Rational equivalence of 0-cycles on $K3$ surfaces and conjectures of Huybrechts and O'Grady

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Abstract

We give a new interpretation of O'Grady's filtration on the CH_0 group of a $K3$ surface. In particular, we get a new characterization of the canonical 0-cycles kc_X : given $k \geq 0$, kc_X is the only 0-cycle of degree k on X whose orbit under rational equivalence is of dimension k. Using this, we extend results of Huybrechts and O'Grady concerning Chern classes of simple vector bundles on $K3$ surfaces.

1 Introduction

Let X be a projective $K3$ surface. In [1], Beauville and the author proved that X carries a canonical 0-cycle c_X of degree 1, which is the class in $CH_0(X)$ of any point of X lying on a (possibly singular) rational curve on X. This cycle has the property that for any divisors D, D' on X, we have
$$D \cdot D' = \deg(D \cdot D') \, c_X \text{ in } CH_0(X).$$

In recent works of Huybrechts [5] and O'Grady [13], this 0-cycle appeared to have other characterizations. Huybrechts proves for example the following result (which is proved in [5] to have much more general consequences on spherical objects and autoequivalences of the derived category of X):

Theorem 1.1. *(Huybrechts [5]) Let X be a projective complex $K3$ surface. Let F be a simple vector bundle on X such that $H^1(\operatorname{End} F) = 0$ (such an F is called spherical in [5]). Then $c_2(F)$ is proportional to c_X in $CH_0(X)$ if one of the following conditions holds.*

1. *The Picard number of X is at least 2.*

2. *The Picard group of X is $\mathbf{Z}H$ and the determinant of F is equal to kH with $k = \pm 1$ mod. $r = \operatorname{rank} F$.*

This result is extended in the following way by O'Grady : In [13], he introduces the following increasing filtration of $CH_0(X)$
$$S_0(X) \subset S_1(X) \subset \ldots \subset S_d(X) \subset \ldots \subset CH_0(X),$$

where $S_d(X)$ is defined as the set of classes of cycles of the form $z + z'$, with z effective of degree d and z' a multiple of c_X. It is also convenient to introduce $S_d^k(X)$ which will be by definition the set of degree k 0-cycles on X which lie in $S_d(X)$. Thus by definition
$$S_d^k(X) = \{z \in CH_0(X), z = z' + (k - d)c_X\},$$

where z' is effective of degree d.

Consider a torsion free or more generally a pure sheaf $\mathcal{F}$ on X which is H-stable with respect to a polarization H. Let $2d(v_{\mathcal{F}})$ be the dimension of the space of deformations of $\mathcal{F}$, where $v_{\mathcal{F}}$ is the Mukai vector of $\mathcal{F}$ (cf. [6]). We recall that $v_{\mathcal{F}} \in H^*(X, \mathbf{Z})$ is the triple

$$(r, l, s) \in H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z}),$$

with $r = \operatorname{rank} \mathcal{F}$, $l = c_1^{top}(\det \mathcal{F})$ and $s \in H^4(X, \mathbf{Z})$ is defined as

$$v_{\mathcal{F}} = ch(\mathcal{F})\sqrt{td(X)}. \tag{1}$$

In particular we get by the Riemann-Roch formula that

$$\sum_i (-1)^i \dim Ext^i(\mathcal{F}, \mathcal{F}) = <v_{\mathcal{F}}, v_{\mathcal{F}}^*> = 2rs - l^2 = 2 - 2d(v_{\mathcal{F}}),$$

where $<,>$ is the intersection pairing on $H^*(X, \mathbf{Z})$, and $v^* = (r, -l, s)$ is the Mukai vector of $\mathcal{F}^*$ (if $\mathcal{F}$ is locally free).

In particular $d(v_{\mathcal{F}}) = 0$ if $\mathcal{F}$ satisfies $\operatorname{End} \mathcal{F} = \mathbb{C}$ and $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) = 0$, so that $\mathcal{F}$ is spherical as in Huybrechts' theorem. Noticing that $S_0(X) = \mathbf{Z}c_X$, one can then rephrase Huybrechts' statement by saying that if $\mathcal{F}$ satisfies $\operatorname{End}(\mathcal{F}) = \mathbb{C}$, $d(v_{\mathcal{F}}) = 0$, then $c_2(\mathcal{F}) \in S_0(X)$, assuming the Picard number of X is at least 2.

O'Grady then extends Huybrechts' results as follows:

Theorem 1.2. *(O'Grady [13]) Assuming $\mathcal{F}$ is H-stable, one has $c_2(\mathcal{F}) \in S_{d(v_{\mathcal{F}})}(X)$, $v_{\mathcal{F}} = (r, l, s)$, if furthermore one of the following conditions holds:*

1. *$l = H$, l is primitive and $s \geq 0$.*

2. *The Picard number of X is at least 2, r is coprime to the divisibility of l and H is v-generic.*

3. *$r \leq 2$ and moreover H is v-generic if $r = 2$.*

In fact, O'Grady's result is stronger, as he also shows that $S_{d(v)}^k(X)$, $k = \deg c_2(v)$, is equal to the set of classes $c_2(\mathcal{G})$ with $\mathcal{G}$ a deformation of $\mathcal{F}$. O'Grady indeed proves, by a nice argument involving the rank of the Mukai holomorphic 2-form on the moduli space of deformations of $\mathcal{F}$, the following result:

Proposition 1.3. *(O'Grady [13, Prop. 1.3]) If there is a H-stable torsion free sheaf $\mathcal{F}$ with $v = v(\mathcal{F})$, and the conclusion of Theorem 1.2 holds for the deformations of $\mathcal{F}$, then*

$$\{c_2(\mathcal{G}), \mathcal{G} \in \overline{\mathcal{M}^{st}}(X, H, v)\} = S_{d(v)}^k(X), \ k = \deg c_2(\mathcal{F}).$$

In this statement, $\overline{\mathcal{M}^{st}}(X, H, v)$ is any smooth completion of the moduli space of H-stable sheaves with Mukai vector v.

Our results in this paper are of two kinds: First of all we provide another description of $S_d^k(X)$ for any $d \geq 0$, $k \geq d$. In order to state this result, let us introduce the following notation: Given an integer $k \geq 0$, and a cycle $z \in CH_0(X)$ of degree k, the subset O_z of $X^{(k)}$ consisting of effective cycles $z' \in X^{(k)}$ which are rationally equivalent to z is a countable union of closed algebraic subsets of $X^{(k)}$ (see [15, Lemma 10.7]). This is the "effective orbit" of z under rational equivalence, and the analogue of $|D|$ for a divisor $D \in CH^1(W)$ on any variety W. We define $\dim O_z$ as the supremum of the dimensions of the components of O_z. This is the analogue of $r(D) = \dim |D|$ for a divisor $D \in CH^1(W)$ on any variety W. We will prove the following:

Theorem 1.4. *Let X be a projective K3 surface. Let $k \geq d \geq 0$. We have the following characterization of $S_d^k(X)$:*

if $k > d$, $S_d^k(X) = \{z \in CH_0(X), O_z \text{ nonempty, } \dim O_z \geq k - d\}$.

Remark 1.5. The inclusion $S_d^k(X) \subset \{z \in CH_0(X),\ O_z$ nonempty, $\dim O_z \geq k-d\}$ is easy since the cycle $(k-d)c_X$ has its orbit of dimension $\geq k-d$, (for example $C^{(k-d)} \subset X^{(k-d)}$, for any rational curve $C \subset X$, is contained in the orbit of $(k-d)c_X$). Hence any cycle of the form $z + (k-d)c_X$ with z effective of degree d has an orbit of dimension $\geq k-d$.

A particular case of the theorem above is the case where $d(v) = 0$. By definition $S_0(X)$ is the subgroup $\mathbf{Z}c_X \subset CH_0(X)$. We thus have:

Corollary 1.6. *For $k > 0$, the cycle kc_X is the unique 0-cycle z of degree k on X such that* $\dim O_z \geq k$.

Remark 1.7. We have in fact $\dim O_z = k$, $z = kc_X$ since by Mumford's theorem [11], any component L of O_z is Lagrangian for the holomorphic symplectic form on $S_{reg}^{(k)}$, hence of dimension $\leq k$ if L intersects $S_{reg}^{(k)}$. If L is contained in the singular locus of $S^{(k)}$, we can consider the minimal multiplicity-stratum of $S^{(k)}$ containing L, which is determined by the multiplicities n_i of the general cycle $\sum_i n_i x_i$, x_i distinct, parametrized by L) and apply the same argument.

Remark 1.8. We will give in Section 2 an alternative proof of Corollary 1.6, using the remark above, and the fact that any Lagrangian subvariety of X^k intersects a product $D_1 \times \ldots \times D_k$ of ample divisors on X.

Our main application of Theorem 1.4 is the following result which generalizes O'Grady's and Huybrechts' theorems 1.2, 1.1 in the case of simple vector bundles (instead of semistable torsion free sheaves). We do not need any of the assumptions appearing in Theorems 1.2, 1.1, but our results, unlike those of O'Grady, are restricted to the locally free case.

Theorem 1.9. *Let X be a projective $K3$ surface. Let F be a simple vector bundle on X with Mukai vector $v = v(F)$. Then*

$$c_2(F) \in S_{d(v)}(X).$$

A particular case of this statement is the case where $d = 0$: The corollary below proves Huybrechts' Theorem 1.1 without any assumption on the Picard group of the $K3$ surface or on the determinant of F. It is conjectured in [5].

Corollary 1.10. *Let F be a simple rigid vector bundle on a $K3$ surface. Then the element $c_2(F)$ of $CH_0(X)$ is a multiple of c_X.*

We also deduce the following corollary, in the same spirit (and with essentially the same proof) as Proposition 1.3:

Corollary 1.11. *Let $v \in H^*(X, \mathbf{Z})$ be a Mukai vector, with $k = c_2(v)$. Assume there exists a simple vector bundle F on X with Mukai vector v. Then*

$$S_d^k(X) = \{c_2(G),\ G \text{ a simple vector bundle on } X,\ v_G = v\},$$

where $k = c_2(v) := c_2(F)$.

These results answer for simple vector bundles on $K3$ surfaces questions asked by O'Grady (see [13, section 5]) for simple sheaves.

The paper is organized as follows: in Section 2, we prove Theorem 1.4. We also show a variant concerning family of subschemes (rather than 0-cycles) of given length in a constant rational equivalence class. In section 3, Theorem 1.9 and Corollary 1.11 are proved.

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J'ai grand plaisir à dédier cet article à Rob Lazarsfeld, avec toute mon estime et ma sympathie. Son merveilleux article [8] redémontrant un grand théorème classique sur les séries linéaires sur les courbes génériques a aussi joué un rôle décisif dans l'étude des fibrés vectoriels et des 0-cycles sur les surfaces K3.

2 An alternative description of O'Grady's filtration

This section is devoted to the proof of Theorem 1.4 which we state in the following form:

Theorem 2.1. *Let $k \geq d$ and let $Z \subset X^{(k)}$ be a Zariski closed irreducible algebraic subset of dimension $k - d$. Assume that all cycles of X parametrized by Z are rationally equivalent in X. Then the class of these cycles belongs to $S_d^k(X)$.*

We will need for the proof the following simple lemma, which already appears in [14].

Lemma 2.2. *Let X be a projective K3 surface and let $C \subset S$ be a (possibly singular) curve such that all points of C are rationally equivalent in X. Then any point of C is rationally equivalent to c_X.*

Proof. Let L be an ample line bundle on X. Then $c_1(L)_{|C}$ is a 0-cycle on C and our assumptions imply that $j_*(c_1(L)_{|C}) = \deg(c_1(L)_{|C})\, c$, for any point c of C.

On the other hand, we have

$$j_*(c_1(L)_{|C}) = c_1(L) \cdot C \text{ in } CH_0(X)$$

and thus, by [1], $j_*(c_1(L)_{|C}) = \deg(c_1(L)_{|C})\, c_X$ in $CH_0(X)$. Hence we have

$$\deg(c_1(L)_{|C})\, c = \deg(c_1(L)_{|C})\, c_X \text{ in } CH_0(X).$$

This concludes the proof, since c is arbitrary, $\deg(c_1(L)_{|C}) \neq 0$ and $CH_0(X)$ has no torsion.
$\square$

Lemma 2.3. *The union of curves C satisfying the property stated in Lemma 2.2 is Zariski dense in X.*

Proof. The 0-cycle c_X is represented by any point lying on a (singular) rational curve $C \subset X$ (see [1]), so the result is clear if one knows that there are infinitely many distinct rational curves contained in X. This result is to our knowledge known only for general K3 surfaces but not for all K3 surfaces (see however [4] for results in this direction). In any case, we can use the following argument which already appears in [7]: By [9], there is a 1-parameter family of (singular) elliptic curves E_t on X. Let C be a rational curve on X which meets the fibers E_t. For any integer N, and any point t, consider the points $y \in \widetilde{E}_t$ (the desingularization of E_t), which are rationally equivalent in $\widetilde{E}_t$ to the sum of a point $x_t \in E_t \cap C$ (hence rationally equivalent to c_X) and a N-torsion 0-cycle on $\widetilde{E}_t$.

As $CH_0(X)$ has no torsion, the images y_t of these points in X are all rationally equivalent to c_X in X. Their images are clearly parametrized for N large enough by a (maybe reducible) curve $C_N \subset X$. Finally, the union over all N of the points y_t above is Zariski dense in each $\widetilde{E}_t$, hence the union of the curves C_N is Zariski dense in X.
$\square$

Proof of Theorem 2.1. The proof is by induction on k, the case $k = 1$, $d = 0$ being Lemma 2.2 (the case $k = 1$, $d = 1$ is trivial). Let Z' be an irreducible component of the inverse image of Z in X^k. Let $p : Z' \to X$ be the first projection. We distinguish two cases and note that they exhaust all possibilities, up to replacing Z' by another component Z'' deduced from Z' by letting the symmetric group $\mathfrak{S}_k$ act.

Case 1. The morphism $p : Z' \to X$ is dominant. For a curve $C \subset X$ parametrizing points rationally equivalent to c_X, consider the hypersurface

$$Z_C' := p^{-1}(C) \subset Z'.$$

Let $q : Z' \to X^{k-1}$ be the projection on the product of the $k-1$ last factors. Assume first that $\dim q(Z_C') = \dim Z_C' = k - d - 1$. Note that all cycles of X parametrized by $q(Z_C')$

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are rationally equivalent in X. Indeed, an element z of Z'_C is of the form (c, z') with $c \in C$ so that $c = c_X$ in $CH_0(X)$. So the rational equivalence class of z' is equal to $z - c_X$ and is independent of $z' \in Z'_C$. Thus the induction assumption applies and the cycles of degree $k - 1$ parametrized by $\operatorname{Im} q$ belong to $S_d^{k-1}(X)$. It follows in turn that the classes of the cycles parametrized by Z' (or Z) belong to $S_d^k(X)$. Indeed, as just mentioned above, a 0-cycle z parametrized by Z' is rationally equivalent to $z = c_X + z'$ where $z' \in S_d^{k-1}(X)$, so z' is rationally equivalent to $(k - d - 1)c_X + z''$, $z'' \in X^{(d)}$. Hence z is rationally equivalent in X to $(k - d)c_X + z''$, for some $z'' \in X^{(d)}$. Thus $z \in S_d^k(X)$.

Assume to the contrary that $\dim q(Z'_C) < \dim Z'_C = k - d - 1$ for any curve C as above. We use now the fact (see Lemma 2.3) that these curves C are Zariski dense in X. We can thus assume that there is a point $x \in Z'_C$ which is generic in Z', so that both Z' and Z'_C are smooth at x, of respective dimensions $k - d$ and $k - d - 1$. The fact that $\dim q(Z'_C) < k - d - 1$ implies that q is not of maximal rank $k - d$ at x and as x is generic in Z', we conclude that q is of rank $< k - d$ everywhere on Z'_{reg}, so that $\dim \operatorname{Im} q \leq k - d - 1$.

Now recall that all 0-cycles parameterized by Z' are rationally equivalent. It follows that for any fiber F of q, all points in $p(F)$ are rationally equivalent in X. This implies that all these points are rationally equivalent to c_X by Lemma 2.2. This contradicts the fact that p is surjective.

Case 2. None of the projections pr_i, $i = 1, \ldots, k$, from X^k to its factors is dominant. Let $C_i := \operatorname{Im} p_i \subset X$ if $\operatorname{Im} pr_i$ is a curve, and any curve containing $\operatorname{Im} p_i$ if $\operatorname{Im} p_i$ is a point. Thus Z' is contained in $C_1 \times \ldots \times C_k$.

Let C be a non necessarily irreducible ample curve such that all points in C are rationally equivalent to c_X. Observe that the line bundle $pr_1^* \mathcal{O}_X(C) \otimes \ldots \otimes pr_k^* \mathcal{O}_X(C)$ on X^k has its restriction to $C_1 \times \ldots \times C_k$ ample and that its $k - d$-th self-intersection on $C_1 \times \ldots \times C_k$ is a complete intersection of ample divisors and is equal to

$$W := (k - d)! \sum_{i_1 < \ldots < i_{k-d}} p_{i_1}^* \mathcal{O}_{C_1}(C) \cdot \ldots \cdot p_{i_{k-d}}^* \mathcal{O}_{C_{k-d}}(C) \tag{2}$$

in $CH^{k-d}(C_1 \times \ldots \times C_k)$, where the p_i are the projections from $\prod_i C_i$ to its factors.

The cycle W of (2) is as well the restriction to $C_1 \times \ldots \times C_k$ of the effective cycle

$$W' := (k - d)! \sum_{i_1 < \ldots < i_{k-d}} pr_{i_1}^* C \cdot \ldots \cdot pr_{i_{k-d}}^* C. \tag{3}$$

As the $k - d$ dimensional subvariety Z' of $C_1 \times \ldots \times C_k$ has a nonzero intersection with W, it follows that the intersection number of Z' with W' is nonzero in X^k, hence that

$$Z' \cap pr_{i_1}^* C \cdot \ldots \cdot pr_{i_{k-d}}^* C \neq \emptyset$$

for some choice of indices $i_1 < \ldots < i_{k-d}$. This means that there exists a cycle in Z which is of the form

$$z = z' + z''$$

with $z' \in C^{(k-d)}$ and $z'' \in X^{(d)}$. As z' is supported on C, it is equal to $(k - d)c_X$ in $CH_0(X)$ and we conclude that $z \in S_d^k(X)$.

$\square$

Let us now prove the following variant of Theorem 2.1. Instead of a family of 0-cycles (that is, elements of $X^{(k)}$), we now consider families of 0-dimensional *subschemes* (that is, elements of $X^{[k]}$) :

Variant 2.4. *Let $k \geq d$ and let $Z \subset X^{[k]}$ be a Zariski closed irreducible algebraic subset of dimension $k - d$. Assume that all cycles of X parametrized by Z are rationally equivalent in X. Then the class of these cycles belongs to $S_d^k(X)$.*

Proof. Let $z \in Z$ be a general point. The cycle $c(z)$ of z, where $c : X^{[k]} \to X^{(k)}$ is the Hilbert-Chow morphism , is of the form $\sum_i k_i x_i$, with $\sum_i k_i = k$, where x_i are k' distinct points of X. We have of course

$$k' = k - \sum_i (k_i - 1). \tag{4}$$

The fiber of c over a cycle of the form $\sum_i k_i x_i$ as above is of dimension $\sum_i (k_i - 1)$ (see for example [2]). It follows that the image Z_1 of Z in $X^{(k)}$ is of dimension $\geq k - d - \sum_i (k_i - 1)$. By definition, Z_1 is contained in a multiplicity-stratum of $X^{(k)}$ where the support of the considered cycles has cardinality $\leq k'$. Let $Z_1' \subset X^{k'}$ be the set of $(x_1, \ldots, x_{k'})$ such that $\sum_i k_i x_i \in c(Z)$. Then the morphism

$$Z_1' \to Z_1, \ (x_1, \ldots, x_{k'}) \mapsto \sum_i k_i x_i$$

is finite and surjective, so that

$$\dim Z_1' = \dim Z_1 \geq k - d - \sum_i (k_i - 1), \tag{5}$$

which by (4) can be rewritten as

$$\dim Z_1' = \dim Z_1 \geq k' - d.$$

Note that by construction, Z_1' parameterizes k'-uples $(x_1, \ldots, x_{k'})$ with the property that $\sum_i k_i x_i$ is rationally equivalent to a constant cycle.

The proof of the variant 2.4 then concludes with the following statement :

Proposition 2.5. *Let l be a positive integer, $k_1 > 0, \ldots, k_l > 0$ be positive multiplicities. Let Z be a closed algebraic subset of X^l. Assume that $\dim Z \geq l - d$ and the cycles $\sum_i k_i x_i$, $(x_1, \ldots, x_l) \in Z$, are all rationally equivalent in X. Then the class of the cycles $\sum_i k_i x_i$, $(x_1, \ldots, x_l) \in Z$, belongs to $S_d^k(X)$, where $k = \sum_i k_i$.*

$\square$

For the proof of Proposition 2.5, we have to start with the following Lemma:

Lemma 2.6. *Let $x_1, \ldots, x_d \in X$ and let $k_i \in \mathbf{Z}$. Then $\sum_i k_i x_i \in S_d^k(X)$, $k = \sum_i k_i$.*

Proof. We use the following characterization of $S_d(X)$ given by O'Grady:

Proposition 2.7. *(O'Grady [13]) A cycle $z \in CH_0(X)$ belongs to $S_d(X)$ if and only if there exists a (possibly singular, possibly reducible) curve $j : C \subset X$, such that the genus of the desingularization of C (or the sum of the genera of its components if C is reducible) is non greater than d and z belongs to $\mathrm{Im}\,(j_* : CH_0(C) \to CH_0(X))$.*

Let now $x_1, \ldots, x_d$ be as above. There exists by [9] a curve $C \subset X$, whose desingularization has genus $\leq d$ and containing $x_1, \ldots, x_d$. Thus for any k_i, the cycle $\sum_i k_i x_i$ is supported on C, which proves the Lemma by Proposition 2.7 $\square$

Proof of Proposition 2.5. Proposition 2.5 is proved exactly as Theorem2.1, by induction on l. In case 1 considered in the induction step, we apply the same argument as in that proof. In case 2 considered in the induction step, using the same notations as in that proof, we conclude that there is in Z a l-uple $(x_1, \ldots, x_l)$ satisfying (up to permutation of the indices)

$$x_{d+1} \ldots, x_l \in C,$$

and as any point of C is rationally equivalent to c_X, we find that

$$\sum_i k_i x_i = (\sum_{i>d} k_i) c_X + \sum_{l \leq i \leq d} k_i x_i.$$

By Lemma 2.6, $\sum_{1 \leq i \leq d} k_i x_i \in S_d(X)$, so that $\sum_i k_i x_i \in S_d(X)$.

$\square$

As mentioned in the introduction, Theorem 2.1 in the case $d = 0$ provides the following characterization of the cycle kc_X, $k > 0$: It is the only degree k 0-cycle z of X, whose orbit $O_z \subset X^{(k)}$ is k-dimensional (cf. Corollary 1.6). Let us give a slightly more direct proof in this case. We use the following Lemma 2.8: Let V be a 2-dimensional complex vector space. Let $\eta \in \bigwedge^2 V^*$ be a nonzero generator, and let $\omega \in \bigwedge_{\mathbf{R}}^{1,1}(V^*)$ be a positive real $(1,1)$-form on V.

Lemma 2.8. *Let $W \subset V^k$ be a k-dimensional complex vector subspace which is Lagrangian for the nondegenerate 2-form $\eta_k := \sum_i pr_i^* \eta$ on V^k, where the pr_i's are the projections from V^k to V. Then $\prod_i pr_i^* \omega$ restricts to a volume form on W.*

Proof. The proof is by induction on k. Let $\pi : W \to V^{k-1}$ be the projector on the product of the last $k - 1$ summands. We can clearly assume up to changing the order of factors, that $\dim \operatorname{Ker} \pi < 2$. As $\dim \operatorname{Ker} \pi \leq 1$, we can choose a linear form μ on V such that the $k - 1$-dimensional vector space $W_\mu := \ker pr_1^* \mu_{|W}$ is sent injectively by π to a $k - 1$-dimensional subspace W' of V^{k-1}. Furthermore, since W is Lagrangian for η_k, W' is Lagrangian for η_{k-1} because $W_\mu \subset \operatorname{Ker} \mu \times V^{k-1}$, and on $\operatorname{Ker} \mu \times V^{k-1}$, $\eta_k = \pi^* \eta_{k-1}$. By the induction hypothesis, the form $\prod_{i>1} pr_i^* \omega$ restricts to a volume form on W', where the projections here are considered as restricted to $0 \times V^{k-1}$, and it follows that

$$pr_1^*(\sqrt{-1}\mu \wedge \overline{\mu}) \wedge \prod_{i>1} pr_i^* \omega$$

restricts to a volume form on W. It immediately follows that $\prod_{i \geq 1} pr_i^* \omega$ restricts to a volume form on W since for a positive number α, we have

$$\omega \geq \alpha \sqrt{-1}\mu \wedge \overline{\mu}$$

as real $(1,1)$-forms on V.

$\square$

Proof of Corollary 1.6. Let $z \in CH_0(X)$ be a cycle of degree k such that $\dim O_z \geq k$. Let $\Gamma \subset X^k$ be an irreducible component of the inverse image of a k-dimensional component of $O_z \subset X^{(k)}$ via the map $X^k \to X^{(k)}$. By Mumford's theorem [11], using the fact that all the 0-cycles parameterized by Γ are rationally equivalent in X, Γ is Lagrangian for the symplectic form $\sum_i pr_i^* \eta_X$ on X^k, where $\eta_X \in H^{2,0}(X)$ is a generator. Let L be an ample line bundle on X such that there is a curve $D \subset X$ in the linear system $|L|$, all of whose components are rational. We claim that

$$\Gamma \cap D^k \neq \emptyset.$$

Indeed, it suffices to prove that the intersection number

$$[\Gamma] \cdot [D^k] \tag{6}$$

is positive. Let $\omega_L \in H^{1,1}(X)$ be a positive representative of $c_1(L)$. Then (6) is equal to

$$\int_{\Gamma_{reg}} \prod_i pr_i^* \omega_L. \tag{7}$$

By Lemma 2.8, the form $\prod_i pr_i^* \omega_L$ restricts to a volume form on Γ at any smooth point of Γ and the integral (7) is thus positive. $\square$

3 Second Chern class of simple vector bundles

This section is devoted to the proof of Theorem 1.9. Recall first from [13], that in order to prove the result for a vector bundle F on X, it suffices to prove it for $F \otimes L$, where L is a line bundle on X. Choosing L sufficiently ample, we can thus assume that F is generated by global sections, and furthermore that

$$H^1(X, F^*) = 0. \tag{8}$$

Let $r = \operatorname{rank} F$. Choose a general $r-1$-dimensional subspace W of $H^0(X, F)$, and consider the evaluation morphism:

$$e_W : W \otimes \mathcal{O}_X \to F.$$

The following result is well-known (cf. [6, 5.1]).

Lemma 3.1. *The morphism e_W is generically injective, and the locus $Z \subset X$ where its rank is $< r-1$ consists of k distinct reduced points, where $k = c_2^{top}(F)$.*

Proof. Let $G = Grass(r-1, H^0(X, F))$ be the Grassmannian of $r-1$-dimensional subspaces of $H^0(X, F)$. Consider the following universal subvariety of $G \times X$:

$$G_{deg} := \{(W, x) \in G \times X, \operatorname{rank} e_{W,x} < r-1\}.$$

Since F is generated by sections, G_{deg} is a fibration over X, with fibers smooth away from the singular locus

$$G_{deg}^{sing} := \{(W, x) \in G \times X, \operatorname{rank} e_{W,x} < r-2\}.$$

Furthermore, we have

$$\dim G_{deg} = \dim (G \times X) - 2 = \dim G$$

and $\dim G_{deg}^{sing} < \dim G$.

Consider the first projection: $p_1 : G_{deg} \to G$. It follows from the observations above and from Sard's theorem that for general $W \in G$, $p_1^{-1}(W)$ avoids G_{deg}^{sing} and consists of finitely many reduced points in X. The statement concerning the number k of points follows from [3, 14.3], or from the following argument that we will need later on: Given a W such that the morphism e_W is generically injective, and the locus Z_W where its rank is $< r-1$ consists of k distinct reduced points, we have an exact sequence

$$0 \to W \otimes \mathcal{O}_X \to F \to \mathcal{I}_{Z_W} \otimes \mathcal{L} \to 0, \tag{9}$$

where $\mathcal{L} = \det F$. Hence $c_2(F) = c_2(\mathcal{I}_Z \otimes \mathcal{L}) = c_2(\mathcal{I}_Z) = Z$, and in particular $c_2^{top}(F) = \deg Z$. This proves the lemma. $\qquad\square$

By Lemma 3.1, we have a rational map

$$\phi : G \dashrightarrow X^{(k)}, \ W \mapsto c(Z_W),$$

where $c : X^{[k]} \to X^{(k)}$ is the Hilbert-Chow morphism.

Proposition 3.2. *If F is simple and satisfies the assumption (8), the rational map ϕ is generically one to one on its image.*

Proof. Let $G^0 \subset G$ be the Zariski open set parameterizing the subspaces $W \subset H^0(X, F)$ of dimension $r-1$ satisfying the conclusions of Lemma 3.1. Note that c is a local isomorphism at a point Z_W of $X^{[k]}$ consisting of k distinct points, so that the dimension of the image of ϕ is equal to the dimension of the image of the rational map $G \dashrightarrow X^{[k]}$, $W \mapsto Z_W$, which we will also denote by ϕ. This ϕ is a morphism on G^0 and it suffices to show that the map $\phi^0 := \phi_{|G^0}$ is injective. Let $W \in G^0$, $Z := \phi(W)$. For any $W' \in \phi^{0-1}(Z)$, we have an exact sequence as in (9):

$$0 \to W' \otimes \mathcal{O}_X \to F \to \mathcal{I}_Z \otimes \mathcal{L} \to 0, \tag{10}$$

so that W' determines a morphism

$$t_{W'} : F \to \mathcal{I}_Z \otimes \mathcal{L},$$

and conversely, we recover W' from the data of $t_{W'}$ up to a scalar as the space of sections of $\operatorname{Ker} t_{W'} \subset F$. We thus have an injection of the fiber $\phi^{0-1}(Z)$ into $\mathbf{P}(\operatorname{Hom}(F, \mathcal{I}_Z \otimes \mathcal{L}))$.

In order to compute $\operatorname{Hom}(F, \mathcal{I}_Z \otimes \mathcal{L})$, we tensor by F^* the exact sequence (9). We then get the long exact sequence:

$$\ldots \to \operatorname{Hom}(F, F) \to \operatorname{Hom}(F, \mathcal{I}_Z \otimes \mathcal{L}) \to H^1(X, F^* \otimes W). \tag{11}$$

By the vanishing (8) we conclude that the map

$$\operatorname{Hom}(F, F) \to \operatorname{Hom}(F, \mathcal{I}_Z \otimes \mathcal{L})$$

is surjective. As F is simple, the left hand side is generated by Id_F, so the right hand side is generated by t_W. The fiber $\phi^{0-1}(Z)$ thus consists of one point.

$\square$

Proof of Theorem 1.9. Let F be a simple nontrivial globally generated vector bundle of rank r, with $h^1(F) = 0$ and with Mukai vector

$$v = v_F = (r, l, s) \in H^*(X, \mathbf{Z}).$$

This means that $r = \operatorname{rank} F$, $l = c_1^{top}(F) \in H^2(X, \mathbf{Z})$ and

$$\chi(X, \operatorname{End} F) = < v, v^* > = 2rs - l^2. \tag{12}$$

The Riemann-Roch formula applied to $\operatorname{End} F$ gives

$$\chi(X, \operatorname{End} F) = 2r^2 + (r-1)l^2 - 2rc_2^{top}(F), \tag{13}$$

hence we get the formula (which can be derived as well from the definition (1)) :

$$s = r + \frac{l^2}{2} - c_2^{top}(F). \tag{14}$$

We also have by definition of $d(v)$

$$\chi(X, \operatorname{End} F) = 2 - 2d(v)$$

and thus by (12),

$$d(v) = 1 - rs + \frac{l^2}{2}. \tag{15}$$

The Riemann-Roch formula applied to F gives on the other hand:

$$\chi(X, F) = 2r + \frac{l^2}{2} - c_2^{top}(F) \tag{16}$$

which by (14) gives

$$\chi(X, F) = r + s. \tag{17}$$

As we assume $h^1(F) = 0$ and we have $h^2(F) = 0$ since F is nontrivial, generated by sections and simple, we thus get

$$h^0(X, F) = r + s. \tag{18}$$

9

With the notations introduced above, we conclude that

$$\dim G = (r-1)(s+1).$$

By Proposition 3.2, as all cycles parameterized by $\operatorname{Im}\phi$ are rationally equivalent in X, the orbit under rational equivalence of $c_2(F)$ in $X^{(k)}$, $k = c_2^{top}(F)$, has dimension greater than or equal to

$$(r-1)(s+1) = rs - s + r - 1.$$

But we have by (14) and (15):

$$k - d(v) = r - s + rs - 1.$$

By Theorem 2.1, we conclude that $c_2(F) \in S_{d(v)}^k(X)$.

$\square$

Remark 3.3. Instead of proving that the general Z_W is reduced and applying Theorem 2.1, we could as well apply directly the variant 2.4 to the family of subschemes Z_W.

For completeness, we conclude this section with the proof of Corollary 1.11, although a large part of it mimics the proof of Proposition 1.3 in [13].

We recall for convenience the statement:

Corollary 3.4. *Let $v \in H^*(X, \mathbf{Z})$ be a Mukai vector. Assume there exists a simple vector bundle F on X with Mukai vector v. Then*

$$S_d^k(X) = \{c_2(G),\ G \text{ a simple vector bundle on } X,\ v_G = v\},$$

where $d = d(v)$, $k = c_2(v) := c_2^{top}(F)$, $v_F = v$.

Proof. The inclusion

$$\{c_2(G),\ G \text{ a simple vector bundle on } X,\ v_G = v\} \subset S_d^k(X) \qquad (19)$$

is the content of Theorem 1.9.

For the reverse inclusion, we first prove that there exists a Zariski open set $U \subset X^{(d)}$ such that

$$cl(U) + (k - d(v))c_X \subset \{c_2(G),\ G \text{ a simple} \qquad (20)$$
$$\text{vector bundle on } X,\ v_G = v\}$$

where $cl : X^{(d)} \to CH_0(X)$ is the cycle map.

As F is simple, the local deformations of F are unobstructed. Hence there exist a smooth connected quasi-projective variety Y, a locally free sheaf $\mathcal{F}$ on $Y \times X$ and a point $y_0 \in Y$ such that $\mathcal{F}_{y_0} \cong F$ and the Kodaira-Spencer map

$$\rho : T_{Y,y_0} \to H^1(X, \operatorname{End} F)$$

is an isomorphism.

As $\mathcal{F}_{y_0}$ is simple, so is $\mathcal{F}_y$ for y in a dense Zariski open set of Y. Shrinking Y if necessary, $\mathcal{F}_y$ is simple for all $y \in Y$. By Theorem 1.9, we have $c_2(\mathcal{F}_y) \in S_{d(v)}(X)$ for all $y \in Y$.

Let $\Gamma := c_2(\mathcal{F}) \in CH^2(Y \times X)$. Consider the following set $R \subset Y \times X^{(d(v))}$

$$R = \{(y, z),\ \Gamma_*(y) = c_2(\mathcal{F}_y) = cl(z) + (k - d(v))c_X \text{ in } CH_0(X)\},$$

where $cl : X^{(d(v))} \to CH_0(X)$ is the cycle map and $k = c_2(v)$.

R is a countable union of closed algebraic subsets of $Y \times X^{(d)}$ and by the above inclusion (19), the first projection

$$R \to Y$$

is surjective. By a Baire category argument, it follows that for some component $R_0 \subset R$, the first projection is dominant.

We claim that the second projection $R_0 \to X^{(d(v))}$ is also dominant. This follows from the fact that by Mumford's theorem, the pull-back to R_0 of the holomorphic 2-forms on Y and $X^{(d(v))}$ are equal. As the first projection is dominant and the Mukai form on Y has rank $2d(v)$, the same is true for its pull-back to R_0 (or rather its smooth locus). Hence the pull-back to R_0 of the symplectic form on $X^{(d(v))}$ by the second projection also has rank $2d(v)$. This implies that the second projection is dominant hence that its image contains a Zariski open set. Thus (20) is proved. The proof of Corollary 3.4 is then concluded with Lemma 3.5 below.

$\square$

Lemma 3.5. *Let X be a K3 surface and $d > 0$ be an integer. Then for any open set (in the analytic or Zariski topology) $U \subset X^{(d)}$, we have*

$$cl(U) = cl(X^{(d)}) \subset CH_0(X).$$

Proof. It clearly suffices to prove the result for $d = 1$. It is proved in [7] that for any point $x \in X$, the set of points $y \in X$ rationally equivalent to x in X is dense in X for the usual topology. This set thus meets U, so that $x \in cl(U)$.

$\square$

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