \mathbf{P}(W^\perp)} \to 0.$$

Twisting with $\mathcal{O}(3)$ and applying the vanishings $H^1(\mathcal{I}_{\Gamma'}(1)) = 0$ and $H^2(\mathcal{I}_{\Gamma'}) = 0$, we get the desired equality $W \cdot I_{\Gamma'}(2) = I_{\Gamma' \cup \mathbf{P}(W^\perp)}(3)$.

To conclude the proof of Sublemma 3.9, it only remains to prove the following claim:

Claim 3.10. *The set of cubic fourfolds in $\mathbf{P}^5_{reg} \setminus D_{rk3}$ satisfying (14) for a pair (W, Γ') with $\Gamma'_W \neq \emptyset$ has codimension ≥ 2.*

□

Proof of Claim 3.10. Let us first assume that along $\mathbf{P}(W^\perp)$, the local lengths of the embedded scheme supported on Δ are at most 2 and the local lengths of Γ' away from Δ are at most 3. We observe that in this situation, if $X, Y \in W$ are generically chosen, and $\mathbf{P}^3_{X,Y} \supseteq \mathbf{P}(W^\perp)$ is defined by X and Y, we have

$$\Gamma' \cap \mathbf{P}^3_{X,Y} = \Gamma' \cap \mathbf{P}(W^\perp) = \Gamma'_W.$$

We want to estimate the dimension of $J^{\Gamma',W}$ and recall that

$$J^{\Gamma',W} = (W \cdot I_{\Gamma'}(2))^{\perp}$$

so that

$$\dim J^{\Gamma',W} = 56 - \dim (W \cdot I_{\Gamma'}(2)).$$

We consider the exact sequence

$$0 \to \langle X, Y\rangle \cdot I_{\Gamma'}(2) \to W \cdot I_{\Gamma'}(2) \to W \cdot I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} \to 0.$$

and observe that $\dim W \cdot I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = \dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}}$. Therefore

$$\dim (W \cdot I_{\Gamma'}(2)) = \dim (\langle X, Y\rangle \cdot I_{\Gamma'}(2)) + \dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}}. \tag{17}$$

Furthermore, if

$$I_{\Gamma'\setminus\mathbf{P}(W^{\perp})}(1) = [I_{\Gamma'}(2) : \langle X, Y\rangle] = \{l \in V | l \cdot \langle X, Y\rangle \subset I_{\Gamma'}(2)\}$$

is the space of linear forms in the ideal of the scheme residual to $\mathbf{P}(W^{\perp})$ in Γ', then multiplication by the matrix $(X, -Y)$ and $(Y, X)^t$ respectively defines an exact sequence

$$0 \to [I_{\Gamma'}(2) : \langle X, Y\rangle] \to I_{\Gamma'}(2) \oplus I_{\Gamma'}(2) \to \langle X, Y\rangle \cdot I_{\Gamma'}(2) \to 0.$$

From this sequence, and the fact (13) that $\dim I_{\Gamma'}(2) = 14$, we get

$$\dim (\langle X, Y\rangle \cdot I_{\Gamma'}(2)) = 2 \dim I_{\Gamma'}(2) - \dim [I_{\Gamma'}(2) : \langle X, Y\rangle] = 28 - \dim [I_{\Gamma'}(2) : \langle X, Y\rangle].$$

Putting this equality together with the equation (17) we get:

$$\dim J^{\Gamma',W} = 28 + \dim [I_{\Gamma'}(2) : \langle X, Y\rangle] - \dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}}.$$

We make now a case-by-case analysis. Recall that if the scheme Γ'_W has finite length, this length is ≤ 3 and if it contains the line Δ, it contains at most one reduced residual point.

(1) If $\Gamma'_W = [l]$ is a reduced point on $\Delta = \mathbf{P}(U)$, which is not the support of an embedded point, then $\dim [I_{\Gamma'}(2) : \langle X, Y\rangle] = 0$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 9$, so we get $\dim J^{\Gamma',W} = 19$. The parameter space for such $(W, \Gamma)'$s has dimension $7 + 28 = 35$, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W, Γ) has dimension $\leq 35 + 18 = 53$.

(2) If $\Gamma'_W = [l]$ is a reduced point on $\Delta = \mathbf{P}(U)$, which is the support of an embedded point, then $\dim [I_{\Gamma'}(2) : \langle X, Y\rangle] = 1$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 9$. Thus $\dim J^{\Gamma',W} = 20$. As Γ' has an embedded point on Δ, the parameter space for Γ' has dimension 27, so the parameter space for such $(W, \Gamma)'$s has dimension $7 + 27 = 34$. Thus the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W, Γ) has dimension $\leq 34 + 19 = 53$.

(3) If $\Gamma'_W = [l]$ is a reduced point not in Δ, then $\dim [I_{\Gamma'}(2) : \langle X, Y\rangle] = 1$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 9$, so we get $\dim J^{\Gamma',W} = 20$. The parameter space for such $(W, \Gamma)'$s has dimension $6 + 28 = 34$, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W, Γ) has dimension $\leq 34 + 19 = 53$.

(4) If Γ'_W is a subscheme of length 2 that intersects Δ in one point, which is not the support of an embedded point, then $\dim [I_{\Gamma'}(2) : \langle X, Y\rangle] = 1$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 8$, so we get $\dim J^{\Gamma',W} = 21$. The parameter space for such $(W, \Gamma)'$s has dimension $4 + 28 = 32$, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W, Γ) has dimension $\leq 32 + 20 = 52$.

(5) If Γ'_W is a subscheme of length 2 that intersects Δ in one point, which is the support of an embedded point, then $\dim[I_{\Gamma'}(2) : \langle X, Y\rangle] = 2$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 8$, so we get $\dim J^{\Gamma',W} = 22$. The parameter space for such $(W,\Gamma)'$s has dimension $3+27 = 30$, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W,Γ) has dimension $\leq 30 + 21 = 51$.

(6) If $\Gamma'_W = z_2$ is a subscheme of length 2 that does not intersect Δ, then $\dim[I_{\Gamma'}(2) : \langle X, Y\rangle] = 2$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 8$, so we get $\dim J^{\Gamma',W} = 22$. The parameter space for such $(W,\Gamma)'$s has dimension $3+28 = 31$, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W,Γ) has dimension $\leq 31 + 21 = 52$.

(7) If $\Gamma'_W = \Delta$, then then $\dim[I_{\Gamma'}(2) : \langle X, Y\rangle] = 2$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 7$, so we get $\dim J^{\Gamma',W} = 23$. The parameter space for such $(W,\Gamma)'$s has dimension $3+28 = 31$, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W,Γ) has dimension $\leq 31 + 22 = 53$.

(8) If Γ'_W is a subscheme of length 3 that does not intersect Δ, then
$$\dim[I_{\Gamma'}(2) : \langle X, Y\rangle] = 3$$
and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 7$, so we get $\dim J^{\Gamma',W} = 24$. The parameter space for such $(W,\Gamma)'$s has dimension 28, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W,Γ) has dimension $\leq 23 + 28 = 51$.

(9) If Γ'_W is a subscheme of length 3 that intersects Δ in a point $[l]$, which is the support of an embedded point, then $\dim[I_{\Gamma'}(2) : \langle X, Y\rangle] = 3$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 7$, so we get $\dim J^{\Gamma',W} = 24$. The parameter space for such $(W,\Gamma)'$s has dimension 27, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W,Γ) has dimension $\leq 27 + 23 = 50$.

(10) If Γ'_W is a subscheme of length 3 that intersects Δ in a point $[l]$, which is not the support of an embedded point, then $\dim[I_{\Gamma'}(2) : \langle X, Y\rangle] = 2$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 7$, so we get $\dim J^{\Gamma',W} = 23$. The parameter space for such $(W,\Gamma)'$s has dimension $1+28 = 29$, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W,Γ) has dimension $\leq 22 + 29 = 51$.

(11) If Γ'_W is the union of the line Δ and an embedded point, then $\dim[I_{\Gamma'}(2) : \langle X, Y\rangle] = 3$ and $\dim I_{\Gamma'}(2)_{|\mathbf{P}^3_{X,Y}} = 6$, so we get $\dim J^{\Gamma',W} = 25$. The parameter space for such $(W,\Gamma)'$s has dimension 28, so the subset of $\mathbf{P}^{55}_{reg}$ satisfying equation (14) with this condition on (W,Γ) has dimension $\leq 24 + 28 = 52$.

This proves the claim, except for the cases where Γ' has an isolated point of length 4 or an embedded scheme supported on the line with local length ≥ 3. In the first case Γ'_W is either empty, or it is as in one of the cases (1),(3),(4),(6),(7),(8), (10). In the second case Γ'_W is either empty, or it is as in one of the cases (2),(5),(7),(9), (11).

□

The proof of Lemma 3.7, hence also of Proposition 3.1, is finished. □

3.2. **Syzygies.** An important tool in the schematic study of $VSP(F, 10)$ is the notion of *cactus rank* of F:

DEFINITION **3.11.** *The cactus rank of a hypersurface $F \subset \mathbf{P}^n$ is the minimal length of a 0-dimensional subscheme Z of $\check{\mathbf{P}}^n$ which is apolar to F.*

REMARK **3.12.** (1) Buczyńska and Buczyński showed in [2, Proposition 2.2, Lemma 2.3] that a finite subscheme Z, that is apolar to F and has length equal to its cactus rank, is locally Gorenstein.

(2) Casnati, Jelisiejew and Notari have shown that any local Gorenstein scheme of length at most 13 is smoothable (cf. [5, Theorem A]).

Thus, since the smooth apolar schemes form an open set in its component of the Hilbert scheme we get:

LEMMA **3.13.** *If F is a general cubic form of rank 10, then the cactus rank is also 10 and $VSP(F,10)$ is irreducible and contains all subschemes of length 10 that are apolar to F.*

Proof. Inside the open set $U \subset \mathrm{Hilb}_{10}(\mathbf{P}^5)$ of length 10 subschemes that impose independant conditions of cubics, the ones that are locally Gorenstein belong, by Remark 3.12, to the irreducible component of the smooth subschemes. Therefore, for general F, by Corollary 3.2, $VSP(F,10)$ contains all apolar subschemes of length 10.

□

Considering the syzygies of the ideal I_f, we give below a partial characterization of cubic hypersurfaces with cactus rank < 10, which we will use to prove Proposition 4.8 in the next section.

For a cubic fourfold $F \subset \mathbf{P}(V^*)$, let $V_{10}(F) \subset \mathbf{P}(V)$ be the union of subschemes of length 10 which are apolar to F. We shall show, in Lemma 3.20, that when F is general and of cactus rank 10, then $V_{10}(F)$ is a hypersurface of degree 9. As suggested to us by Hans Christian von Bothmer, to find the equation of $V_{10}(F)$, when it is a hypersurface, we study the syzygies of the apolar ideal I_f and compare it with syzygies of the ideal of subschemes of length 9 and 10.

We are interested in the graded Betti numbers for the minimal free resolution of the ideal I_f for a general f, and for the ideal of a general set of 9 and 10 points.

Example 3.14. *The Betti numbers in the following examples have been computed with Macaulay2 [13].*

(1) *Let $f \in \mathbf{C}[x_0,...,x_5]$ be the cubic form*

$$f = 2x_1^2x_2 - 2x_0x_2^2 - 2x_1^2x_3 - 2x_3^2x_4 - x_0x_1x_5 + 2x_1x_2x_5 \\ + x_2^2x_5 + x_2x_3x_5 + 3x_1x_4x_5 + x_4^2x_5 + 3x_0x_5^2 + x_3x_5^2$$

Then the resolution of I_f has Betti numbers:

$$\begin{matrix} 1 & - & - & - & - & - & - \\ - & 15 & 35 & 21 & - & - & - \\ - & - & - & 21 & 35 & 15 & - \\ - & - & - & - & - & - & 1 \end{matrix}.$$

(2) *Let Z_6 be the 6 coordinate points in $\mathbf{P}^5$, then the resolution of the ideal of the 9 points*

$$Z_9 = Z_6 \cup \{(1:1:1:1:0:0), (0:0:1:-1:-1:1), (1:-1:0:0:1:1)\}$$

in $\mathbf{P}^5$ has Betti numbers

$$\begin{matrix} 1 & - & - & - & - & - \\ - & 12 & 25 & 15 & - & - \\ - & - & - & 6 & 10 & 3 \end{matrix},$$

(3) *The resolution of the ideal of the* 10 *points* $Z_9 \cup \{(0:1:-1:-1:0:1)\}$ *in* $\mathbf{P}^5$ *has Betti numbers*

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 11 & 20 & 5 & - & - \\ - & - & - & 16 & 15 & 4 \end{array}.$$

REMARK **3.15.** *By the minimality of Betti numbers in Example 3.14 we conclude that the Betti numbers are the same as in these examples for a general cubic form, and for the ideal of* 9 *(resp.* 10*) general points in* $\mathbf{P}^5$.

LEMMA **3.16.** *If* f *is a cubic form with no partials of rank* ≤ 3*, then* f *has cactus rank* ≥ 9*. If furthermore the minimal free resolution of the apolar ideal* I_f *has Betti numbers*

$$\begin{array}{ccccccc} 1 & - & - & - & - & - & - \\ - & 15 & 35 & 21 & - & - & - \\ - & - & - & 21 & 35 & 15 & - \\ - & - & - & - & - & - & 1 \end{array}$$

and f *has cactus rank* 9*, then the* (35×21)*-matrix* M_2 *of linear second order syzygies has generic rank at most* 20*. In other words, if* f *has no partial derivative of rank* ≤ 3 *and the matrix* M_2 *has generic rank* 21*,* f *has cactus rank* 10*.*

Proof. Since f has no partial derivatives of rank ≤ 3, the map $q_f : \mathbf{P}(V) \to X_f$ is a smooth embedding and X_f has no trisecant lines, by Lemma 3.4. Let Z be an apolar subscheme of length at most 8. Since $I_Z(2) \subset Q_f = I_f(2)$ and $Q_f \subset S^2V$ has codimension 6, the rank of the restriction map $Q_f \to H^0(\mathcal{O}_Z(2))$ is at most 2. Hence $Z_f = q_f(Z)$ is contained in a line, and X_f would have a trisecant line, a contradiction. Therefore f has cactus rank at least 9.

Assume next, f has cactus rank 9 computed by an apolar subscheme $Z \subset \mathbf{P}(V)$ that consists of 9 general points. Then the Gale transform Z' of Z is a set of 9 points in a plane, and Z' is general since Z is general. In particular we may assume that Z' lies on a unique smooth cubic curve. By [8, Corollary 3.2], the set of 9 points Z itself lies on this curve reembedded as an elliptic sextic curve C_Z in $\mathbf{P}(V)$. The Betti numbers of the minimal free resolution of the ideal of C_Z are

$$\begin{array}{ccccc} 1 & - & - & - & - \\ - & 9 & 16 & 9 & - \\ - & - & - & - & 1 \end{array}.$$

Since $I_{C_Z} \subset I_Z$ and $I_Z \subset I_f$, by assumption, we get that $I_{C_Z} \subset I_f$, i.e. the elliptic sextic curve C_Z is apolar to F. The inclusion of the resolution of I_{C_Z} in the resolution of I_f displays a third order syzygy of the ideal I_{C_Z} that is a third order syzygy for the linear strand of the resolution of the ideal I_f. In the resolution of I_f the matrix M_2 therefore has generic rank at most 20.

It remains to consider any cubic form f of cactus rank 9 and no partials of rank at most 3. Let Z be a length 9 subscheme apolar to F. Now, by Remark 3.12, the scheme Z is locally Gorenstein and the limit of smooth schemes of length 9, the form f is likewise a limit of forms of cactus rank 9 with a smooth apolar scheme of length 9. Therefore, by the previous argument, the matrix M_2 in the resolution of I_f is the limit of matrices of generic rank at most 20, so M_2 also has generic rank at most 20.

□

In this proof the existence of an elliptic normal sextic curve through 9 general points in $\mathbf{P}(V)$ played a crucial role. We take this a step further in analyzing curvilinear schemes of length 10 that are apolar to a cubic fourfold F of rank 10. If Z is a set of 10 general points apolar to F, then a subset $Z_0 \subset Z$ of 9 points lies in an elliptic normal sextic curve C. We will show that the ideal of $C \cup Z$, and hence also I_Z, has a second order syzygy that vanishes in the point $p = Z \setminus Z_0$ and conclude that the matrix M_2 in the resolution of I_f has rank at most 20 at p.

Furthermore, an elliptic normal sextic curve C lies in a smooth Veronese surface: any of the four linear systems $|D|$ of degree 3 on C such that $|2D|$ is the linear system of hyperplane sections of $C \subset \mathbf{P}(V)$, is the linear system of conic sections of C in a smooth Veronese surface in $\mathbf{P}(V)$. When the set $Z_0 \subset C$ of nine points in $\mathbf{P}(V)$ is the base locus of a pencil of elliptic sextic curves on the Veronese surface, then the above argument yields a pencil of second order syzygies for I_Z that vanishes at $p = Z \setminus Z_0$. So we conclude that the matrix M_2 in the resolution of I_f has rank at most 19 at p.

LEMMA **3.17.** *If C is an elliptic sextic curve and p is a general point in $\mathbf{P}(V)$, then the ideal $I_{C\cup\{p\}}$ has a second order linear syzygy vanishing in p. In particular, if p is a point in a curvilinear scheme Z of length 10 with Betti numbers*

$$\begin{matrix} 1 & - & - & - & - & - \\ - & 11 & 20 & 5 & - & - \\ - & - & - & 16 & 15 & 4 \end{matrix},$$

then the ideal I_Z has a second order linear syzygy that vanishes at p.

If the scheme Z_0, residual to p in Z, is contained in a pencil of elliptic curves on a Veronese surface, then I_Z has a pencil of second order linear syzygies that vanishes at p.

Proof. Let p be a general point in $\mathbf{P}(V)$, in particular a point outside the secant variety of C. The minimal free resolution of I_C is symmetric with Betti numbers

$$\begin{matrix} 1 & - & - & - & - \\ - & 9 & 16 & 9 & - \\ - & - & - & - & 1 \end{matrix}.$$

The third order syzygy is therefore nonzero at the point p outside C. Therefore one of the second order syzygies vanishes at p. This syzygy is a syzygy among at most 5 first order linear syzygies that also vanishes at p, and finally, these first order syzygies are linear syzygies among quadrics in the ideal of C that vanish at p. Therefore the ideal of $I_{C\cup\{p\}}$ has a second order linear syzygy vanishing at p. By the following example we may assume that there is exactly one such second order linear syzygy for a general C and p.

Example 3.18. *The* $2x2$ *minors of the matrix*

$$\begin{bmatrix} x_0 + 4x_3 & 6x_1 + 3x_4 & 5x_2 + 2x_5 \\ 2x_1 + 5x_4 & x_2 + 4x_5 & 6x_3 + 3x_0 \\ 3x_2 + 6x_5 & 2x_3 + 5x_0 & x_4 + 4x_1 \end{bmatrix}$$

generate the ideal of an elliptic normal curve C in $\mathbf{P}^5$. The ideal of the union of C and the point $(1:1:1:1:0:0)$ *has a minimal free resolution with Betti numbers (computed with Macaulay2* [13]*)*

$$\begin{matrix} 1 & - & - & - & - & - \\ - & 8 & 11 & 1 & - & - \\ - & - & 2 & 10 & 6 & 1 \end{matrix}.$$

A length 10 curvilinear scheme Z with the given syzygies is the limit of a set Z_g of 10 general points that includes the point p. The union of an elliptic sextic curve C_g through $Z_g \setminus p$ and p has a linear second order syzygy vanishing at p, hence so does also Z_g, and by semicontinuity also Z. A pencil of elliptic curves through the length 9 subscheme Z_0 residual to p in Z, gives of course, rise to a pencil of second order linear syzygies vanishing at p. □

Let F be a cubic fourfold defined by a form f of rank 10 and consider the incidence

$$I_{VSP} = \{(p,[Z])|p \in Z\} \subset \mathbf{P}(V) \times VSP(F,10).$$

Then, by definition, $V_{10}(F) \subset \mathbf{P}(V)$ is the image of I_{VSP} under the first projection $I_{VSP} \to \mathbf{P}(V)$.

COROLLARY **3.19.** *Let f be a cubic form of rank 10 with no partial derivatives of rank ≤ 3 and Betti numbers*

1	–	–	–	–	–	–
–	15	35	21	–	–	–
–	–	–	21	35	15	–
–	–	–	–	–	–	1

for the ideal I_f of forms apolar to f. Then the (35×21)-matrix M_2 of linear second order syzygies has rank at most 20 along $V_{10}(F)$. Furthermore, M_2 has rank at most 19 at every point $p \in Z \subset \mathbf{P}(V)$ with Z an apolar subscheme of length 10 such that the subscheme Z_0, residual to p in Z, is contained in a pencil of elliptic sextic curves on a Veronese surface.

Proof. For a general length 10 curvilinear apolar subscheme Z to f, and every point $p \in Z$, the ideal I_Z has, by Lemma 3.17, a second order linear syzygy vanishing at p, so M_2 has rank at most 20 in p. Thus M_2 has rank at most 20 along a Zariski open set of $V_{10}(F)$, hence everywhere along $V_{10}(F)$. Similarly, I_Z has, by Lemma 3.17, a pencil of second order linear syzygies vanishing at p if the scheme Z_0, residual to p in Z, is contained in a pencil of elliptic sextic curves on a Veronese surface, so the second part of the Corollary follows.

□

LEMMA **3.20.** *(von Bothmer [3]) Let f be a cubic form whose apolar ideal I_f has a minimal free resolution with Betti numbers*

1	–	–	–	–	–	–
–	15	35	21	–	–	–
–	–	–	21	35	15	–
–	–	–	–	–	–	1

.

Then the (35×21)-matrix M_2 of linear second order syzygies has rank at most 20 either on all of $\mathbf{P}(V)$ or on a hypersurface Y_F of degree 9. In the second case, $V_{10}(F)$ is equal to this hypersurface.

Proof. Consider the linear strand of the resolution of I_f with Betti numbers

1	–	–	–	–
–	15	35	21	–

evaluated at a general point. The first map has kernel of dimension 14. Therefore the corank of the third map φ_{M_2} is at least 14. If the linear strand is exact at a general point, then the rank of the third map drops along a hypersurface. We compute the degree of this hypersurface by restricting the linear strand to a general line $L \subset \mathbf{P}(V)$. This restriction of the linear strand is a complex

$$0 \leftarrow \mathcal{O}_L \leftarrow 15\mathcal{O}_L(-2) \leftarrow 35\mathcal{O}_L(-3) \leftarrow 21\mathcal{O}_L(-4) \leftarrow 0$$

that is exact, except at $35\mathcal{O}_L(-3)$. The kernel of the first map is a vector bundle E_1 of rank 14 and first chern class $c_1(E_1) = -30$ on L. Therefore, the second map factors into a surjective map $E_1 \leftarrow 35\mathcal{O}_L(-3)$ with kernel a vector bundle E_2 of rank 21 with first chern class $c_1(E_2) = 35 \cdot (-3) - (-30) = -75$ on L. The third map of the linear strand, defined by the restriction of φ_{M_2} to L, factors through a vector bundle map $E_2 \leftarrow 21\mathcal{O}_L(-4)$ between two bundles of rank 21. The determinant of this bundle map, since it is assumed to be nonzero, defines a divisor whose class is the difference of the first chern classes of the two bundles, i.e. of degree $-75 + 21 \cdot (-4) = 9$ on L. So φ_{M_2} either has rank at most 20 on all of $\mathbf{P}(V)$ or it has rank at most 20 on a hypersurface of degree 9.

For the last statement, we already proved in Corollary 3.19 that the hypersurface $V_{10}(F) \subset \mathbf{P}(V)$ is contained in the determinental hypersurface Y_F of points where M_2 has rank at most 20. On the other hand, one can exhibit F for which Y_F is irreducible (an explicit such example is given in the proof of Proposition 4.8, see Remark 4.9). Hence for such an F, $V_{10}(F)$ must be equal to Y_F, which implies the same result for any F. □

LEMMA **3.21.** *Let $f \in S^3V$ be a cubic form whose apolar ideal I_f has a minimal free resolution with Betti numbers*

$$\begin{matrix} 1 & - & - & - & - & - & - \\ - & 15 & 35 & 21 & - & - & - \\ - & - & - & 21 & 35 & 15 & - \\ - & - & - & - & - & - & 1 \end{matrix}.$$

If the (35×21)-matrix M_2 of linear second order syzygies has rank 21 at a general point, then $V_{10}(F)$ is singular along the set of points $[l] \in \mathbf{P}(V)$ for which $f - l^3$ has rank 9 and the matrix M_2 has rank at most 19, in particular at the points $[l]$ for which $f - l^3$ is apolar to a pencil of elliptic normal sextic curves on a Veronese surface.

Proof. Indeed, by Lemma 3.20, the (35×21)-matrix M_2 has rank at most 20 along the determinantal hypersurface $V_{10}(F) = Y_F$, so it is singular where the rank is at most 19. The lemma therefore follows from Corollary 3.19. □

REMARK **3.22.** We have computed with *Macaulay2* [13] for certain cubic forms f, that $V_{10}(F)$ is a hypersurface of degree 9 whose singular locus is a surface of degree 140 that coincides with the locus where M_2 has rank at most 19. Therefore we conjecture that this holds for a general f.

We conclude this section with a criterion for $VSP(F, 10)$ to avoid the singular locus of $\mathrm{Hilb}_{10}\mathbf{P}(V)$, that we will also need in the next section.

PROPOSITION **3.23.** *Let $V = \mathbf{C}^6$, and let F be a fourfold defined by a cubic form $f \in \mathrm{Sym}^3V$ with no partial derivative of rank ≤ 3. If f has cactus rank 10 and Z is apolar to f for every $[Z] \in VSP(F, 10)$, then $VSP(F, 10)$ does not intersect the singular locus of $\mathrm{Hilb}_{10}(\mathbf{P}(V))$.*

Proof. Let $[Z] \in VSP(F, 10)$, then, by Remark 3.12, the scheme Z is locally Gorenstein. Consider the morphism $q_f : \mathbf{P}(V) \to \mathbf{P}(Q_f^*)$ defined by the space of quadrics Q_f that are apolar to F. Then the linear span of the image $q_f(Z)$ has, by Lemma 3.5, dimension 2 or

3. Since f has no partial of rank ≤ 3, the morphism q_f is, by Lemma 3.4, an embedding, so the scheme Z is embeddable in $\mathbf{P}^3$. By [12] and [6, Corollary 2.6], the corresponding point $[Z]$ is smooth in the Hilbert scheme. $\square$

4. The divisor of cubic fourfolds apolar to a Veronese

In the first part of this section we show (Corollary 4.3) that for a general cubic fourfold F apolar to a Veronese surface Σ, i.e. in the set D_{V-ap}, the variety $VSP(F,10)$ is singular along a $K3$ surface, and then (Corollary 4.6) that the hypersurface $V_{10}(F)$ introduced and studied in the previous section is singular along Σ. Subsequently, we show (Proposition 4.8) that the general F in D_{V-ap} is apolar to finitely many Veronese surfaces, by exhibiting an F in D_{V-ap} such that the singular locus of $V_{10}(F)$ cannot contain a one-dimensional family of Veronese surfaces. Next, we extend results in Section 3 to show (Corollary 4.10 and Propositions 4.11 and 4.12) that D_{V-ap} is a divisor different from D_{rk3} and D_{copl}, and that (Corollary 4.14) for a general $[f] \in D_{V-ap}$, the fourfold $VSP(F,10)$ does not meet the singular locus of the Hilbert scheme. In the final part of the section we show (Proposition 4.15) that D_{V-ap} is not a Noether Lefschetz divisor in the moduli space of smooth cubic fourfolds.

The Propositions 4.8, 4.11 and 4.15 and Corollary 4.14 are applied in Section 5 to show that for a general $[f] \in D_{V-ap}$, the fourfold $VSP(F,10)$ is smooth outside a surface along which it has quadratic singularities.

Let W be a vector space of rank 3, and $V = S^2W$, and recall the linear map (cf. (2))

$$s : S^6W \to S^3V \qquad \text{s.t. } s(a^6) = (a^2)^3 \tag{18}$$

For $g \in S^6W$, we consider the cubic form $f = s(g) \in S^3V$. Formula (18) shows that there is a natural embedding

$$\phi : VSP(C,10) \to VSP(F,10)$$

from the variety of sums of powers $VSP(C,10)$ of the plane sextic curve C defined by g to the variety of sums of powers $VSP(F,10)$ of the cubic F defined by f. Indeed, if $g = \sum_i a_i^6$, then $f = s(g) = \sum_i (a_i^2)^3$. For distinct $[a_i] \in \mathbf{P}(W)$, the morphism ϕ sends the length 10 subscheme $\{[a_i] | i = 1, ..., 10\}$ to the length 10 subscheme $\{[a_i^2] | i = 1, ..., 10\}$ of $\mathbf{P}(V)$. More generally, ϕ associates to a length 10 apolar subscheme Z of g in $\mathbf{P}(W)$ the length 10 apolar subscheme to f in $\mathbf{P}(V)$ which is the image of Z under the Veronese embedding.

Remark **4.1.** *When g is general sextic ternary form and $C = V(g)$, then Mukai showed in* [14] *that $VSP(C,10)$ is a smooth $K3$ surface. We shall often use the notation $S_g := VSP(C,10)$.*

We have the following general criterion for singularities of the variety of power sums of a hypersurface:

Lemma **4.2.** *Assume that k is the rank of a general hypersurface of degree d in $\mathbf{P}^n$. Let $F \subset \mathbf{P}^n$ be a hypersurface of degree d and rank k, and let $[Z] \in Hilb_k(\check{\mathbf{P}}^n)$ be an apolar subscheme to F such that $Z = \{l_1, \ldots, l_k\}$ consists of k distinct points imposing independent conditions to polynomials of degree d. Then $VSP(F,k)$ is singular at $[Z]$, if there is a hypersurface of degree d in $\check{\mathbf{P}}^n$ which is singular along Z.*

Proof. Consider the universal family

$$\mathcal{V}SP = \{([Z],[f]) | [Z] \in VSP(F,k)\} \subset \mathrm{Hilb}_k(\check{\mathbf{P}}^n) \times \mathbf{P}(S^dV)$$

where $\check{\mathbf{P}^n} = \mathbf{P}(V)$. The fiber of the second projection over a point $[f] \in \mathbf{P}(S^dV)$ is $VSP(F,k)$, where $F = V(f)$. The fiber of the first projection over a point $[Z] \in \mathrm{Hilb}_k(\check{\mathbf{P}^n})$ is a linear space, the linear span $\langle\rho_d(Z)\rangle$ of the image $\rho_d(Z)$ in $\mathbf{P}(S^dV)$ under the $d-$uple Veronese embedding ρ_d. Now, the set of subschemes Z that impose independent conditions to polynomials of degree d is open in the Hilbert scheme, and $\langle\rho_d(Z)\rangle$ is a $\mathbf{P}^{k-1}$. Since $\mathrm{Hilb}_k(\check{\mathbf{P}^n})$ is smooth of dimension kn near Z, we conclude that $\mathcal{V}SP$ is smooth of dimension $kn+k-1$ near $([Z],[f])$. Since F has rank k, the second projection $\mathcal{V}SP \to \mathbf{P}(S^dV)$ is onto, so the variety $VSP(F,k)$ is singular at a point $[Z]$ if the rank of the second projection $\mathcal{V}SP \to \mathbf{P}(S^dV)$ at the point $([Z],[f])$ is less than $\dim(\mathbf{P}(S^dV))$. If $Z = \{[l_1],...,[l_k]\}$, then this rank is the dimension of the span $T_Z = \langle[l_i^{d-1}y_j] | 1 \le i \le k, 0 \le j \le n\rangle$ where $\langle y_0,\dots,y_n\rangle = V$. In fact, from the expansion of $(l_i+y_j)^d$, we see that $l_i^{d-1}y_j$ defines a tangent direction at the point $[l_i^d]$, so T_Z is the span of the tangent spaces to the d-uple embedding $\rho_d(\mathbf{P}(V))$ at the points $[l_i^d]$ (this is a special case of Terracini's Lemma (cf. [16], [19]). Hence $VSP(F,k)$ is singular at $[Z]$ if these tangent spaces do not span $\mathbf{P}(S^dV)$. But hyperplanes in $\mathbf{P}(S^dV)$ correspond to hypersurfaces of degree d in $\check{\mathbf{P}^n}$, and a hyperplane contains the tangent space at $[l_i^d]$ if and only if the corresponding hypersurface is singular at $[l_i]$. Therefore $VSP(F,k)$ is singular at $[Z]$ if there is a hypersurface in $\check{\mathbf{P}^n}$ of degree d singular in the points $[l_1],\dots,[l_k]$. $\square$

COROLLARY **4.3.** *If g, f are as above, $VSP(F,10)$ is singular along S_g.*

Proof. Assume $[Z] \in S_g$ consists of 10 distinct points imposing independent conditions to S^3V^*, (which is the case for general $[Z] \in S_g$). According to Lemma 4.2, the variety $VSP(F,10)$ is singular at $[Z]$ if there exists a cubic fourfold singular along Z. This condition is satisfied in our situation since Z is contained in the Veronese surface $\Sigma \subset \mathbf{P}(V)$. In fact, the Veronese surface is the singular locus of the discriminant cubic hypersurface parameterizing singular conics in $\mathbf{P}(W^*)$. $\square$

The next couple of lemmas are used to prove that if F is a general cubic fourfold apolar to a Veronese surface Σ, then the hypersurface $V_{10}(F)$ is singular along Σ (Corollary 4.6).

LEMMA **4.4.** *Let F be a fourfold defined by a general cubic form f, and let*

$$I_{VSP} = \{([l],[Z]) | [l] \in Z\} \subset V_{10}(F) \times VSP(F,10)$$

be the natural incidence variety. Then the projection onto the first factor is $2:1$. Furthermore, the conclusion holds as soon as f has cactus rank 10 and for the general point $[l] \in V_{10}(F)$ and general $[Z] \in VSP(F,10)$ with $[l] \in Z$, the residual subscheme $Z \setminus [l]$ lies in an elliptic sextic curve.

Proof. First we show, as in the proof of Lemma 3.16, that any 9-tuple of points apolar to a general cubic f' of rank 9 lies on a unique elliptic normal curve E of degree 6 and that for general such f', there are exactly two such sets on E. Let $p_1,...,p_9$ be a set of points apolar to f'. We may assume that these are in general position. Then the Gale dual to these points are 9 points in general position in $\mathbf{P}^2$. The Gale transform (cf. [8]) embeds the unique elliptic cubic curve through these points in $\mathbf{P}^2$ as an elliptic normal curve E of degree 6 through the points $p_1,...,p_9$ in $\mathbf{P}^5$. The curve E is apolar to f', and we claim that any 9-tuple of points apolar to f' lies on this curve.

By Terracini's Lemma, (cf. [16], [19]), the tangent space to the 9-th secant variety of the 3-uple embedding W_3 of $\mathbf{P}^5$ at the point $[f']$ is the span of the tangent spaces of any 9 points in W_3 whose span contains $[f']$. The tangent space to the 9-th secant variety at $[f]$ is therefore defined by the linear space of cubic hypersurfaces that are singular at $p_1, ..., p_9$. The curve E is contained in four Veronese surfaces, corresponding to the four square roots of the hyperplane line bundle of degree 6. The secant varieties of these Veronese surfaces generate a pencil of cubic hypersurfaces singular along the elliptic curve. Their intersection is precisely the union of secant lines to E, so there are no other cubics singular along E, and E is the common singular locus of this pencil. We will show that these hypersurfaces are precisely the cubic hypersurfaces singular at $p_1, ..., p_9$. Since the divisor which is twice the sum of 9 general double points on E is not linearly equivalent to a cubic hypersurface divisor, a cubic hypersurface singular in 9 general points must contain the curve E. Furthermore, on any smooth intersection S of three quadrics containing the curve, the curve E has trivial normal bundle. Therefore, the residual of a cubic hypersurface section of S that contains E, meets the curve in a divisor equivalent to a cubic section. Hence, a cubic that is singular at 9 general points, must contain the doubling of the curve in the three quadrics. Varying the complete intersection surface S, we may conclude that the cubic must be singular along the curve.

Summing up we see that tangent space of the 9-th secant variety of W_3 at the point $[f']$ has codimension 2 and that any 9-tuple of points on W_3 whose span contains $[f']$ is contained in E.

The 3-uple embedding of the curve E in W_3 is an elliptic normal curve of degree 18. By [4, Proposition 5.2], a general point on the 8-th secant variety of this curve is contained in the span of exactly two sets of nine points on the curve. Therefore f' is apolar to exactly two subschemes of length 9 supported on E.

Coming back to the general cubic form f, for a general l^3 appearing as a summand in a power sum presentation of f, the form $f' = f - l^3$ is a general cubic form of rank 9, so the previous conclusion applies to f' and shows that the number of preimages of the projection

$$I_{VSP} \to V_{10}(F) \qquad ([l],[Z]) \mapsto [l]$$

above $[l]$ is 2.

□

LEMMA **4.5.** *Let C be a plane curve defined by a general sextic form g, and let*

$$I_{VSP} = \{([l],[Z]) | [l] \in Z\} \subset \mathbf{P}^2 \times VSP(C,10)$$

be the natural incidence variety. Then the projection onto the first factor is $2:1$.

Proof. As above, let $p = [l] \in Z \subset \mathbf{P}^2$ be a point in an apolar subscheme of length 10. Then $Z - p$ is apolar to $g - l^6$ and lies in a cubic curve. For p and Z general, we may assume that the curve is smooth, and the argument of the proof of the previous lemma applies to show that p is contained in two distinct subschemes on this curve that both are apolar to g. □

COROLLARY **4.6.** *If a cubic fourfold F has cactus rank* 10 *and is apolar to a Veronese surface Σ, then $V_{10}(F)$ is singular along Σ.*

Proof. By Corollary 4.3, the variety $VSP(F,10)$ is singular along the $K3$ surface $S_g = VSP(C,10)$, the variety of power sums of the plane sextic curve C defined by g such that $f = s(g)$.

Now, assume first that g_t is a general plane sextic curve. Then any colength one subscheme of a general apolar subscheme of length 10 of g_t is contained in a elliptic sextic curve on Σ. Therefore the same holds for a general apolar subscheme of length 10 of any f_t of cactus rank 10 such that $f_t = s(g_t)$. The map $I_{VSP} \to VSP(F_t, 10)$ and its restriction over S_{g_t} are both finite and of degree 10, and likewise, by Lemmas

4.4 and 4.5, the map $I_{VSP} \to V_{10}(F_t)$ and its restriction over Σ both have degree 2. An analytic neighborhood in $VSP(F_t, 10)$ of a general point in S_{g_t} is therefore isomorphic to a suitable neighborhood in $V_{10}(f_t)$ of any of the corresponding points in Σ. Therefore $V_{10}(f_t)$ is singular along Σ if and only if $VSP(F_t, 10)$ is singular along S_{g_t}.

Finally, we may specialize f_t to f and conclude that $V_{10}(F)$ is singular along Σ. □

REMARK **4.7.** In computations we have found forms f apolar to a Veronese surface Σ, such that $V_{10}(F)$ is singular along the union of Σ and a surface of degree 140, the locus of points where the matrix M_2 of second order linear syzygies has rank at most 19. The matrix M_2 has rank 20 generically on Σ.

By a direct calculation in an example we now prove:

PROPOSITION **4.8.** *Let F be a general cubic fourfold apolar to a Veronese surface.*
(i) F has cactus rank 10*. Hence no length* 9 *subscheme of* $\mathbf{P}(V)$ *is apolar to F.*
(ii) The form f defining F has no partial derivatives of rank 3*.*
(iii) F is apolar to finitely many Veronese surfaces.

Proof. We find with *Macaulay2* [13] a cubic form apolar to a Veronese surface Σ, and compute the resolution of its annihilator (apolar ideal). Let Σ be the Veronese surface defined by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

So the ideal of Σ is generated by

$$\langle x_0x_3 - x_1^2, x_0x_5 - x_2^2, x_3x_5 - x_4^2, x_0x_4 - x_1x_2, x_1x_4 - x_2x_3, x_1x_5 - x_2x_4. \rangle$$

By differentiation one may check that each of these quadratic forms annihilates the following space of cubic forms:

$$\begin{aligned} f(a,b,c,d,e) =& a(2y_2y_4y_5 + y_1y_5^2) + b(y_2y_3^2 + 2y_1y_3y_4) + c(2y_1y_2y_3 + y_1^2y_4 + 2y_0y_3y_4) \\ &+ d(y_2^3 + 6y_0y_2y_5) + e(y_1y_2^2 + 2y_0y_2y_4 + 2y_0y_1y_5) \end{aligned}$$

with $a, b, c, d, e \in \mathbf{C}$. So they are all apolar to the Veronese surface Σ. The apolar ideal of the cubic form $f(1, -1, 1, -1, 1)$ has Betti numbers

$$\begin{matrix} 1 & - & - & - & - & - & - \\ - & 15 & 35 & 21 & - & - & - \\ - & - & - & 21 & 35 & 15 & - \\ - & - & - & - & - & - & 1 \end{matrix} .$$

Its 35×21 matrix of second order linear syzygies restricted to the plane with coordinates

$$x_0 = z_0 + z_1 + z_2, x_1 = z_0 + z_1, x_2 = z_1, x_3 = z_0, x_4 = z_0 + z_2, x_5 = z_2$$

has rank 20 along a curve of degree 9. Reduced modulo 5 the defining form for this curve is

$$\begin{aligned}
&z_0^9 - 2z_0^8z_1 + 2z_0^7z_1^2 - z_0^6z_1^3 - z_0^5z_1^4 - z_0^3z_1^6 - 2z_0^2z_1^7 + 2z_0^7z_1z_2 - 2z_0^6z_1^2z_2\\
&+ z_0^5z_1^3z_2 + z_0^3z_1^5z_2 - z_0z_1^7z_2 + 2z_0^7z_2^2 - z_0^6z_1z_2^2 + 2z_0^5z_1^2z_2^2 - z_0^4z_1^3z_2^2\\
&- z_0^3z_1^4z_2^2 + 2z_0^2z_1^5z_2^2 - z_0z_1^6z_2^2 + z_1^7z_2^2 - 2z_0^5z_1z_2^3 + z_0^4z_1^2z_2^3\\
&+ z_0^3z_1^3z_2^3 + 2z_0^2z_1^4z_2^3 - z_0z_1^5z_2^3 - 2z_1^6z_2^3 + z_0^5z_2^4 + z_0^4z_1z_2^4\\
&+ z_0^3z_1^2z_2^4 + 2z_0^2z_1^3z_2^4 - z_0z_1^4z_2^4 + 2z_1^5z_2^4 + z_0^4z_2^5 - z_0^3z_1z_2^5 + 2z_0^2z_1^2z_2^5\\
&- z_0z_1^3z_2^5 - 2z_1^4z_2^5 - 2z_0^3z_2^6 - 2z_0^2z_1z_2^6 - z_0z_1^2z_2^6 - z_1^3z_2^6 + z_0^2z_2^7 + z_0z_1z_2^7\\
&+ 2z_1^2z_2^7 + 2z_0z_2^8 - z_2^9
\end{aligned}$$

It has a unique singular point at $z_0 = z_2 = 0$ which is an ordinary quadratic singularity. In particular the generic rank of the matrix M_2 is 21 for $f(1,-1,1,-1,1)$. Therefore, by Lemma 3.16, the cactus rank of $f(1,-1,1,-1,1)$ is 10, which proves (i).

A direct computation shows that the cubic form $f(1,-1,1,-1,1)$ has no partials of rank 3, which proves (ii).

Finally, the union of a $1-$dimensional family of Veronese surfaces is a threefold of degree at least 3, since it must span $\mathbf{P}(V)$. So, if $f(1,-1,1,-1,1)$ is apolar to a $1-$dimensional family of Veronese surfaces, then, by Lemmas 3.20 and 4.6, the degree 9 determinantal hypersurface $V_{10}(f(1,-1,1,-1,1))$ would be singular along a threefold of degree ≥ 3. But then its intersection with a $\mathbb{P}^2$ could not have a single ordinary quadratic singularity. This proves (ii). $\square$

REMARK **4.9.** The same computation also shows that for the cubic F defined by the form $f(1,-1,1,-1,1)$, the determinantal hypersurface $Y_F = V_{10}(f(1,-1,1,-1,1))$ has irreducible restriction to a plane, hence it is irreducible.

COROLLARY **4.10.** *The set of cubic forms that are apolar to some Veronese surface is a hypersurface D_{V-ap} in $\mathbf{P}(S^3V)$.*

Proof. The fact that we obtain a hypersurface follows from a dimension count: plane sextics have $28-9=19$ parameters, while cubic fourfolds have $56-36=20$ parameters. The map s induces a rational map

$$s_{mod} : S^6W//Gl(W) \dashrightarrow S^3V//Gl(V).$$

Let $g = s(f)$. The image of s_{mod} is the locus of cubics apolar to a Veronese, and to prove that it is a divisor, it suffices to show that s_{mod} has generically finite fiber. The fiber of s_{mod} over a point parameterizing a cubic F may be identified with the set of Veronese surfaces which are apolar to F. The result thus follows from Proposition 4.8(iii). $\square$

PROPOSITION **4.11.** *A general cubic fourfold F which is apolar to a Veronese surface satisfies the conclusion of Proposition 3.1, namely, any element $[Z] \in VSP(F,10)$ corresponds to a length* 10 *subscheme Z which imposes independent conditions on cubics and is apolar to F.*

Proof. By Proposition 3.1, the divisorial part of the set of cubic fourfolds not satisfying this conclusion is contained in the union of the irreducible divisors D_{rk3} and D_{copl} introduced in Section 2. As we know that the set of cubics apolar to a Veronese surface is an irreducible divisor which is different from D_{rk3} by Proposition 4.8 (ii), the result follows from the following Proposition 4.12. $\square$

PROPOSITION **4.12.** *The divisors D_{copl} and D_{V-ap} are different.*

Proof. Recall from Section 2 that D_{IR} denotes the set of cubic fourfolds $F(S)$ associated to a $K3$ surface section S of $G(2,6)$. We shall distinguish D_{copl} and D_{V-ap} by proving that their intersections $D_{V-ap} \cap D_{IR}$ and $D_{copl} \cap D_{IR}$ with D_{IR} are different.

LEMMA **4.13.** *A cubic form in $D_{V-ap} \cap D_{IR}$ has cactus rank at most* 9.

Proof. We argue by contradiction and assume that $[f] \in D_{V-ap} \cap D_{IR}$ has cactus rank 10. Assume Z is an apolar subscheme of length 10 to $[f] \in D_{V-ap} \cap D_{IR}$ that is contained in a Veronese surface Σ. On the Veronese surface $\Sigma \cong \mathbf{P}^2$, Z is apolar to the sextic ternary form g, the restriction of the cubic form f to Σ. Since f has rank 10, the sextic form g has rank 10, so the general such Z cannot be contained in a cubic curve on $\Sigma \cong \mathbf{P}^2$. Therefore its Hilbert Birch matrix is a 4×5 matrix of linear forms. By [8], the Gale transform of Z is a subscheme in $\mathbf{P}^3$ defined by the 3-minors of the 3×5 matrix adjoint to the Hilbert Birch matrix. But $[f] \in D_{IR}$ so by Lemma 2.4, the subscheme Z is also contained in a quartic surface scroll, which means that its Gale transform in $\mathbf{P}^3$ is contained in a quadric surface. But a length 10 subscheme defined by the 3-minors of a 3×5 matrix of linear forms, is not contained in any quadric surface, and the lemma follows. □

We now exhibit a cubic form in D_{copl} that is apolar to a quartic surface scroll and has cactus rank 10. By Lemma 2.4, it belongs to $D_{copl} \cap D_{IR}$. As it has cactus rank 10, we conclude by the above lemma that it does not belong to $D_{V-ap} \cap D_{IR}$.

The cubic form

$$\begin{aligned} f = {} & 2x_1^2x_2 - 2x_0x_2^2 - 2x_1^2x_3 - 2x_3^2x_4 - x_0x_1x_5 + 2x_1x_2x_5 \\ & + x_2^2x_5 + x_2x_3x_5 + 3x_1x_4x_5 + x_4^2x_5 + 3x_0x_5^2 + x_3x_5^2 \end{aligned}$$

is apolar to the conic section $V(x_0x_2 - x_1^2, x_3, x_4, x_5)$ and the six points defined by the 2-minors of

$$\begin{pmatrix} x_0 & x_1 & x_3 & x_4 & x_0+x_3+x_5 & x_1+x_2+x_4 \\ x_1 & x_2 & x_4 & x_5 & x_1+x_3+x_4 & x_0+x_2+x_5 \end{pmatrix}$$

Notice that the points and the conic section lie in the quartic surface scroll defined by the 2-minors of

$$\begin{pmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix}$$

The resolution of the apolar ideal I_f has Betti numbers

$$\begin{matrix} 1 & - & - & - & - & - & - \\ - & 15 & 35 & 21 & - & - & - \\ - & - & - & 21 & 35 & 15 & - \\ - & - & - & - & - & - & 1 \end{matrix}.$$

and the matrix 35×21 matrix M_2 of second order linear syzygies of I_f restricted to the plane $V(x_3, x_4, x_5)$ has no syzygies. So we conclude that f has rank 10 by Lemma 3.16. □

COROLLARY **4.14.** *For a general cubic fourfold F which is apolar to a Veronese surface, $VSP(F,10)$ does not meet $Sing(Hilb_{10}(\mathbf{P}(V))$.*

Proof. This follows from Proposition 4.8(ii) which guarantees that the form f that defines F has no partial derivative of rank ≤ 3, Proposition 4.11 and Proposition 3.23. □

We conclude this section with the following result concerning the divisor D_{V-ap}.

PROPOSITION **4.15.** *The divisor D_{V-ap} is not a Noether-Lefschetz divisor.*

Here by a Noether-Lefschetz divisor (or components of the Hodge loci, see [17]), we mean a divisor D along which a locally constant nonzero primitive rational cohomology class in $H^4(F_b, \mathbf{Q})$, $b \in D$, remains a Hodge class. Equivalently, as the Hodge conjecture is satisfied by cubic fourfolds, the cubic fourfolds F_b parameterized by such a divisor carry a codimension 2 cycle whose cohomology class is not proportional to the class h^2, $h = c_1(\mathcal{O}_{F_b}(1))$. Hodge theory shows that in the case of cubic fourfolds, the Hodge loci are hypersurfaces in the moduli space, as a consequence of the equality $h^{3,1}(F) = 1$ (see [17]).

Proof of Proposition 4.15. First of all, we recall that in the moduli stack $\mathcal{M}$ of smooth cubic fourfolds (or in the local universal family of deformations), Noether-Lefschetz divisors have a smooth normalization. More precisely, each local branch $\mathcal{M}_\alpha$ near a cubic fourfold $[F]$ is defined by a class $\alpha \in H^4(F, \mathbf{Q})_{prim}$, where $\mathcal{M}_\alpha$ is the "locus of points $t \in \mathcal{M}$ where the class $\alpha_t \in H^4(F_t, \mathbf{Q})_{prim}$ deduced from α by parallel transport is a Hodge class", and the statement is that $\mathcal{M}_\alpha$ is smooth. We refer to [17] for various local descriptions of these Hodge loci and their local study. The smoothness follows from [17, Corollary 3.3], and from the following fact:

LEMMA **4.16.** *Let F be a nonsingular cubic fourfold, and $0 \neq \alpha \in H^2(F, \Omega_F^2)_{prim}$. Then the cup-product-contraction map*

$$\lrcorner\alpha : H^1(F, T_F) \to H^3(F, \Omega_F)$$

is surjective.

This lemma can be proved directly using Griffiths' description of the infinitesimal variations of Hodge structures of hypersurfaces, or by using the Beauville-Donagi isomorphism between the variation of Hodge structures on $H^4(F, \mathbf{Q})_{prim}$ and the variation of Hodge structures on $H^2(L(F), \mathbf{Q})_{prim}$, where $L(F)$ is the Fano variety of lines of F, together with general properties of the period map for hyper-Kähler manifolds.

The universal family of deformations of the cubic Fermat hypersurface $F_{Fermat} = V(f_{Fermat})$ in $\mathbf{P}^5$ can be obtained as follows: in S^3V we choose a linear subspace T which is transverse to the tangent space at the point f_{Fermat} to the orbit of f_{Fermat} under $Gl(V)$, and we restrict the universal hypersurface in $S^3V \times \mathbf{P}^5$ to $T \times \mathbf{P}^5$, where T is embedded in an affine way in S^3V, by $t \mapsto f_{Fermat} + t$. Since the differential at $(Id, 0)$ of the map

$$Gl(V) \times T \to S^3V,$$

$$(\gamma, t) \mapsto \gamma(f_{Fermat}) + t,$$

is an isomorphism, it is a local isomorphism in the analytic topology, hence there is a neighborhood U' of f_{Fermat} in S^3V and a holomorphic retraction $\pi : U' \to U \subset T$ with the property that $\pi(g)$ is the unique point of intersection of $U' \cap O_g$ with T (where O_g is the orbit of $g \in U$ under $Gl(V)$).

It is well-known (see [18, Remark 6.16]) that the tangent space to the orbit of f_{Fermat} at f_{Fermat} is the degree 3 part of the Jacobian ideal of f_{Fermat}, generated by the partial derivatives of f_{Fermat}. If we write $f_{Fermat} = \sum_{i=0}^{i=5} X_i^3$, the Jacobian ideal $J_{f_{Fermat}}$ is generated by the X_i^2, so there is a natural such complementary subspace T; the vector subspace of S^3V generated by the $X_iX_jX_k$ for i, j, k all distinct.

As the map $s_{mod} : S^6W//Gl(W) \dashrightarrow S^3V//Gl(V)$ is induced by the linear map $s : S^6W \to S^3V$, the divisor $D_{V-ap} \subset S^3V//Gl(V)$ comes from a divisor D_U in $U \subset T \subset S^3V$ (U is a analytic open set which will be the basis of a universal family of deformations of F_{Fermat}), where D_U is obtained as the image of the composition of the linear map $s : S^6W \to S^3V$ with $\pi : U' \to U \subset T$, where it is defined.

The following proposition implies that D_{V-ap} is not a Noether-Lefschetz divisor, thus concluding the proof of the proposition. □

PROPOSITION **4.17.** *The local branches of the divisor D_U at the origin are singular.*

REMARK **4.18.** We cannot identify here D_U with an open set of D_{V-ap}. Indeed, D_U is a divisor in the universal family of deformations of F_{Fermat}, and its image D_{V-ap} is obtained by taking the quotient by the big group of automorphisms of F_{Fermat}. The criterion involving the smoothness of the local branches of the considered divisor can be applied only to the universal family of deformations, which is itself smooth.

Proof of Proposition 4.17. We wish to exploit the following observation:

LEMMA **4.19.** *For a generic sextic polynomial $g \in S^6W$ which is the sum of six 6-th powers of elements of W, $f = s(g)$ is (conjugate to) the Fermat polynomial $g_F = \sum_{i=0}^{i=5} X_i^3$.*

Proof. This follows immediately from formula (2), which says that if $g = \sum_{1=0}^{i=5} a_i^6$ then $f = \sum_{i=0}^{i=5} (a_i^2)^3$. On the other hand, for a generic choice of the a_i's, the a_i^2 provide a basis $X_i, i = 0, ..., 5$ of $V = S^2W$. □

We fix $a_0, \dots, a_5$ providing a basis $X_i = a_i^2, i = 0, ..., 5$ of V. For any $b_\bullet = (b_0, \dots, b_5) \in V^6$ and $b \in V$, we consider the curve in S^6W parameterized by the coordinate t, of the form

$$t \mapsto g_{b_\bullet,b,t} := \sum_{i=0}^{i=5} b_i^6 + tb^6 \in S^6W.$$

At $t = 0$, the corresponding curve $t \mapsto s(g_{b_\bullet,b,t}) \in S^3V$ passes through $s(\sum_{i=0}^{i=5} b_i^6)$, which is equal to $\sum_{i=0}^{i=5} (b_i^2)^3 \in S^3V$. The later polynomial is not equal for generic $b_\bullet$ to the Fermat polynomial $f_{Fermat} = \sum_i X_i^3$ but it is canonically conjugate to it, namely, let $\gamma_{b_\bullet} \in Gl(V)$ be determined by

$$\gamma_{b_\bullet}(b_i^2) = X_i, \; i = 0, \dots, 5.$$

Then we have

$$\gamma_{b_\bullet}(s(\sum_{i=0}^{i=5} b_i^6)) = f_{Fermat},$$

and may conclude that the curve

$$t \mapsto f_{b_\bullet,b,t} := \gamma_{b_\bullet}(s(g_{b_\bullet,b,t})) \in S^3V, \; t \in \mathbf{C}$$

passes through f_{Fermat} at $t = 0$. By definition, its image in $S^3V//Gl(V)$ is contained in $\operatorname{Im} s_{mod}$. Furthermore, for small t, $f_{b_\bullet,b,t}$ belongs to the small open set where the holomorphic retraction $\pi : U \to T$ is defined, so that $\pi(f_{b_\bullet,b,t}) \in D_U$ for any such $(b_\bullet, t)$. Thus there must be one branch D'_U of D_U such that $\pi(f_{b_\bullet,b,t})) \in D'_U$ for any $(b_\bullet, t)$, since the parameter space for the family $f_{b_\bullet,b,t}$ is smooth hence in particular normal. Let us now prove that D'_U is not smooth at the point f_{Fermat}. The derivative at 0 with respect to t of the holomorphic map

$$t \mapsto \pi(f_{b_\bullet,t}) \in T$$

is obtained by applying the projection

$$p : S^3V \to T \cong S^3V/J_{f_{Fermat}}$$

to $\gamma_{b_\bullet}(s(b^6)) = \gamma_{b_\bullet}((b^2)^3)$. The above reasoning shows that all these elements lie in the Zariski tangent space $T_{D'_U,0}$ at the point 0 (parameterizing the Fermat equation). The proof that D'_U is not smooth is thus concluded with the following lemma:

LEMMA **4.20.** *The set S of elements $p(\gamma_{b_\bullet}((b^2)^3)) \in T$ generates T as a vector space.*

Proof. Choose two independent elements Y_0, Y_1 of W. Then the three elements Y_0^2, Y_1^2, $(Y_0 + Y_1)^2$ are independent in S^2W. For a generic choice of a_3, a_4, $a_5 \in W$, the set

$$Y_0^2, Y_1^2, (Y_0 + Y_1)^2, a_3^2,\ a_4^2,\ a_5^2$$

forms a basis of V. We choose

$$b_\bullet = (Y_0, Y_1, Y_0 + Y_1, a_3, a_4, a_5).$$

Then

$$\gamma_{b_\bullet}(Y_0^2) = X_0,\ \gamma_{b_\bullet}(Y_1^2) = X_1,\qquad (19)$$
$$\gamma_{b_\bullet}((Y_0 + Y_1)^2) = X_2,\ \gamma_{b_\bullet}(a_i^2) = X_i,\ i = 3, 4, 5.$$

Choose now for b a generic linear combination of Y_0 and Y_1. Then we can write $b^2 = \alpha Y_0^2 + \beta Y_1^2 + \gamma(Y_0 + Y_1)^2$, with the coefficients α, β, γ all nonzero. It follows that

$$(b^2)^3 = 6\alpha\beta\gamma(Y_0)^2(Y_1)^2(Y_0 + Y_1)^2 + P((Y_0)^2,\ (Y_1)^2,\ (Y_0 + Y_1)^2)$$

where the cubic polynomial P contains all monomials in $(Y_0)^2$, $(Y_1)^2$, $(Y_2)^2$ containing at least one quadratic power of one of the variables. Applying the transformation $\gamma_{b_\bullet}$ of (19) and the projection p, we get

$$p(\gamma_{b_\bullet}((b^2)^3)) = 6\alpha\beta\gamma X_0X_1X_2$$

since all the monomials in the X_j containing at least a quadratic power of the variables are in $J^3_{f_{Fermat}}$. We thus proved that the set S contains $X_0X_1X_2$, and the same proof would show that S contains $X_iX_jX_k$ for arbitrary distinct indices i, j, k. Thus S generates T as a vector space.

□

The proof of Proposition 4.17 is finished. □

5. LOCAL STRUCTURE OF VSP FOR A CUBIC FOURFOLD APOLAR TO A VERONESE SURFACE

Let W be a 3-dimensional vector space, and let $g \in S^6W$, $f = s(g) \in S^3(S^2W)$ be as in the previous section, i.e. $C = V(g)$ is a plane sextic curve, and $F = V(f)$ is a cubic fourfold. Our goal in this section is to prove the following theorem (from which Theorem 1.6 of the introduction immediately follows):

THEOREM **5.1.** *Assume that g is a general ternary sextic form, and let $f = s(g)$.*

(i) The variety $VSP(F, 10)$ is smooth of dimension 4 away from the $K3$ surface $S_g = VSP(C, 10)$. In particular, there is only one Veronese surface apolar to f, so we may denote by S_f the surface S_g.

(ii) The singularities of $VSP(F, 10)$ are quadratic nondegenerate in the normal direction to S_f at any point of S_f.

Proof of Theorem 5.1, (i). We know, by Corollary 4.10, that the set of cubics apolar to a Veronese surface is a divisor D_{V-ap} in the space parameterizing all cubics. Let

$$\mathcal{V}SP_{V-ap} := \{([Z],[f]) \in \mathrm{Hilb}_{10}(\mathbf{P}^5) \times D_{V-ap},\ I_Z(3) \subset H_f\} \tag{20}$$

be the universal family of VSP's of cubics apolar to a Veronese surface.

We prove the following:

Lemma **5.2.** *There is a dense Zariski open set $D^0_{V-ap} \subset D_{V-ap}$ such that for $[f] \in D^0_{V-ap}$, there is only one Veronese surface that is apolar to f, thus determining a unique curve C defined by a ternary sextic form g such that $S_g = VSP(C,10) \subset VSP(F,10)$. In this case we denote by S_f this surface S_g.*

Furthermore, denoting by $\mathcal{V}SP_{V-ap,0}$ the restriction to D^0_{V-ap} of the family $\mathcal{V}SP_{V-ap}$, $\mathcal{V}SP_{V-ap,0}$ is nonsingular away from the family $\mathcal{S} \subset \mathcal{V}SP_{V-ap,0}$ of surfaces S_f.

Proof. The Zariski open set D^0_{V-ap} is defined as the set of points $[f]$ in the smooth locus of D_{V-ap} such that Corollary 4.14 and Propositions 4.11 and 4.8, (i) are satisfied. Let $K \subset \mathrm{Hom}(H_f, S^3V^*/H_f)$ be the tangent space of D_{V-ap} at $[f]$, with $f = s(g)$ for some $g \in H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(6))$ and some Veronese embedding $\Sigma \subset \mathbf{P}(V)$ of $\mathbf{P}^2$. We denote again by $h \in S^3V^*$ the discriminant cubic form, such that $V(h)$ is singular along Σ. First of all, we claim that $K^\perp$ is generated by h. As K is a hyperplane, it suffices to show that h belongs to $K^\perp$. In fact, as $\mathcal{V}SP_{V-ap}$ contains the smooth family $\mathcal{S}$ of surfaces near a point $([Z],[f])$, with $f = s(g)$ and $[Z] \in S_g$, it even suffices to prove that for $[Z] \in S_g$, the discriminant cubic form h belongs to the orthogonal of $\mathrm{Im}\,(pr_{2*} : T_{\mathcal{V}SP_{V-ap},([Z],[f])} \to T_{\mathbf{P}(S^3V),f})$. We identify f, as before, with a hyperplane H_f in S^3V^*, which contains $I_Z(3)$. We may assume that Z consists of ten distinct points that impose independent conditions on cubics, so we can identify $T_{\mathrm{Hilb}_{10}(\mathbb{P}(V)),[Z]}$ with $H^0(T_{\mathbf{P}^5|Z})$, and furthermore $H^0(T_{\mathbf{P}^5|Z})$ with $\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}(V)}}(\mathcal{I}_Z, \mathcal{O}_Z)$. We have then the following description of the tangent space of $\mathcal{V}SP$ at $([Z],[f])$:

$$\begin{aligned} T_{([Z],[f])} := \{(u,\gamma) \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}(V)}}(\mathcal{I}_Z, \mathcal{O}_Z) \times \mathrm{Hom}(H_f, S^3V^*/H_f),\\ \gamma_{|I_Z(3)} = p \circ d_u : I_Z(3) \to S^3V^*/H_f\}, \end{aligned} \tag{21}$$

where $d_u : I_Z(3) \to H^0(\mathcal{O}_Z(3))$ is the map induced by $u \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}(V)}}(\mathcal{I}_Z, \mathcal{O}_Z)$ on global sections, and $p : H^0(\mathcal{O}_Z(3)) \to S^3V^*/H_f$ is deduced from the quotient map $S^3V^* \to S^3V^*/H$, using the fact that the restriction map $S^3V^* \to H^0(\mathcal{O}_Z(3))$ is surjective and that its kernel $I_Z(3)$ is contained in H_f. We just have to prove that for γ satisfying the equation (21), we have

$$\gamma(h) = 0. \tag{22}$$

But as $h \in I_Z(3)$, we get $\gamma(h) = d_u(h)$ modulo H_f, and since h is singular along Z, $d_u(h) = 0$, which proves (22). The claim is thus proved.

Note that the claim proves in particular that for $[f] \in D^0_{V-ap}$, there is a unique Veronese surface apolar to f since it says that the cubic h is determined by $K = T_{D^0_{V-ap},[f]}$ and on the other hand it determines Σ, because Σ is the singular locus of $V(h)$.

The proof of the smoothness of $\mathcal{V}SP_{V-ap,0}$ away from $\mathcal{S}$ will now use the fact that the discriminant cubic with equation h is smooth away from Σ. The argument goes as follows: Let $[f] \in D^0_{V-ap}$, $[Z] \in VSP(F,10)$, $[Z] \notin S_f$ and $K = T_{D^0_{V-ap},[f]}$. Recall that the conclusion of Corollary 4.14 holds, so that $[Z]$ is a smooth point of $\mathrm{Hilb}_{10}(\mathbf{P}(V))$. Furthermore, Proposition 4.11 also holds, so Z is apolar to f and imposes independent conditions on cubics. Hence $I_Z(3) \subset H_f$, and this property gives us the local equations for

$VSP(F,10)$ inside $\mathrm{Hilb}_{10}(\mathbf{P}(V))_{reg}$. Differentiating these equations, the Zariski tangent space to $\mathcal{V}SP_{V-ap}$ at $([Z],[f])$ is thus given as before by

$$T_{\mathcal{V}SP_{V-ap},([Z],[f])} := \{(u,\alpha) \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}(V)}}(\mathcal{I}_Z,\mathcal{O}_Z) \times K, \quad \alpha_{|I_Z(3)} = p \circ d_u : I_Z(3) \to S^3V^*/H_f\}, \tag{23}$$

where K is the hyperplane in $\mathrm{Hom}(H_f, S^3V^*/H_f)$ of linear forms vanishing on h.

The variety $\mathcal{V}SP_{V-ap}$ is smooth at $([Z],[f])$ if the restriction map

$$K \to \mathrm{Hom}(I_Z(3), S^3V^*/H_f)$$

is surjective, since this implies that the linear equations in (23) defining the Zariski tangent space to $\mathcal{V}SP_{V-ap}$ at $([Z],[f])$, which are nothing but the differentials of the equations defining $\mathcal{V}SP_{V-ap}$, are linearly independent.

1) If h does not vanish identically on Z, then the map $K \to \mathrm{Hom}(I_Z(3), S^3V^*/H_f)$ is surjective.

2) If h vanishes on Z, the image of the map $K \to \mathrm{Hom}(I_Z(3), S^3V^*/H_f)$ is the set of linear forms on $I_Z(3)$ vanishing on $h \in I_Z(3)$. For the smoothness of $\mathcal{V}SP_{V-ap}$ at (Z,f), it suffices then to know that the map

$$\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}(V)}}(\mathcal{I}_Z,\mathcal{O}_Z) \to \mathbf{C},$$

$$u \mapsto d_u(h) \bmod H_f$$

is nonzero since this implies as before that the linear equations in (23) are linearly independent. We argue as follows: The map

$$u \mapsto u(h) \in H^0(\mathcal{O}_Z(3)) \tag{24}$$

is $H^0(\mathcal{O}_Z)$-linear. So if its image is contained in $H_f/I_Z(3)$, it provides a sub-$H^0(\mathcal{O}_Z)$-module of $H^0(\mathcal{O}_Z(3))$ which is the ideal of a subscheme of Z apolar to f. By Proposition 4.8, (i), this implies that this ideal is equal to 0, that is, the map (24) is 0. In conclusion, if $\mathcal{V}SP_{V-ap}$ is not smooth at $([Z],[f])$, then Z is contained in the singular locus of h, hence in Σ. In other words, $[Z]$ belongs to S_f.

□

Lemma 5.2 implies (i) by a Sard type argument and this concludes the proof of Theorem 5.1, (i).

□

Proof of Theorem 5.1, (ii). We first prove the following result:

LEMMA **5.3.** *For general g and $f = s(g)$, the embedding dimension of $VSP(F,10)$ is* 5 *at any point of S_f.*

Proof. We know that the universal family $\mathcal{VSP}$ is smooth and that the hypersurface $\mathcal{V}SP_{V-ap}$ contains the family of surfaces $\mathcal{S}$ which has generically smooth fibers. If S_f is smooth, the corank of the map $pr_{2*} : T_{\mathcal{VSP},([Z],[f])} \to T_{\mathbf{P}(S^3V),[f]}$ is 1 everywhere along S_f. This implies that the embedding dimension of $VSP(F,10)$ is 5 at any point of S_f. □

This lemma shows that for general g and $f = s(g)$, the variety $VSP(F,10)$ has locally hypersurface singularities along S_f, and our goal now is to show that the Hessian of the local defining equation, which is a homogeneous quadratic polynomial on the normal bundle N_{S_f}, is everywhere nondegenerate. Here the bundle N_{S_f} is defined as the quotient of $T_{VSP(F,10)|S_f}$ by its subbundle T_{S_f}. The bundle N_{S_f} is thus locally free of rank 3 by Lemma 5.3.

We first have the following:

LEMMA **5.4.** *The determinant of N_{S_f} is trivial.*

Proof. We recall that by Proposition 4.11, $VSP(F,10)$ is defined as the following set:

$$VSP(F,10) = \{[Z] \in \mathrm{Hilb}_{10}(\mathbf{P}(V)),\ I_Z(3) \subset H_f\}. \tag{25}$$

The variety $VSP(F,10)$ is contained in the smooth part of $\mathrm{Hilb}_{10}(\mathbf{P}(V))$ and defined according to (25) as the 0-locus of a section σ of the bundle $\mathcal{F}$ with fiber $I_z(3)^*$ over the point $[Z] \in \mathrm{Hilb}_{10}(\mathbf{P}(V))$. (More precisely, since we assumed that $[f] \in D^0_{V-ap}$, the conclusion of Proposition 4.11 holds and thus $VSP(F,10)$ is contained in the open set of $\mathrm{Hilb}_{10}(\mathbf{P}(V))$ where $\mathcal{F}$ is locally free.) For a general $f \in S^3V$, we know by [11] that $VSP(F,10)$ is a smooth Hyper-Kähler manifold, hence in particular has trivial canonical bundle. This means that the line bundle

$$\det\left(T_{\mathrm{Hilb}_{10}(\mathbf{P}^5)|VSP(F,10)}\right) \otimes (\det \mathcal{F})^{-1}$$

has trivial restriction to $VSP(F,10)$, which implies that it has trivial restriction to $VSP(F,10)$ when f is a general cubic apolar to a Veronese surface, because the family $\mathcal{V}SP \to \mathbf{P}(S^3V)$ is flat over a neighborhood of $[f]$ by our previous results.

On the other hand, the proof of Lemma 5.3 shows that the cokernel of the differential $d\sigma$ along S_f is the trivial line bundle with fiber $\mathrm{Hom}\,(\mathbf{C}h, S^3V^*/H_f)$ at any point $[Z]$ of S_f.

The exact sequence

$$0 \to T_{VSP(F,10)|S_f} \to T_{\mathrm{Hilb}_{10}(\mathbf{P}^5)|S_f} \to \mathcal{F}_{|S_f} \to \mathrm{Coker}\, d\sigma \to 0$$

thus implies the triviality of $\det T_{VSP(F,10)|S_f}$, hence the triviality of $\det N_{S_f}$ since $\det T_{S_f}$ is trivial. □

Using the fact that the cokernel of the map $d\sigma$ is the trivial line bundle on S_f, we conclude that the Hessian of σ is a section of $S^2N^*_{S_f}$. Here we use the following notion of Hessian for a section σ of a vector vector bundle E of rank r on a smooth variety Y, at a point y where $d\sigma$ is not of maximal rank. The Hessian is then intrinsically an element of $(\mathrm{Coker}\, d\sigma_y) \otimes S^2\Omega_{Y,y,\sigma}$, where $\Omega_{Y,y,\sigma} = (\mathrm{Ker}\, d\sigma_y)^*$. (Note that $d\sigma_y : T_{Y,y} \to E_y$ is not intrisically defined but $\mathrm{Ker}\, d\sigma_y$ and $\mathrm{Coker}\, d\sigma_y$ are.) This Hessian is related to the usual Hessian as follows: In an adequate local trivialization of E near y, σ is given by a r-tuple $(\sigma_1, \ldots, \sigma_r)$ of functions on Y, and we can assume that if k is the rank of $d\sigma$ at y, then $d\sigma_1, \ldots, d\sigma_k$ are independent at the point y, while $d\sigma_{k+1}, \ldots, d\sigma_r$ vanish at y. Let Y' be the smooth codimension k submanifold of Y defined by σ_i, $i \leq k$. Then $\Omega_{Y,y,\sigma} = \Omega_{Y',y}$ and the restriction $\sigma_{|Y'}$ has zero differential at y. Then the Hessian of σ at y is the $r-k$-tuple of quadratic forms $(\mathrm{Hess}(\sigma_{k+1|Y'}), \ldots, \mathrm{Hess}(\sigma_{r|Y'}))$. If furthermore we know that the vanishing locus of σ has ordinary quadratic singularities along a submanifold $Z \subset Y$, then near y, we have $Z \subset Y'$ and the Hessians $\mathrm{Hess}(\sigma_{i|Y'}) \in S^2\Omega_{Y',y}$ appearing above in fact belong to $S^2N^*_{Z/Y'}$. In our case, Z is S_f and what we denoted by N_{S_f} is naturally isomorphic to $N_{Z/Y'}$.

As the determinant of N_{S_f} is trivial, this quadric is nondegenerate everywhere along S_f if and only if it is nondegenerate generically along S_f. The last property can be shown as follows: Recall that f is a generic cubic apolar to a Veronese surface and $[Z] \in S_f$. The pair $([Z],[f])$ can be constructed starting from a general subscheme of length 10 of the Veronese surface Σ, and taking for H_f a general hyperplane of S^3V^* containing $I_Z(3)$.

Take for Z a reduced subscheme consisting of ten distinct points $x_1, \ldots, x_{10}$ in general position. Then the hyperplane H_f is determined by a linear form $p : S^3V^* \to S^3V^*/H_f$. This form is the composite of the projection map $S^3V^* \to S^3V^*/I_Z(3)$ and a linear form

$$p' : H^0(\mathcal{O}_Z(3)) \to \mathbf{C}.$$

After trivialization of $\mathcal{O}_Z(3)$ we may write

$$p' = \sum_i p_i \mathrm{ev}_{x_i}$$

for some scalars p_i which can be chosen arbitrarily. Recalling that the cokernel of $d\sigma$ is generated by $\mathrm{Hom}\,(\mathbf{C}h, S^3V^*/H_f)$, it is clear that the Hessian $\mathrm{Hess}(\sigma)$ at the point $[Z]$ is obtained by restricting the sum $\sum_i p_i d^2h_{x_i}$ to

$$N_{S_f,[Z]} \subset H^0(N_{\Sigma/\mathbf{P}^5|Z}) = \oplus_i N_{\Sigma/\mathbf{P}^5,x_i}.$$

Here we use the same trivialization of $\mathcal{O}_Z(3)$ as above to see the Hessian $d^2h_{x_i}$ of h at x_i as an element of $S^2N^*_{\Sigma/\mathbf{P}^5,x_i}$. Since h has nondegenerate quadratic singularities along Σ, each of the quadrics $d^2h_{x_i}$ is nondegenerate. We now have:

LEMMA **5.5.** *The 3-dimensional vector space $N_{S_f,[Z]}$ is the orthogonal complement with respect to the quadratic form $\sum_i p_i d^2h_{x_i}$ of the subspace Im $(H^0(\Sigma, N_{\Sigma/\mathbf{P}^5}) \to \oplus_i N_{\Sigma/\mathbf{P}^5,x_i})$.*

Proof. Indeed, the space $N_{S_f,[Z]}$ is equal to the kernel of the composite map

$$H^0(N_{\Sigma/\mathbf{P}^5|Z}) \to \mathrm{Hom}\,(I_\Sigma(3), H^0(\mathcal{O}_Z(3)) \xrightarrow{p'} \mathrm{Hom}\,(I_\Sigma(3), S^3V^*/H_f),$$

where $H^0(N_{\Sigma/\mathbf{P}^5|Z}) \cong \oplus_i N_{\Sigma/\mathbf{P}^5,x_i}$ and $p' = \sum_i p_i \mathrm{ev}_{x_i}$.

Let now $u \in H^0(\Sigma, N_{\Sigma/\mathbf{P}^5})$ and $v = (v_i) \in H^0(N_{\Sigma/\mathbf{P}^5|Z})$. Then

$$(\sum_i p_i d^2h_{x_i})(u_{|z}, v_{|z}) = \sum_i p_i d^2h_{x_i}(u_i, v_i).$$

The section $u \in H^0(\Sigma, N_{\Sigma/\mathbf{P}^5})$ lifts to a section $U \in H^0(\mathbf{P}^5, T_{\mathbf{P}^5})$, and the degree 3 polynomial $d_U(h)$ belongs to $I_\Sigma(3)$. Furthermore we have

$$d^2h_{x_i}(u_i, v_i) = d(d_U(h))(v_i)$$

for any i. It follows that

$$\sum_i p_i d^2h_{x_i}(u_i, v_i) = \sum_i p_i d(d_U(h))(v_i).$$

If now (v_i) belongs to $N_{S_f,[Z]}$, we find that $\sum_i p_i d(d_U(h))(v_i) = 0$ and thus

$$\sum_i p_i d^2h_{x_i}(u_i, v_i) = 0.$$

Hence we proved that $\mathrm{Im}\,(H^0(\Sigma, N_{\Sigma/\mathbf{P}^5}) \to \oplus_i N_{\Sigma/\mathbf{P}^5,x_i})$ is perpendicular with respect to $\sum_i p_i d^2h_{x_i}$ to the space $N_{S_f,[Z]}$. As the space $H^0(\Sigma, N_{\Sigma/\mathbf{P}^5})$ is of dimension 27, the map $H^0(\Sigma, N_{\Sigma/\mathbf{P}^5}) \to \oplus_i N_{\Sigma/\mathbf{P}^5,x_i}$ is injective of maximal rank 27 for a general choice of the x_i's. As the space $N_{S_f,[Z]}$ is of dimension 3, we conclude that $\mathrm{Im}\,(H^0(\Sigma, N_{\Sigma/\mathbf{P}^5}) \to \oplus_i N_{\Sigma/\mathbf{P}^5,x_i})$ is the orthogonal complement with respect to $\sum_i p_i d^2h_{x_i}$ of the space $N_{S_f,[Z]}$ since the quadratic form $\sum_i p_i d^2h_{x_i}$ on the 30-dimensional vector space $\oplus_i N_{\Sigma/\mathbf{P}^5,x_i}$ is nondegenerate. ☐

It follows that the quadratic form $\mathrm{Hess}(\sigma)$, that is the restriction of $\sum_i p_i d^2 h_{x_i}$ to $N_{S_f,[Z]}$, is nondegenerate if and only if the quadratic form $\sum_i p_i d^2 h_{x_i}$ has a nondegenerate restriction to $\mathrm{Im}\,(H^0(\Sigma, N_{\Sigma/\mathbf{P}^5}) \to \oplus_i N_{\Sigma/\mathbf{P}^5,x_i})$. The last property may be achieved because the points x_i being general, the map

$$H^0(\Sigma, N_{\Sigma/\mathbf{P}^5}) \to \oplus_{1\leq i\leq 9} N_{\Sigma/\mathbf{P}^5,x_i}$$

is injective (hence an isomorphism). Hence any combination $\sum_{1\leq i\leq 9} p_i d^2 h_{x_i}$ with $p_i \neq 0$ for any $i \leq 9$ has a nondegenerate restriction to $\mathrm{Im}\,(H^0(\Sigma, N_{\Sigma/\mathbf{P}^5}) \to \oplus_i N_{\Sigma/\mathbf{P}^5,x_i})$ and thus a general combination $\sum_{1\leq i\leq 10} p_i d^2 h_{x_i}$ has a nondegenerate restriction to $\mathrm{Im}\,(H^0(\Sigma, N_{\Sigma/\mathbf{P}^5}) \to \oplus_i N_{\Sigma/\mathbf{P}^5,x_i})$.

In conclusion, we proved that, for general g and $f = s(g)$, at a general point $[Z] \in S_f = S_g \subset VSP(F,10)$, the Hessian of the local defining equation of $VSP(F,10)$ has rank 3, and as explained above, this implies that it is everywhere nondegenerate in the normal direction to S_f. □

6. Proof of Theorem 1.4

We first recall the statement of the result:

THEOREM **6.1.** *Let F be a very general cubic fourfold. Then there is no nonzero morphism of Hodge structures between $H^4(F,\mathbf{Q})_{prim}$ and $H^2(VSP(F,10),\mathbf{Q})_{prim}$.*

Proof. Let B be the Zariski open set of $\mathbf{P}(H^0(\mathbf{P}^5, \mathcal{O}_{\mathbf{P}^5}(3)))$ parameterizing smooth cubics. We have the universal family $\pi : \mathcal{X} \to B$ of cubic hypersurfaces, where the morphism π is smooth and projective. We also have the family $\pi' : \mathcal{V}SP \to B$ which is projective over B but is not smooth. The base B contains the divisor D_{V-ap} parameterizing cubic fourfolds apolar to a Veronese surface. We proved in Theorem 5.1 that for $[f]$ in an open subset D^0_{V-ap}, the fiber $VSP(F,10) = \pi'^{-1}([f])$ has for singularities nondegenerate quadratic singularities along the surface S_f which is a smooth $K3$ surface. Let $[f]$ be a point of D^0_{V-ap} and let B^0 be a Zariski open set of B containing $[f]$ and such that $D_{V-ap} \cap B^0 \subset D^0_{V-ap}$. Let $B' \to B^0$ be the double cover ramified along D^0_{V-ap}. Then B' is smooth, and the pulled-back family $\tilde{\pi}' : \mathcal{V}SP' \to B'$ is smooth except along the family of surfaces $\mathcal{S} \to D^0_{V-ap}$, which has codimension 3 in $\mathcal{V}SP'$, and along which $\mathcal{V}SP'$ has quadratic nondegenerate singularities. The family $\mathcal{V}SP' \to B'$ can be modified after passing to a degree 2 étale cover of B' to a family of smooth complex projective manifolds by a small resolution: For this we first blow-up $\mathcal{V}SP'$ along $\mathcal{S}$ to get $\mathcal{V}SP'' \to B$. The exceptional divisor E of the blow-up is a bundle over $\mathcal{S}$ with fibers smooth two-dimensional quadrics. There is an étale double cover $\widetilde{\mathcal{S}} \to \mathcal{S}$ parameterizing the rulings in the fibers of $E \to \mathcal{S}$. As a $K3$ surface is simply connected, this double cover comes from a double cover $\widetilde{D^0_{V-ap}} \to D^0_{V-ap}$. We may assume this étale double cover is induced by an étale double cover $\widetilde{B}^0 \to B^0$. Performing this base change, the pulled-back family $\widetilde{\mathcal{V}SP''} \to \widetilde{B}^0$ has the property that the inverse image $\widetilde{E}$ of E admits two morphisms to a $\mathbf{P}^1$-bundle over $\widetilde{\mathcal{S}}$. We choose one of them, and as is well-known, we can contract $\widetilde{E}$ to $\widetilde{\mathcal{S}}$ along this morphism. The resulting family $\phi : \widetilde{\mathcal{V}SP} \to \widetilde{B}^0$ is smooth proper over $\widetilde{B}^0$.

We now have two families

$$\phi : \widetilde{\mathcal{V}SP} \to \widetilde{B},\ \psi : \widetilde{\mathcal{X}} \to \widetilde{B}$$

of smooth proper complex manifolds, where $\widetilde{\mathcal{X}} := \mathcal{X} \times_B \widetilde{B}^0$. The fibers of both families are projective, and in particular Kähler, although it is not clear if both morphisms are projective. We thus get two associated variations of Hodge structures on $\widetilde{B}$, one of weight 2 on the primitive cohomology of degree 2 of the fibers of the first family with associated local system H^2, the other of weight 4 on the primitive cohomology of degree 4 of the fibers of the second family with associated local system H^4. The locus of points $b \in \widetilde{B}$ where there is a nonzero morphism of Hodge structures $H^4(\widetilde{\mathcal{X}}_b, \mathbf{Q})_{prim} \to H^2(\widetilde{\mathcal{V}SP}_b, \mathbf{Q})_{prim}$ is the Hodge locus for the induced variation of Hodge structure on the local system $\mathrm{Hom}\,(H^4, H^2)$. The Hodge locus is a countable union of closed algebraic subsets of the base $\widetilde{B}$ (cf. [17]). In order to prove Theorem 6.1, it thus suffices to prove that there is a point of $\widetilde{B}$ where there is no nonzero morphism of Hodge structures between $H^4(\widetilde{\mathcal{X}}_b, \mathbf{Q})_{prim}$ and $H^2(\widetilde{\mathcal{V}SP}_b, \mathbf{Q})_{prim}$.

By Proposition 4.15, the divisor D_{V-ap} is not a Noether-Lefschetz locus for the family $\mathcal{X} \to B$. This means that there exists a point $b \in D_{V-ap}$, that we may assume to be in D^0_{V-ap}, such that there is no nonzero Hodge class in $H^4(\mathcal{X}_b, \mathbf{Q})_{prim}$. This fact implies that the Hodge structure on $H^4(\mathcal{X}_b, \mathbf{Q})_{prim}$ is simple. Indeed, since $h^{3,1}(\mathcal{X}_b) = 1$, any proper sub-Hodge structure has $h^{3,1}$-number 0 or its orthogonal complement for the intersection pairing satisfies this property. In both cases, the existence of a proper sub-Hodge structure implies the existence of a nonzero Hodge class. Note also that it has $h^{2,2}$-number equal to 20.

On the other hand, we claim that the transcendental part of $H^2(\widetilde{\mathcal{V}SP}_b, \mathbf{Q})_{prim}$ has $h^{1,1}$-number ≤ 19. Here the transcendental part is defined as the minimal sub-Hodge structure containing the $H^{2,0}$-component.

The claim follows from the fact that $\widetilde{\mathcal{V}SP}_b$ is hyper-Kähler, being a fiber of a family of Kähler manifolds whose general member is hyper-Kähler, and on the other hand it is the blow-up of $\mathcal{V}SP_b$ along the $K3$ surface S_b. It thus contains the exceptional divisor E_b over S_b and the morphism of Hodge structures $H^2(\widetilde{\mathcal{V}SP}_b, \mathbf{Q}) \to H^2(E_b, \mathbf{Q})$ does not vanish on $H^{2,0}(\widetilde{\mathcal{V}SP}_b)$ because a symplectic form on a fourfold cannot vanish on a divisor. On the other hand, this morphism sends $H^2(\widetilde{\mathcal{V}SP}_b, \mathbf{Q})_{tr}$ to $H^2(E_b, \mathbf{Q})_{tr}$ which is equal to $H^2(S_b, \mathbf{Q})_{tr}$. The induced morphism

$$H^2(\widetilde{\mathcal{V}SP}_b, \mathbf{Q})_{tr} \to H^2(S_b, \mathbf{Q})_{tr}$$

must be injective by the same simplicity argument as above, and thus

$$h^{1,1}(\widetilde{\mathcal{V}SP}_b, \mathbf{Q})_{prim} \leq h^{1,1}(S_b)_{prim} \leq 19.$$

As the Hodge structure on $H^4(\mathcal{X}_b, \mathbf{Q})_{prim}$ is simple with $h^{2,2}$-number equal to 20, any morphism of Hodge structures between $H^4(\mathcal{X}_b, \mathbf{Q})_{prim}$ and a weight 2 Hodge structure with $h^{1,1}$-number ≤ 19 is identically 0, which concludes the proof of Theorem 1.4.

□

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