CONSTRUCTION OF DISCRETE SERIES FOR CLASSICAL p-ADIC GROUPS

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Introduction

The goal of this paper is to complete (after [M2]) the classification of irreducible square integrable representations of classical p-adic groups, under the assumption that one basic assumption holds (see below). This classification also implies a parameterization of irreducible tempered representations of these groups. Therefore, it implies a classification of the non-unitary duals of these groups.

The classical groups whose classification of irreducible square integrable representations we give, are symplectic, orthogonal and unitary groups over a non-archimedean local field F. For simplicity, in this introduction we shall explain the classification in the case of symplectic and odd-orthogonal groups (in the case of unitary groups, the Galois interpretation of the classification is substantially more complicated).

Denote by G a symplectic or odd-orthogonal group on the space (of the corresponding type) of dimension $2n$ or $2n+1$ respectively. The classification of irreducible square integrable representations is directly related to the Langland's classification of irreducible square integrable representations in terms of dual objects. These objects consist of two parts. The first is a semi-simple morphism

$$\phi \ : \ W_F \times SL(2,\mathbb{C}) \ \rightarrow \ {}^LG,$$

where ${}^LG$ is the dual group of G (in the case that we consider, even for a non-split orthogonal group, we can replace ${}^LG$ by its connected component; see [M3]). The morphism ϕ has to be algebraic and discrete (by discrete we mean that it does not factor through a proper Levi factor). The second part is a morphism

$$\epsilon : \mathrm{Cent}_{{}^LG}\phi \rightarrow \{\pm 1\},$$

with the following restriction to the center of ${}^LG$ if G is odd-orthogonal: the restriction of ϵ to the center of ${}^LG$ is trivial if the anisotropic kernel has dimension 1, and is -1 if it has dimension 3. In fact, to take a complete care of the anisotropic kernel, we need to require a condition on $\det(\phi)$, which is not now important to us. It is explained in [M3] in the even case. The condition on ϵ and $\det(\phi)$ enables us to avoid to require in the definition of being discrete, that the Levi factor is defined over F (the restrictions that we shall consider are restrictions of both ϕ and ϵ, and they must also to satisfy the above properties, because the anisotropic kernel does not change.)

A simple form of the Langlands classification of irreducible square integrable representations would be a bijection between the set of (all isomorphism classes of) irreducible square integrable representations of G and conjugacy classes of pairs (ϕ, ϵ) as above. This parameterization has to satisfy a number of properties to be unique (these properties concern liftings and local harmonic analysis). Our elementary method does not give information in that direction. Further, although our classification is on the line of the Langlands classification of irreducible square integrable representations, it does not prove that such a simple form of the Langlands classification of irreducible square integrable representations exists. The problem is caused by the cuspidal representations, as we shall explain later. A pair (ϕ, ϵ), which should correspond to a cuspidal representation, should have a description in terms of cuspidal sheaves à la Lusztig. In our case, we can give a much more elementary description of such a pair.

First of all, it is easy to describe $\mathrm{Cent}_{{}^LG}(\phi)$, specially in the discrete case. Consider the decomposition of ϕ into irreducible components:

$$\phi \;=\; \underset{(\rho,a)}{\oplus}\; \rho\otimes E_a.$$

This decomposition must be multiplicity free (this follows from the property that ϕ is discrete). In the above direct sum, ρ are irreducible representations of W_F (necessarily orthogonal or symplectic; this follows again from the property that ϕ is discrete) and E_a is the irreducible (complex) algebraic representation of $SL(2,\mathbb{C})$ of the dimension $a \in \mathbb{N}$. The parity of a is uniquely determined by ρ and G. Denote by

$$Jord(\phi)$$

the set of all indexes (ρ, a) in the above direct sum decomposition. In particular, for $(\rho, a) \in Jord(\phi)$, $\rho \otimes E_a$ factors through an orthogonal subgroup of ${}^LG$ if G is symplectic, and symplectic subgroup if G is orthogonal. In any case, this subgroup has a center isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and further

$$\mathrm{Cent}_{{}^LG}(\phi) \;\simeq\; \prod_{(\rho,a)\in Jord(\phi)} (\mathbb{Z}/2\mathbb{Z}),$$

where one needs to take elements of determinant one if G is symplectic (in the orthogonal case, we have no restrictions).

Now we shall consider certain simple morphisms $\epsilon : Jord(\phi) \to \{\pm 1\}$. It is convenient to define

$$Jord^+(\phi) := Jord(\phi) \cup \{(\rho, 0);\ \text{there exists } a \in 2\mathbb{N} \ \text{ such that } (\rho, a) \in Jord(\phi)\}.$$

We shall extend ϵ as above to ϵ^+ on $Jord^+(\phi)$ by defining $\epsilon^+(\rho, 0) = 1$. We say that (ϕ, ϵ) is cuspidal if and only if, for any $(\rho, a) \in Jord^+(\phi)$ such that $a \geq 2$, we have $(\rho, a-2) \in Jord^+(\phi)$ and

$$\epsilon(\rho, a) \neq \epsilon^+(\rho, a-2).$$

Cuspidal pairs should correspond to cuspidal representations. The problem with the cuspidal case is that we have not a lot of evidence that the set of all equivalence classes of irreducible cuspidal representations is in a bijection with the set of all conjugacy classes of the cuspidal pairs (ϕ, ϵ). A case which is well understood, is the case where all ρ in $Jord(\phi)$ are quadratic characters (see [M3]). But we can expect progress in the level 0 case.

To avoid a hypothesis regarding the cuspidal case, we proceed in the following way. On the side of the irreducible representations of G, we have the notion of the cuspidal support. A weaker notion of the cuspidal support is the notion of a partial cuspidal support. The definition is following. Let π be an irreducible representation of G. Denote by π_{cusp} the unique irreducible cuspidal representation of a subgroup H of G such that, for a suitable integer k, $GL(k,F) \times H$ is a Levi factor of a parabolic subgroup P of G and there exists a representation τ of $GL(k,F)$ such that π is a subquotient of the parabolically induced representation $\tau \rtimes \pi_{cusp} = \mathrm{Ind}_P^G(\tau \rtimes \pi_{cusp})$.

The partial cuspidal support has an analogue for the pair (ϕ, ϵ). A combinatorial exercise (see section 14) implies that for each fixed (ϕ, ϵ), there exists a unique cuspidal pair $(\phi_{\phi,\epsilon,cusp}, \epsilon_{\phi,\epsilon,cusp})$ related to some subgroup H of G as above, which satisfies one of the following conditions. We consider two possibilities.

We shall say that (ϕ, ϵ) is alternated if $\epsilon(\rho, a_-) \neq \epsilon(\rho, a)$ for each pair $(\rho, a_-), (\rho, a) \in Jord(\phi)$, $a_- < a$, which satisfies: $(\rho, b) \notin Jord(\phi)$ for any $a_- < b < a$. If (ϕ, ϵ) is alternated, the cuspidal support $(\phi_{\phi,\epsilon,cusp}, \epsilon_{\phi,\epsilon,cusp})$ needs to satisfy

(1) There exists a bijection

$$\psi : Jord(\phi) \to Jord(\phi_{\phi,\epsilon,cusp}) \text{ or } Jord^+(\phi_{\phi,\epsilon,cusp})$$

of the form:

$$\psi(\rho, a) = (\rho, \psi_\rho(a)),$$

where each ψ_ρ is monotone (i.e. preserves the ordering of $\mathbb{Z}_+$), and

$$\epsilon(\rho, a) = \epsilon_{\phi,\epsilon,cusp}(\rho, \psi_\rho(a)).$$

(The cuspidal condition imposes the choice in "or".)

Suppose that (ϕ, ϵ) is not alternated. Then there exist $(\rho, a_-), (\rho, a) \in Jord(\phi)$, $a_- < a$, satisfying the following two properties

$$\text{if } a_- < b < a, \text{ then } (\rho, b) \notin Jord(\phi);$$
$$\epsilon(\rho, a_-) = \epsilon(\rho, a).$$

Define

$$\phi_1 \ := \bigoplus_{(\rho',a') \in Jord(\phi) \setminus \{(\rho,a_-),(\rho,a)\}} \rho' \times E_{a'}.$$

This is a parameter for a subgroup G_1 of G of the same type. There is an obvious way to define a restriction ϵ_1 of ϵ corresponding to a group G_1. Applying this construction several times, we shall come from (ϕ, ϵ) in a finitely many steps to some (ϕ', ϵ') of alternated type.

(2) We define $(\phi_{\phi,\epsilon,cusp}, \epsilon_{\phi,\epsilon,cusp})$ to be the cuspidal support of (ϕ', ϵ') (as it is defined in (1)).

The definition of $\phi_{\phi,\epsilon,cusp}$ depends on both ϕ and ϵ. The fibers of the mapping $(\phi,\epsilon)\mapsto(\phi_{\phi,\epsilon,cusp},\epsilon_{\phi,\epsilon,cusp})$ have a following simple property. To any (ϕ,ϵ), we can associate (ϕ,Δ_ϵ) where Δ_ϵ is the morphism:

$$Jord^+(\phi)\times Jord^+(\phi)\to\{\pm 1\},$$

defined only on the pairs $(\rho,a),(\rho',a')$ for which $\rho=\rho'$. On such a pair, Δ_ϵ is defined by

$$\Delta_\epsilon((\rho,a),(\rho,a'))=\epsilon(\rho,a)\epsilon(\rho,a')^{-1}.$$

An easy combinatorial exercise (see section 14) shows that the fibers of the mapping

$$(\phi,\epsilon)\mapsto(\phi,\Delta_\epsilon)$$

are subsets of the fibers of the mapping

$$(\phi,\epsilon)\mapsto\phi_{\phi,\epsilon,cusp}.$$

Now we can explain our classification of (equivalence classes of) irreducible square integrable representations of G. Recall that in [M2] is associated (using elementary techniques) to each irreducible square integrable representation π of G a set $Jord(\pi)$ (see section 2) of a similar type as $Jord(\phi)$. We use the local Langlands correspondence for general linear groups to identify these two type of sets ($Jord(\pi)$'s and $Jord(\phi)$'s). There is still a problem with the parity: parity of a for $(\rho,a)\in Jord(\phi)$ depends on G, and on the fact if ρ is orthogonal or symplectic. On the other side, the parity of a for $(\rho,a)\in Jord(\pi)$ depends on G, and on the pole of certain L-function. Conjecturaly, the pole (or holomorphy) reflects exactly the orthogonality or the symplecticity property. This is true on the Galois side, but it is not yet proved that the Langlands correspondence preserves this property. In this introduction, we shall assume this property. In the rest of the paper, we don't use the L-group explicitly, and therefore we do not (need to) assume this. In [M2], to π is associated a mapping Δ_π of the same type as the Δ_ϵ's which we considered above (actually, in [M2] we use the notation ϵ_π instead of Δ_π; it would be more consistent if we had used Δ_π there). With our elementary techniques, we are not able to prove the dimension relation:

$$\sum_{(\rho,a)\in Jord(\pi)} a\dim\rho=\begin{cases}2n, & G=SO(2n+1);\\ 2n+1, & G=Sp(2n);\end{cases}$$

(we do not use this equality in the paper, except here in the introduction). This dimension property is proved only if we assume some Arthur's conjectures to hold. Therefore, we can take this dimension equality as a hypothesis (in the introduction).

The principal result of [M2] is the following: assuming the basic assumption (which will be discussed later), the mapping

$$\pi\mapsto(\pi_{cusp},Jord(\pi),\Delta_\pi)$$

which is defined on the set of all equivalence classes of irreducible square integrable representations of G is injective, and it has the property that

$$Jord(\pi_{cusp}) = Jord(\phi_{\phi,\epsilon,cusp})$$

for any (ϕ, ϵ) such that $Jord(\phi) = Jord(\pi)$ and $\Delta_\epsilon = \Delta_\pi$ (we use the identification of $Jord(\phi)$'s and $Jord(\pi)$'s resulting from the local Langlands correspondences for general linear groups). The last assertion is the admissibility condition from the introduction of [M2].

The aim of this paper, is to prove that the above mapping $\pi \mapsto (\pi_{cusp}, Jord(\pi), \Delta_\pi)$ is surjective. This implies that we have proved that the equivalence classes of irreducible square integrable representations of G are classified by the triples

$$(\pi_{cusp}, \phi, \Delta_\epsilon).$$

In a triple, π_{cusp} is a cuspidal representation of a subgroup of G of the same type (symplectic or orthogonal). Further, ϕ and Δ_ϵ are coming in the way that we have explained above, from a pair ϕ, ϵ which satisfies the following two conditions

$$\sum_{(\rho,a)\in Jord(\phi)} a \dim \rho = \begin{cases} 2n, & G = SO(2n+1); \\ 2n+1, & G = Sp(2n); \end{cases}$$

and

$$Jord(\pi_{cusp}) = Jord(\phi_{\phi,\epsilon,cusp}).$$

The property that ϕ is discrete has a simple translation: $Jord(\phi)$ is multiplicity free.

The surjectivity is proved under the basic assumption (BA) (see the second section), which describes the reducibility points of a representation induced by an irreducible cuspidal representation, in terms of the Jordan bloc. This basic assumption was supposed to hold in [M2]. In [M1] it is proved that (BA) follows from a weak form of the Arthur's conjectures. Our basic assumption provides us with a way to compute Plancherel measures (modulo holomorphic invertible functions) in terms of Jordan blocs. This point of view was already present in [Sh1]. There F. Shahidi proves (BA) for generic cuspidal representations. Therefore, if π_{cusp} is generic, we do not need to assume in our paper (BA) that holds (i.e. our paper has no hypothesis if π_{cusp} is generic).

In fact, in this paper, we avoid to use the dual group in order to minimize the assumptions. We proceed much more technically (and more directly). As it is explained above, our classification starts from cuspidal representations. We classify all irreducible square integrable subquotients of the (generalized) principal series (i.e. all non-cuspidal irreducible square integrable representations). A generalized principal series is an induced representation:

$$\left(\prod_{k\in\mathcal{K}} \nu^{z_k}\rho_k\right) \rtimes \pi_{cusp},$$

where $\mathcal{K}$ is a set of indexes, and for each $k \in \mathcal{K}$, ρ_k is an irreducible unitarizable representation of some general linear group, ν is $|\det|_F$ and $z_k \in \mathbb{R}$. Further, π_{cusp} is an irreducible cuspidal representation of some subgroup H of G of the same type (see the first section for details regarding the notation). We do the classification using only a (natural) assumption about reducibility points of parabolically induced representations of the following type:

$$\nu^x \rho_k \rtimes \pi_{cusp},$$

where $k \in \mathcal{K}$ and $x \in \mathbb{R}$. Our basic assumption is that this induced representation is irreducible except when one of the following two possibilities occur:

(i) $x = 0$ or $\pm 1/2$, $(\rho_k, a) \notin Jord(\pi_{cusp})$ for all $a \in \mathbb{N}$ and $\nu^x \rho_k \rtimes 1$ reduces (here we assume for simplicity that the group is split);

(ii) $|x| \geq 1$ and $(\rho_k, 2|x| - 1) \in Jord(\pi_{cusp})$ but $(\rho_k, 2|x| + 1) \notin Jord(\pi_{cusp})$

(see section 2, and also section 12 for another interpretation). Moreover, we assume that a reducibility point as above, for fixed ρ_k, is unique up to a sign. G. Muić and F. Shahidi have explained us that this uniqueness follows from the work of Silberger [Si].

We shall now briefly describe the content of the paper. First, let us recall that in the paper we shall never use the L-group explicitly. Thus, there is no morphisms ϕ in the paper. It is replaced by its Jordan blocs, i.e. by a set $Jord$ satisfying a parity condition (and a dimension condition). This explains why ϕ's disappear in the rest of the paper. There is also a significant difference between the use of the notation ϵ in this introduction, and the use of this notation in the rest of the paper. In the rest, ϵ will denote the partial function which we have denoted by Δ_ϵ in this introduction. Nevertheless, in our description of the content of the paper, we continue with the notation which we have used above.

The first section introduces the notation. To simplify the exposition of the paper, after introducing notation for classical groups, we restrict ourselves until the end of section 14. to the case of symplectic and odd-orthogonal groups. The necessary modifications and comments regarding unitary and even-orthogonal groups are given in sections 15. and 16. The second section recalls in detail the basic assumption and the admissible triples. Following three sections collect preliminary results that we shall use in the proof of the square integrability. The sixth section states the main result of the paper. In the seventh section, we prove surjectivity for triples $(\pi_{cusp}, \phi, \Delta_\epsilon)$ in the case that ϕ, ϵ are alternated, i.e. satisfy condition (1) with respect to its partial cuspidal support (this is clearly a property of Δ_ϵ, it does not depend on a choice of ϵ). We call this the alternated case. The other case is called the mixed case. The proof in the mixed case proceeds by induction.

Suppose that π_{cusp}, $Jord$ (i.e. ϕ) and ϵ are fixed, and that

$$Jord(\pi_{cusp}) = Jord(\phi_{\phi,\epsilon,cusp}).$$

Let $(\rho, a_-), (\rho, a) \in Jord$, $a_- < a$, be such that $\epsilon(\rho, a_-) = \epsilon(\rho, a)$ and $(\rho, b) \notin Jord$ for $a_- < b < a$. Define ϕ_1, ϵ_1 (and G_1) in the same way as we did in the definition of $(\phi_{\phi,\epsilon,cusp}, \epsilon_{\phi,\epsilon,cusp})$. By the inductive assumption, we know that there exists an irreducible square integrable representation π_1 of G_1 corresponding to $(\pi_{cusp}, \phi_1, \Delta_{\epsilon_1})$. Denote by $\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho])$ the irreducible essentially square integrable representation of $GL(d_\rho(a + a_-)/2, F)$ (where ρ is a representation of $GL(d_\rho, F)$), which is a unique irreducible subrepresentation of the induced representation

$$\nu^{(a-1)/2}\rho \times \nu^{(a-1)/2-1} \times \cdots \times \nu^{-(a_- -1)/2}\rho.$$

It is proved in [M2] (and recalled here), that the induced representation

$$\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi_1$$

has exactly two irreducible subrepresentations, and that these subrepresentations are not isomorphic. For the proof of the surjectivity of the mapping $\pi \mapsto (\pi_{cusp}, Jord(\pi), \Delta_\pi)$, it is enough to prove that these two subrepresentation are square integrable. This is enough for the following reason. If we know that these subrepresentations are square integrable, then their parameters are computed in [M2]. They are extensions of ϕ_1, ϵ_1. Since there exist exactly two parameters which extend ϕ_1, ϵ_1 for G in this case, we get that that they correspond to these irreducible square integrable subrepresentations, and one of them must correspond to ϕ, ϵ.

The square integrability of these irreducible subrepresentations is the most difficult part of the paper. We solve first the case where π_1 is cuspidal in the ninth section (before, in the eighth section we prove a very useful lemma about tempered representations with the same infinitesimal characters). In section 10, we prove the square integrability in a case when all the Jacquet modules of π_1 have good properties relatively to a and a_- (this includes for example the case when a is the greatest element such that $(\rho, a) \in Jord$). Section 11 completes the proof, arguing with an inductive argument with respect to a.

In section 12, we discuss very briefly the basic assumption. Section 13 describes the irreducible tempered representations. Tempered representations can be also classified with triples as above. We have only to suppress the condition that $Jord$'s are multiplicity free. We have not developed this point of view here, which is under our basic assumption, a direct consequence of the Harish-Chandra's result on the intertwining algebras. Section 14 proves two exercises which we have mentioned earlier in this introduction, and gives simple examples of admissible triples. At the end, sections 15. and 16. give modifications and comments which are necessary that the classifications obtained in previous sections, also holds for unitary and even-orthogonal groups.

1. Notation

In this section, first we shall recall some notation for general linear groups (more details can be found in [Z]).

We fix a local non-archimedean field F. We do not assume any restriction on the characteristic, but the reader has to be aware that our basic assumption (BA) has only been verified in [M]) under the assumption of some Arthur's conjectures, which need the hypothesis that the characteristic of the field is 0.

By F' we shall denote in this paper either F or a separable quadratic extension of F. This will depend on the following: if we are working with symplectic or orthogonal groups, then F' will denote F, and if we are working with unitary groups, then F' will denote the separable quadratic extension which enters the definition of the unitary groups.

If F' is a separable quadratic extension, we shall denote by θ the non-trivial element of the Galois group $Gal(F'/F)$ of F' over F. Otherwise (i.e. if $F' = F$), θ will denote the identity mapping on F.

The modulus character of F' will be denoted by $| \ |_{F'}$.

By a representations of a reductive group G over F, we shall always mean in this paper a smooth representation. We denote by $\mathfrak{R}(G)$ the Grothendieck group of the category of all representations of G of finite length. There is a natural ordering $\leq$ on $\mathfrak{R}(G)$. These orderings determine also natural ordering on a direct sums of Grothendieck groups of categories, for different reductive groups. For a finite length representation π of G, we shall denote its semi simplification by s.s.(τ), and consider it as an element of $\mathfrak{R}(G)$. For two finite length representations π_1, π_2 of G, the fact s.s.$(\pi_1) \leq$ s.s.(π_2) we shall write shorter $\pi_1 \leq \pi_2$.

For two representations π_1, π_2, of general linear groups over F', Bernstein and Zelevinsky defined a parabolically induced representation $\pi_1 \times \pi_2$ in a natural way (see [Z]). Denote $R = \oplus_{n\geq 0}\mathfrak{R}(GL(n,F'))$. Then $\times$ factors in a natural way to $\times : R\times R \longrightarrow R$. The multiplication factors through $R\otimes R$ in a unique way by a map, which will be denoted by $m : R\times R \longrightarrow R$. Recall that Bernstein and Zelevinsky defined also a comultiplication $m^* : R \longrightarrow R\otimes R$ using Jacquet modules. With these two operations, R is a Hopf algebra.

The character $|\det|_{F'}$ of $GL(n,F')$ will be denoted by ν. For an irreducible essentially square integrable representation δ of $GL(n,F')$, there exists unique $e(\delta) \in \mathbb{R}$ and irreducible unitarizable square integrable representation δ^u such that

$$\delta = \nu^{e(\delta)}\delta^u.$$

Let ρ be an irreducible cuspidal representation of a general linear group (over F'). For $n \in \mathbb{Z}_+$, denote

$$[\rho, \nu^n\rho] = \{\rho, \nu\rho, \ldots, \nu^n\rho\}$$

($\mathbb{Z}_+$ denotes the set of all non-negative integers). We shall call $\Delta = [\rho, \nu^n\rho]$ a segment (in irreducible cuspidal representations of general linear groups). The representation $\rho\times\nu\rho\times \ldots\times\nu^n\rho$ has a unique irreducible quotient (resp. irreducible subrepresentation), which we denote by $\delta(\Delta)$ (resp. $\mathfrak{s}(\Delta)$). The representation $\delta(\Delta)$ is essentially square integrable. For $n < 0$, we shall assume $[\rho, \nu^n\rho]$ to be $\emptyset$. We take $\delta(\emptyset) = 1$ (identity in R, which is the trivial representation of $GL(0,F')$).

We shall recall now some notations for classical p-adic groups. More details regarding this notations can be found in [MViW] and [T5]. We shall consider the following series S_n of classical groups over F (and we shall fix one of these series of classical groups).

Symplectic groups $Sp(2n,F)$ form a series of groups where n denotes the split semisimple rank of the group. The group $Sp(2n,F)$ will be denoted by S_n. We have here the notion of the Witt tower (we shall denote by V_n the symplectic space of dimension $2n$ in this tower; we shall denote V_0 also by Y_0 in the case of symplectic groups).

In the case of odd orthogonal groups, we also have Witt tower: this means that we fix an anisotropic orthogonal vector space Y_0 over F of odd dimension dimension (1 or 3), and we look at the Witt tower based on Y_0. For each n such that $2n+1 \geq \dim Y_0$, there is exactly one space V_n in the Witt tower whose dimension is $2n+1$. We denote by S_n the special orthogonal group of this space (recall that in this case, the orthogonal group of V_n is a product of its center and S_n).

In the case of even-orthogonal groups, we fix an anisotropic orthogonal space Y_0 over F of even dimension, and consider the Witt tower based on Y_0. If $2n \geq \dim_F(Y_0)$, then there is exactly one space V_n in the Witt tower of dimension $2n$. The orthogonal group of V_n will be denoted by S_n.

In the case of symplectic or orthogonal series S_n of groups, F' will denote F.

In the case of unitary groups, F' will denote a separable quadratic extension of F. We shall fix an anisotropic unitary space Y_0 over F', and consider the Witt tower of unitary spaces V_n based on Y_0.

Suppose that $\dim_{F'}(Y_0)$ is odd (i.e. 1). Then for each $2n+1 \geq \dim_{F'}(Y_0)$, there exists a unique space V_n in the Witt tower of dimension $2n+1$. The unitary group of this space will be denoted by S_n.

If $\dim_{F'}(Y_0)$ is even, then for each $2n \geq \dim_{F'}(Y_0)$ one takes a unique space V_n in the Witt tower of dimension $2n$, and denote its unitary group by S_n.

We fix a minimal parabolic subgroup in S_n. We shall consider in this paper only standard parabolic subgroups, i.e. the parabolic subgroups which contain the fixed minimal parabolic subgroup. For more information regarding convenient matrix realizations of groups S_n and description of their standard parabolic subgroups, one can consult [T5].

Fix one of the series S_n of groups that we have defined. Let n' be the Witt index of V_n (n' is $n - 1/2\dim_{F'}(Y_0)$ if V_n is symplectic or even-orthogonal or even-unitary group, otherwise n' is $n - 1/2(\dim_{F'}(Y_0) - 1)$. For each $0 \leq k \leq n'$, there exists a standard parabolic subgroup $P_{(k)} = M_{(k)}N_{(k)}$ of S_n, whose Levi subgroup $M_{(k)}$ is naturally isomorphic to $GL(k, F') \times S_{n-k}$ (see [T5] and [B] for the series of symplectic and split orthogonal groups; the isomorphism in the case of other series of groups is defined in analogous way). This parabolic subgroup is the stabilizer of an isotropic space of dimension k. For a representations π and σ of $GL(k, F')$ and S_{n-k} respectively, the representation parabolically induced by $\pi \otimes \sigma$ is denoted by

$$\pi \rtimes \sigma.$$

A simple but very useful property of $\rtimes$ is that for two representations π_1 and π_2 of general linear groups (over F') we have

$$\pi_1 \rtimes (\pi_2 \rtimes) \cong (\pi_1 \times \pi_2) \rtimes \sigma$$

(see [T5]).

For π as above, denote by $\check{\pi}$

$$g \mapsto \tilde{\pi}(\theta(g)),$$

where $\tilde{\pi}$ denotes the contragredient representation of π. If π and σ are representations of finite length, then

$$\text{s.s.}(\pi \rtimes \sigma) = \text{s.s.}(\check{\pi} \rtimes \sigma).$$

In particular, if $\pi \rtimes \sigma$ is irreducible, then $\pi \rtimes \sigma \cong \check{\pi} \rtimes \sigma$.

We shall say that a representation π of a general linear group over F' is F'/F-selfdual if

$$\pi \cong \check{\pi}.$$

If $F' = F$, then we shall say also simply that π is selfdual.

For a representation τ of S_n, we denote by $s_{(k)}(\tau)$ the normalized Jacquet module of τ with respect to $P_{(k)}$. All the Jacquet modules that we shall consider in this paper, will be normalized Jacquet modules with respect to standard parabolic subgroups (and obvious Levi decompositions).

Denote by $R(S) = \oplus\mathfrak{R}(S_n)$, where the sum runs over all the groups from the series S_n of the classical groups that we have fixed (in other words, the sum runs over all integers $n \geq 1/2\,(\dim_{F'}(Y_0) - 1)$ if we consider odd-orthogonal or odd-unitary groups, and over all $n \geq 1/2\dim_{F'}(Y_0)$ otherwise). Then $\rtimes$ defines in a natural way a mapping $\rtimes : R \times R(S) \to R(S)$. Let

$$\mu^*(\tau) = \sum_{k=0}^{n'} \text{s.s.}\left(s_{(k)}(\tau)\right),$$

where τ is an irreducible representation of S_n (n' is the Witt index of V_n). Extend μ^* additively to $\mu^* : R(S) \to R \otimes R(S)$.

Let $M^* = (m\otimes 1)\circ(\,\check{}\,\otimes m^*)\circ\kappa\circ m^* : R \to R\otimes R$, where $\check{}\, : R \to R$ is a homomorphism defined by $\pi \mapsto \check{\pi}$ on irreducible representations, and $\kappa : R \times R \to R \times R$ maps $\sum x_i \otimes y_i$ to $\sum y_i \otimes x_i$. Note that $M^*(\pi)$ depends also on the series S_n of the groups that we have fixed.

For a finite length representation π of $GL(k, F')$, the component of $M^*(\pi)$ which is in $\mathfrak{R}(GL(k, F')) \otimes \mathfrak{R}(GL(0, F'))$, will be denoted by

$$M^*_{GL}(\pi) \otimes 1.$$

For a finite length representation τ of S_q, $\mu^*(\tau)$ will denote $\mu^*(\text{s.s.}(\tau))$. The similar convention we will be used for M^* and M^*_{GL}.

Let π be a representation of $GL(k, F')$ of finite length, and let σ be an irreducible cuspidal representation of S_n. Suppose that τ is a subquotient of $\pi \rtimes \sigma$. Then we shall denote $s_{(k)}(\tau)$ also by

$$s_{GL}(\tau).$$

In the sequel till the end of section 14., we shall assume that the series of groups S_n consists of symplectic or odd-orthogonal groups. The main reason for this restriction, is to simplify the exposition in these sections. In sections 15. and 16. we shall describe the case of unitary and even-orthogonal groups respectively.

Recall that for symplectic and orthogonal groups we have $F' = F$ and $\check{\pi} = \tilde{\pi}$. Nevertheless, we shall use also in sections 2.–14. the notations F' and $\check{\pi}$ (instead of F and π, which we could also use). The reason to do it, is that with such choice of the notation, sections 2.–14. will apply after a few comments also to unitary groups without any significant changes.

For $\pi \in R$ and $\sigma \in R(S)$ we have

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma). \tag{1-1}$$

For split groups S_n, this follows from Theorems 5.4 and 6.5 of [T5]. For a non-split odd-orthogonal series of groups S_n, we have the same root system of type B as in the split case, and the same Weyl group. Now in the same way as in section 6. of [T5], follows the above formula for non-split odd-orthogonal groups. More precisely, the calculations in section 4. of [T5] are independent of the groups (but depend on the root systems), and they imply (an analogue) of Lemma 5.1 for non-split odd-orthogonal groups. Now the formula (1-1) follows from this lemma in a completely same way as in [T5] Theorem 5.2 follows from Lemma 5.1, by a formal computations.*

Let π be a finite length representation of a general linear group, and let τ be a similar representation of S_n. Then (1-1) implies

$$\text{s.s.}(s_{GL}(\pi \rtimes \tau)) = M^*_{GL}(\pi) \times \text{s.s.}(s_{GL}(\tau)) \tag{1-2}$$

($\times$ in the above formula denotes multiplication in R of $M^*(\pi)$ with the factors on the left hand side of $\otimes$ in s.s.$(s_{GL}(\tau))$).

In this paper, we shall several times use formulas for $M^*(\delta(\Delta))$ and $M^*_{GL}(\delta(\Delta))$. This is the reason that we shall write these formulas here for the segments that we shall consider most often. Let $a_-, a \in \mathbb{N}$ ($\mathbb{N}$ denotes the positive integers). Suppose $a - a_- \in 2\mathbb{N}$. Let ρ be an irreducible F'/F-selfdual cuspidal representation of a general linear group. Then the formula for $m^*(\delta(\Delta))$ implies

$$M^*\left(\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho])\right) = \sum_{i=-(a_--1)/2-1}^{(a-1)/2} \sum_{j=i}^{(a-1)/2} \delta([\nu^{-i}\rho, \nu^{(a_--1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]). \tag{1-3}$$

To get M^*_{GL}, we need to take $j = i$ in the above formula. Thus

$$M^*_{GL}\left(\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho])\right) = \sum_{i=-(a_--1)/2-1}^{(a-1)/2} \delta([\nu^{-i}\rho, \nu^{(a_--1)/2}\rho]) \times \delta([\nu^{i+1}\rho, \nu^{(a-1)/2}\rho]). \tag{1-4}$$

Denote by γ the part of the sum (1-4) corresponding to indexes

$$-(a_- - 1)/2 - 1 \leq i \leq (a_- - 1)/2 \tag{1-5}$$

$$\big(\text{resp.} - (a_- - 1)/2 - 1 \leq i \leq (a_- - 1)/2 - 1\big). \tag{1-6}$$

Then γ satisfies the condition (3-9) (resp. (3-10)) of Lemma 3.5.

At the end, we have

$$M^*\left(\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho])\right) = \sum_{i'=-(a_--1)/2-1}^{(a_--1)/2} \sum_{j'=i'}^{(a_--1)/2} \delta([\nu^{-i'}\rho, \nu^{(a_--1)/2}\rho]) \times \delta([\nu^{j'+1}\rho, \nu^{(a_--1)/2}\rho]) \otimes \delta([\nu^{i'+1}\rho, \nu^{j'}\rho]). \tag{1-7}$$

Clearly, $M^*_{GL}\left(\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho])\right)$ satisfies the condition (3-9) of Lemma 3.5.

*The second author is thankful to T. Springer who informed him in 1997. about the paper "An application of Hopf-algebra techniques to representations of finite classical groups" of Marc A. A. van Leeuwen (Journal of Algebra 140(1991), 210-246). In this paper is computed formula (1-1) in the case of finite classical groups (Theorem 3.2.1). Note that the formula is simpler in the finite field case. The reason for it is the cocommutativity of the Hopf algebra attached to representations of finite general linear groups.

2. Invariants of square integrable representations of classical groups

In this section we shall recall of the invariants attached in [M2] to square integrable representations, and some of their properties.

Let $y \in \mathbb{N}$. If we are considering the series of groups $Sp(2n)$ (resp. $SO(2n+1)$), then R_y will denote the representation of $GL(y, \mathbb{C})$ on $\wedge^2 \mathbb{C}^y$ (resp. $\mathrm{Sym}^2 \mathbb{C}^y$).

To any irreducible square integrable representation π of S_n, [M2] associates three objects:

$$Jord(\pi), \pi_{cusp} \text{ and } \epsilon_\pi.$$

By definition, $(\rho, a) \in Jord(\pi_0)$ if and only if ρ is an irreducible F'/F-selfdual cuspidal representation of a general linear group $GL(d_\rho, F')$ (this defines d_ρ) and $a \in \mathbb{N}$ such that

$$\text{(J-1)} \qquad a \text{ is even if } L(\rho, R_{d_\rho}, s) \text{ has a pole at } s = 0, \text{ and odd otherwise,}$$

and

$$\text{(J-2)} \qquad \delta(\rho, a) \rtimes \pi_0 \text{ is irreducible,}$$

where $\delta(\rho, a)$ denotes $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho])$.

For an irreducible F'/F-selfdual cuspidal representation of a general linear group ρ, denote

$$Jord_\rho(\pi) = \{a; (\rho, a) \in Jord(\pi)\}.$$

Let π_{cusp} be an irreducible cuspidal representation of some S_q and let ρ be an irreducible F'/F-selfdual cuspidal representation of a general linear group. In this paper we shall assume that the following basic assumption holds:

(BA) $\quad \nu^{\pm(a_\rho+1)/2}\rho \rtimes \pi_{cusp}$ reduces for

$$a_\rho = \begin{cases} \max Jord_\rho(\pi_{cusp}) & \text{if } Jord_\rho(\pi_{cusp}) \neq \emptyset, \\ 0 & \text{if } L(\rho, R_{d_\rho}, s) \text{ has a pole at } s = 0 \text{ and } Jord_\rho(\pi_{cusp}) = \emptyset, \\ -1 & \text{otherwise;} \end{cases}$$

moreover, there are no other reducibility points in $\mathbb{R}$.

Note that a part of the assumption (BA) is that $Jord_\rho(\pi_{cusp})$ are finite sets. An additional comment regarding this conjecture can be found in the section 12.

The partial cuspidal support, π_{cusp} of π is defined to be the cuspidal irreducible representation of $S_{n'}$, with $n' \leq n$, such that there exists an irreducible representation σ of $GL(n-n', F')$ and an embedding

$$\pi \hookrightarrow \sigma \times \pi_{cusp}.$$

Note that the notion of partial cuspidal support is well defined in this way for any irreducible representation of S_n.

Let π be a an irreducible square integrable representation of S_n and let π_{cusp} be the partial support of π, where π_{cusp} is a cuspidal representation of some $S_{n'}$ ($n' \leq n$). We shall assume that the following formula holds:

$$\text{(A)} \qquad \sum_{(\rho,a)\in Jord(\pi_{cusp})} a\, d_\rho \ = \ \begin{cases} 2n' & \text{if } S_{n'} = SO(2n'+1); \\ 2n'+1 & \text{if } S_{n'} = Sp(2n). \end{cases}$$

This assumption is introduced only to have a simpler and more natural exposition in the paper. The main results of this paper does not depend on (A) (see Remark 14.5 for interpretations of the classification of irreducible square integrable representations without assuming that (A) holds). It is proved in [M2] (see [M2] 4.1 and 4.2) that

$$\text{(2-1)} \qquad \sum_{(\rho,a)\in Jord(\pi)} a\, d_\rho \ - \sum_{(\rho',a')\in Jord(\pi_{cusp})} a' d_{\rho'} \ = \ 2(n-n').$$

At the end, [M2] associates to π a partially defined function ϵ_π from $Jord(\pi_0)$ into $\{\pm 1\}$ (we shall explain below precisely what we mean by partially defined function). The definition of ϵ_π is in terms of Jacquet modules of π, except in one case, in which normalized intertwining operators are used (see [M2] for the definition of ϵ_π).

The first author proved in [M2] that $Jord(\pi_0), \pi_{cusp}, \epsilon_{\pi_0}$ form an admissible triple. A general definition of an admissible triple will be recalled below. The fundamental result of [M2] is that the mapping

$$\text{(2-2)} \qquad \pi \mapsto (Jord(\pi), \pi_{cusp}, \epsilon_\pi)$$

is an injective mapping from the set of all equivalence classes of irreducible square integrable representations of groups S_n into the set of all admissible triples (assuming that (BA) holds). Before we give the definition of an admissible triple, we shall give two remarks which are related to the admissibility condition.

The first one is a result which we shall use later, which tells that for an irreducible cuspidal representation π_{cusp} of S_n holds:

$$\text{(2-3)} \qquad (\rho, a) \in Jord(\pi_{cusp}) \text{ and } a > 2 \quad \Rightarrow \quad (\rho, a-2) \in Jord(\pi_{cusp}).$$

In [M1], this assertion and (BA) are obtained simultaneously as consequences of some Arthur's conjectures. One can check easily that (BA) implies (2-3) (one can find the proof of this implication in section 12). In fact, (2-3) follows from the fact that there is only one reducibility point in $\{x \in \mathbb{R}; x \geq 0\}$ in the case of the parabolic induction from a maximal parabolic subgroup by an irreducible cuspidal representation. We are sure that a proof of (BA) will give directly (2-3).

For an irreducible square integrable representation π of S_n define the multiset

$$\text{Supp}(Jord(\pi)) = \sum_{(\rho,a)\in Jord(\pi)} [\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]$$

(multiset means that elements are counted with multiplicity; see [Z]). It is evident how to reconstruct $Jord(\pi)$ from the multiset Supp($Jord(\pi)$). Now

$$\text{Supp}(Jord(\pi_{cusp})) \subseteq \text{Supp}(Jord(\pi)). \tag{2-4}$$

The above inclusion follows from the admissibility of the triple $(Jord(\pi), \pi_{cusp}, \epsilon_\pi)$.

Now we shall define the notion of an admissible triple.

Fix an irreducible cuspidal representation π_{cusp} of some S_q. Let $Jord$ be a finite set of pairs (ρ, a), where ρ are F'/F-selfdual irreducible cuspidal representations of general linear groups and $a \in \mathbb{N}$ is such that condition (J-1) holds for each of these pairs. We shall say that $Jord$ has degree n if (2-1) holds (the sum in (2-1) runs over $Jord$ instead of $Jord(\pi)$). We define the multiset Supp($Jord$) in the same way as we did for $Jord(\pi)$. For an F'/F-selfdual irreducible cuspidal representations ρ of a general linear group, one defines $Jord_\rho = \{a; (\rho, a) \in Jord\}$. For $a \in Jord_\rho$ we denote by

$$a_- = \max\{b \in Jord_\rho; b < a\},$$

if $\{b \in Jord_\rho; b < a\} \neq \emptyset$ (and then we say that a_- exists, or that it is defined). Otherwise, we shall say that a does not have a_-, or that a is minimal in $Jord_\rho$.

Let ϵ be a partially defined function from a subset of $Jord \cup (Jord \times Jord)$ into $\{\pm 1\}$ in the following sense. For $(\rho, a) \in Jord$, $\epsilon(\rho, a)$ is not defined if and only if a is odd and $(\rho, a') \in Jord(\pi_{cusp})$ for some $a' \in \mathbb{N}$. Further ϵ is defined on a pair $(\rho, a), (\rho', a') \in Jord$ if and only if $\rho = \rho'$ and $a \neq a'$.

We shall require that ϵ satisfies the following properties. Let $(\rho, a), (\rho, a') \in Jord, a \neq a'$. If $\epsilon(\rho, a)$ is defined, then ϵ on this pair has the value

$$\epsilon(\rho, a)\epsilon(\rho, a')^{-1}.$$

If $\epsilon(\rho, a)$ is not defined, then the value of ϵ on this pair we shall denote also formally by $\epsilon(\rho, a)\epsilon(\rho, a')^{-1}$ (although $\epsilon(\rho, a)$ and $\epsilon(\rho, a')$ are not defined). In this case (when $\epsilon(\rho, a)$ is not defined), for different $a, a', a'' \in Jord_\rho$ we require that

$$\epsilon(\rho, a)\epsilon(\rho, a'')^{-1} = \left(\epsilon(\rho, a)\epsilon(\rho, a')^{-1}\right)\left(\epsilon(\rho, a')\epsilon(\rho, a'')^{-1}\right)$$

(recall that the notation in the above formula is formal). In this case, we also require for different $a, a' \in Jord_\rho$:

$$\epsilon(\rho, a)\epsilon(\rho, a')^{-1} = \epsilon(\rho, a')\epsilon(\rho, a)^{-1}.$$

In the case that $\epsilon(\rho, a)\epsilon(\rho, a')^{-1} \neq 1$ (resp. $\epsilon(\rho, a)\epsilon(\rho, a')^{-1} = 1$), we shall write formally

$$\epsilon(\rho, a) \neq \epsilon(\rho, a') \quad (\text{resp. } \epsilon(\rho, a) = \epsilon(\rho, a')),$$

even if $\epsilon(\rho, a)$ and $\epsilon(\rho, a')$ are not defined.

Suppose that we have triple $Jord, \pi_{cusp}, \epsilon$ as above. Let $(\rho, a) \in Jord$ be such that $a_- \in Jord_\rho$ is defined and

$$\epsilon(\rho, a) = \epsilon(\rho, a_-).$$

Denote $Jord' = Jord \setminus \{a, a_-\}$. Let ϵ' be the partially define function on $Jord'$ which we get by restricting ϵ to $Jord'$. Then ϵ' again satisfies the above requirements with respect to $Jord'$ and π_{cusp}. If we have two triples $Jord, \pi_{cusp}, \epsilon$ and $Jord', \pi_{cusp}, \epsilon'$ as above, then we shall say that $Jord', \pi_{cusp}, \epsilon'$ is subordinated to $Jord, \pi_{cusp}, \epsilon$.

Let $Jord, \pi_{cusp}, \epsilon$ be a triple as above. For $(\rho, a) \in Jord$ denote

$$Jord'_\rho(\pi_{cusp}) = \begin{cases} Jord_\rho(\pi_{cusp}) \cup \{0\} & \text{if } a \text{ is even and } \epsilon(\rho, \min Jord_\rho) = 1; \\ Jord_\rho(\pi_{cusp}) & \text{otherwise.} \end{cases}$$

The two notations $Jord'$ and $Jord'(\pi_{cusp})$ will not be used simultaneously, to avoid confusion.

Now we shall define the admissible triple. Let $Jord, \pi_{cusp}, \epsilon$ be a triple as above. First we have the following definition.

We shall say that $Jord, \pi_{cusp}, \epsilon$ is an admissible triple of the alternated type if for each $(\rho, a) \in Jord$ the following two conditions hold:

$$\epsilon(\rho, a) \neq \epsilon(\rho, a_-) \quad \text{if } a_- \text{ is defined, and} \tag{2-5}$$

$$\operatorname{card} Jord'_\rho(\pi_{cusp}) = \operatorname{card} Jord_\rho. \tag{2-6}$$

We shall say that a triple $Jord, \pi_{cusp}, \epsilon$ is admissible, if there exists a sequence of triples $Jord_i, \pi_{cusp}, \epsilon_i$, $1 \leq i \leq k$ such that

$$\begin{aligned} &(Jord, \pi_{cusp}, \epsilon) = (Jord_1, \pi_{cusp}, \epsilon_1), \\ &Jord_{i+1}, \pi_{cusp}, \epsilon_{i+1} \text{ is subordinated to } Jord_i, \pi_{cusp}, \epsilon_i \text{ for } 1 \leq i \leq k-1, \text{ and} \\ &Jord_k, \pi_{cusp}, \epsilon_k \text{ is an admissible triple of the alternated type.} \end{aligned}$$

An admissible triple which is not of the alternated type, will be called an admissible triple of the mixed type.

Suppose that $Jord, \pi_{cusp}, \epsilon$ is an admissible triple of the alternated type. Then (2-6) implies that for each $(\rho, a) \in Jord$ there exists a unique monotone bijection

$$\phi_\rho : Jord_\rho \longrightarrow Jord'_\rho(\pi_{cusp}). \tag{2-7}$$

We shall prove that to each admissible triple $Jord, \pi_{cusp}, \epsilon$ of degree n is attached an irreducible square integrable representation π of S_n with that invariants, i.e. that $(Jord(\pi), \pi_{cusp}, \epsilon_\pi) = (Jord, \pi_{cusp}, \epsilon)$ (where π_{cusp} is the partial cuspidal support of π).

To illustrate the notion of $Jord(\pi)$, we shall give now a proposition which is the basic method for computing $Jord(\pi)$ from $Jord(\pi_{cusp})$. This result is already contained in [M2]. It will be used several times in the sequel.

2.1. Proposition. *Let π' be an irreducible square integrable representation of S_q and let $x, y \in (1/2)\mathbb{Z}$ such that $x - y \in \mathbb{Z}_+$. Let ρ be an F'/F-selfdual cuspidal unitarizable representation of $GL(d_\rho)$. We assume that x, $y \in \mathbb{Z}$ if and only if $L(\rho, R_{d_\rho}, s)$ has not a pole at $s = 0$ (see the beginning of this section for definition of R_{d_ρ}). Further, suppose* *that there is an irreducible square integrable representation π embedded in the induced representation*

$$\pi \hookrightarrow \nu^x\rho \times \cdots \times \nu^{x-i+1}\rho \times \cdots \times \nu^y\rho \times \pi'. \tag{2-8}$$

Then:
(i) If $y > 0$, then $2y-1 \in Jord_\rho(\pi')$ and

$$Jord_\rho(\pi) = (Jord_\rho(\pi') \setminus \{(\rho, 2y-1)\}) \cup \{(\rho, 2x+1)\}.$$

(ii) If $y \leq 0$, then

$$Jord_\rho(\pi) = Jord_\rho(\pi') \cup \{(\rho, 2x+1), (\rho, -2y+1)\}.$$

In particular, $2x+1$ and $-2y+1$ are not in $Jord_\rho(\pi')$.

2.2. Remark. If x does not satisfy the condition regarding the holomorphy of the L-function of the proposition, then $(\rho, 2x+1)$ and $(\rho, 2y+1)$ are not in $Jord_\rho(\pi)$ and $Jord_\rho(\pi')$ by the definition of the Jordan blocks. Further, in this case, the embedding (2-8) cannot happen (recall that π' and π are square integrable).

Proof. To compute $Jord_\rho(\pi)$ in terms of $Jord_\rho(\pi')$, we have to compare Plancherel measures. For $z \in (1/2)\mathbb{N}$, we denote by $\mu(z,\pi)(s)$ the meromorphic function (in $s \in \mathbb{C}$), which is the composite of two standard intertwining operators:

$$\nu^s\delta(\rho, z) \rtimes \pi' \to \nu^{-s}\delta(\rho, z) \rtimes \pi' \to \nu^s\delta(\rho, z) \rtimes \pi'$$

(here $\delta(\rho, z)$ denotes $\delta([\nu^{-(z-1)/2}\rho, \nu^{(z-1)/2}\rho])$, as before). We are interested in $\mu(z,\pi)(s)$ modulo holomorphic invertible functions of s only. We define $\mu(z,\pi')(s)$ in an analogous way for π'. By the results of Shahidi, the computation of the standard intertwining operators for $GL(d)$ is very well known for with arbitrary d. This gives the following equality modulo a holomorphic invertible function of s

$$\begin{aligned}
&\mu(z,\pi)(s) = \mu(z,\pi') \\
&\quad L(\delta(\rho,z)\times\rho, s-x)\; L(\delta(\rho,z)\times\rho, s-y+1)^{-1} \\
&\qquad\qquad L(\delta(\rho,z)\times\rho, y+s)\; L(\delta(\rho,z)\times\rho, x+s+1)^{-1} && \text{(2-9)}\\
&\quad L(\delta(\rho,z)\times\rho, -s-x)\; L(\delta(\rho,z)\times\rho, -s-y+1)^{-1} \\
&\qquad\qquad L(\delta(\rho,z)\times\rho, y-s)\; L(\delta(\rho,z)\times\rho, x-s+1)^{-1}. && \text{(2-10)}
\end{aligned}$$

The product of L-functions is just the product of the normalizing factors in the intertwining operator corresponding to the GL's factors. Usual simplifications are used to get this formula (see [MW], I). By a general result of Harish-Chandra ([W1]) we know that at $s=0$, $\mu(z,\pi)$ has order 0 or 2. Moreover, by the definition of $Jord_\rho(\pi)$:

$$z \in Jord_\rho(\pi) \Leftrightarrow \mathrm{ord}_{s=0}(\mu(z,\pi)) = 2.$$

We need to look at the different L-factors using a general result of [JPSS] (Theorem 8.2 there):

$$L(\delta(\rho, z) \times \rho, s') = L(\rho \times \rho, s' + (z-1)/2).$$

For $s'' \in \mathbb{R}$, the function $L(\rho \times \rho, s'')$ is non-zero and the only pole here is at $s'' = 0$. The pole is simple. It is obvious that at $s = 0$, (2-9) and (2-10) above have the same order.

Now $s = 0$ is a pole for $\mu(z, \pi)(s)$ if and only if either it is a pole of $\mu(z, \pi')(s)$ and not a zero of the factor (2-9) or it is a pole of the factor (2-9). Moreover, if $s = 0$ is a pole of the factor (2-9), it cannot be a pole for $\mu(z, \pi')(s)$ by the result of Harish-Chandra.

To prove the lemma, we first need to observe the following facts:

$$L(\delta(\rho, z) \times \rho, s - x)$$

has a (simple) pole at $s = 0$ if and only if $x = (z-1)/2$;

$$L(\delta(\rho, z) \times \rho, s - y + 1)^{-1}$$

has a (simple) zero at $s = 0$ if and only if $y - 1 = (z-1)/2$;

$$L(\delta(\rho, z) \times \rho, y + s)$$

has a (simple) pole at $s = 0$ if and only if $y = -(z-1)/2$;

$$L(\delta(\rho, z) \times \rho, x + s + 1)^{-1}$$

has a (simple) zero at $s = 0$ if and only if $x + 1 = -(z-1)/2$ (this cannot happen since $x + 1 = -(z-1)/2$ implies $z = -2x - 1 \leq -2$, which is impossible).

These observations imply that $2x + 1 \in Jord_\rho(\pi)$ and $2x + 1 \notin Jord_\rho(\pi')$. Further, $a' \neq 2x + 1$ is an element of $Jord_\rho(\pi')$ if and only if it is an element of $Jord_\rho(\pi)$, except if $a' = -2y + 1$ (which implies $y \leq 0$) or $a' = 2y - 1$ (which implies $y > 0$). In the case $a' = -2y + 1$, we have $a' \notin Jord_\rho(\pi')$ but $a' \in Jord_\rho(\pi)$. The case $a' = -2y + 1$ gives that $a' \notin Jord_\rho(\pi)$ and $a' \in Jord_\rho(\pi')$, to avoid a zero of $\mu(a', \pi)(s)$ at $s = 0$. □

2.3. Remark. In this remark we shall suppose that S_n are split groups.

(i) By [Sh1], the condition (J-1) is equivalent to

(J-1') a is even if $\nu^{1/2}\rho \rtimes 1$ reduces and odd otherwise
(i.e. if $\rho \rtimes 1$ or $\nu\rho \rtimes 1$ reduces).

(ii) If $a > 1$, then by the fourth section of [T2], (J-1) is equivalent to the fact that

$$\delta(\rho, a) \rtimes 1 \text{ reduces}$$

(here, 1 denotes the trivial representation of S_0).

3. General technical lemmas

In this section we shall collect some facts from the representation theory. Most of the results are for classical groups. Some of them are very simple and well-known. First we shall recall of the following simple fact.

3.1. Lemma. *Let σ be an irreducible cuspidal subquotient of a Jacquet module $r_M^G(\pi)$ of an irreducible representation π of a connected reductive p-adic group G with respect to a parabolic subgroup $P = MN$ of G. Then*

$$\pi \hookrightarrow Ind_P^G(\sigma).$$

Proof. Decompose $r_M^G(\pi)$ into a direct sum with respect to the action of the Bernstein center of M (into generalized eigenspaces for the action; it means that all irreducible subquotients in one component have the same infinitesimal characters, and that for different components we have different infinitesimal characters). Consider the component whose infinitesimal character is equal to the infinitesimal character of σ. The assumption that σ is a subquotient of the Jacquet module implies that this component is non-zero. Further, since σ is irreducible cuspidal, σ must be also (isomorphic to) a quotient of this component. Now the Frobenius reciprocity implies the lemma. $\square$

3.2. Lemma. *Let π be an irreducible representation of a reductive p-adic group and let $P = MN$ be a parabolic subgroup of G. Suppose that M is a direct product of two reductive subgroups M_1 and M_2. Let τ_1 be an irreducible representation of M_1, and let τ_2 be a representation of M_2. Suppose*

$$\pi \hookrightarrow Ind_P^G(\tau_1 \otimes \tau_2).$$

Then there exists an irreducible representation τ_2' such that

$$\pi \hookrightarrow Ind_P^G(\tau_1 \otimes \tau_2').$$

Proof. First note that there is a non-zero intertwining from the Jacquet module $r_M^G(\pi)$ of π (with respect to $P = MN$) into $\tau_1 \otimes \tau_2$. The image X has a finite length as a representation of M. Therefore, X has an irreducible quotient, say $\tau_1' \otimes \tau_2'$. To see the lemma, it is enough to show $\tau_1' \cong \tau_1$.

Now $\tau_1 \otimes \tau_2$ (resp. $\tau_1' \otimes \tau_2'$) is, as a representation of M_1, a sum of copies of τ_1 (resp. τ_1'). From the existence of a non-zero M_1-intertwining from the subspace X of $\tau_1 \otimes \tau_2$ into $\tau_1' \otimes \tau_2'$, we get $\tau_1' \cong \tau_1$ (since X is semi simple representation of M_1, and it is isomorphic to a direct sum of copies of τ_1). $\square$

3.3. Lemma. *Let π be an irreducible representation of S_q. The following three sets of irreducible cuspidal representations τ of general linear groups coincide*

(1) The set of all τ for which there exists an irreducible subquotient $\sigma \otimes \pi_{cusp}$ of $s_{GL}(\pi)$, such that τ is in the support of σ (the support is defined in Proposition 1.10 of [Z]).

(2) The set of all τ *such that there exists an irreducible cuspidal subquotient* $\rho_1\otimes\ldots\otimes\rho_k\otimes\sigma$ *of a standard Jacquet module of* π *and index* i *such that* $\tau\cong\rho_i$ *(standard Jacquet module means that it is a Jacquet module with respect to a standard parabolic subgroup).*

(3) The set of all τ *such that there exist irreducible cuspidal representations* $\rho_1,\ldots,\rho_k$ *and* σ *of general linear groups and of some* $S_{q'}$ *respectively, and an index* i *such that:*

$$\pi\hookrightarrow\rho_1\times\ldots\times\rho_k\rtimes\sigma \tag{3-1}$$

and $\tau\cong\rho_i$.

Irreducible cuspidal representations of general linear groups characterized by one of the above descriptions will be called factors of π.

Proof. For $i=1,2,3$, denote the set described in (i) by X_i. Proposition 1.10 of [Z] and induction in stages implies $X_1\subseteq X_2$. The transitivity of Jacquet modules and the definition of the support in Proposition 1.10 of [Z] imply $X_2\subseteq X_1$. Frobenius reciprocity implies $X_3\subseteq X_2$. At the end, $X_2\subseteq X_1$ follows from Lemma 3.1. □

The following lemma is essentially claim about the representations of general linear groups.

3.4. Lemma. *Let* π *be an irreducible representation of* S_q *and let* τ *be a factor of* π *such that* $\nu^{-1}\tau$ *is not a factor of* π.
(i) Then there exists $z\in\mathbb{Z}_+$ *and an irreducible representation* σ *of some* $S_{q'}$, $q'<q$, *such that*

$$\pi\hookrightarrow\delta([\tau,\nu^z\tau])\rtimes\sigma.$$

(ii) Let $\rho_1'\otimes\rho_2'\otimes\ldots\otimes\rho_l'\otimes\pi_{cusp}$ *be an irreducible cuspidal subquotient of a Jacquet module of* π, *such that* $\tau\cong\rho_{j'}'$. *Suppose that there exists* $j\in\mathbb{Z}_+$ *such that* $\nu^{j+1}\tau\not\cong\rho_i'$ *for all* $i<j'$. *Then one can find an embedding in (i) for which* $z\le j$.

(iii) Let $\rho_1'\otimes\rho_2'\otimes\ldots\otimes\rho_l'\otimes\pi_{cusp}$ *be an irreducible cuspidal subquotient of a Jacquet module of* π. *Suppose that* $\tau\cong\rho_m'$ *for some* m. *Then there exists* $k\in\mathbb{Z}_+$, $k\le m-1$, *and irreducible cuspidal representations* ρ_i, $k+2\le i\le l$, *such that*

(1) $\pi\hookrightarrow\delta([\tau,\nu^k\tau])\times\rho_{k+2}\times\ldots\times\rho_l\rtimes\pi_{cusp}$;

(2) $\rho_i'\cong\rho_i$ *for* $m+1\le i\le l$;

(3) $\nu^k\tau,\nu^{k-1}\tau,\ldots,\tau,\rho_{k+2},\rho_{k+3},\ldots,\rho_m$ *is a permutation of* $\rho_1',\rho_2',\ldots,\rho_m'$.

Proof. Obviously, (ii) implies (i). Now we shall show that (iii) implies (ii). Suppose that we have proved (iii). Applying (iii) to (ii) for $m=j'$, we get $\pi\hookrightarrow\delta([\tau,\nu^k\tau])\times\rho_{k+2}\times\ldots\times\rho_l\rtimes\pi_{cusp}$, where (ii) and (iii) imply $k\le j$. Now Lemma 3.2 imply (ii).

It remains to prove (iii). We shall do it now.

Look at all possible embeddings like

$$\pi\hookrightarrow\left(\prod_{j=1}^{l}\rho_j\right)\rtimes\pi_{cusp}, \tag{3-2}$$

such that (2) and (3) hold. By Lemma 3.1, there exists at least one such embedding. For such an embedding, we know that $\rho_i \cong \tau$ for at least one $i \leq j$. Among all such embeddings, chose one with minimal possible index i such that $\rho_i \cong \tau$. Clearly, $i \leq m$ The condition of the lemma implies

$$\rho_j \not\cong \nu^{-1}\rho_i \text{ for } 1 \leq j \leq i-1. \tag{3-3}$$

Denote

$$\pi' = \left(\prod_{j=i+1}^{l} \rho_j\right) \rtimes \pi_{cusp} \quad \text{and} \quad \tau = \left(\otimes_{j=i+1}^{l}\rho_j\right) \otimes \pi_{cusp}.$$

Let $j' \in \{0, 1, \ldots, i-1\}$ be maximal such that

$$\rho_{i-j} = \nu^j \rho_i, \text{ for } j = 0, 1, \ldots, j', \tag{3-4}$$

and

$$\pi \hookrightarrow \left(\prod_{j=1}^{i-j'-1} \rho_j\right) \times \delta([\rho_i, \nu^{j'}\rho_i]) \rtimes \pi'. \tag{3-5}$$

Clearly, there is at least one such $j' \geq 0$ satisfying (3-4) and (3-5) (it is $j' = 0$). We shall prove $j' = i-1$. This will complete the proof of the lemma.

Suppose $j' < i-1$. Look at $\rho_{i-j'-1}$. If $\rho_{i-j'-1} \times \delta([\rho_i, \nu^{j'}\rho_i])$ is irreducible, then $\rho_{i-j'-1} \times \delta([\rho_i, \nu^{j'}\rho_i]) \cong \delta([\rho_i, \nu^{j'}\rho_i]) \times \rho_{i-j'-1}$. This would imply the existence of an embedding of π like in (3-2), with ρ_i at the $(i-1)$-th place. Therefore, $\rho_{i-j'-1} \times \delta([\rho_i, \nu^{j'}\rho_i])$ reduces. There are two possibility for that. The possibility $\rho_{i-j'-1} \cong \nu^{-1}\rho_i$ cannot happen by the assumption of the lemma. Thus,

$$\rho_{i-j'-1} = \nu^{j'+1}\rho_i. \tag{3-6}$$

This implies that (3-4) holds for $j'+1$. Further, (3-5), (3-6) and the Frobenius reciprocity imply that we have a non-zero intertwining ϕ from the corresponding Jacquet module $r_M^{S_q}(\pi)$ of π into

$$\left(\otimes_{j=1}^{i-j'-2}\rho_j\right) \otimes \nu^{j'+1}\rho_i \times \delta([\rho_i, \nu^{j'}\rho_i]) \otimes \tau. \tag{3-7}$$

Note that (3-7) is a length two representation with a unique irreducible subrepresentation, which is $\left(\otimes_{j=1}^{i-j'-2}\rho_j\right) \otimes \delta([\rho_i, \nu^{j'+1}\rho_i]) \otimes \tau$. Since j' is maximal, the image of ϕ can not be this irreducible subrepresentation (otherwise, the Frobenius reciprocity would imply that (3-5) holds also for $j'+1$, and this would contradict to the assumption that j' is maximal, since we have seen already that (3-4) holds for $j'+1$). Therefore, the image is the whole (3-7). From this (and transitivity of Jacquet modules) we conclude that

$$\left(\otimes_{j=1}^{i-j'-2}\rho_j\right) \otimes \delta([\rho_i, \nu^{j'}\rho_i]) \otimes \nu^{j'+1}\rho_i \otimes \tau$$

is a subquotient of a corresponding Jacquet module of π. Now Lemma 3.1 implies that ρ_i can show up in embeddings like (3-2) at $(i-1)$-th place, which contradicts our choice of i. This contradiction ends the proof that $j' = i-1$. □

Now we shall recall of a notion of a strongly positive (or strictly positive) irreducible representation of S_q, defined in [M2]. One says that π is strongly positive if for any factor τ of π we have

$$e(\tau) > 0.$$

The Casselman's square integrability criterion implies that each strongly positive irreducible representation is square integrable.

In this paper, we shall use several times the following

3.5. Lemma. *Let π and τ be irreducible representations of S_q and $S_{q'}$ respectively. Suppose that τ is tempered (resp. square integrable) and $q > q'$. Let γ be a representation of $GL(q-q', F')$ of finite length. Suppose*

$$s_{GL}(\pi) \leq \gamma \times s_{GL}(\tau), \tag{3-8}$$

and suppose that for each irreducible cuspidal subquotient $\rho_1 \otimes \ldots \otimes \rho_k$ of a standard Jacquet module (with respect to the subgroup of upper triangular matrices; see [Z]) of γ, we have

$$\sum_{i=1}^{j} e(\rho_i) d_{\rho_i} \geq 0 \tag{3-9}$$

$$\Big(\textit{resp. } \sum_{i=1}^{j} e(\rho_i) d_{\rho_i} > 0\Big) \tag{3-10}$$

for all $j = 1, \ldots, k$ (recall that d_{ρ_i} is defined by the condition that ρ_i is a representation of $GL(d_{\rho_i}, F')$). Then π is tempered (resp. square integrable).

Proof. First note $\pi_{cusp} = \tau_{cusp}$. Further, (3-8) implies that for each irreducible cuspidal subquotient

$$\rho_1 \otimes \ldots \otimes \rho_n \otimes \pi_{cusp} \tag{3-11}$$

of a standard Jacquet module of π, there exists a partition of $\{1, \ldots, n\}$ into two subsets

$$i_1 < i_2 < \cdots < i_l \quad \text{and} \quad j_1 < j_2 < \cdots < j_m,$$

such that

$$\rho_{i_1} \otimes \ldots \otimes \rho_{i_l}$$

is a subquotient of a standard Jacquet module of γ and

$$\rho_{j_1} \otimes \ldots \otimes \rho_{i_m} \otimes \pi_{cusp}$$

is a subquotient of a standard Jacquet module of τ. Then transitivity of Jacquet modules implies that $\rho_{j_1} \otimes \ldots \otimes \rho_{i_m} \otimes \pi_{cusp}$ is a subquotient of a standard Jacquet module of $s_{GL}(\tau)$.

By the Casselman's square integrability criterion, π is square integrable if and only if for each subquotient (3-11) we have

$$\sum_{i=1}^{j} e(\rho_i)\, d_{\rho_i} > 0 \quad \text{for every } j = 1, \dots, n. \tag{3-12}$$

Suppose that τ is square integrable and γ satisfies (3-10). Then (3-10), square integrability criterion (3-12) applied to τ, and the above description of the subquotients (3-11) in terms of Jacquet modules of γ and τ, imply that π is square integrable. In the same way one proves the claim of the lemma in the tempered case. □

Now we shall write a direct consequence of 5.1.2 from [M2].

3.6. Lemma. *Suppose that π is an irreducible square integrable representation of S_q. Let $\nu^x\rho \otimes \tau$ be an irreducible subquotient of a standard Jacquet module of π, where ρ is an irreducible F'/F-selfdual cuspidal representation of $GL(p, F')$, $x \in \mathbb{R}$, and τ is an irreducible representation of $S_{q'}$. Then*

$$(\rho, 2x + 1) \in Jord(\pi).$$

Proof. Take irreducible cuspidal representations $\rho_1, \dots, \rho_l$ and σ of general linear groups and (some) $S_{q''}$ respectively, such that

$$\tau \hookrightarrow \rho_1 \times \dots \times \rho_l \rtimes \sigma.$$

By the transitivity of Jacquet modules (and the Frobenius reciprocity), $\nu^x\rho\otimes\rho_1\times\dots\otimes\rho_l\otimes\sigma$ is (an irreducible cuspidal) subquotient of a standard Jacquet module of π. Lemma 3.1 implies

$$\pi \hookrightarrow \nu^x\rho \times \rho_1 \times \dots \times \rho_l \rtimes \sigma.$$

Lemma 3.2 implies the existence of an irreducible representation τ' such that

$$\pi \hookrightarrow \nu^x\rho \rtimes \tau'.$$

Now 5.1.2 of [M2] implies the lemma. □

4. On irreducible subrepresentations

In this section we shall present some facts related to irreducible subrepresentations and their uniqueness. These facts shall be used frequently later.

4.1. Lemma. *Let $n \in \mathbb{N}$, let ρ be an irreducible cuspidal F'/F-selfdual representation of a general linear group and let π_{cusp} be an irreducible cuspidal representation of S_q. Suppose that $\alpha, \beta : \{1, \dots, n\} \longrightarrow \mathbb{R}$ are functions which satisfy*

$$\beta(i) - \alpha(i) \in \mathbb{Z}_+ \text{ for all } i, \tag{4-1}$$

$$\alpha(i) > 0 \text{ for all } i, \quad \text{and} \tag{4-2}$$

$$\beta(1) < \beta(2) < \dots < \beta(n). \tag{4-3}$$

Then the representation

$$\left(\prod_{i=1}^{n}\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho])\right)\rtimes\pi_{cusp}$$

has a unique irreducible subrepresentation.

Proof. For the proof of the lemma it is enough to show that the multiplicity of

$$\left(\otimes_{i=1}^{n}\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho])\right)\otimes\pi_{cusp} \tag{4-4}$$

in the corresponding standard Jacquet module of $\left(\prod_{i=1}^{n}\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho])\right)\rtimes\pi_{cusp}$ is one.

First note

$$s_{GL}\left(\left(\prod_{i=1}^{n}\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho])\right)\rtimes\pi_{cusp}\right) = \left(\prod_{i=1}^{n}\left(\sum_{j_i=\alpha(i)}^{\beta(i)+1}\delta([\nu^{j_i}\rho,\nu^{\beta(i)}\rho])\times\delta([\nu^{-j_i+1}\rho,\nu^{-\alpha(i)}\rho])\right)\right)\otimes\pi_{cusp}.$$

Now (4-2) and the above formula imply that it is enough to see that the multiplicity of (4-4) in a corresponding Jacquet module of $\left(\prod_{i=1}^{n}\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho])\right)\otimes\pi_{cusp}$ is one (since to get (4-4) for a subquotient of a corresponding Jacquet module of a term in the above formula, one needs to take $j_i=\alpha(i)$ for all i, by (4-2)). To see this, it is enough to show that the multiplicity of

$$\otimes_{i=1}^{n}\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho]) \tag{4-5}$$

in a corresponding Jacquet module of

$$\prod_{i=1}^{n}\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho])$$

is one.

Note that

$$m^*\left(\prod_{i=1}^{n}\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho])\right) = \prod_{i=1}^{n}\left(\sum_{j_i=\alpha(i)-1}^{\beta(i)}\delta([\nu^{j_i+1}\rho,\nu^{\beta(i)}\rho])\otimes\delta([\nu^{\alpha(i)}\rho,\nu^{j_i}\rho])\right). \tag{4-6}$$

Let $\prod_{i=1}^{n}\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho])$ and $\delta([\nu^{\alpha(i)}\rho,\nu^{\beta(i)}\rho])$ be representations of $GL(p',F')$ and $GL(k_i,F')$ respectively.

Now we shall show that the multiplicity of $\otimes_{i=1}^n \delta([\nu^{\alpha(i)}\rho, \nu^{\beta(i)}\rho])$ in a corresponding Jacquet module of $\prod_{i=1}^n \delta([\nu^{\alpha(i)}\rho, \nu^{\beta(i)}\rho])$ is one, by induction with respect to n. For $n = 1$ the claim obviously holds. Suppose $n > 1$.

We shall compute the multiplicity of $\otimes_{i=1}^n \delta([\nu^{\alpha(i)}\rho, \nu^{\beta(i)}\rho])$ in a corresponding Jacquet module of $\prod_{i=1}^n \delta([\nu^{\alpha(i)}\rho, \nu^{\beta(i)}\rho])$ using the formula (4-6) and transitivity of Jacquet modules. First we consider the Jacquet module with respect to $GL(p' - k_n, F') \times GL(k_n, F')$. To get $\otimes_{i=1}^n \delta([\nu^{\alpha(i)}\rho, \nu^{\beta(i)}\rho])$ for a subquotient of corresponding Jacquet module of a term in (4-6), we must take $j_n = \beta(n)$ (note that $\nu^{\beta(n)}\rho$ shows up in the support of the last tensor factors of $\otimes_{i=1}^n \delta([\nu^{\alpha(i)}\rho, \nu^{\beta(i)}\rho])$, and then use (4-3)). Therefore, $\delta([\nu^{\alpha(n)}\rho, \nu^{\beta(n)}\rho])$ shows up on the right hand side of $\otimes$. Since we are considering the Jacquet module with respect to $GL(p' - k_n, F') \times GL(k_n, F')$, we must have $j_i = \alpha(i) - 1$ for $1 \leq i \leq n-1$. Now applying the inductive assumption in the case of $n-1$, one get the claim for n. □

The following evident fact we shall use a number of times.

4.2. Remark. *Let π and π' be representations of finite length. Suppose that π has at most n irreducible subrepresentations. Let $\pi' \hookrightarrow \pi$. Then π' has also at most n irreducible subrepresentations.*

The following lemma holds in a much bigger generality.

4.3. Lemma. *Let π be an irreducible square integrable representation of S_q and let ρ be an irreducible F'/F-selfdual cuspidal representation of $GL(p, F')$. Let $\alpha, \beta \in (1/2)\mathbb{Z}_+$. Then:*
(i) The multiplicity of $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \otimes \pi$ in $\mu^*(\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \pi)$ is 2.
(ii) Suppose $\beta > \alpha$. Then the multiplicity of $\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \otimes \delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \otimes \pi$ in a corresponding Jacquet module of $\delta([\nu^{-\beta}\rho, \nu^{\beta}\rho]) \times \delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \pi$ is 4.

Proof. We have

$$\begin{aligned} \text{(4-7)} \quad & \mu^* \left(\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \pi\right) \\ &= \left(\sum_{i=-\alpha-1}^{\alpha} \sum_{j=i}^{\alpha} \delta([\nu^{-i}\rho, \nu^{\alpha}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{\alpha}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \right) \rtimes \mu^*(\pi). \end{aligned}$$

If $i = j$, then one gets directly that $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \otimes \pi$ can be a subquotient of corresponding term only if $i = -\alpha - 1$ and α. In each term the multiplicity is one (each of these terms is just $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \otimes \pi$).

Suppose that we can get $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \otimes \pi$ from a term corresponding to $i < j$. Then $-\alpha - 1 \leq i < j$ implies $j + 1 > -\alpha$. Also $-i > -\alpha$ (if $-i = -\alpha$, then $i = \alpha$ and thus $j = i = \alpha$ what contradicts to $i < j$). Thus $\nu^{-\alpha}\rho$ must come from $\mu^*(\pi)$. From the above discussion and (4-7) we get that $\delta([\nu^{-\alpha}\rho, \nu^{\alpha-k}\rho]) \otimes \tau \leq \mu^*(\pi)$ for some $k \in \mathbb{Z}_+$, $k \leq 2\alpha$. The Casselman's square integrability criterion implies that π is not square integrable. This contradiction completes the proof of (i).

Suppose $\alpha < \beta$. Write

$$\text{(4-8)}\quad \mu^*\left(\delta([\nu^{-\beta}\rho,\nu^{\beta}\rho])\times\delta([\nu^{-\alpha}\rho,\nu^{\alpha}\rho])\rtimes\pi\right)$$
$$=\Big(\sum_{i=-\beta-1}^{\beta}\sum_{j=i}^{\beta}\delta([\nu^{-i}\rho,\nu^{\beta}\rho])\times\delta([\nu^{j+1}\rho,\nu^{\beta}\rho])\otimes\delta([\nu^{i+1}\rho,\nu^{j}\rho])\Big)$$
$$\times M^*(\delta([\nu^{-\alpha}\rho,\nu^{\alpha}\rho]))\rtimes\mu^*(\pi).$$

We analyze when one can get $\delta([\nu^{-\beta}\rho,\nu^{\beta}\rho])\otimes\tau$ for a subquotient of (4-8) (where τ is arbitrary irreducible representation). One can get it for $i=j=-\beta-1$ or β, and these are the only cases when one can get it if $i=j$. Then the corresponding term in the above sum is $\delta([\nu^{-\beta}\rho,\nu^{\beta}\rho])\otimes M^*(\delta([\nu^{-\alpha}\rho,\nu^{\alpha}\rho]))\rtimes\mu^*(\pi)=\delta([\nu^{-\beta}\rho,\nu^{\beta}\rho])\otimes\mu^*(\delta([\nu^{-\alpha}\rho,\nu^{\alpha}\rho])\rtimes\pi)$. Now (i) tells that the multiplicity of $\delta([\nu^{-\beta}\rho,\nu^{\beta}\rho])\otimes\delta([\nu^{-\alpha}\rho,\nu^{\alpha}\rho])\otimes\pi$ in each of these two terms is two. Therefore, to prove (ii), one needs to show that one can not get $\delta([\nu^{-\beta}\rho,\nu^{\beta}\rho])\otimes\tau$ for a subquotient of (4-8) if $i<j$.

Suppose that we can get it for some $i<j$. We shall have now similar arguments as in the proof of (i). To get $\delta([\nu^{-\beta}\rho,\nu^{\beta}\rho])\otimes\tau$, one needs to get $\nu^{-\beta}\rho$ on the left hand side of $\otimes$. Note that $\nu^{-\beta}\rho$ can not come from $M^*(\delta([\nu^{-\alpha}\rho,\nu^{\alpha}\rho]))$ since $\alpha<\beta$. Suppose now that $M^*(\delta([\nu^{-\alpha}\rho,\nu^{\alpha}\rho]))$ gives a non-trivial contribution to the left hand side of $\otimes$ of the term where $\delta([\nu^{-\beta}\rho,\nu^{\beta}\rho])\otimes\tau$ is a subquotient. Then a short discussion gives $\delta([\nu^{-\beta}\rho,\nu^{\beta-k}\rho])\otimes\tau\leq\mu^*(\pi)$ for some $k\in\mathbb{N}$, $k\leq 2\beta$. This contradicts to the square integrability of π. Thus, $M^*(\delta([\nu^{-\alpha}\rho,\nu^{\alpha}\rho]))$ does not give a non-trivial contribution to the left hand side of $\otimes$. Now one can repeat the argument from (i) without change, and get that we cannot get $\delta([\nu^{-\beta}\rho,\nu^{\beta}\rho])\otimes\tau$ for for a subquotient if $i<j$. This completes the proof. □

The following lemma (which holds in a much bigger generality) points out a well-known property of the Langlands classification. We state it in the setting in which we shall use the property later.

4.4. Lemma. *Suppose that τ is an irreducible tempered representation of S_q and suppose that ρ is an irreducible cuspidal F'/F-selfdual representation of a general linear group $GL(p)$. Let $x,y\in\mathbb{R}$ be such that $y-x\in\mathbb{Z}_+$ and $x+y>0$. Then the representation*

$$\delta([\nu^{x}\rho,\nu^{y}\rho])\rtimes\tau$$

has a unique irreducible quotient. Denote it by π. The multiplicity of $\delta([\nu^{-y}\rho,\nu^{-x}\rho])\otimes\tau$ in $\mu^*(\delta([\nu^{x}\rho,\nu^{y}\rho])\rtimes\tau)$ and $\mu^*(\pi)$ is one.

Proof. It is a well-known property of the Langlands classification that the multiplicity of $\delta([\nu^{-y}\rho,\nu^{-x}\rho])\otimes\tau$ in $\mu^*(\delta([\nu^{x}\rho,\nu^{y}\rho])\rtimes\tau)$ is one (this implies the uniqueness of the irreducible quotient). One can also get this fact easily using the formula (1-1) (and (1-3), where we allow a_- in (1-3) also to be negative). Further, $\pi\hookrightarrow\delta([\nu^{-y}\rho,\nu^{-x}\rho])\rtimes\tau$ (this follows from the fact that π is an image of the long intertwining operator in the Langlands classification). The Frobenius reciprocity now implies that the multiplicity of $\delta([\nu^{-y}\rho,\nu^{-x}\rho])\otimes\tau$ in $\mu^*(\pi)$ is one. □

4.5. Lemma. *Let π be an irreducible square integrable representation of S_q and let ρ be an irreducible F'/F-selfdual cuspidal representation of $GL(p,F')$. Fix $a \in \mathbb{Z}_+$ and fix positive numbers $\alpha_1,\dots,\alpha_k$. Suppose*

(i) $\alpha_i > (a-1)/2$ for all $i=1,\dots,k$, or $\alpha_i < (a-1)/2$ for all $i=1,\dots,k$;
(ii) $\alpha_i \neq \alpha_j$ for $i \neq j$ in $\{1,\dots,k\}$;
(iii) $(\rho, 2\alpha_i+1) \notin Jord(\pi)$ for $i=1,\dots,k$.

Let T be any irreducible subrepresentation of $\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho]) \rtimes \pi$ (this is a unitarizable representation). Then the multiplicity of $\left(\otimes_{i=1}^k \nu^{\alpha_i}\rho\right)\otimes T$ in corresponding standard Jacquet modules of $\left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right)\rtimes T$ and $\left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right)\times\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi$ is one. In particular, $\left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right)\rtimes T$ contains a unique irreducible subrepresentation.

Further, the multiplicity of $\left(\otimes_{i=1}^k \nu^{\alpha_i}\rho\right)\otimes\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\otimes\pi$ in a corresponding standard Jacquet module of $\left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right)\times\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi$ is two, and the last representation has at most two irreducible subrepresentations. If we have two irreducible subrepresentations, they are not isomorphic.

Proof. The Frobenius reciprocity implies that the multiplicity of $\left(\underset{i=1}{\overset{k}{\otimes}}\nu^{\alpha_i}\rho\right)\otimes T$ in a corresponding Jacquet module of $\left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right)\rtimes T$ is at least one. Now write

$$\text{(4-9)}\quad \mu^*\left(\left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right)\times\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi\right)$$

$$\text{(4-10)}\qquad = \left(\prod_{i=1}^k \left(1\otimes\nu^{\alpha_i}\rho+\nu^{\alpha_i}\rho\otimes 1+\nu^{-\alpha_i}\rho\otimes 1\right)\right)$$

$$\text{(4-11)}\qquad \times\Bigg(\sum_{i'=-(a-1)/2-1}^{(a-1)/2}\ \sum_{j'=i'}^{(a-1)/2}\delta([\nu^{-i'}\rho,\nu^{(a-1)/2}\rho])\times\delta([\nu^{j'+1}\rho,\nu^{(a-1)/2}\rho])$$

$$\text{(4-11)}\qquad \otimes\delta([\nu^{i'+1}\rho,\nu^{j'}\rho])\Bigg)\rtimes\mu^*(\pi).$$

Denote $\Psi=\left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right)\times\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\rtimes\pi$.

Let τ be an irreducible representation of $GL(kp,F')$ such that $\otimes_{i=1}^k\nu^{\alpha_i}\rho$ is a subquotient of a corresponding Jacquet module of τ. Then the support of τ is $\nu^{\alpha_1}\rho,\nu^{\alpha_2}\rho,\dots,\nu^{\alpha_k}\rho$. Let σ be an irreducible representation of S_{q-kp}. Each of $\left(\otimes_{i=1}^k\nu^{\alpha_i}\rho\right)\otimes T$, $\left(\otimes_{i=1}^k\nu^{\alpha_i}\rho\right)\otimes\delta([\nu^{-(a-1)/2}\rho,\nu^{(a-1)/2}\rho])\otimes\pi$, which is a subquotient of a corresponding Jacquet module of Ψ, is a subquotient of some $\tau\otimes\sigma$ as above (we use the transitivity of Jacquet modules).

First we shall analyze which terms in the above sum (4-10) – (4-11) can give (after further multiplication) $\tau\otimes\sigma$ as above, for a subquotient. We shall first discuss how one can chose terms in the sum in (4-11). Suppose that for some indexes i', j' we have $-i' \le (a-1)/2$ or $j'+1\le(a-1)/2$. Then we would have $\nu^{(a-1)/2}\rho$ in the support of

τ, which contradict (i). Thus we always have $i' = -(a-1)/2 - 1$ and $j' = (a-1)/2$. Therefore, there is only one possibility for the term in the sum in (4-11). This term is

$$1 \otimes \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]).$$

Now we shall discuss how one can chose the terms in the product (4-10) . First of all, since $\alpha_i > 0$ for all i, we must not take $\nu^{-\alpha_i}\rho \otimes 1$, since $\nu^{-\alpha_i}\rho$ are not in the support of τ. Thus, we must take either $1 \otimes \nu^{\alpha_i}\rho$ or $\nu^{\alpha_i}\rho \otimes 1$.

Suppose that for some index i, we have chosen $1 \otimes \nu^{\alpha_i}\rho$ (which gives $\tau \otimes \sigma$ for a subquotient after further multiplication). Since $\nu^{\alpha_i}\rho$ is in the support of τ, and we have seen that in (4-11) we must take the term $1 \otimes \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho])$, to be able to get $\tau \otimes \sigma$ for a subquotient, we must have an irreducible representation $\tau' \otimes \sigma'$ such that

$$\tau' \otimes \sigma' \leq \mu^*(\pi), \ \mathrm{Supp}(\tau') \subseteq \{\nu^{\alpha_1}\rho, \dots, \nu^{\alpha_k}\rho\} \text{ and } \nu^{\alpha_i}\rho \in \mathrm{Supp}(\tau')$$

((ii) implies that $\mathrm{Supp}(\tau)$ and $\mathrm{Supp}(\tau')$ are actually sets). The above discussion and Lemma 3.2 imply that $\nu^{\alpha_j}\rho \otimes \sigma'' \leq \mu^*(\pi)$ for some $\nu^{\alpha_j}\rho \in \mathrm{Supp}(\tau')$ and some irreducible representation σ''. Lemma 3.6 implies $(\rho, 2\alpha_j + 1) \in Jord(\pi)$. This contradicts (iii). Therefore, we must always take terms $\nu^{\alpha_i}\rho \otimes 1$ in the product.

Thus, $\tau \otimes \sigma$ must be a subquotient $\left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right) \otimes \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi$. Note that $\prod_{i=1}^k \nu^{\alpha_i}\rho$ is a regular representation of $GL(kp, F')$. Since $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi$ is a multiplicity one representation by (i) of Lemma 4.3 and the Frobenius reciprocity, the multiplicity one of $\left(\otimes_{i=1}^k \nu^{\alpha_i}\rho\right) \otimes T$ claimed in the lemma directly follows. Further, the claim of the lemma about multiplicity two of $\left(\otimes_{i=1}^k \nu^{\alpha_i}\rho\right) \otimes \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \otimes \pi$ follows from (i) of Lemma 4.3. The claims about the number of irreducible subrepresentations follow from the Frobenius reciprocity and the above multiplicities which we have calculated.

The previous part of the proof implies that if $\left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right) \times \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi$ has two irreducible subrepresentations, say π_1 and π_{-1}, then $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi$ reduces into a sum of irreducible representations $T_1 \oplus T_{-1}$. Further, $\pi_i \hookrightarrow \left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right) \rtimes T_i$ for $i = \pm 1$, or $\pi_i \hookrightarrow \left(\prod_{i=1}^k \nu^{\alpha_i}\rho\right) \rtimes T_{-i}$ for $i = \pm 1$. This implies that $\left(\otimes_{i=1}^k \nu^{\alpha_i}\rho\right) \otimes T_i$ is a subquotient of a corresponding Jacquet module of π_i for $i = \pm 1$, or $\left(\otimes_{i=1}^k \nu^{\alpha_i}\rho\right) \otimes T_{-i}$ is a subquotient of a corresponding Jacquet module of π_i for $i = \pm 1$. Now the first part of the proof implies $\pi_1 \not\cong \pi_{-1}$. $\square$

Note that for the proof of the above lemma we could suppose instead (i) a weaker condition: $(a-1)/2 \notin \{a_1, \alpha_2, \dots, \alpha_k\}$. Actually, the lemma holds in a much bigger generality, but we prove a version which is adapted for our most often applications.

4.6. Lemma. *Let ρ be an irreducible cuspidal F'/F-selfdual representations of $GL(p, F')$ and let π be an irreducible square integrable representation of S_q. Let $a_-, a \in \mathbb{N}$, $a_- < a$. Suppose that $\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \rtimes \pi$ reduces. Write*

$$\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \rtimes \pi = T_1 \oplus T_{-1},$$

where T_η are irreducible and $T_1 \not\cong T_{-1}$ (we can do this by (i) of Lemma 4.3). Suppose that $(\rho, b) \notin Jord(\pi)$ for $a_- \leq b \leq a$. For $\eta \in \{\pm 1\}$ denote

$$\sigma_\eta = \left(\bigotimes_{i=0}^{(a-1)/2-((a_--1)/2+1)} \nu^{(a-1)/2-i}\rho \right) \otimes T_\eta,$$

$$\pi_0 = \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi,$$

$$\pi_\eta = \left(\prod_{i=0}^{(a-1)/2-((a_--1)/2+1)} \nu^{(a-1)/2-i}\rho \right) \rtimes T_\eta,$$

$$\pi'_\eta = \delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \rtimes T_\eta.$$

Then:
(i) The multiplicity of σ_η in corresponding Jacquet modules of π_0, π_η and π'_η is one.
(ii) The multiplicity of σ_η in corresponding Jacquet modules of $\pi_{-\eta}$ and $\pi'_{-\eta}$ is zero.

Proof. Let $k \in \mathbb{Z}_+$. For an irreducible representation γ of S_q and $\tau \otimes \sigma$ of $M_{(p')} \cong GL(p', F') \times S_{q-p'}$ we shall write

$$m(\tau \otimes \sigma, \mu^*(\gamma)) = k$$

if $k(\tau \otimes \sigma) \leq \mu^*(\gamma)$ and $(k+1)(\tau \otimes \sigma) \not\leq \mu^*(\gamma)$.

Denote
$$\sigma'_\eta = \delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \otimes T_\eta.$$

Then the multiplicity of σ'_η in corresponding Jacquet modules of π_0, π_η, $\pi_{-\eta}$, π'_η and $\pi'_{-\eta}$ is equal to multiplicity of σ_η in corresponding Jacquet modules of π_0, π_η, $\pi_{-\eta}$, π'_η and $\pi'_{-\eta}$ respectively (this follows from the well-known characterization of $\delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho])$ in terms of Jacquet modules).

We have

$$\text{(4-13)} \quad \mu^* \left(\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi \right) = \left(\sum_{i=-(a_--1)/2-1}^{(a-1)/2} \sum_{j=i}^{(a-1)/2} \delta([\nu^{-i}\rho, \nu^{(a_--1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]) \right) \rtimes \mu^*(\pi).$$

Note

$$\pi'_\eta \leq \pi_\eta. \tag{4-14}$$

Since

$$\delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \leq M^*(\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]))$$

(take in (4-13) indexes $i = -(a_- - 1)/2 - 1$, $j = (a_- - 1)/2$), we get

$$1 \le m(\sigma'_\eta : \mu^*(\pi_0)). \tag{4-15}$$

The Frobenius reciprocity implies

$$1 \le m(\sigma'_\eta : \mu^*(\pi'_\eta)). \tag{4-16}$$

Now (4-14) implies

$$1 \le m(\sigma'_\eta : \mu^*(\pi_\eta)). \tag{4-17}$$

Note

$$\pi_0 \le \pi'_1 + \pi'_{-1} \le \pi_1 + \pi_{-1}. \tag{4-18}$$

Lemma 4.5 implies

$$m(\sigma_\eta, \mu^*(\pi_1 + \pi_{-1})) = m(\sigma_\eta, \mu^*(\pi_\eta)) = 1. \tag{4-19}$$

Now from (4-14) - (4-19) we can conclude all the claims of the lemma. □

From the above lemma follows directly the following corollary, which is already proved in [M2] (Remark 5.1.1).

4.7. Corollary. *Let the notation and the assumptions be the same as in the above lemma. Then*

(i) The representation π_0 has exactly two irreducible subrepresentations. They are not isomorphic.

(ii) The representation π_η has a unique irreducible subrepresentation. This subrepresentation is a unique irreducible subrepresentation of π'_η. The corresponding two irreducible subrepresentations, for $\eta = \pm 1$, are irreducible subrepresentations of π_0.

(iii) The representations π'_η are reducible.

Proof. We get directly (i) and (ii) from $\pi_0 \hookrightarrow \pi_1 \oplus \pi_{-1}$ and the multiplicities in the above lemma. Namely, π_1 and π_{-1} have unique irreducible subrepresentations and they are not isomorphic (this follows from the above lemma). Further, these irreducible subrepresentations must show up in π_0 because of the multiplicities calculated in the above lemma. It remains to prove (iii).

Suppose that π'_η is irreducible. Then $\pi'_\eta \le \pi_0$ by (ii). This implies $\mu^*(\pi'_\eta) \le \mu^*(\pi_0)$. Therefore, $\delta([\nu^{-(a-1)/2}\rho, \nu^{-(a_- -1)/2-1}\rho]) \otimes T_\eta$ is a subquotient of (4-13) (see Lemma 4.4). Since $a, a_- > 0$, to be able to get $\delta([\nu^{-(a-1)/2}\rho, \nu^{-(a_- -1)/2-1}\rho]) \otimes T_\eta$ for a subquotient of a term in the sum (4-13), we need to have $-i > (a_- - 1)/2$ and $j + 1 > (a-1)/2$. This implies $i = -(a_- - 1)/2 - 1$ and $j = (a-1)/2$. Thus, $\delta([\nu^{-(a-1)/2}\rho, \nu^{-(a_- -1)/2-1}\rho]) \otimes T_\eta$ must be a subquotient of $1 \otimes \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \mu^*(\pi)$. Therefore, we must have $\delta([\nu^{-(a-1)/2}\rho, \nu^{-(a_- -1)/2-1}\rho]) \otimes \tau \le \mu^*(\pi)$ for some irreducible τ. This contradicts the square integrability of π. The proof is now complete. □

5. Factors of square integrable representations and *Jord*

We shall use very often the following lemma, which follows from Remark 5.1.3 of [M2].

5.1. Lemma. *Let π be an irreducible square integrable representation of S_q and let ρ be an irreducible F'/F-selfdual cuspidal representation of $GL(p, F')$. Suppose that we have $a \in Jord_\rho(\pi)$ such that a_- is defined and that holds*

$$\epsilon_\pi(\rho, a) = \epsilon_\pi(\rho, a_-).$$

Denote $Jord' = Jord(\pi) \setminus \{(\rho, a), (\rho, a_-)\}$. Let $Jord', \pi_{cusp}, \epsilon'$ be the subordinated triple to $Jord(\pi), \pi_{cusp}, \epsilon_\pi$. Then there exists an irreducible square integrable representation π' of $S_{q-p(a+a_-)/2}$ such that $(Jord(\pi'), \pi'_{cusp}, \epsilon_{\pi'}) = (Jord', \pi_{cusp}, \epsilon')$. Further

$$\pi \hookrightarrow \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'. \tag{5-1}$$

Proof. Since $\epsilon_\pi(\rho, a) = \epsilon_\pi(\rho, a_-)$, we have $\pi \hookrightarrow \delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \rtimes \tau$ for some irreducible τ. The first lemma in the fifth section of [M2] implies

$$\tau \hookrightarrow \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \rtimes \pi'$$

for the irreducible square integrable representation π' satisfying $(Jord(\pi'), (\pi')_{cusp}, \epsilon_{\pi'}) = (Jord', \pi_{cusp}, \epsilon')$. Thus

$$\pi \hookrightarrow \delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \rtimes \pi'. \tag{5-2}$$

Note that

$$\begin{aligned} &\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi' \\ &\quad \hookrightarrow \delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \rtimes \pi'. \end{aligned} \tag{5-3}$$

By Lemma 4.5 (and Remark 4.2), the representation

$$\delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \rtimes \pi'$$

has exactly two irreducible subrepresentations. Since Corollary 4.7 implies that the representation $\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'$ has also exactly two irreducible subrepresentations, (5-2) and (5-3) imply (5-1). □

Let π be an irreducible square integrable representation of S_q. If π is strongly positive, then by definition

$$e(\tau) > 0$$

for each factor τ of π. Further, the admissible triple $Jord(\pi), \pi_{cusp}, \epsilon_\pi$ is of alternated type. Suppose that π is not strongly positive. Then $Jord(\pi), \pi_{cusp}, \epsilon_\pi$ is of mixed type. The the following lemma gives us a general information about factors of square integrable representations which are not strongly positive.

To simplify discussion in some cases, we shall often in the sequel restrict us to the case where the admissible triple $Jord, \pi_{cusp}, \epsilon$ satisfies the following condition: there exists an irreducible cuspidal F'/F-selfdual representation ρ of $GL(p, F')$ such that

$$\text{(L)} \qquad Jord \setminus \{(\rho, a); a \in Jord_\rho\} \subseteq Jord(\pi_{cusp}).$$

This means that we are looking only at the irreducible square integrable representations which are subquotients of the parabolically induced representations of the following type:

$$\left(\prod_{x \in \mathcal{R}} \nu^x \rho\right) \rtimes \pi_{cusp},$$

where $\mathcal{R}$ is a finite subset of $\mathbb{R}$.

We shall say that an irreducible square integrable representation π of S_q satisfies (L), if $Jord(\pi), \pi_{cusp}, \epsilon_\pi$ satisfies (L). The representation ρ in the condition (L) will be always fixed in such a way, that it is clear which ρ is considered.

5.2. Lemma. *Suppose that π is an irreducible square integrable representation of S_q which is not strongly positive. Take any $(\rho, a) \in Jord(\pi)$ such that $a_- \in Jord_\rho(\pi)$ is defined, and that $\epsilon_\pi(\rho, a) = \epsilon_\pi(\rho, a_-)$. Suppose that (L) holds. Define $a_\rho(\pi) \in Jord_\rho(\pi)$ to be maximal such a in $Jord_\rho(\pi)$ (then $a_\rho(\pi)_- \in Jord_\rho(\pi)$ is defined and $\epsilon_\pi(\rho, a_\rho(\pi)) = \epsilon_\pi(\rho, a_\rho(\pi)_-)$). Now if $\tau = \nu^k \rho$, $k \in \mathbb{R}$, is a factor of π, then*

$$\text{(5-4)} \qquad k = e(\tau) \geq -(a_\rho(\pi)_- - 1)/2.$$

Proof. The proof proceeds by induction with respect to q. Denote

$$a = a_\rho(\pi).$$

By the above lemma, there exists an irreducible square integrable representation π' such that

$$\text{(5-5)} \qquad \pi \hookrightarrow \delta([\nu^{-(a_- - 1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'$$

where

$$\text{(5-6)} \qquad Jord_\rho(\pi') = Jord_\rho(\pi) \setminus \{a, a_-\}$$

and $Jord(\pi'), \pi'_{cusp}, \epsilon_{\pi'}$ is subordinated to $Jord(\pi), \pi_{cusp}, \epsilon_\pi$. Note that (5-6) and the inductive assumption imply

(5-7) either π' is strongly positive, or $a_\rho(\pi')$ is defined and $a_\rho(\pi')_- < a_- = a_\rho(\pi)_-$.

Thus, the by inductive assumption by the fact that π' is strongly positive if it is strongly positive, imply that $\nu^\alpha \rho$ is not a factor of π' if $\alpha \leq -(a_- - 1)/2$.

Suppose that (5-4) does not hold. Let $\tau = \nu^k\rho$ be a factor for which (5-4) does not hold. Fix such a factor $\tau = \nu^k\rho$ with minimal possible k. Note that (5-5), the fact that $\nu^\alpha\rho$ is not a factor of π' if $\alpha \leq -(a_- - 1)/2$, (1-2) and (1-4) imply that

$$k \in [-(a-1)/2, -(a_- - 1)/2 - 1] \tag{5-8}$$

(and $k + (a-1)/2 \in \mathbb{Z}$). Clearly $k \leq -1$. By the choice of k, $\tau = \nu^k\rho$ satisfies the assumption of Lemma 3.4 (and also (5-8)). Let $c = -2k + 1$ (then $k = -(c-1)/2$ and $c \geq 3$). Now (5-8) implies $-(a-1)/2 \leq -(c-1)/2 < -(a_- - 1)/2$. Thus

$$a_- < c \leq a. \tag{5-9}$$

Lemma 3.4 implies

$$\pi \hookrightarrow \delta([\nu^{-(c-1)/2}\rho, \nu^{(b-1)/2}\rho]) \rtimes \pi'' \tag{5-10}$$

for some $b \in c + 2\mathbb{Z}$ which satisfies $-(c-1)/2 \leq (b-1)/2$ (i.e. $b + c - 2 \geq 0$). The square integrability of π and (5-10) (and the Frobenius reciprocity), imply

$$c < b. \tag{5-11}$$

Since (by our choice) k is minimal such that $\nu^k\rho$ is a factor of π, there is no embedding $\pi \hookrightarrow \delta([\nu^{-(c'-1)/2}\rho, \nu^{(b-1)/2}\rho]) \rtimes \pi'''$, for some irreducible π''' with $-(c'-1)/2 < (c-1)/2$. Therefore, Remark 3.2 of [M2] implies that π'' is square integrable. Proposition 2.1 now implies

$$b, c \in Jord_\rho(\pi). \tag{5-12}$$

From (5-9), (5-11) and (5-12) we get

$$a \leq c < b. \tag{5-13}$$

This implies that $b_- \in Jord_\rho(\pi)$ is defined and (5-13) implies $b_- \geq c$. Now (5-10), Lemma 3.2 and the definition of ϵ_π imply

$$\epsilon_\pi(\rho, b) = \epsilon_\pi(\rho, b_-) \tag{5-14}$$

(see also Lemma 5.2 in [M2]).

Note that (5-14) implies $b \leq a = a_\rho(\pi)$. This contradicts to (5-13). This contradiction completes the proof. $\square$

We shall include here the following lemma, although we shall not use the lemma in this paper. This lemma complements Lemma 5.4.1 of [M2].

5.3. Lemma. *Let π be an irreducible square integrable representation. Suppose $a \in Jord_\rho(\pi)$ and $a+2 \notin Jord_\rho(\pi)$. Then*

$$\nu^{(a+1)/2}\rho \rtimes \pi \tag{10-1}$$

reduces. Further, it contains a unique irreducible subrepresentation

Proof. Suppose that $\nu^{(a+1)/2}\rho \rtimes \pi$ is irreducible. Let τ be an irreducible subrepresentation of $\delta([\nu^{-(a+1)/2}\rho, \nu^{(a+1)/2}\rho]) \rtimes \pi$. Then

$$\begin{aligned}
\tau \hookrightarrow \delta([\nu^{-(a+1)/2}\rho, \nu^{(a+1)/2}\rho]) \rtimes \pi \\
&\hookrightarrow \delta([\nu^{-(a+1)/2+1}\rho + 1, \nu^{(a+1)/2}\rho]) \times \nu^{-(a+1)/2}\rho \rtimes \pi \\
&\cong \delta([\nu^{-(a+1)/2+1}\rho + 1, \nu^{(a+1)/2}\rho]) \times \nu^{(a+1)/2}\rho \rtimes \pi \\
&\cong \nu^{(a+1)/2}\rho \times \delta([\nu^{-(a+1)/2+1}\rho + 1, \nu^{(a+1)/2}\rho]) \rtimes \pi \\
&\hookrightarrow \nu^{(a+1)/2}\rho \times \nu^{(a+1)/2}\rho \times \delta([\nu^{-(a+1)/2+1}\rho + 1, \nu^{(a+1)/2-1}\rho]) \rtimes \pi.
\end{aligned}$$

Now Frobenius reciprocity implies that

$$\tau' = \nu^{(a+1)/2}\rho \times \nu^{(a+1)/2}\rho \otimes \delta([\nu^{-(a-1)/2}\rho + 1, \nu^{(a-1)/2}\rho]) \rtimes \pi$$

is a subquotient of a corresponding Jacquet module of π (note that τ' is irreducible).

Now we get directly that the multiplicity of τ' in $\mu^*(\delta([\nu^{-(a+1)/2}\rho, \nu^{(a+1)/2}\rho]) \rtimes \pi)$ is one (one uses $a+2 \notin Jord_\rho(\pi)$). This proves irreducibility of $\delta([\nu^{-(a+1)/2}\rho, \nu^{(a+1)/2}\rho]) \rtimes \pi$, which contradicts our assumption $a+2 \notin Jord_\rho(\pi)$ (note that $a \in Jord_\rho(\pi)$ and therefore a and also $a+2$ satisfy (J-1)). Thus, we have proved reducibility. From Lemma 4.5 follows the uniqueness of irreducible subrepresentation. □

Having in mind Lemma 5.4.1 of [M2] and the above (simple) lemma, it remains to describe a criterion for reducibility of $\nu^{(a+1)/2}\rho \rtimes \pi$ when $a \in Jord_\rho(\pi)$ and $a+2 \in Jord_\rho(\pi)$. The criterion is the following: we have irreducibility if and only if

$$\epsilon_\pi(\rho, a+2) \neq \epsilon_\pi(\rho, a).$$

This is proved using intertwining operators. The composition of the two standard intertwining operators:

$$\nu^{(a+1)/2+s}\rho \rtimes \pi \to \nu^{-(a+1)/2-s}\rho \rtimes \pi \to \nu^{(a+1)/2+s}\rho \rtimes \pi$$

is holomorphic and non-zero at $s=0$ under the hypothesis that (ρ, a) and $(\rho, a+2)$ are in $Jord(\pi)$. Moreover the first one is holomorphic at $s=0$ by a general result of Harish-Chandra. Therefore, we need to prove that the second one is also holomorphic. This can be done by an inductive argument with respect to π (since the argument is longer for writing, and we shall not use this result in this paper, we shall not write it down here).

6. Theorem

In [M2] is proved that the mapping $\pi \longrightarrow (Jord(\pi), \pi_{cusp}, \epsilon_\pi)$ is an injective mapping from the set of all equivalence classes of irreducible square integrable representation of S_n into the set of all admissible triples of degree n. In this paper we shall prove that this mapping is surjective. In this way we shall prove the following

6.1. Theorem. *Assume that (BA) and (A) hold. Then the map $\pi \longrightarrow (Jord(\pi), \pi_{cusp}, \epsilon_\pi)$ defines a bijection of the set of all equivalence classes of irreducible square integrable representation of S_n onto the set of all admissible triples of degree n.*

This classification theorem can be formulated without assuming (A). For such formulations, see Remark 14.5.

In the next five sections we shall prove this theorem.

7. Strongly positive irreducible representations

Let $Jord$, π_{cusp}, ϵ be an admissible triple of alternated type. In the first part of this section we shall assume that (L) holds for this triple. When we shall work with $Jord$ which satisfies (L), then for $a \in Jord_\rho$ we shall usually denote $\epsilon(\rho, a)$ simply by $\epsilon(a)$. Also the function ϕ_ρ will be denoted simply by ϕ. Note that if $\rho' \not\cong \rho$, then $\phi_{\rho'}(b) = b$ for $b \in Jord_{\rho'} = Jord_{\rho'}(\pi_{cusp})$.

Write the elements of $Jord_\rho$ as

(7-1) $$a_1 < a_2 < \cdots < a_k$$

(the possibility $k = 0$ is not excluded). Recall that $Jord_\rho \subseteq 2\mathbb{N}$ or $1 + 2\mathbb{Z}_+$. We shall say that we are in the even case if $Jord_\rho \subseteq 2\mathbb{N}$, and in the odd case if $Jord_\rho \subseteq 1 + 2\mathbb{Z}_+$.

If we are in the odd case, then from (2-3) and (2-6) follows

(7-2) $$Jord'_\rho(\pi_{cusp}) = \{1, 3, \dots, 2k-1\},$$

(7-3) $$\phi : \{a_1, a_2, \dots, a_k\} \longrightarrow \{1, 3, \dots, 2k-1\}, \quad a_i \mapsto 2i - 1.$$

In the even case we have two possibilities.

Suppose $\epsilon(a_1) = -1$. Then

(7-4) $$Jord'_\rho(\pi_{cusp}) = \{2, 4, \dots, 2k\},$$

(7-5) $$\phi : \{a_1, a_2, \dots, a_k\} \longrightarrow \{2, 4, \dots, 2k\}, \quad a_i \mapsto 2i.$$

If $\epsilon(a_1) = 1$, then

(7-6) $$Jord'_\rho(\pi_{cusp}) = \{0, 2, 4, \dots, 2(k-1)\},$$

(7-7) $$\phi : \{a_1, a_2, \dots, a_k\} \longrightarrow \{0, 2, 4, \dots, 2(k-1)\}, \quad a_i \mapsto 2(i-1).$$

Suppose that we have some $Jord$ and π_{cusp} (we do not assume that they form an admissible triple in the moment). Assume that they satisfy (L). Write $Jord_\rho$ as in (7-1).

Let $Jord_\rho \subseteq 1 + 2\mathbb{Z}$. Suppose we have a function ϕ as in (7-3). Then it is easy to see that there is a unique partially defined function ϵ such that $Jord, \pi_{cusp}, \epsilon$ form an admissible triple of alternated type, and that the corresponding function ϕ_ρ is ϕ.

suppose $Jord_\rho \subseteq 2\mathbb{Z}$ and suppose that a function ϕ as in (7-5) (resp. (7-7)), is given. Then again there exists a unique partially defined function ϵ such that $Jord, \pi_{cusp}, \epsilon$ form an admissible triple of alternated type such that the corresponding function ϕ_ρ is ϕ. One need to take $\epsilon(a_1) = -1$ (resp. $\epsilon(a_1) = 1$).

Therefore, we do not need to work with ϵ's. Instead we can work directly with functions ϕ.

Note that (7-1) – (7-7) imply

$$a_i \geq \phi(a_i) \tag{7-8}$$

$$a_{i'} > \phi(a_{i'}) \quad \Rightarrow \quad a_i \geq \phi(a_i) + 2 \quad \text{for all } i \geq i'. \tag{7-9}$$

Lemma 4.1 implies that the representation

$$\sigma_0 = \left(\prod_{i=1}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho]) \right) \rtimes \pi_{cusp} \tag{7-10}$$

has a unique irreducible subrepresentation. We shall denote this subrepresentation by

$$\pi = \pi_{(Jord, \pi_{cusp}, \epsilon)} = \pi_{(Jord, \pi_{cusp}, \phi)}. \tag{7-11}$$

In this section we shall prove

7.1. Proposition. *The representation π is strongly positive (square integrable representation).*

If $\phi(a_i) = a_i$ for all i, then $\pi = \pi_{cusp}$ and the proposition holds. If this is not the case, then $a_k \geq \phi(a_k) + 2$. Therefore, we shall assume

$$a_k \geq \phi(a_k) + 2.$$

We shall prove the proposition by induction with respect to the degree of $Jord$.

Let

$$j = \min\{i \ ; \ 1 \leq i \leq k \text{ and } a_i > \phi(a_i)\}.$$

In the proof of the proposition we shall need the following

7.2. Lemma. *(i) The multiplicity of*

$$\nu^{(a_j-1)/2}\rho \otimes \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-1)/2-1}\rho]) \otimes \left(\otimes_{i=j+1}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho]) \right) \otimes \pi_{cusp}$$

in a corresponding standard Jacquet module of

$$\begin{aligned} \sigma_1 = \nu^{(a_j-1)/2}\rho \times \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-1)/2-1}\rho]) \\ \times \left(\prod_{i=j+1}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho]) \right) \rtimes \pi_{cusp} \end{aligned} \tag{7-12}$$

is one.
(ii) Suppose $a_j \geq \phi(a_j) + 4$. *Then the multiplicity of*

$$\delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) \otimes \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-2)/2-2}\rho]) \otimes \left(\otimes_{i=j+1}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho])\right) \otimes \pi_{cusp}$$

in a corresponding standard Jacquet module of

$$\text{(7-13)} \quad \sigma_2 = \delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) \times \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-1)/2-2}\rho]) \times \left(\prod_{i=j+1}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho])\right) \rtimes \pi_{cusp}$$

is one.
(iii) Suppose $a_j = \phi(a_j) + 2$ *and* $j < n$. *Then the multiplicity of*

$$\nu^{(a_j-1)/2}\rho \otimes \nu^{(a_{j+1}-1)/2}\rho \otimes \delta([\nu^{(\phi(a_{j+1})+1)/2}\rho, \nu^{(a_{j+1}-1)/2-1}\rho]) \otimes \left(\otimes_{i=j+2}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho])\right) \otimes \pi_{cusp}$$

in a corresponding standard Jacquet module of

$$\text{(7-14)} \quad \sigma_3 = \nu^{(a_j-1)/2}\rho \times \nu^{(a_{j+1}-1)/2}\rho \times \delta([\nu^{(\phi(a_{j+1})+1)/2}\rho, \nu^{(a_{j+1}-1)/2-1}\rho]) \times \left(\prod_{i=j+2}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho])\right) \rtimes \pi_{cusp}$$

is one.

Proof. (i) In the same way as in Lemma 4.1, the proof of (i) reduces to the fact that the multiplicity of $\nu^{(a_j-1)/2}\rho \otimes \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-2)/2-1}\rho])$ in a corresponding Jacquet module of $\nu^{(a_j-1)/2}\rho \times \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-1)/2-1}\rho])$ is one. This is true, since we are in the regular situation.

(ii) Again, as in the proof of Lemma 4.1, the proof of (ii) reduces to the fact that the multiplicity of $\delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) \otimes \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-2)/2-2}\rho])$ in a corresponding Jacquet module of $\delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) \times \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-1)/2-2}\rho])$ is one. This again follow from the regularity.

(iii) The proof of (iii) proceeds in a similar way. The proof reduces to the fact that the multiplicity of $\nu^{(a_j-1)/2}\rho \otimes \nu^{(a_{j+1}-1)/2}\rho \otimes \delta([\nu^{(\phi(a_{j+1})+1)/2}\rho, \nu^{(a_{j+1}-1)/2-1}\rho])$ in a corresponding Jacquet module of $\nu^{(a_j-1)/2}\rho \times \nu^{(a_{j+1}-1)/2}\rho \times \delta([\nu^{(\phi(a_{j+1})+1)/2}\rho, \nu^{(a_{j+1}-1)/2-1}\rho])$ is one.

Note that $a_j = \phi(a_j) + 2 < \phi(a_{j+1}) + 2$. This implies $(a_j - 1)/2 < (\phi(a_{j+1}) + 1)/2$. Thus, we are again in the regular situation. Therefore, the above multiplicity one holds and further, the multiplicity one claimed in (iii) holds. □

Proof of Proposition 7.1. We use the notation from the beginning of this section and after Proposition 7.1. Recall that σ_0 is defined in (7-10).

First we shall define a new triple with the same π_{cusp}.

Denote $Jord' = (Jord \setminus \{(\rho, a_j)\}) \cup \{(\rho, a_j - 2)\}$ if $a_j \neq 2$. Let ϕ' be the function on $Jord'_\rho$ defined by $\phi'(a_j - 2) = \phi(a_j)$ and $\phi'(a_i) = \phi(a_i)$ otherwise.

Suppose $a_j = 2$ (then $j = 1$ and we are in the setting of (7-6) and (7-7)). Denote $Jord' = Jord \setminus \{(\rho, 2)\}$. In this case take ϕ' to be the restriction of ϕ.

Now the representation

$$\sigma_1' = \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-1)/2-1}\rho]) \times \left(\prod_{i=j+1}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho]) \right) \rtimes \pi_{cusp}$$

contains by Lemma 4.1 a unique irreducible subrepresentation $\pi' = \pi_{(Jord', \pi_{cusp}, \phi')}$. By the inductive assumption, π' is strongly positive (square integrable representation).

Note that $\sigma_1 = \nu^{(a_j-1)/2}\rho \rtimes \sigma_1'$, and that this representation by (i) of Lemma 7.2 contains a unique irreducible subrepresentation. Clearly $\nu^{(a_j-1)/2}\rho \rtimes \pi' \hookrightarrow \nu^{(a_j-1)/2}\rho \rtimes \sigma_1'$. Since $\sigma_0 \hookrightarrow \nu^{(a_j-1)/2}\rho \rtimes \sigma_1'$ and $\pi \hookrightarrow \sigma_0$, we get

$$\pi \hookrightarrow \nu^{(a_j-1)/2}\rho \rtimes \pi'. \tag{7-15}$$

This implies

$$s_{GL}(\pi) \leq (\nu^{(a_j-1)/2}\rho + \nu^{-(a_j-1)/2}\rho) \times s_{GL}(\pi'). \tag{7-16}$$

Suppose that π' is cuspidal. Then

$$\phi(a_j) = a_j - 2 \quad \text{and} \quad j = k. \tag{7-17}$$

Thus, $\pi \hookrightarrow \nu^{(a_k-1)/2}\rho \rtimes \pi_{cusp}$. Suppose that we are in the even case. If $\epsilon(a_1) = 1$, then k must be 1, and then $a_1 = 2$. The reducibility is at $1/2$ by (BA). This implies that the irreducible subrepresentation of $\nu^{1/2}\rho \rtimes \pi_{cusp}$ is square integrable and strongly positive. Suppose $\epsilon(a_1) = -1$. Then $a_k = 2k + 2$. Now we have reducibility at $(2k + 1)/2 = (a_k - 1)/2$. Therefore, we get again that π is strongly positive. Suppose that we are in the odd case. Now (7-16) and the definition of j imply that we have reducibility at $((2k - 1) + 1)/2 = k = (a_k - 1)/2$. Thus, π is again strongly positive.

It remains to consider the case of non-cuspidal π'. There are two possibilities.

Assume $a_j > \phi(j)+2$. Thus $a_j \geq \phi(j)+4$ (note that $a_{j-1} = \phi(a_{j-1})$ if $j > 1$). If $a_j > 4$, define $Jord'' = (Jord \setminus \{(\rho, a_j)\}) \cup \{(\rho, a_j - 4)\}$, and define ϕ'') by $\phi''(a_j - 4) = \phi(a_j)$ and $\phi''(a_i) = \phi(a_i)$ otherwise. If $a_j = \phi(j) + 4$, define $Jord'' = Jord \setminus \{(\rho, a_j)\}$ and take ϕ'' to be the restriction of ϕ.

Now the representation

$$\sigma_2' = \delta([\nu^{(\phi(a_j)+1)/2}\rho, \nu^{(a_j-1)/2-2}\rho]) \times \left(\prod_{i=j+1}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho]) \right) \rtimes \pi_{cusp}$$

contains a unique irreducible subrepresentation $\pi'' = \pi_{(Jord'',\pi_{cusp},\phi'')}$. By the inductive assumption, π'' is strongly positive. Since $\sigma_0 \hookrightarrow \delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) \rtimes \sigma_2'$, $\delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) \rtimes \pi'' \hookrightarrow \delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) \rtimes \sigma_2'$, $\pi \hookrightarrow \sigma_0$ and $\sigma_2 = \delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) \rtimes \sigma_2'$ contains a unique irreducible subrepresentation, we get

$$\pi \hookrightarrow \delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) \rtimes \pi''. \tag{7-18}$$

This implies

$$s_{GL}(\pi) \leq (\delta([\nu^{(a_j-1)/2-1}\rho, \nu^{(a_j-1)/2}\rho]) + \nu^{-(a_j-1)/2-1}\rho \times \nu^{(a_j-1)/2}\rho + \delta([\nu^{-(a_j-1)/2}\rho, \nu^{-(a_j-1)/2+1}\rho])) \times s_{GL}(\pi''). \tag{7-19}$$

Since π' and π'' are strongly positive, (7-19) applied to (7-16) gives

$$s_{GL}(\pi) \leq \nu^{(a_j-1)/2}\rho \times s_{GL}(\pi'). \tag{7-20}$$

Since π' is strongly positive, (7-20) implies that π is strongly positive.

It remains to consider the case $a_j = \phi(a_j) + 2$. Since π' is not cuspidal, we have $j < k$.

Suppose $a_{j+1} = \phi(a_{j+1}) + 2$. Note that $a_{j+1} = \phi(a_{j+1}) + 2 = \phi(a_j) + 4 = a_j + 2$. Then in the same way as we defined strongly positive representation π' from π, we can repeat that construction and get that there exists an irreducible subrepresentation π''' of

$$\sigma_3' = \left(\prod_{i=j+2}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho]) \right) \rtimes \pi_{cusp},$$

which is by the inductive assumption strongly positive, and that

$$\pi' \hookrightarrow \nu^{(a_j+1)/2}\rho \rtimes \pi''', \tag{7-21}$$

which implies

$$\pi \hookrightarrow \nu^{(a_j-1)/2}\rho \rtimes \pi' \hookrightarrow \nu^{(a_j-1)/2}\rho \times \nu^{(a_j+1)/2}\rho \rtimes \pi''' \hookrightarrow \nu^{(a_j-1)/2}\rho \times \nu^{(a_j+1)/2}\rho \rtimes \sigma_3'. \tag{7-22}$$

Since

$$\mathfrak{s}([\nu^{(a_j-1)/2}\rho, \nu^{(a_j-1)/2+1}\rho]) \rtimes \pi''' \hookrightarrow \nu^{(a_j-1)/2}\rho \times \nu^{(a_j-1)/2+1}\rho \rtimes \sigma_3'$$

and $\sigma_3 = \nu^{(a_j-1)/2}\rho \times \nu^{(a_j+1)/2}\rho \rtimes \sigma_3'$ has a unique irreducible subrepresentation by (iii) of Lemma 7.2, we get

$$\pi \hookrightarrow \mathfrak{s}([\nu^{(a_j-1)/2}\rho, \nu^{(a_j-1)/2+1}\rho]) \rtimes \pi'''.$$

This implies

$$s_{GL}(\pi) \leq \Big(\mathfrak{s}([\nu^{(a_j-1)/2}\rho, \nu^{(a_j-1)/2+1}\rho]) + \nu^{-(a_j-1)/2-1}\rho \times \nu^{(a_j-1)/2}\rho + \mathfrak{s}([\nu^{-(a_j-1)/2-1}\rho, \nu^{-(a_j-1)/2}\rho]) \Big) \rtimes s_{GL}(\pi''').$$

From this and (7-16) we get again (7-20) (using that π''' is strongly positive). Thus, that π is strongly positive.

It remains to consider the case of $a_{j+1} > \phi(a_{j+1}) + 2 = \phi(a_j) + 4 = a_j + 2$. Note that π embeds into σ_3, which is isomorphic to

$$\nu^{(a_{j+1}-1)/2}\rho \times \nu^{(a_j-1)/2}\rho \times \delta([\nu^{(\phi(a_{j+1})+1)/2}\rho, \nu^{(a_{j+1}-1)/2-1}\rho]) \times \left(\prod_{i=j+2}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho])\right) \rtimes \pi_{cusp} \tag{7-23}$$

(since $a_{j+1} > a_j + 2$). Recall that σ_3 (and also the above representation) has a unique irreducible subrepresentation by (iii) of Lemma 7.2. Consider $Jord'''' = (Jord \setminus \{(\rho, a_{j+1})\}) \cup \{(\rho, a_{j+1} - 2)\}$ and ϕ'''' defined by $\phi''''(a_{j+1} - 2) = \phi(a_{j+1})$ and $\phi''''(a_i) = \phi(a_i)$ otherwise. By Lemma 4.1 we know that the representation

$$\sigma_3'' = \nu^{(a_j-1)/2}\rho \times \delta([\nu^{(\phi(a_{j+1})+1)/2}\rho, \nu^{(a_{j+1}-1)/2-1}\rho]) \times \left(\prod_{i=j+2}^{k} \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho])\right) \rtimes \pi_{cusp}$$

has a unique irreducible subrepresentation, which we denote by π''''. Further, the inductive assumption implies that π'''' is strongly positive. Since $\nu^{(a_j-1)/2}\rho \rtimes \pi'''' \hookrightarrow \nu^{(a_j-1)/2}\rho \rtimes \sigma_3'' \cong \sigma_3$, $\pi \hookrightarrow \sigma_0 \hookrightarrow \sigma_3$ and σ_3 has a unique irreducible subrepresentation, we get

$$\pi \hookrightarrow \nu^{(a_{j+1}-1)/2}\rho \rtimes \pi''''.$$

This implies

$$s_{GL}(\pi) \leq (\nu^{(a_{j+1}-1)/2}\rho + \nu^{-(a_{j+1}-1)/2}\rho) \times s_{GL}(\pi'''').$$

Since $(a_{j+1} - 1)/2 \neq (a_j - 1)/2$, the above inequality and (7-16) imply (7-20) (using that π'''' is strongly positive). Thus, π is strongly positive. Now the proof of the square integrability claimed in the proposition is complete. □

At the end of this section, we shall not assume more that (L) holds for our admissible triple. Denote

$$\sigma = \left(\prod_{\rho}\left(\prod_{a \in Jord_\rho} \delta([\nu^{(\phi_\rho(a)+1)/2}\rho, \nu^{(a-1)/2}\rho])\right)\right) \rtimes \pi_{cusp},$$

where the first product runs over all ρ for which $(\rho, a) \in Jord$ for some $a \in \mathbb{N}$, and the second product is taken in an order which follows the ordering of $Jord_\rho$. Then in a similar way as in Lemma 4.1 follows that σ has a unique irreducible subrepresentation. Denote it by $\pi = \pi_{(Jord, \pi_{cusp}, \epsilon)}$. Now Proposition 7.1 imply that π is square integrable. There are several arguments for that. Maybe the most elementary is a simple elementary Lemma 4.7 of [T4].

7.3. Lemma. *Let* $Jord, \pi_{cusp}, \epsilon$ *be an admissible triple of the alternated type. We have* $(\pi_{(Jord,\pi_{cusp},\epsilon)})_{cusp} = \pi_{cusp}$. *Denote* $\pi = \pi_{(Jord,\pi_{cusp},\epsilon)}$. *Then*

$$(Jord(\pi), \pi_{cusp}, \epsilon_\pi) = (Jord, \pi_{cusp}, \epsilon).$$

Proof. It is easy to see from the proof that the partial cuspidal support of π is π_{cusp}. The fact that $Jord(\pi)$ coincides with $Jord$ is proved using 2.1 (i). Now the representation π coincides with the strongly positive representation in 4.1 of [M2]. The strongly positive property is equivalent (see 5.3 of [M2]) to the fact that ϵ_π is the alternated. One easily sees that alternated partially defined function on $Jord$, if exists, is unique. This implies $\epsilon_\pi = \epsilon$. □

This ends the proof that each admissible triple of the alternated type comes from a (strongly positive) square integrable representation. Therefore, for a proof of Theorem 6.1 we need to settle the case of admissible triples of the mixed type. Before we go to the proof in this case, we shall prove a useful (essentially combinatorial) result in the following section.

The simplest examples of the strongly positive irreducible square integrable representations are Steinberg representations. Further simple examples of such representations can be found in [T1]. Let π be a strongly positive irreducible square integrable representation. Then it is easy to show that $s_{GL}(\pi)$ is irreducible (one can also write directly the Langlands parameters of these Jacquet modules). Other Jacquet modules do not need to be always irreducible (for the difference of the case of Steinberg representations). A very interesting representations from the point of view of the unitary duals of general linear groups, can be tensor factors in $s_{GL}(\pi)$.

8. Tempered and square integrable representations with the same infinitesimal character.

Let $Jord$, π_{cusp}, ϵ be an admissible triple of the alternated type, and let π be the strongly positive representations with these invariants. Suppose

$$\pi \leq \tau_1 \times \ldots \times \tau_n \rtimes \pi_{cusp}, \tag{8-1}$$

where τ_i are irreducible cuspidal representations. Then by the induction with respect to the number of elements in $Jord \setminus Jord(\pi_{cusp})$ we get

$$\mathrm{Supp}(Jord) = \mathrm{Supp}(Jord(\pi)) = \mathrm{Supp}(Jord(\pi_{cusp})) + \sum_{i=1}^{n} \{\tau_i, \check{\tau}_i\}. \tag{8-2}$$

Suppose now that the triple $Jord, \pi_{cusp}, \epsilon$ is of the mixed type. Then there exists a sequence of triples $Jord_i, \pi_{cusp}, \epsilon_i$, $1 \leq i \leq k$ such that: $(Jord, \pi_{cusp}, \epsilon) = (Jord_1, \pi_{cusp}, \epsilon_1)$, $Jord_{i+1}, \pi_{cusp}, \epsilon_{i+1}$ is subordinated to $Jord_i, \pi_{cusp}, \epsilon_i$ for $1 \leq i \leq k-1$, and $Jord_k, \pi_{cusp}, \epsilon_k$ is admissible of alternated type. Therefore, there exists $a(i), a(i)_- \in (Jord_i)_{\rho_i}$, $i = 1, \ldots, k-1$ such that

$$Jord_{i+1} = Jord_i \setminus \{(\rho_i, a(i)), (\rho_i, a(i)_-)\}, \quad i = 1, \ldots, k-1. \tag{8-3}$$

Denote by π_+ the strongly positive representation determined by $Jord_k, \pi_{cusp}, \epsilon_k$. Now

(8-4) $\quad \mathrm{Supp}(Jord)$

$$= \mathrm{Supp}(\pi_+) + \sum_{i=1}^{k-1} \left(\delta([\nu^{-(a(i)-1)/2}\rho, \nu^{(a(i)-1)/2}\rho]) + \delta([\nu^{-(a(i)_- -1)/2}\rho, \nu^{(a(i)_- -1)/2}\rho]) \right)$$

$$= \mathrm{Supp}(\pi_+) + \sum_{i=1}^{k-1} \left(\delta([\nu^{-(a(i)_- -1)/2}\rho, \nu^{(a(i)-1)/2}\rho]) + \delta([\nu^{-(a(i)-1)/2}\rho, \nu^{(a(i)_- -1)/2}\rho]) \right).$$

Suppose that $\pi_+ \leq \tau_1 \times \ldots \times \tau_n \rtimes \pi_{cusp}$, where τ_i are irreducible cuspidal representations. Consider the following element of the Grothendieck group

$$\Pi = \left(\prod_{i=1}^{k-1} \left(\prod_{j_i = -(a(i)_- -1)/2}^{(a(i)-1)/2} \nu^{j_i} \rho_i \right) \right) \times \tau_1 \times \ldots \times \tau_n \rtimes \pi_{cusp}. \tag{8-5}$$

Write

$$\Pi = \sigma_1 \times \ldots \sigma_m \rtimes \pi_{cusp}, \tag{8-6}$$

where σ_i are irreducible cuspidal representations (this is just another way of writing (8-5)). Then (8-2) and (8-4) imply

$$\mathrm{Supp}(Jord) = \mathrm{Supp}(Jord(\pi_{cusp})) + \sum_{i=1}^{m} \{\sigma_i, \check{\sigma}_i\}. \tag{8-7}$$

One reconstructs $Jord$ from $\mathrm{Supp}(Jord)$ in the following simple way. One can write $\mathrm{Supp}(Jord) = \sum_{i=1}^{l} \Delta_i$ as a sum of segments in a such a way that among segments Δ_i there is no linking (Δ_i are sets, so one can consider them as multisets). First,

$$\Delta_i \neq \Delta_j \text{ for } i \neq j. \tag{8-8}$$

Write $\Delta_i = [\nu^{-(b_i-1)/2}\gamma_i, \nu^{(b_i-1)/2}\gamma_i]$. Then

$$Jord = \{(\gamma_1, b_1), \ldots, (\gamma_l, b_l)\}. \tag{8-9}$$

The following essentially combinatorial lemma will be very useful to us.

8.1. Lemma. *Let the notation be as above. Then if π is irreducible tempered representation and $\pi \leq \Pi$, then π is square integrable.*

Proof. Suppose that π is not square integrable. Then there exit irreducible unitarizable square integrable representations $\delta(\Delta_i)$ of $GL(n_i, F')$ for $i = 1, \ldots, s$, where $s \geq 1$ and all $n_i \geq 1$, and an irreducible unitarizable square integrable representation π' of $S_{q'}$ such that

$$\pi \leq \delta(D_1) \times \ldots \times \delta(D_s) \rtimes \pi'.$$

Since $\pi \leq \Pi$, we get that $\check{\Delta}_i = \Delta_i$ for $i = 1, \dots, s$. From the formula (8-7) we get

$$\text{(8-10)} \quad \mathrm{Supp}(Jord) = \mathrm{Supp}(Jord(\pi')) + \sum_{i=1}^{s} \left(\Delta_i + \check{\Delta}_i\right) = \mathrm{Supp}(Jord(\pi')) + 2\sum_{i=1}^{s} \Delta_i.$$

Since π' is square integrable, from [M2] follows that $Jord(\pi')$, π_{cusp}, $\epsilon_{\pi'}$ form an admissible triple. Thus for $Jord(\pi')$ holds the properties which we described above for $Jord$. Now (8-10) implies that (8-8) does not hold. This contradiction completes the proof. □

The following direct consequence of the above lemma we shall not use, but it is interesting to note it.

8.2. Corollary. *Let τ_1 and τ_2 be irreducible tempered representations of S_q with the same infinitesimal character. Then τ_1 is square integrable if and only if τ_2 is square integrable.* □

9. Square integrability I

In this and the following two sections, π will denote an irreducible square integrable representation of S_q, ρ will be an irreducible cuspidal F'/F-selfdual representation of $GL(p, F')$, $a, a_- \in \mathbb{N}$ will be such that: $a - a_- \in 2\mathbb{N}$. We shall assume that (ρ, a) satisfies (J-1) and that

$$\text{(9-1)} \qquad [a_-, a] \cap Jord_\rho(\pi) = \emptyset.$$

Then by Corollary 4.7, the representation

$$\text{(9-2)} \qquad \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi$$

has exactly two irreducible subrepresentations. They are not equivalent. Write

$$\text{(9-3)} \qquad \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \rtimes \pi = T_1 \oplus T_{-1}$$

as a sum of two irreducible (tempered) representations (T_1 and T_{-1} are not equivalent). Then

$$\delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \rtimes T_\eta$$

contains a unique irreducible subrepresentation, which we denote by π_η. Further, $\pi_1 \ncong \pi_{-1}$, and these representations are the irreducible subrepresentations of (9-2). Note that

$$\text{(9-4)} \quad \pi_\eta \hookrightarrow \delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \rtimes T_\eta \\ \hookrightarrow \nu^{(a-1)/2}\rho \times \nu^{(a-1)/2-1}\rho \times \ldots \times \nu^{(a_- -1)/2+1}\rho \rtimes T_\eta.$$

Denote

$$\text{(9-5)} \qquad \Pi_\eta = \nu^{(a-1)/2}\rho \times \nu^{(a-1)/2-1}\rho \times \ldots \times \nu^{(a_- -1)/2+1}\rho \rtimes T_\eta.$$

Then Π_η has also a unique irreducible subrepresentation by Lemma 4.5 (use (9-1)).

The aim of this and the following two sections is to prove the following

9.1. Proposition. *Representations π_η are square integrable.*

The importance of the above proposition follows from the following

9.2. Lemma. *Suppose that the claim of the above proposition holds for all π, ρ, a and a_- as above, which satisfy*

$$q + p(a + a_-)/2 \leq n.$$

Then for each admissible triple $Jord, \pi_{cusp}, \epsilon$ of degree $\leq n$, there exists an irreducible square integrable representation with these invariants.

Therefore, the above lemma shows that Proposition 9.1 implies Theorem 6.1.

Proof. We shall prove the lemma by induction. The basis of induction is provided by Proposition 7.1. Let $Jord, \pi_{cusp}, \epsilon$ be an admissible triple of degree $\leq n$. If it is of the alternated type, then Proposition 7.1 implies the existence of an irreducible square integrable representation with these invariants.

Suppose now that $Jord, \pi_{cusp}, \epsilon$ is of the mixed type. Then we can choose $a, a_- \in Jord_\rho$ for some ρ, such that $\epsilon(\rho, a) = \epsilon(\rho, a_-)$. Denote $Jord' = Jord \setminus \{(\rho, a), (\rho, a_-)\}$. Let ϵ' be the restriction of ϵ to $Jord'$ (ϵ' is partially defined). Then the degree of $Jord'$ is strictly lower than the degree of $Jord$, which is $\leq n$. Since $Jord', \pi_{cusp}, \epsilon'$ is an admissible triple, the inductive assumption implies that there exists an irreducible square integrable representation π' with these invariants. Now, the assumption of the lemma implies that π_1 and π_{-1} are square integrable. Clearly, $(\pi_\eta)_{cusp} = \pi_{cusp}$ for $\eta = \pm 1$. Further, Lemma 5.2 of [M2] implies $Jord(\pi_\eta) = Jord$ for $i = \pm 1$. Therefore, the first two invariants of π_1 and π_{-1} are the same. Since $\pi_1 \not\cong \pi_{-1}$, we have $\epsilon_{\pi_1} \neq \epsilon_{\pi_{-1}}$ (this implies the main result of [M2], the injectivity of (2-2)). Further, by (9-1) we can apply Lemma 5.4, Lemma 5.5 and (ii) of Proposition 6.1 (all) from [M2]. They give that the restriction of $\epsilon_{\pi_\eta}, \eta = \pm 1$, to $Jord(\pi') = Jord'$ is $\epsilon_{\pi'} = \epsilon'$.

It is easy to see that ϵ' can be extended to a partially defined function on $Jord$, to make an admissible triple with $Jord$ and π_{cusp}, exactly in two ways. Denote these extensions by ϵ_1 and ϵ_{-1}. The above discussion implies $\epsilon_{\pi_\eta} = \epsilon_\eta$ for $\eta = \pm 1$ or $\epsilon_{\pi_\eta} = \epsilon_{-\eta}$ for $\eta = \pm 1$. Since $\epsilon \in \{\epsilon_{\pm 1}\}$, we get that there exists $\eta \in \{\pm 1\}$ such that $\epsilon_{\pi_\eta} = \epsilon$. This completes the proof. □

Now we shall start the proof of Proposition 9.1.

To prove the proposition, it is enough to prove it for π which satisfy (L). There are several ways to see this. Technically, the simplest way seems to be to apply an elementary Lemma 4.7 of [T4].

Note that the proof of the proposition that we shall present actually does not require (L), but assuming (L), the proof (which is relatively complicated) will became a little bit shorter and it shall use more simple notation.

Therefore, we shall assume that π satisfies (L) (this assumption is not important for this section, since it is automatically satisfied here).

First we shall prove the proposition for cuspidal π.

9.2. Lemma. *If π is cuspidal, then π_η are square integrable.*

Proof. We shall prove the lemma by induction with respect to $a - a_-$.

Suppose that σ is an irreducible subquotient of (9-2). From (9-2) follows

$$(9\text{-}6)\qquad s_{GL}(\sigma) \leq \sum_{i=-(a_- -1)/2}^{(a-1)/2+1} \delta([\nu^i\rho, \nu^{(a-1)/2}\rho]) \times (\delta([\nu^{-i+1}\rho, \nu^{(a_- -1)/2}\rho]) \otimes \pi_{cusp}.$$

This implies that the factors of σ are contained in

$$\{\nu^{-(a-1)/2}\rho, \nu^{-(a-1)/2+1}\rho, \dots, \nu^{(a-1)/2}\rho\}.$$

Suppose that $\nu^{-(a-1)/2}\rho$ is a factor of σ. Then we see from (9-6) that the only possibility to get $\nu^{-(a-1)/2}\rho$ as a factor, is to take the index $i = (a-1)/2$. This implies that $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \otimes \pi_{cusp} \leq s_{GL}(\sigma)$. Now Lemma 4.4 implies that σ is an irreducible (Langlands) quotient of (9-2). Since (9-2) reduces, we obtain that $\sigma \not\cong \pi_\eta$ for $\eta = \pm 1$.

Fix $\eta \in \{\pm 1\}$. We have just shown that

(9-7) the factors of π_η are contained in $\{\nu^{-(a-1)/2+1}\rho, \nu^{-(a-1)/2+2}\rho, \dots, \nu^{(a-1)/2}\rho\}$.

Suppose $a = a_- + 2$. Then $\pi_\eta \hookrightarrow \nu^{(a-1)/2}\rho \rtimes T_\eta$. Now (1-2) implies

$$s_{GL}(\pi_\eta) \leq s_{GL}(\nu^{(a-1)/2}\rho \rtimes T_\eta) = (\nu^{(a-1)/2}\rho + \nu^{-(a-1)/2}\rho) \times s_{GL}(T_\eta).$$

Further, the above inequality and (9-7) imply

$$s_{GL}(\pi_\eta) \leq \nu^{(a-1)/2}\rho \times s_{GL}(T_\eta).$$

From this and Lemma 3.5, we get that π_η is tempered (recall that T_η is tempered). Lemma 8.1 implies now that π_η is square integrable.

It remains to consider the case $a > a_- + 2$. Consider the unique irreducible subrepresentation of $\delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2-1}\rho]) \rtimes T_\eta$, which we denote by π'_η. By the inductive assumption, π'_η is square integrable. Since

$$\nu^{(a-1)/2}\rho \rtimes \pi'_\eta \hookrightarrow \nu^{(a-1)/2}\rho \rtimes \delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2-1}\rho]) \rtimes T_\eta \hookrightarrow \Pi_\eta,$$

and Π_η has a unique irreducible subrepresentation, which is π_η, we get

$$\pi_\eta \hookrightarrow \nu^{(a-1)/2}\rho \rtimes \pi'_\eta.$$

Now, in the same way as above, from (9-7) we get

$$s_{GL}(\pi_\eta) \leq \nu^{(a-1)/2}\rho \times s_{GL}(\pi'_\eta).$$

Lemma 3.5 implies the square integrability. $\square$

A number of additional information about representations π_η and their Jacquet modules in the case of cuspidal π, can be found in [T3].

10. Square integrability II

In this section, we shall continue to use the notation introduced in the last section.

For the proof of Proposition 9.1, it remains to show the square integrability of π_η in the case of non-cuspidal π. We shall prove the proposition by induction. The basis of the induction is provided by Proposition 7.1. The inductive assumption is: the claim of Proposition 9.1 holds for all π, ρ, a and a_- satisfying

$$(a + a_-)/2 + \deg Jord(\pi) < n.$$

Then Lemma 9.2 implies that each admissible triple $Jord, \pi_{cusp}, \epsilon$ of degree $< n$ corresponds to some square integrable representation.

In this section we shall suppose

$$(a + a_-)/2 + \deg Jord(\pi) = n.$$

Denote

(10-1) $$b = \max\Big\{ a' \in Jord_\rho(\pi);\quad a' - 2 \notin Jord_\rho(\pi), \text{ or } a' - 2 \in Jord_\rho(\pi) \text{ and } \epsilon_\pi(a') = \epsilon_\pi(a' - 2), \text{ or } a' = 2 \text{ and } \epsilon_\pi(a') = 1 \Big\}.$$

Suppose that the set on the right hand side of (10-1) is empty. This assumption first implies that π is strongly positive. Further, this and (2-6) imply $Jord_\rho(\pi) = Jord_\rho(\pi_{cusp})$. Thus, $\phi(a_i) = a_i$ for all i. This implies that the representation (7-10) is cuspidal. Thus, π must be cuspidal (see the seventh section). This contradicts out assumption. Therefore, since π is not cuspidal, b is well defined.

Now we shall consider all possible relations among a, a_- and b.

10.1. First we shall analyze the case

(10-2) $$b < a_-.$$

The definition of b implies

(10-3) $$\max\, Jord_\rho(\pi) \leq a_- - 2$$

(if $m = \max\, Jord_\rho(\pi) \geq a_-$, then (9-1) implies $m > a$, which implies that the set $\{a' \in Jord_\rho(\pi); a' > a\}$ is non-empty; denote the minimum of it by m'; then clearly $m' - 2 \notin Jord_\rho(\pi)$, which implies $m' \leq b$, and further $a_- < a < m' \leq b$; this contradicts (10-2)).

From this follows directly:

(10-4) factors of π are contained in $\{\nu^{-(a_- - 1)/2+1}\rho, \nu^{-(a_- - 1)/2+2}\rho, \dots, \nu^{(a_- - 1)/2-1}\rho\}$.

We can get this also from Lemma 5.2 (this lemma can give a more precise information about factors).

First we shall prove the following

Lemma. *Suppose $b < a_-$. Then π_η has no factors in the set $X = \{\nu^{-(a-1)/2-z}\rho, z \in \mathbb{Z}_+\}$*

Proof. First note that (10-4) implies that π has no factors in X. This, (1-2) and (1-4) imply that π_η has no factors in the set $\{\nu^{-(a-1)/2-z}\rho, z \in \mathbb{N}\}$. Therefore, it remains to show that $\nu^{-(a-1)/2}\rho$ is not a factor of π_η. Suppose that it is a factor.

Since $\nu^{-(a-1)/2-1}\rho$ is not a factor of π_η by the above discussion, we can apply Lemma 3.4 to π_η for $\tau = \nu^{-(a-1)/2}\rho$. Then

$$\pi_\eta \hookrightarrow \delta([\nu^{-(a-1)/2}\rho, \nu^{(a'-1)/2}\rho]) \rtimes \sigma \tag{10-5}$$

for some irreducible representation σ, and for $a' \in \mathbb{Z}$ such that $a + a' \in 2\mathbb{Z}$ and $(a'-1)/2 - (-(a-1)/2) \geq 0$ (i.e. $a' \geq -a+2$). Now (10-5) implies that

$$\nu^{(a'-1)/2}\rho \otimes \nu^{(a'-1)/2-1}\rho \otimes \dots \nu^{-(a-1)/2}\rho \otimes \rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_l \otimes \pi_{cusp} \tag{10-6}$$

is an irreducible subquotient (actually a quotient) of a corresponding Jacquet module of π_η, for some irreducible cuspidal representations ρ_i.

Recall

$$\pi_\eta \hookrightarrow \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi. \tag{10-7}$$

We shall consider two possible relation of a_- and a'. Suppose first

$$a' \leq a_-.$$

The transitivity of Jacquet modules implies that (10-6) is a subquotient of a standard Jacquet module of $s_{((a+a_-)/2)}(\pi_\eta)$. Therefore, there exists an irreducible subquotient $\sigma \otimes \tau$ of $s_{((a+a_-)/2)}(\pi_\eta)$ such that (10-6) is a subquotient of a corresponding standard Jacquet module of $\sigma \otimes \tau$. Now $a' \leq a_-$ implies that $\nu^{-(a-1)/2}\rho$ must be in the support of σ (the support is defined in Proposition 1.10 of [Z], as we already noted).

Note that (10-7) implies

$$\sigma \otimes \tau \leq s_{((a+a_-)/2)}(\pi_\eta) \leq s_{((a+a_-)/2)}(\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi). \tag{10-8}$$

Write

$$\mu^*(\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi) = \tag{10-9}$$
$$\Bigg(\sum_{i=-(a_--1)/2-1}^{(a-1)/2} \sum_{j=i}^{(a-1)/2} \delta([\nu^{-i}\rho, \nu^{(a_--1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \Bigg) \rtimes \mu^*(\pi).$$

Because of (10-4), the only terms in the above sum which can have an irreducible subquotient $\sigma' \otimes \tau'$ such that $\nu^{-(a-1)/2}\rho$ is in the support of τ' are $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a_--1)/2}\rho]) \times \sigma'' \otimes \tau''$, where $\sigma'' \otimes \tau'' \leq \mu^*(\pi)$ (then must be $i = j = (a-1)/2$ in the above sum). Since $\sigma \otimes \tau$ is a subquotient of $s_{((a+a_-)/2)}(\pi_\eta)$, there is only one possibility for $\sigma'' \otimes \tau''$: we must have $\sigma'' = 1$ and $\tau'' = \pi$. Thus $\sigma \otimes \tau = \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \otimes \pi$. Now Lemma 4.4 implies that π_η is a Langlands quotient of $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \rtimes \pi$. This and (10-7) contradicts to the reducibility of $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \rtimes \pi$.

Therefore, we must have

$$a_- < a'.$$

Note that the fact that (10-6) is a subquotient of a Jacquet module of π_η, Lemma 3.1 and Lemma 3.2 imply that

$$\nu^{(a'-1)/2}\rho \otimes \gamma \leq s_{(p)}(\pi_\eta),$$

for some irreducible γ, such that $\nu^{-(a-1)/2}\rho$ is a factor of γ. Further, (10-7) implies

$$\nu^{(a'-1)/2}\rho \otimes \gamma \leq s_{(p)}(\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi). \tag{10-10}$$

Write

$$s_{(p)}(\pi) = \sum_i \mu_i \otimes \lambda_i \tag{10-11}$$

as a sum of irreducible representations. Then (10-9) implies (recall that π satisfies (L))

$$s_{(p)}(\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi) = \sum_i \mu_i \otimes \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \lambda_i$$

$$+\nu^{(a-1)/2}\rho \otimes \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2-1}\rho]) \rtimes \pi \tag{10-12}$$

$$+\nu^{(a_- -1)/2}\rho \otimes \delta([\nu^{-(a_- -1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi. \tag{10-13}$$

Since $a_- < a'$, we see that $\nu^{(a'-1)/2}\rho \otimes \gamma$ cannot be a subquotient of (10-13). From (1-2), (1-4) and (10-4) we get that $\nu^{-(a-1)/2}\rho$ is not a factor of $\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2-1}\rho]) \rtimes \pi$ (this term shows up in (10-12)). This implies that $\nu^{(a'-1)/2}\rho \otimes \gamma$ is not a subquotient of (10-12). Therefore, $\nu^{(a'-1)/2}\rho \otimes \gamma$ is a subquotient for some i of $\mu_i \otimes \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \lambda_i$. This implies $\nu^{(a'-1)/2}\rho \cong \mu_i$. Now (10-11) and Lemma 3.6 imply $a' \in Jord_\rho(\pi)$. Since $a_- < a'$, we get a contradiction with (10-3). This contradiction ends the proof. □

We shall prove now square integrability of π_η in the case $b < a_-$. Recall (9-4)

$$\begin{aligned} \pi_\eta &\hookrightarrow \delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \times T_\eta \\ &\hookrightarrow \nu^{(a-1)/2}\rho \times \delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2-1}\rho]) \times T_\eta. \end{aligned} \tag{10-14}$$

Suppose $a = a_- + 2$. Then (10-14) implies

$$s_{GL}(\pi_\eta) \leq \left(\nu^{(a-1)/2}\rho + \nu^{-(a-1)/2}\rho\right) \times s_{GL}(T_\eta). \tag{10-15}$$

Now the above lemma implies

$$s_{GL}(\pi_\eta) \leq \nu^{(a-1)/2}\rho \times s_{GL}(T_\eta). \tag{10-16}$$

Since T_η is a tempered representation, Lemma 3.5 implies that π_η is tempered. Now Lemma 8.1 implies that π_η is square integrable.

Suppose now $a > a_- + 2$. Then denote by π'_η the irreducible subrepresentation of

$$\delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2-1}\rho]) \rtimes T_\eta,$$

(which is unique; see section 9). Then the inductive assumption implies that π'_η is square integrable. Since the representation Π_η from (9-5) has a unique irreducible subrepresentation, and both π_η and $\nu^{(a-1)/2-1}\rho \rtimes \pi'_\eta$ embed into it, we conclude

$$\pi_\eta \hookrightarrow \nu^{(a-1)/2}\rho \rtimes \pi'_\eta.$$

Applying s_{GL} to both sides and using (1-2), we get from the above lemma

$$s_{GL}(\pi_\eta) \leq \nu^{(a-1)/2}\rho \times s_{GL}(\pi'_\eta). \tag{10-17}$$

Now Lemma 3.5 implies the square integrability of π_η. This ends the proof of the square integrability of π_η in the case $b < a$.

It remains to consider the case $a_- < b$. The condition (9-1) implies $a < b$.

10.2. We shall now assume

$$a < b. \tag{10-18}$$

Then

$$b > 4.$$

10.2.1. First we shall consider the case

$$b - 2 \in Jord_\rho(\pi). \tag{10-19}$$

Then clearly $b_- = b - 2$. The definition (10-1) of b implies

$$\epsilon_\pi(b) = \epsilon_\pi(b_-). \tag{10-20}$$

Note that (10-19) and (9-1) imply

$$a < b_-.$$

Therefore,

$$a + 2 < b.$$

Note that (10-19) and the definition (10-1) of b imply that b is the maximum of all $a' \in Jord_\rho(\pi)$ such that $a'_- \in Jord_\rho(\pi)$ is defined and $e_\pi(a') = e_\pi(a'_-)$. Therefore, Lemma 5.2 implies that

$$\pi \text{ has no factors in } \{\nu^{-(b-1)/2-z}\rho; z \in \mathbb{Z}_+\}. \tag{10-21}$$

By Lemma 5.1, there exists an irreducible square integrable representation π' such that

$$\pi \hookrightarrow \delta([\nu^{-(b_- -1)/2}\rho, \nu^{(b-1)/2}\rho]) \rtimes \pi'. \tag{10-22}$$

Proposition 2.1 implies

$$Jord(\pi') = Jord(\pi) \setminus \{(\rho, b), (\rho, b_-)\}. \tag{10-23}$$

Further, (10-7), (10-22), the fact that $(a-1)/2+1 < (b-1)/2$, $b = b_- + 2$ and $(a-1)/2 \le (b_- - 1)/2$ (which implies $-(b_- - 1)/2 \le -(a_- - 1)/2$), imply

$$\pi_\eta \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu^{-(b_- -1)/2}\rho, \nu^{(b-1)/2}\rho]) \rtimes \pi' \hookrightarrow \tag{10-24}$$

$$\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \times \nu^{(b-1)/2}\rho \times \delta([\nu^{-(b_- -1)/2}\rho, \nu^{(b_- -1)/2}\rho]) \rtimes \pi' \tag{10-25}$$

$$\cong \nu^{(b-1)/2}\rho \times \delta([\nu^{-(b_- -1)/2}\rho, \nu^{(b_- -1)/2}\rho]) \times \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'. \tag{10-26}$$

Now we know that $\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'$ has exactly two irreducible subrepresentations. Denote them by $\pi'_{\eta'}$, $\eta' \in \{\pm 1\}$. Thus

$$\bigoplus_{\eta' \in \{\pm 1\}} \pi'_{\eta'} \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'. \tag{10-27}$$

Applying the inductive assumption to $\pi'_{\eta'}$ (these representations satisfy the conditions of the section 9.), we get that they are square integrable. From proposition 2.1, we get

$$\begin{aligned} Jord(\pi'_{\eta'}) &= Jord(\pi') \cup \{(\rho, a), (\rho, a_-)\} \\ &= (Jord(\pi) \setminus \{(\rho, b), (\rho, b_-)\}) \cup \{(\rho, a), (\rho, a_-)\}. \end{aligned} \tag{10-28}$$

Since $b_- \notin Jord_\rho(\pi'_{\eta'})$, the representation

$$\delta([\nu^{-(b_- -1)/2}\rho, \nu^{(b_- -1)/2}\rho]) \times \pi'_{\eta'} \tag{10-29}$$

reduces into a sum of two irreducible tempered representations (which are not equivalent). Further, (10-27) implies that

$$\bigoplus_{\eta' \in \{\pm 1\}} \nu^{(b-1)/2} \times \delta([\nu^{-(b_- -1)/2}\rho, \nu^{(b_- -1)/2}\rho]) \times \pi'_{\eta'} \tag{10-30}$$

embeds into (10-26). The above discussion implies that (10-30) has at least four irreducible subrepresentations (since (10-29) reduces into a sum of two irreducible subrepresentation for each $\eta' \in \{\pm 1\}$).

Now we shall show that (10-26) has at most four irreducible subrepresentations (the above discussion will imply that it has exactly four irreducible subrepresentations). First recall that (10-26) is isomorphic to (10-25). Therefore, it is enough to see that (10-25) has at most four irreducible subrepresentations. Note that (10-25) embeds into

$$\begin{aligned}
&\nu^{(b-1)/2} \times \delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \\
&\qquad \times\delta([\nu^{-(b_--1)/2}\rho, \nu^{(b_--1)/2}\rho]) \times \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \rtimes \pi' \\
&\hookrightarrow \nu^{(b-1)/2} \times \nu^{(a-1)/2}\rho \times \nu^{(a-1)/2-1}\rho \times \ldots \times \nu^{(a_--1)/2+1}\rho \\
&\qquad \times\delta([\nu^{-(b_--1)/2}\rho, \nu^{(b_--1)/2}\rho]) \times \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \rtimes \pi'
\end{aligned} \tag{10-31}$$

It is enough to prove that (10-31) has at most four irreducible subrepresentations. Since each irreducible subrepresentation of (10-31) has

$$\begin{aligned}
t = {}&\nu^{(b-1)/2} \otimes \nu^{(a-1)/2}\rho \otimes \nu^{(a-1)/2-1}\rho \otimes \ldots \otimes \nu^{(a_--1)/2+1}\rho \\
&\otimes\delta([\nu^{-(b_--1)/2}\rho, \nu^{(b_--1)/2}\rho]) \otimes \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \otimes \pi'
\end{aligned} \tag{10-32}$$

as a quotient of corresponding Jacquet module, the fact that (10-31) has at most four irreducible subrepresentations will follow from

Lemma. *The multiplicity of t (defined in (10-32)), in a corresponding standard Jacquet module of (10-31), is four.*

Proof. Denote the representation (10-31) by Ψ. Then

$$\begin{aligned}
\mu^*(\Psi) = {}&\left(1 \otimes \nu^{(b-1)/2}\rho + \nu^{(b-1)/2}\rho \otimes 1 + \nu^{-(b-1)/2}\rho \otimes 1\right) \\
&\times \left(\prod_{i=(a_--1)/2+1}^{(a-1)/2} \left(1 \otimes \nu^{i}\rho + \nu^{i}\rho \otimes 1 + \nu^{-i}\rho \otimes 1\right)\right) \\
&\times \Bigg(\sum_{i'=-(a_--1)/2-1}^{(a_--1)/2} \sum_{j'=i'}^{(a_--1)/2} \delta([\nu^{-i'}\rho, \nu^{(a_--1)/2}\rho]) \times \delta([\nu^{j'+1}\rho, \nu^{(a_--1)/2}\rho]) \\
&\qquad \otimes \delta([\nu^{i'+1}\rho, \nu^{j'}\rho])\Bigg) \\
&\times \Bigg(\sum_{i''=-(b_--1)/2-1}^{(b_--1)/2} \sum_{j''=i''}^{(b_--1)/2} \delta([\nu^{-i''}\rho, \nu^{(b_--1)/2}\rho]) \times \delta([\nu^{j''+1}\rho, \nu^{(b_--1)/2}\rho]) \\
&\qquad \otimes \delta([\nu^{i''+1}\rho, \nu^{j''}\rho])\Bigg) \rtimes \mu^*(\pi').
\end{aligned} \tag{10-33}$$

If t is a subquotient of a corresponding Jacquet module of Ψ, by the transitivity of Jacquet modules, there exists an irreducible subquotient $\tau \otimes \sigma$ of $s_{(p(a-a_-+2)/2)}(\Psi)$, such that t is a subquotient of a corresponding standard Jacquet module of $\tau \otimes \sigma$. Now we shall examine (10-33), to find all such $\tau \otimes \sigma$. We shall analyze which indexes and terms in the products and sums we can take, to get such $\tau \otimes \sigma$ for a subquotient.

Note that the support of τ is $\{\nu^{(b-1)/2}, \nu^{(a-1)/2}\rho, \nu^{(a-1)/2-1}\rho, \dots, \nu^{(a_--1)/2+1}\rho\}$ (in general, the support is a multiset, but here is actually a set; therefore we are in the regular situation).

First, to get $\tau \otimes \sigma$, we must not take $\nu^{-(b-1)/2}\rho \otimes 1$ and $\nu^{-i}\rho \otimes 1$, since $\nu^{-(b-1)/2}\rho$ and $\nu^{-i}\rho$ are not in the support of τ.

Suppose that we get $\tau \otimes \sigma$ from some term where we take $1 \otimes \nu^{i}\rho$ in the right hand side of (10-33). Then to get it in support of τ, it cannot come from $\nu^{(b-1)/2}\rho \otimes 1$ and it cannot come from indexes i', j' since $i > (a_- - 1)/2$. It cannot come also from indexes i'', j'', since then we would have in the support of τ also $(b_- - 1)/2$, which is not the case. From this we conclude that there exists irreducible $\tau' \otimes \sigma' \leq \mu^*(\pi')$, such that the support of τ' is non-empty and contained in the support of τ. This implies (using Lemma 3.6) that either $b \in Jord_\rho(\pi')$ or $2l+1 \in Jord_\rho(\pi')$ for some $(a_- - 1)/2 + 1 \leq l \leq (a-1)/2$. But (10-23) and (9-1) imply that this is impossible.

Further, we must not take $1 \otimes \nu^{(b-1)/2}\rho$. Otherwise, since $(b-1)/2$ is greater than each of $(a-1)/2, (a_- - 1)/2, (b_- - 1)/2$, to get $\nu^{(b-1)/2}\rho$ in the support of τ, we must have some $\tau' \otimes \sigma'$ with the same properties as in the above paragraph. We have seen that this implies the contradiction.

Therefore, we have proved that τ must be subquotient of

$$\nu^{(b-1)/2} \times \nu^{(a-1)/2}\rho \times \nu^{(a-1)/2-1}\rho \times \dots \times \nu^{(a_--1)/2+1}\rho \tag{10-34}$$

multiplied by other terms. But since $\tau \otimes \sigma$ is a subquotient of $s_{(p(a-a_-+2)/2)}(\Psi)$, we see that all other terms are equal to 1. Thus, τ is a subquotient of (10-34). This implies that σ must be a subquotient of

$$\delta([\nu^{-(b_--1)/2}\rho, \nu^{(b_--1)/2}\rho]) \times \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \rtimes \pi'. \tag{10-35}$$

At the end, note that the multiplicity of $\nu^{(b-1)/2} \otimes \nu^{(a-1)/2}\rho \otimes \nu^{(a-1)/2-1}\rho \otimes \dots \otimes \nu^{(a_--1)/2+1}\rho$ in corresponding (standard) Jacquet module of (10-34) is one. From the other side, (ii) of Lemma 4.3 implies that the multiplicity of $\delta([\nu^{-(b_--1)/2}\rho, \nu^{(b_--1)/2}\rho]) \otimes \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \otimes \pi'$ in a corresponding Jacquet module of (10-35) is four. Therefore the multiplicity of t in a corresponding standard Jacquet module of Ψ is $1 \cdot 4 = 4$. This ends the proof. $\square$

Since (10-30) embeds into (10-26), and since we have just shown that both representations have exactly four irreducible subrepresentations, (10-24) – (10-26) imply that π_η embeds into (10-30). This and (1-2) imply

$$s_{GL}(\pi_\eta) \leq \left(\nu^{(b-1)/2} + \nu^{-(b-1)/2}\right) \times s_{GL}\left(\bigoplus_{\eta' \in \{\pm 1\}} \delta([\nu^{-(b_--1)/2}\rho, \nu^{(b_--1)/2}\rho]) \times \pi'_{\eta'}\right). \tag{10-36}$$

From (10-21), (10-7), $b > a+2$ (and (1-2)), we get that $\nu^{-(b-1)/2}\rho$ is not a factor of π_η. Thus, (10-36) implies

$$s_{GL}(\pi_\eta) \leq \nu^{(b-1)/2} \times s_{GL}\left(\bigoplus_{\eta' \in \{\pm 1\}} \delta([\nu^{-(b_--1)/2}\rho, \nu^{(b_--1)/2}\rho]) \times \pi'_{\eta'}\right). \tag{10-37}$$

Now Lemma 3.5 implies that π_η is tempered (note that we did not use in the proof of Lemma 3.5 that τ is irreducible). Lemma 8.1 implies now that π_η is square integrable. This ends the proof of the square integrability in the case $b-2 \in Jord_\rho(\pi)$ (and $a<b$).

10.2.2. It remains to consider the case $b-2 \notin Jord_\rho(\pi)$ (we continue to assume $a<b$). First we have a general

Lemma. *If $b-2 \notin Jord_\rho(\pi)$, then there exists an irreducible square integrable representation π'' such that*

$$\pi \hookrightarrow \nu^{(b-1)/2}\rho \rtimes \pi'' \tag{10-38}$$

and (then)

$$Jord(\pi'') = (Jord(\pi) \setminus \{(\rho,b)\}) \cup \{(\rho,b-2)\}. \tag{10-39}$$

Remark. The proof of the lemma holds if one takes any $a' \in Jord_\rho(\pi)$ instead b, which satisfies

$$a' > 2 \quad \text{and} \quad a'-2 \notin Jord_\rho(\pi).$$

Note that b satisfies the above condition.

Proof. We prove the lemma, and more general the remark, by induction (for our fixed π, assuming that our general inductive assumption holds). We need to show the existence of irreducible square integrable representation π'' such that

$$\pi \hookrightarrow \nu^{(a'-1)/2}\rho \rtimes \pi'', \tag{10-38'}$$

$$Jord(\pi'') = (Jord(\pi) \setminus \{(\rho,a')\}) \cup \{(\rho,a'-2)\}. \tag{10-39'}$$

Since (10-38') and Proposition 2.1 imply (10-39'), for the proof is enough to show the existence of the embedding (10-38').

We discuss several possibilities.

First suppose that π is strongly positive. Now we shall repeat a part of construction in the seventh section, using the notation that we were using there. First, π embeds into some representation σ_0 defined in (7-10). Let $a' = a_i$ (then $i=1$ and $a_i \geq 3$, or $i>1$ and $a_{i-1} \leq a_i - 4$). Denote by σ_0' the representation that one gets by substituting in (7-10) $\nu^{(a_i-1)/2}\rho \times \delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2-1}\rho])$ instead of $\delta([\nu^{(\phi(a_i)+1)/2}\rho, \nu^{(a_i-1)/2}\rho])$ (all other terms leaving the same, as well as the order of the multiplication). Let σ_0'' be the representation that one obtains from σ_0' dropping $\nu^{(a_i-1)/2}\rho$ from the definition of σ_0'. Now $a'-2 \notin Jord_\rho(\pi)$ implies

$$\sigma_0' \cong \nu^{(a_i-1)/2}\rho \rtimes \sigma_0''.$$

Further, by the seventh section, σ_0'' has an irreducible square integrable subrepresentation, which we shall denote by π'' (for this one again uses $a'-2 \notin Jord_\rho(\pi)$). Since σ_0' has a unique irreducible subrepresentation by Lemma 4.1, then $\pi \hookrightarrow \sigma_0 \hookrightarrow \sigma_0'$ and $\nu^{(a_i-1)/2}\rho \rtimes \pi'' \hookrightarrow \nu^{(a_i-1)/2}\rho \rtimes \sigma_0'' \cong \sigma_0'$ imply (10-38').

Suppose now that π is not strongly positive. Then we can choose $c, c_- \in Jord_\rho(\pi)$ such that $\epsilon_\pi(c) = \epsilon_\pi(c_-)$ and by Lemma 5.1 there exists an irreducible square integrable representation π''' such that

$$\pi \hookrightarrow \delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi'''. \tag{10-40}$$

Proposition 2.1 implies

$$Jord(\pi''') = Jord(\pi) \setminus \{(\rho, c), (\rho, c_-)\}. \tag{10-41}$$

Note that

$$[c_-, c] \cap Jord_\rho(\pi''') = \emptyset.$$

We shall consider three cases.

First consider the case $a' = c$. Now $a' = c \geq c_- + 4$. This and (10-40) imply

$$\pi \hookrightarrow \delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi''' \tag{10-42}$$

$$\hookrightarrow \nu^{(c-1)/2}\rho \times \delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2-1}\rho]) \rtimes \pi''' \tag{10-43}$$

$$\hookrightarrow \nu^{(c-1)/2}\rho \times \delta([\nu^{(c_--1)/2+1}\rho, \nu^{(c-1)/2-1}\rho]) \times\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_--1)/2-1}\rho]) \rtimes \pi'''. \tag{10-44}$$

Now $\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2-1}\rho]) \rtimes \pi'''$ has two irreducible subrepresentation, which are square integrable by our general inductive assumption (see (10-41)). Denote them by $\pi'''_{\eta'}$, $\eta' \in \{\pm 1\}$. By Lemma 4.5 and Remark 4.2, (10-44) has at most two irreducible subrepresentations. Therefore (10-43) has at most two irreducible subrepresentation. This implies $\pi \hookrightarrow \nu^{(c-1)/2}\rho \rtimes \pi'''_{\eta'}$ for some η'. This proves (10-38').

Let now $a' = c_-$. Then

$$\begin{aligned} \pi &\hookrightarrow \delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi''' \\ &\hookrightarrow \delta([\nu^{-(c_--1)/2+1}\rho, \nu^{(c-1)/2}\rho]) \times \nu^{-(c_--1)/2}\rho \rtimes \pi''' \\ &\cong \delta([\nu^{-(c_--1)/2+1}\rho, \nu^{(c-1)/2}\rho]) \times \nu^{(c_--1)/2}\rho \rtimes \pi'''. \end{aligned}$$

The last equivalence follows from Lemma 5.4.1 of [M2] (since $(c_- - 1)/2 = (d+1)/2$ for some $d \in Jord_\rho(\pi)$ would imply $d = c_- - 2 = a' - 2 \in Jord_\rho(\pi)$, which contradicts to our assumptions). From the above embeddings, we get

$$\begin{aligned} \pi &\hookrightarrow \nu^{(c_--1)/2}\rho \times \delta([\nu^{-(c_--1)/2+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi''' \\ &= \nu^{(c_--1)/2}\rho \times \delta([\nu^{-(c_--2-1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi'''. \end{aligned}$$

Note

$$[c_- - 2, c] \cap Jord_\rho(\pi''') = \emptyset$$

(since $a' - 2 = c_- - 2 \notin Jord_\rho(\pi)$). Therefore

$$\delta([\nu^{-(c_- -2-1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi'''$$

has exactly two irreducible subrepresentations. Denote them by $\pi'''_{\eta'}$, $\eta' \in \{\pm 1\}$. They are square integrable by the general inductive assumption. Further

$$\begin{aligned}\pi &\hookrightarrow \nu^{(c_- -1)/2}\rho \times \delta([\nu^{-(c_- -1)/2+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi''' \\ &\hookrightarrow \nu^{(c_- -1)/2}\rho \times \delta([\nu^{(c_- -1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{-(c_- -1)/2+1}\rho, \nu^{(c_- -1)/2-1}\rho]) \rtimes \pi'''.\end{aligned}$$

Using $[c_-, c] \cap Jord_\rho(\pi''') = \emptyset$, in a similar way as in the proof of Lemma 4.5, one gets that the last representation has at most two irreducible subrepresentations (the only difference from the proof of Lemma 4.5 is that one does not use regularity, but the well known fact that $\nu^{(c_- -1)/2}\rho \times \delta([\nu^{(c_- -1)/2}\rho, \nu^{(c-1)/2}\rho])$ has multiplicity one in $\nu^{(c_- -1)/2}\rho \times \prod_{i=0}^{(c-c_-)/2} \nu^{(c_- -1)/2+i}\rho$. From this and the above embeddings we conclude that

$$\pi \hookrightarrow \nu^{(c_- -1)/2}\rho \rtimes \pi'''_{\eta'}$$

for some $\eta' \in \{\pm 1\}$. This shows (10-38') in the case $a' = c_-$.

Suppose at the end $a' \notin \{c_-, c\}$. Then $a' \notin [c_-, c]$. Further note that

$$\nu^{(a'-1)/2}\rho \times \delta([\nu^{-(c_- -1)/2}\rho, \nu^{(c-1)/2}\rho]) \cong \delta([\nu^{-(c_- -1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \nu^{(a'-1)/2}\rho$$

since either $a' < c_-$ or $(c-1)/2 + 1 < (a'-1)/2$ by our assumptions. Then $c \leq a' - 4$. Further, $a' \in Jord_\rho(\pi''')$ by (10-41) and $a' - 2 \notin Jord_\rho(\pi''')$. Therefore, we can apply the inductive assumption. Applying it, we get that there exists a square integrable representation π'''' such that

$$\pi''' \hookrightarrow \nu^{(a'-1)/2}\rho \rtimes \pi'''', \tag{10-45}$$

$$\begin{aligned} Jord(\pi'''') &= (Jord(\pi''') \setminus \{(\rho, a')\}) \cup \{(\rho, a'-2)\} \\ &= (Jord(\pi) \setminus \{(\rho, c), (\rho, c_-), (\rho, a')\}) \cup \{(\rho, a'-2)\}. \end{aligned} \tag{10-46}$$

Now

$$\pi \hookrightarrow \delta([\nu^{-(c_- -1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi''' \tag{10-47}$$

$$\hookrightarrow \delta([\nu^{-(c_- -1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \nu^{(a'-1)/2}\rho \rtimes \pi'''' \tag{10-48}$$

$$\cong \nu^{(a'-1)/2}\rho \times \delta([\nu^{-(c_- -1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi'''' \tag{10-49}$$

$$\begin{aligned}&\hookrightarrow \nu^{(a'-1)/2}\rho \times \delta([\nu^{(c_- -1)/2+1}\rho, \nu^{(c-1)/2}\rho]) \\ &\qquad \times \delta([\nu^{-(c_- -1)/2}\rho, \nu^{(c_- -1)/2}\rho]) \rtimes \pi''''.\end{aligned} \tag{10-50}$$

Now (10-46) implies that $\delta([\nu^{-(c_- -1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi''''$ has exactly two irreducible subrepresentations, and they are square integrable by the general inductive assumption.

Denote them by $\pi''''_{\eta'}$, $\eta' \in \{\pm\}$. Lemma 4.5, Remark 4.2 and (4-46) imply that (10-50) has at most two irreducible subrepresentations. Then, the same holds for (10-49). This implies that $\pi \hookrightarrow \nu^{(a'-1)/2}\rho \times \pi''''_{\eta'}$ for some η'. This proves (10-38'). Now the proof of the lemma and the remark is complete. $\square$

Using this lemma, we get

$$\text{(10-51)} \quad \pi_\eta \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi \\ \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \times \nu^{(b-1)/2}\rho \rtimes \pi''.$$

Recall $a < b$. We shall consider separately the cases of $a+2 < b$ and $a+2 = b$.

10.2.2.1. First we shall consider the case

$$a + 2 < b.$$

Since $(a-1)/2 + 1 < (b-1)/2$, (10-51) implies

$$\text{(10-52)} \quad \pi_\eta \hookrightarrow \nu^{(b-1)/2}\rho \times \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'' \hookrightarrow \\ \nu^{(b-1)/2}\rho \times \delta([\nu^{(a_- -1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \times \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \rtimes \pi''.$$

Now (10-39) implies that $\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi''$ has exactly two irreducible subrepresentations. They are square integrable by the inductive assumption. Denote them by $\pi''_{\eta'}$, $\eta'\{\pm 1\}$. Further, (10-39), Lemma 4.5 and Remark 4.2 imply that the representation in the second row of (10-52) has at most two irreducible subrepresentations. This implies

$$\text{(10-53)} \qquad \pi_\eta \hookrightarrow \nu^{(b-1)/2}\rho \rtimes \pi''_{\eta'}$$

for some η'. Note that $\pi_\eta \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi$ and Lemma 5.2 applied to π, imply that $\nu^{(b-1)/2}\rho$ is not a factor of π_η (similarly as in 10.2.1; see (10-21)). Now (1-2) and (10-53) imply $s_{GL}(\pi_\eta) \leq \nu^{(b-1)/2}\rho \times s_{GL}(\pi''_{\eta'})$. Now Lemma 3.5 implies that π_η is square integrable.

Therefore, we have settled the case $a+2 < b$ (recall that we assumed $b-2 \notin Jord_\rho(\pi)$). At the end, it remains to consider only the case $b = a+2$.

11. End of proof of square integrability

We continue with the notation of the last section, and assume additionally

$$a + 2 = b.$$

This is the only case where we have not proved yet the square integrability of π_η.

We shall suppose now that π_η is not square integrable.

If $b_- \in Jord_\rho(\pi)$ is not defined, then the definition of b implies that π is strongly positive. Suppose that $b_- \in Jord_\rho(\pi)$ is defined. Then $a+2 = b$ and (9-1) imply

$b_- \leq a_- - 2$. Now Lemma 3.6 implies that $\nu^{-(b_- -1)/2-1}, \nu^{-(b_- -1)/2-2}, \ldots$ are not factors of π. Note that $-(a_- - 1)/2 \leq -(b_- - 1)/2 - 1$. Therefore, we have

$$\text{(11-1)} \qquad \pi \text{ has no factors in the set } \{\nu^{-(a_- -1)/2-z}\rho; z \in \mathbb{Z}_+\},$$

regardless if b_- is defined or not. Recall further

$$\text{(11-2)} \qquad \pi_\eta \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi.$$

This, (1-2), (1-4) and (11-1) imply that

$$\text{(11-3)} \qquad \pi_\eta \text{ has no factors in the set } \{\nu^{-(a-1)/2-1-z}\rho; z \in \mathbb{Z}_+\}.$$

Suppose that $\nu^{-(a-1)/2}$ is not a factor of π_η. Then from (11-2) follows

$$\pi_\eta \hookrightarrow \nu^{(a-1)/2}\rho \times \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2-1}\rho]) \rtimes \pi$$

Using the inductive assumption, from this we get in the same way as before (for example in 10.1) that π_η is square integrable.

Therefore, $\nu^{-(a-1)/2}$ is a factor of π_η.

Recall that Lemma 10.2 implies

$$\text{(11-4)} \qquad \pi \hookrightarrow \nu^{(b-1)/2}\rho \rtimes \pi'',$$

where

$$\text{(11-5)} \qquad Jord_\rho(\pi'') = (Jord_\rho(\pi) \setminus \{b\}) \cup \{a\}.$$

We need to keep all the time in mind that $b = a+2$, which implies $(b-1)/2 = (a-1)/2+1$. Now

$$\text{(11-6)} \qquad \pi_\eta \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \nu^{(b-1)/2}\rho \rtimes \pi''.$$

Now we two technical lemmas. The first one is very simple.

11.1. Lemma. *If $\nu^z\rho \otimes \sigma \leq \mu^*(\pi)$ for some irreducible representation σ and $z \in (1/2)\mathbb{Z}$, then $z \leq (b-1)/2$.*

Proof. Suppose $z > (b-1)/2$. This implies $2z+1 > b$. Lemma 3.6 implies $2z+1 \in Jord_\rho(\pi)$. Lemma 3.1 and Lemma 3.2 imply the existence of an irreducible representation σ', such that $\pi \hookrightarrow \nu^{2z+1}\rho \rtimes \sigma'$. By the definition of b, $2z-1 \in Jord_\rho(\pi)$ (since $2z+1 > b$). Now the definition of the partial function ϵ implies $\epsilon(2z-1, 2z+1) = 1$. This contradicts the definition of b (since $2z+1 > b$). □

11.2. Lemma. *(i) If there exists an embedding*

$$\pi_\eta \hookrightarrow \delta([\nu^{-(a-1)/2}, \nu^l]) \rtimes \sigma', \tag{11-7}$$

with σ' irreducible and $l \in \{-(a-1)/2 + z; z \in \mathbb{Z}_+\}$, then $l = (b-1)/2$.

(ii) There exists an embedding

$$\pi_\eta \hookrightarrow \delta([\nu^{-(a-1)/2}, \nu^{(b-1)/2}]) \rtimes \sigma', \tag{11-8}$$

with irreducible σ'.

(iii) Any representation σ' which satisfies (ii), must be square integrable. Further,

$$Jord_\rho(\sigma') = (Jord_\rho(\pi) \setminus \{b\}) \cup \{a_-\}.$$

Proof. The proof of (i) and (ii) proceeds in a similar way as the proof of Lemma 10.1.

Since $\nu^{-(a-1)/2-1}\rho$ is not a factor of π_η, Lemma 3.4 implies

$$\pi_\eta \hookrightarrow \delta([\nu^{-(a-1)/2}\rho, \nu^{(a'-1)/2}\rho]) \rtimes \sigma, \tag{11-9}$$

for some irreducible σ and $a' \in \mathbb{Z}$ such that $a + a' \in 2\mathbb{Z}$ and $(a'-1)/2 - (-(a-1)/2) \geq 0$. The Frobenius reciprocity implies that

$$\nu^{(a'-1)/2}\rho \otimes \nu^{(a'-1)/2-1}\rho \otimes \dots \nu^{-(a-1)/2}\rho \otimes \rho_1 \otimes \rho_2 \times \dots \otimes \rho_l \otimes \pi_{cusp} \tag{11-10}$$

is an irreducible subquotient of a corresponding Jacquet module of π_η (ρ_i are irreducible cuspidal representations). Now (11-60) must be a subquotient of a corresponding Jacquet module of some irreducible subquotient $\tau \otimes \sigma$ of $s_{((a+a_-)/2)}(\pi_\eta)$.

Suppose first $a' \leq a_-$. This implies that $\nu^{-(a-1)/2}\rho$ must be in the support of τ. Clearly, $\tau \otimes \sigma \leq s_{((a+a_-)/2)}(\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi)$. Now (10-9) and (11-1) imply $\sigma \otimes \tau = \delta([\nu^{-(a-1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \otimes \pi$. Now Lemma 4.4 implies that π_η is a Langlands quotient of $\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi$, which contradicts to the reducibility of $\delta([\nu^{(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi$.

Suppose now $a_- < a'$. Since (11-10) is a subquotient of a Jacquet module of π_η, Lemma 3.1 and Lemma 3.2 imply that $\nu^{(a'-1)/2}\rho \otimes \gamma \leq s_{(p)}(\pi_\eta)$, for some irreducible representation γ, such that $\nu^{-(a-1)/2}\rho$ is a factor of γ. Now (11-2) implies

$$\nu^{(a'-1)/2}\rho \otimes \gamma \leq s_{(p)}(\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi).$$

Write $s_{(p)}(\pi) = \sum_i \mu_i \otimes \lambda_i$ as a sum of irreducible representations (as we did in (10-11)). In 10.1 we have computed the formula for

$$s_{(p)}(\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi)$$

(the first displayed formula after (10-11)). We shall use this formula now. The present assumption $a_- < a'$ implies that $\nu^{(a'-1)/2}\rho \otimes \gamma$ cannot be a subquotient of (10-13). Further (1-2), (1-4) and (11-1) imply that $\nu^{-(a-1)/2}\rho$ is not a factor of the representation $\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2-1}\rho]) \rtimes \pi$. This implies that $\nu^{(a'-1)/2}\rho \otimes \gamma$ is not a subquotient of (10-12). Therefore, $\nu^{(a'-1)/2}\rho \cong \mu_i$ for some i. Lemma 3.6 now implies $a' \in Jord_\rho(\pi)$. Since $a_- < a'$, (9-1) implies $a < a'$. Now (9-1) implies $b \leq a'$.

Suppose $b < a'$. Then, to get $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a'-1)/2-1}\rho]) \otimes \sigma'$ as a subquotient of (10-9) we first need to take $i = (a-1)/2$ in (10-9) (since $\nu^{-(a-1)/2}\rho$ is not a factor of π). Therefore, we must have $\delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a'-1)/2-1}\rho]) \otimes \sigma'' \leq \mu^*(\pi)$ for some irreducible representation σ''. This contradicts the above lemma (since we would have $\nu^{(a'-1)/2-1}\rho \otimes \sigma''' \leq \mu^*(\pi)$ for some irreducible representation σ''', with $(a'-1)/2 > (b-1)/2$). This proves (i).

Now (ii) follows from (i) and (i) of Lemma 3.4.

It remains to prove (iii).

From 10.2.2, we know that there exists an irreducible square integrable representation π'' and an embedding:

$$\pi \hookrightarrow \nu^{(b-1)/2}\rho \times \pi''.$$

Moreover we know by Proposition 2.1 that:

$$Jord_\rho(\pi'') = (Jord_\rho(\pi) \setminus \{b\}) \cup \{b-2\}.$$

Further, since $a = b-2 \in Jord_\rho(\pi'')$ and $(a-2) \notin Jord_\rho(\pi'')$, we can again use Remark 10.2.2. Continuing to use this remark several times (in the cases when we can), we shall get

$$\pi \hookrightarrow \nu^{(a_--1)/2+1}\rho \times \ldots \times \nu^{(a-1)/2}\rho \times \nu^{(b-1)/2}\rho \rtimes \pi', \tag{11-11}$$

where

$$Jord_\rho(\pi') = (Jord_\rho(\pi) \setminus \{b\}) \cup \{a_-\}. \tag{11-12}$$

In particular, if $a' \in Jord_\rho(\pi')$:

$$(a'-1)/2 \notin [(a+1)/2, (a_-+1)/2] \tag{11-13}$$

Now (11-11) and the definition of π_η imply that we have the embedding

$$\pi_\eta \hookrightarrow \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \times \nu^{(a_--1)/2+1}\rho \times \ldots \times \nu^{(a-1)/2}\rho \times \nu^{(a+1)/2}\rho \rtimes \pi'. \tag{11-14}$$

By (ii), we have embedding

$$\pi_\eta \hookrightarrow \delta([\nu^{-(a-1)/2}\rho, \nu^{(a+1)/2}\rho]) \times \sigma'. \tag{11-15}$$

Consider any such embedding. The Frobenius reciprocity implies that:

$$\delta([\nu^{-(a-1)/2}\rho, \nu^{(a+1)/2}\rho]) \otimes \sigma' \tag{11-16}$$

is a subquotient of a corresponding Jacquet module of

$$\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \times \nu^{(a_- -1)/2+1}\rho \times \ldots \times \nu^{(a-1)/2}\rho \times \nu^{(a+1)/2}\rho \rtimes \pi'. \tag{11-17}$$

We shall show now $\sigma' \cong \pi'$. This (and (11-12)) will imply (iii).

First we shall write μ^* of (11-17). It is

$$\left(\prod_{k=(a_- -1)/2+1}^{(a+1)/2} \left(\nu^k\rho \otimes 1 + \nu^{-k}\rho \otimes 1 + 1 \otimes \nu^k\rho\right) \right) \times \tag{11-18}$$
$$\left(\sum_{i=-(a_- -1)/2-1}^{(a-1)/2} \sum_{j=i}^{(a-1)/2} \delta([\nu^{-i}\rho, \nu^{(a_- -1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(a-1)/2}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \right)$$
$$\rtimes \mu^*(\pi').$$

Now we shall analyze when we can get (11-16) for a subquotient of (11-18).

The first conclusion is that we need to take for $k = (a+1)/2$ the term $\nu^{(a+1)/2}\rho \otimes 1$, since $b \notin Jord_\rho(\pi')$.

Now look at $k = (a-1)/2$. Suppose that we have taken the term $1 \otimes \nu^{(a-1)/2}\rho$. We consider two possibilities. Suppose $j+1 \leq (a-1)/2$. Then $i \leq (a-1)/2 - 1$. Therefore, in the term $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a+1)/2}\rho])$ in (11-16), $\nu^{-(a-1)/2}\rho$ must come from $\mu^*(\pi')$. This directly implies that π' is not square integrable. Thus, $j = (a-1)/2$. This would imply that $a \in Jord_\rho(\pi')$. This cannot happen. Therefore, for $k = (a-1)/2$ we mast not take the term $1 \otimes \nu^{(a-1)/2}\rho$ (if we want to get (11-16) for a subquotient). The two possibilities remain.

Suppose that we have taken the term $\nu^{-(a-1)/2}\rho \otimes 1$. Then we need to take in (11-18) $-i \geq -(a-1)/2 + 1$. Suppose $-i > -(a-1)/2 + 1$. Since $a \notin Jord_\rho(\pi')$, we get $j+1 \leq (a-1)/2$. This implies that π' is not square integrable. Thus $-i = -(a-1)/2+1$, i.e. $i = (a-1)/2 - 1$. Now $j = (a-1)/2 - 1$ or $(a-1)/2$. If $j = (a-1)/2$, then $a \in Jord_\rho(\pi')$, which is impossible. Thus, $j = (a-1)/2 - 1$. Now for other k's (i.e. when $k < (a-1)/2$), we must take terms $\nu^k\rho \otimes 1$. This implies $\sigma' \cong \pi'$.

The other possibility is that we have taken the term $\nu^{(a-1)/2}\rho \otimes 1$. This implies $j = (a-1)/2$. Suppose $-i > -(a-1)/2$ (i.e. $i < (a-1)/2$). This easily implies that π' is not square integrable. Therefore, $i = (a-1)/2$. This implies $j = (a-1)/2$. Again for remaining k's, one must take the terms $\nu^k\rho \otimes 1$. This implies again $\sigma' \cong \pi'$.

The proof of (iii) is now complete. □

Suppose $b = \max(Jord_\rho)$. Now (iii) of the above lemma and the tenth section imply the contradiction. Thus, $b < \max(Jord_\rho)$. Nevertheless, using the above lemma, in a finitely many steps we come to the contradiction.

This contradiction implies that our assumption that π_η is not square integrable cannot hold. Thus, π_η is square integrable. This ends the proof of the square integrability.

Note that the representations π_η are constructed recursively. There are the cases when one can define them more directly. In [T4] are examples of constructions of families of such representations (and proof of their square integrability, together with explicit estimates of their Jacquet modules and description of some other properties).

12. Jordan blocks and cuspidal reducibility

In this section only, we shall not assume that (BA) holds (and also we shall not assume that (A) holds). For simplicity, we shall assume in this section that groups S_n are split.

Let ρ be an irreducible F'/F-selfdual cuspidal representation of $GL(p)$ and let σ be an irreducible cuspidal representation of S_q. We are interested in the reducibility points of the family $\nu^\alpha\rho \rtimes \sigma$, $\alpha \in \mathbb{R}$. First, $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha \in \mathbb{R}$. In all the known cases when the reducibility points of such families are computed, holds

(HI) if $\nu^\alpha\rho \rtimes \sigma$ reduces, then $\alpha \in (1/2)\mathbb{Z}$ and $\nu^b\rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R} \setminus \{\pm\alpha\}$.

This is expected to hold in general. The first general result in this direction is done by Shahidi ([Sh1]). Shahidi proved that (HI) holds if σ is generic (he showed much more; see bellow). Assume that (HI) holds for ρ and σ (note that (BA) implies (HI)). Then the non-negative reducibility point is unique. We shall denote it by

$$\alpha(\rho, \sigma).$$

Shahidi proved

(12-1) $$\text{if } \sigma \text{ is generic, then } \alpha(\rho, \sigma) \in \{0, \pm 1/2, \pm 1\}.$$

In particular, this implies for the simplest case, when σ is the trivial representation 1 of the trivial group:

(12-2) $$\alpha(\rho, 1) \in \{0, \pm 1/2, \pm 1\}$$

In the lemma bellow, we shall see that (BA), which comes from a study of Arthur's conjectures, implies that holds:

(D) if $\nu^\alpha\rho \rtimes \sigma$ reduces, then $\alpha - \alpha(\rho, 1) \in \mathbb{Z}$
and $\nu^\beta\rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R} \setminus \{\pm\alpha\}$.

Obviously, (D) implies (HI) (using (12-2)).

12.1. Lemma. *The assumptions (BA) and (D) are equivalent.*

Proof. Assume that (D) holds.

Suppose that $\rho \rtimes \sigma$ reduces. Then by (D) and (12-2), $\rho \rtimes \sigma$ or $\nu\rho \rtimes \sigma$ reduces. This implies that $L(\rho, R_{d_\rho}, s)$ has not a pole at $s = 0$. Thus if $a \in Jord_\rho(\sigma)$, then a is odd. For any odd $a \in \mathbb{N}$, $\delta(\rho, a) \rtimes \sigma$ reduces by Proposition 4.4 of [T2]. Thus, $Jord_\rho(\sigma) = \emptyset$. Therefore, (BA) holds for this pair.

Let $\nu^{1/2}\rho \rtimes \sigma$ be reducible. Then $\nu^{1/2}\rho \rtimes 1$ reduces by (D). This implies that $L(\rho, R_{d_\rho}, s)$ has a pole at $s = 0$. Thus, each $a \in Jord_\rho(\sigma)$ must be even. Now Proposition 4.3 of [T2] implies $Jord_\rho(\sigma) = \emptyset$. Thus, (BA) holds also in this case.

Suppose now that $\nu^{b/2}\rho \rtimes \sigma$ reduces for some $b \in \mathbb{N}$, $b > 1$. We shall consider first the case $b \in 2\mathbb{N}$. Then $\rho \rtimes \sigma$ or $\nu\rho \rtimes \sigma$ reduces by (D). Therefore, $L(\rho, R_{d_\rho}, s)$ has not a pole at $s = 0$. Therefore, $a \in Jord_\rho(\sigma)$ must be odd. Now Theorem 13.2 of [T2] (see also the remark below) implies

$$Jord_\rho(\sigma) = \{1, 3, \dots, b-1\}.$$

Therefore, (BA) holds again. Suppose now that b is odd. Then $\nu^{1/2}\rho \rtimes \sigma$ reduces by (D). This implies that $L(\rho, R_{d_\rho}, s)$ has a pole at $s = 0$, and further $a \in Jord_\rho(\sigma)$ must be even. Theorem 13.2 of [T2] implies

$$Jord_\rho(\sigma) = \{2, 4, \dots, b-1\}.$$

Therefore, (BA) holds.

Assume now that (BA) holds. Let $\nu^{b/2}\rho \rtimes \sigma$ reduces for some $b \in \mathbb{Z}_+$. First consider the case of even b. Suppose $Jord_\rho(\sigma) = \emptyset$. Then (BA) implies that $L(\rho, R_{d_\rho}, s)$ has not pole at $s = 0$. This implies that $\rho \rtimes 1$ or $\nu\rho \rtimes 1$ reduces. Therefore, (D) holds in this case. Suppose now $Jord_\rho(\sigma) \neq \emptyset$. Then by (BA), from $(b-1+1)/2 = b/2$ follows $b-1 \in Jord_\rho(\sigma)$. Since $b-1$ is odd, (J-1') implies that $\rho \rtimes 1$ or $\nu\rho \rtimes 1$ reduces. Again, (D) holds. Now consider the case of odd b. Suppose first $Jord_\rho(\sigma) = \emptyset$. Now (BA) implies that $L(\rho, R_{d_\rho}, s)$ has a pole at $s = 0$. This implies that $\nu^{1/2}\rho \rtimes 1$ reduces. Thus, (D) holds. Suppose now $Jord_\rho(\sigma) \neq \emptyset$. Then again $b-1 \in Jord_\rho(\sigma)$. Since $b-1$ is even, $L(\rho, R_{d_\rho}, s)$ has a pole at $s = 0$, and therefore $\nu^{1/2}\rho \rtimes 1$ reduces, which implies that (D) holds also in this case. This completes the proof. □

12.2. Remark. Note that in Theorem 13.2 of [T2] we have assumption $\mathrm{char}(F') = 0$. This assumption is used in the proof of that theorem to prove irreducibility. Note that the irreducibility that we needed in the proof of the last lemma is in the unitarizable case. This irreducibility follows in the same way as in the proof of Propositions 4.1 and 4.2. Namely, let $\delta(\Delta)$ be an F'/F-selfdual irreducible (unitarizable) square integrable representation of a general linear group such that $\rho' \rtimes \sigma$ is irreducible for every $\rho' \in \Delta$. Then proofs of Propositions 4.1 and 4.2 imply that $\delta(\Delta) \rtimes \sigma$ is irreducible. These proofs does not require $\mathrm{char}(F') = 0$.

We shall write now one direct consequence of the above proof. From the (non-negative) reducibility point $\alpha(\rho, \sigma) = b/2 \in (1/2)\mathbb{Z}_+$, we can write down directly

$$Jord_\rho(\sigma) = \{b - 1 - 2i \ ; \ i \in \mathbb{Z}_+ \text{ and } b - 1 - 2i \in \mathbb{N}\}. \tag{12-3}$$

Observe that for trivial representation we have

$$Jord(1_{SO(1)}) = \emptyset,$$
$$Jord(1_{Sp(0)}) = \{(1_{GL(1)}, 1)\},$$

(depending on the series of the groups with which we are working).

There is one type of ρ for which the problem of computing of the reducibility points of induced representation with irreducible cuspidal representations σ can be in principle solved using the known facts. It is the case where ρ is a quadratic characters. One has to use ideas of Adams, Kudla and Rallis, to interpret the reducibility points in terms of the Howe duality ([W2] is an example of this kind). This interpretation is a local analogue of a more difficult global results explained in [KR] (in particular of 6.1 in [KR]; see also [M4]). A particular case which is completely written, is the case where the cuspidal representation σ is quadratic unipotent ([M3]).

13. Tempered representations

We continue to assume (BA) again (till the end of the paper).

Fix an irreducible (unitarizable) square integrable representation

$$\delta(\rho, a)$$

of a general linear group (recall that $\delta(\rho, a)$ denotes $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho])$). Let π be a similar representation of S_q.

If ρ is not F'/F-selfdual, it is well-known that

$$\delta(\rho, a) \rtimes \pi$$

is irreducible (one can get also this easily from (1-1) and Theorem 4.9 of [T1]).

Therefore, to describe the reducibility of $\delta(\rho, a) \rtimes \pi$, it remains to consider the case of F'/F-selfdual irreducible cuspidal representations ρ. We shall now assume that ρ is F'/F-selfdual.

The composition of the standard intertwining operators

$$\nu^s\delta(\rho, a) \rtimes \pi \to \nu^{-s}\delta(\rho, a) \rtimes \pi \to \nu^s\delta(\rho, a) \rtimes \pi$$

can be computed in terms of the L-functions using (BA) (such computations are carried out in [M1]). To have more simple notations, we will, here, write $\delta(\rho, a)$ instead of $L(\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho])$; this is the generalized Steinberg representation of $GL(d_\rho\, a)$ based on ρ. The result is the following product:

(13-1)
$$\prod_{(\rho',a')\in Jord(\pi)} L(\delta(\rho, a) \times \delta(\rho', a'), s)\ L(\delta(\rho, a) \times \rho', a'), s+1)^{-1}$$

(13-2)
$$\prod_{(\rho',a')\in Jord(\pi)} L(\delta(\rho, a) \times \delta(\rho', a'), -s)\ L(\delta(\rho, a) \times \rho', a'), -s+1)^{-1}$$

(13-3)
$$L(\delta(\rho, a), R_{a\,d_\rho}, 2s)\ L(\delta(\rho, a), R_{a\,d_\rho}, 2s+1)^{-1}$$

(13-4)
$$L(\delta(\rho, a), R_{a\,d_\rho}, -2s)\ L(\delta(\rho, a), R_{a\,d_\rho}, -2s+1)^{-1}$$

Now (J-1) of section 2 is equivalent to the fact that the two last products (13-3) and (13-4) have no pole at $s = 0$. The two first products (13-1) and (13-2), are easy to analyze using Theorem 8.2 of [JPSS]:

$$L(\delta(\rho, a) \times \delta(\rho, a'), s) = \prod_{k=|a-a'|/2}^{(a+a')/2-1} L(\rho \times \rho', s+k).$$

Denominators have no pole and numerators have pole of order one exactly when $(\rho, a) \in Jord(\pi)$.

This explains the following result: the induced representation

$$\delta(\rho, a) \rtimes \pi$$

is irreducible if and only if either (ρ, a) does not satisfy (J-1) of section 2 or $(\rho, a) \in Jord(\pi)$.

Therefore, $Jord_\rho(\pi)$ determines completely if $\delta(\rho, a) \rtimes \pi$ is reducible or not (for arbitrary irreducible unitarizable cuspidal representation ρ).

A more elementary arguments can be applied in the following way to obtain the same result. If a satisfies the condition (J-1) of section 2, then $Jord_\rho(\pi)$ by the definition tells exactly when $\delta(\rho, a) \rtimes \pi$ reduces (this is a part of the definition of $Jord_\rho(\pi)$). Suppose that a does not satisfy (J-1). Let a be odd. Then (BA) implies that $\nu^x \rho \rtimes \sigma$ reduces for some $x \in (1/2) + \mathbb{Z}$. Now Proposition 4.2 of [T2] (together with Remark 12.2 in this paper, and (BA)) implies that $\delta(\rho, a) \rtimes \pi$ is irreducible. If we suppose that a is even, we get in a similar way that $\delta(\rho, a) \rtimes \pi$ is irreducible (using Proposition 4.1 of [T2] and Remark 12.2 of this paper).

The computation of the product of standard intertwining operators can be generalized replacing π by a representation induced from an irreducible square integrable representation. This computation can be made since the case of the general linear groups is already known by the Shahidi's results. Using the result of Harish-Chandra, we can compute the intertwining algebra of a representation induced by an irreducible square integrable representation in terms of poles of the standard intertwining operators. From above description of reducibility of $\delta(\rho, a) \rtimes \pi$, we obtain in that way the following

13.1. Theorem. *(i) Let $\rho_1, \dots, \rho_n$ be a set of (equivalence classes of) irreducible unitarizable cuspidal representations of general linear groups $GL(k_i), k_i \geq 1$ and let $a_1, \dots, a_n \in \mathbb{N}$. Suppose that π be an irreducible square integrable representation of some S_q. Then the induced representation:*

$$\Pi = \left(\prod_{i=1}^{n} \delta(\rho_i, a_i) \right) \rtimes \pi \tag{13-1}$$

is a multiplicity one representation of length 2^m, where m is the cardinal number of the following set:

$$\{(\rho_i, a_i);\ 1 \leq i \leq n,\ (\rho_i, a_i) \text{ satisfies (J-1) and } (\rho_i, a_i) \notin Jord(\pi)\}$$

(note that we count only different (ρ_i, a_i)'s, not the different indexes).

(ii) Suppose that we have another collection $\rho'_1, \dots, \rho'_{n'}, a'_1, \dots, a'_{n'}$ and π' as above. We define the representation Π' for this collection in the same way as we have defined Π in (13-1) for $\rho_1, \dots, \rho_n, a_1, \dots, a_n$ and π. Then the representations Π and Π' have an irreducible subquotient in common, if and only if they are equivalent. This happens if and only if $\pi \cong \pi'$, $n = n'$ and $(\rho_1, a_1), \dots, (\rho_n, a_n)$ is a permutation of $(\rho'_1, a'_1), \dots, (\rho'_n, a'_n)$.

The first part of the theorem can be obtained from the discussion which precedes the theorem, also using the Goldberg's result from [G] (this requires char $F' = 0$ assumption). The second part of the theorem follows from Proposition III.4.1 of [W1].

The above theorem gives a reduction of irreducible tempered representations to cuspidal representations and cuspidal reducibilities. Therefore, it implies also the same type of reduction of the non-unitary duals (i.e. of the parameters in the Langlands classification).

14. Exercises of the introduction and examples of admissible triples

We shall first fix $Jord$ and

$$\epsilon : Jord \to \{\pm 1\}.$$

We shall prove the exercises which we have mentioned in the introduction. Therefore, we shall use the notation ϕ of the introduction, which is uniquely determined by $Jord$ in such a way that $Jord(\phi) = Jord$. Using the notation of the introduction, we shall construct directly

$$\phi_{\phi,\epsilon,cusp}, \epsilon_{\phi,\epsilon,cusp}.$$

To do that, we fix ρ such that $Jord_\rho \neq \emptyset$, and we decompose $Jord_\rho$ into a partition

$$\cup_{i=1}^{\ell} S_i = Jord_\rho$$

of non-empty sets S_i, where $S_1, \cdots, S_\ell$ are subsets of $\mathbb{N}$, in such a way that for all $1 \leq i < \ell$ and for all $a \in S_i$, $a' \in S_{i+1}$ we have $a < a'$ and $\epsilon(\rho, a) \neq \epsilon(\rho, a')$.

Note that the last condition implies that ϵ is constant on each S_i for any $i \in [1, \ell]$. Clearly, one can do the above decomposition, and there is only one way to do that.

Define $I \subseteq [1, \ell]$ by: $i \in I$ if and only if $\mathrm{card}\,(S_i)$ is odd and $i \neq 1$ if $Jord_\rho \subseteq 2\mathbb{N}$ and $\epsilon|S_1$ is trivial. If $I = \emptyset$, we take $Jord_{\rho,cusp} = \emptyset$. If $I \neq \emptyset$, we denote by ψ the unique ordering preserving bijection between I and $[1, \mathrm{card}\,(I)]$. Define

$$Jord_{\rho,cusp} := \{(\rho, 2j - \eta);\ j \in [1, \mathrm{card}\,(I)]\},$$

where $\eta = 1$ (resp. 0) if $Jord_\rho$ contains odd (resp. even) elements. Define

$$\epsilon_{\rho,cusp} : Jord_{\rho,cusp} \to \{\pm 1\}$$

with

$$\epsilon_{\rho,cusp}(\rho, 2j - \eta) = \epsilon(\rho, a_j),$$

where a_j is any element in $S_{\psi^{-1}(j)}$.

We define $\phi_{\phi,\epsilon,cusp}$ uniquely by:

$$Jord(\phi_{\phi,\epsilon,cusp}) = \cup_\rho Jord_{\rho,cusp}.$$

Further, $\epsilon_{\phi,\epsilon,cusp}$ comes from all the $\epsilon_{\rho,cusp}$ in the obvious way. We need now to prove that $\phi_{\phi,\epsilon,cusp}, \epsilon_{\phi,\epsilon,cusp}$ is the cuspidal support of ϕ, ϵ. One proves this directly.

The first case is when all S_i (as above) have cardinality 1. This is exactly the case when the first condition of the introduction is satisfied (i.e. we are in the alternated case). In the other case, we argue by induction. The observation here is that the subsets associated to ϕ_1, ϵ_1 (the notation is the same as in the introduction), are obtained from those associated to ϕ, ϵ by deleting two elements in the same subset.

The second exercise of the introduction is also easy to conclude from our construction. Fix ϕ, ϵ as above, and let ϵ' be such that ϕ, ϵ' is a Langlands parameter. We assume that $\Delta_\epsilon = \Delta_{\epsilon'}$ and we have to prove that

$$\phi_{\phi,\epsilon,cusp} = \phi_{\phi,\epsilon',cusp}.$$

The assumption $\Delta_\epsilon = \Delta_{\epsilon'}$ on the connection between ϵ and ϵ' implies that, for all ρ, the decomposition of $Jord_\rho$ into intersections with segments as above, is the same if we use ϵ instead of ϵ'. This and the above construction imply $\phi_{\phi,\epsilon,cusp} = \phi_{\phi,\epsilon',cusp}$.

In the sequel of this section, we shall write few examples of admissible triples. To simplify discussion, we shall assume that our triples in this section satisfy the condition (L) of the fifth section.

For a given $Jord$ and π_{cusp} (or $Jord(\pi_{cusp})$), we shall say that a partially defined function ϵ is admissible if ϵ, together with $Jord$ and π_{cusp} forms an admissible triple.

14.1. We shall first consider the case when $Jord_\rho(\pi_{cusp}) = \emptyset$ and $L(\rho, R_{d_\rho}, s)$ has a pole at $s = 0$ (the last condition is equivalent to the fact that we are in the even case). Then $\nu^{1/2}\rho \rtimes \pi_{cusp}$ reduces. We shall now discuss some possibilities for $Jord_\rho$.

14.1.0. $Jord_\rho = \emptyset$.

Here is only one ϵ. We are in the alternated case. The attached representation is π_{cusp}.

14.1.1. $Jord_\rho = \{2k_1\}$, $k_1 \in \mathbb{N}$.

Then there are two possible partial functions ϵ, which we can describe by the following table

(14-1-1)

$Jord_\rho$	ϵ_1	ϵ_2
$2k_1$	1	-1.

First ϵ_2 is not admissible (ϵ_2 can not be in the mixed case because card($Jord_\rho$)=1, and ϵ_2 can not be alternated since card($Jord_\rho$)=1 and card($Jord'_\rho(\pi_{cusp})$)=0). Further,

$$\epsilon_1$$

is admissible and we are in the alternated case.

14.1.2. $Jord_\rho = \{2k_1, 2k_2\}$, $k_1 < k_2 \in \mathbb{N}$.

Then there exist the following functions ϵ on $Jord_\rho$:

(14-1-2)

$Jord_\rho$	ϵ_1	ϵ_2	ϵ_3	ϵ_4
$2k_1$	1	1	-1	-1
$2k_2$	1	-1	1	-1.

We cannot have alternated ϵ (since card($Jord_\rho$) = 2 and card($Jord_\rho(\pi_{cusp})$)=0). Obviously,

$$\epsilon_1, \epsilon_4$$

are admissible (see 14.1.0).

14.1.3. $Jord_\rho = \{2k_1, 2k_2, 2k_3\}$, $k_1 < k_2 < k_3 \in \mathbb{N}$.

We have the following possibilities for ϵ

(14-1-3)

$Jord_\rho$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7	ϵ_8
$2k_1$	1	1	1	1	−1	−1	−1	−1
$2k_2$	1	1	−1	−1	1	1	−1	−1
$2k_3$	1	−1	1	−1	1	−1	1	−1.

We cannot have alternated ϵ (since card($Jord_\rho$)=3 and card($Jord_\rho(\pi_{cusp})$)=0). Thus ϵ_3 and ϵ_6 cannot be admissible. Further, if we have odd number of −1's, then from 14.1.1 we see that ϵ can not be admissible. Thus, it remains

$$\epsilon_1, \epsilon_4, \epsilon_7.$$

From 14.1.1 we see that they are admissible.

14.1.4. $Jord_\rho = \{2k_1, 2k_2, 2k_3, 2k_4\}$, $k_1 < k_2 < k_3 < k_4 \in \mathbb{N}$.

We have the following partial functions ϵ

(14-1-4)

$Jord_\rho$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7	ϵ_8	ϵ_9	ϵ_{10}	ϵ_{11}	ϵ_{12}	ϵ_{13}	ϵ_{14}	ϵ_{15}	ϵ_{16}
$2k_1$	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
$2k_2$	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
$2k_3$	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
$2k_4$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1.

We have not alternated ϵ (since card($Jord_\rho$)=4 and card($Jord_\rho(\pi_{cusp})$)=0). Thus ϵ_6 and ϵ_{11} cannot be admissible. Again, if we have odd number of −1's, then from 14.1.2. we see that ϵ can not be admissible. Thus, $\epsilon_2, \epsilon_3, \epsilon_5, \epsilon_8, \epsilon_9, \epsilon_{12}, \epsilon_{14}$ and ϵ_{15} are not admissible. it remains

$$\epsilon_1, \epsilon_4, \epsilon_7, \epsilon_{10}, \epsilon_{13}, \epsilon_{16}.$$

From 14.1.2 follows that they are admissible.

14.2. Now we shall consider the case $Jord_\rho(\pi_{cusp}) = \{2\}$. Then $\nu^{3/2}\rho \rtimes \pi_{cusp}$ reduces. We shall list now some examples for $Jord_\rho$. In paragraphs 14.2.i below, ϵ_j will denote the function ϵ_j from the table 14.1.i.

14.2.0. $Jord_\rho = \emptyset$.

This case cannot have admissible ϵ (since there is no bijection between the sets $\emptyset$ and $Jord_\rho(\pi_{cusp}) = \{2\}$).

14.2.1. $Jord_\rho = \{2k_1\}$, $k_1 \in \mathbb{N}$.

First, ϵ_1 is not admissible (since ϵ_1 can not be in the mixed case, and further ϵ_1 can not be alternated since card($Jord_\rho$)=1 and card($Jord'_\rho(\pi_{cusp})$)=2). For

$$\epsilon_2$$

we are in the alternated case. If $k_1 = 1$, the attached representation is π_{cusp}.

14.2.2. $Jord_\rho = \{2k_1, 2k_2\}$, $k_1 < k_2 \in \mathbb{N}$.

Here we can not have ϵ of mixed type, because of 14.2.0. Now consider alternated ϵ. Suppose that ϵ_3 is admissible. Then we would have a bijection of $Jord_\rho$ onto $\{2\}$, what is impossible. For ϵ_2, we have a bijection of $Jord_\rho$ onto $\{0, 2\}$. Thus

$$\epsilon_2$$

is admissible and we are in the alternated case.

14.2.3. $Jord_\rho = \{2k_1, 2k_2, 2k_3\}$, $k_1 < k_2 < k_3 \in \mathbb{N}$.

We cannot have alternated ϵ. Therefore, ϵ_3 and ϵ_6 cannot be admissible. Further, if we have odd number of 1's, then from 14.2.1 we see that ϵ can not be admissible. Thus, it remains

$$\epsilon_2, \epsilon_5, \epsilon_8.$$

Now 14.2.1 implies that they are admissible.

14.2.4. $Jord_\rho = \{2k_1, 2k_2, 2k_3, 2k_4\}$, $k_1 < k_2 < k_3 < k_4 \in \mathbb{N}$.

We do not have here alternated ϵ. Thus ϵ_6 and ϵ_{11} cannot be admissible. Again, if we have even number of 1's, then from 14.2.2 we see that ϵ can not be admissible. Thus, $\epsilon_1, \epsilon_4, \epsilon_6, \epsilon_7, \epsilon_{10}, \epsilon_{11}, \epsilon_{13}$ and ϵ_{16} are not admissible. Among ϵ's which remain, 14.2.2 implies that after deleting 1, 1, or -1, -1, we need to have 1, -1 left (in this order). Therefore, all the candidates for admissible ϵ's are among

$$\epsilon_2, \epsilon_5, \epsilon_8, \epsilon_{14}.$$

Their admissibility follows from 14.2.2.

14.3. Now suppose that $L(\rho, R_{d_\rho}, s)$ has not pole at $s = 0$ (i.e. we are in the odd case). Let $Jord_\rho(\pi_{cusp}) = \emptyset$ (then $\rho \rtimes \pi$ reduces). Now if one changes $2k_i$ into $2k_i - 1$ in examples 14.1.i, one gets admissible ϵ's in this case, assuming that i is even (i.e. $Jord_\rho$ have even number of elements). If $Jord_\rho$ has odd number of elements, then there are no admissible partial functions in this case.

14.4. If $Jord_\rho(\pi_{cusp}) = \{1, 3, \ldots, 2l-1\}$ for some $l \in \mathbb{N}$, then $\nu^l \rho \rtimes \sigma$ reduces. We shall now consider the case $l = 1$, i.e. $Jord_\rho(\pi_{cusp}) = \{1\}$.

14.4.0. $Jord_\rho = \emptyset$.

Here we cannot have admissible ϵ (otherwise we would have a bijection between the sets $\emptyset$ and $Jord_\rho(\pi_{cusp}) = \{2\}$).

14.4.1. $Jord_\rho = \{2k_1 - 1\}$, $k_1 \in \mathbb{N}$.

There is only one

$$\epsilon = \emptyset,$$

and it is admissible of alternated type.

14.4.2. $Jord_\rho = \{2k_1 - 1, 2k_2 - 1\}$, $k_1 < k_2 \in \mathbb{N}$.

We have the following partial functions ϵ which we shall describe in the following way

$Jord_\rho$	ϵ_1	ϵ_2
$2k_1 - 1$		
	1	-1
$2k_2 - 1$		

.

Here ϵ_1 and ϵ_2 denote the following partial functions

$$\epsilon_1(2k_1 - 1)\epsilon_1(2k_2 - 1)^{-1} = 1,$$

$$\epsilon_2(2k_1 - 1)\epsilon_2(2k_2 - 1)^{-1} = -1.$$

Note that we can not have ϵ of mixed type (because of 14.4.0.). Further, ϵ_2 is not alternated (since card($Jord_\rho$)=2 and card($Jord_\rho(\pi_{cusp})$)=1). Therefore, in this case we do not have admissible ϵ's.

14.4.3. $Jord_\rho = \{2k_1 - 1, 2k_2 - 1, 2k_3 - 1\}$, $k_1 < k_2 < k_3 \in \mathbb{N}$.

We have the following partial functions ϵ:

$Jord_\rho$	ϵ_1	ϵ_2	ϵ_3	ϵ_4
$2k_1 - 1$				
	1	1	-1	-1
$2k_2 - 1$				
	1	-1	1	-1
$2k_3 - 1$				

.

The interpretation of the table is analogous to the interpretation of the table in 14.4.2. First, we cannot have ϵ of alternated type by the usual argument. Thus, it remains

$$\epsilon_1, \epsilon_2, \epsilon_3.$$

As before, 14.4.1. implies that above ϵ's are admissible.

14.4.4. $Jord_\rho = \{2k_1 - 1, 2k_2 - 1, 2k_3 - 1, 2k_4 - 1\}$, $k_1 < k_2 < k_3 < k_4 \in \mathbb{N}$.

Obviously, we can not have alternated ϵ here. Now 14.4.2 (or 14.4.0) implies that we do not have here also ϵ of the mixed type.

14.4.5. $Jord_\rho = \{2k_1 - 1, 2k_2 - 1, 2k_3 - 1, 2k_4 - 1, 2k_5 - 1\}$, $k_1 < k_2 < k_3 < k_4 < k_5 \in \mathbb{N}$.

We have the following partial functions ϵ

$Jord_\rho$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7	ϵ_8	ϵ_9	ϵ_{10}	ϵ_{11}	ϵ_{12}	ϵ_{13}	ϵ_{14}	ϵ_{15}	ϵ_{16}
$2k_1-1$																
	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
$2k_2-1$																
	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
$2k_3-1$																
	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
$2k_4-1$																
	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
$2k_5-1$																.

First, we do have not alternated ϵ. Thus, ϵ_{16} is not admissible. Further, if we remove 1, the resulting restricted partial function that we obtain then, cannot be -1, -1 by 14.4.2. Therefore, the candidates for admissible ϵ's reduce to

$$\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_9, \epsilon_{11}, \epsilon_{12}, \epsilon_{14}.$$

Their admissibility follows from 14.4.2.

14.4.6. $Jord_\rho = \{2k_1 - 1, 2k_2 - 1, \dots, 2k_{2n} - 1\}$, $k_1 < k_2 < \cdots < k_{2n} \in \mathbb{N}$.

Again, we do not have here ϵ of the alternated type. As before, we see that we do not have here ϵ of the mixed type.

14.5. Remark. In this remark we shall give interpretations of the classification of irreducible square integrable representations, without assuming (A).

(i) The first possibility is to make the following modification of the definition of $Jord$ in admissible triple: instead of requiring that $Jord$ is finite, to require that $Jord \backslash Jord(\pi_{cusp})$ is finite. Then the classification of irreducible square integrable representations done in this paper holds in unchanged form (we do not need to assume (A)).

(ii) Fix an irreducible cuspidal representation π_{cusp} of $S_{n'}$. Let $\rho_1, \dots, \rho_k$ be inequivalent F'/F-selfdual irreducible cuspidal representations of general linear groups. Let $\nu^{\mathbb{R}}\rho_i = \{\nu^x\rho_i; x \in \mathbb{R}\}$. Denote by $\mathcal{D}(\rho_1, \dots, \rho_k; \pi_{cusp})$ the set of all equivalence classes of irreducible square integrable representation of groups S_n, whose all factors are contained in $\cup_{i=1}^k \nu^{\mathbb{R}}\rho_i$ and whose cuspidal support is π_{cusp}.

Let $\pi \in \mathcal{D}(\rho_1, \dots, \rho_k; \pi_{cusp})$. For each j, there exists an irreducible representation π_j of some S_{n_j} whose all factors are contained in $\nu^{\mathbb{R}}\rho_j$, and an irreducible representation τ_j of a general linear group whose cuspidal support contains of representations from $\left(\cup_{i=1}^k \nu^{\mathbb{R}}\rho_i\right) \backslash \nu^{\mathbb{R}}\rho_j$, such that $\pi \hookrightarrow \tau_j \rtimes \pi_j$. By [J1], representations $\pi_1, \dots, \pi_k$ are uniquely determined by π, they are all square integrable and $\pi \mapsto (\pi_1, \dots, \pi_k)$ is a bijection from $\mathcal{D}(\rho_1, \dots, \rho_k; \pi_{cusp})$ onto the Cartesian product $\prod_{i=1}^k \mathcal{D}(\rho_i; \pi_{cusp})$. In this way one gets a reduction of the problem of classification of irreducible square integrable representations to the problem of classification of sets $\mathcal{D}(\rho;\pi_{cusp})$ of irreducible square integrable representations (this reduction is implicit in our construction in this paper). Now the pairs $Jord_\rho$ $(\subseteq \mathbb{N})$, ϵ_ρ, where ϵ_ρ is a partially defined function on $Jord_\rho$ which makes with $Jord_\rho(\pi_{cusp})$ an admissible triple, parameterize $\mathcal{D}(\rho;\pi_{cusp})$.

(iii) Now we shall describe more explicitly the parameters of $\mathcal{D}(\rho;\pi_{cusp})$. Suppose that $\nu^{\pm\alpha}\rtimes\pi_{cusp}$ reduces ($\alpha\geq 0$). Now (BA) implies $\alpha\in(1/2)\mathbb{Z}$ (then $Jord_\rho(\pi_{cusp}) = \{2\alpha-1-2i;\ i\in\mathbb{Z}_+ \text{ and } 2\alpha-1-2i\in\mathbb{N}\}$).

Then $Jord_\rho^+$ of alternated type is a subset of $\mathbb{N}$ consisting of the element of the same parity as $2\alpha-1$, of cardinality α if $\alpha\in\mathbb{Z}_+$ and $\alpha\pm 1/2$ if $\alpha\notin\mathbb{Z}_+$. In this case, ϵ_ρ is uniquely determined with $Jord_\rho^+$ (and the fact that we are in the alternated case).

Fix alternated $Jord_\rho = Jord_\rho^+$ (and ϵ_ρ). Take any two numbers $a_- < a\in\mathbb{N}$ of the same parity as $2\alpha-1$ such that $[a_-,a]\cap Jord_\rho=\emptyset$. Set $Jord_\rho^{(1)} = Jord_\rho\cup\{a_-,a\}$. Denote by $\epsilon_\rho^{(1)}$ any extension of ϵ_ρ to $Jord_\rho^{(1)}$ such that $\epsilon_\rho(a_-) = e_\rho(a)$ (there are precisely two such extensions). Repeating this construction, we get $Jord_\rho^{(2)}$, $\epsilon_\rho^{(2)}$. We can continue this construction. By this simple construction, each admissible $Jord_\rho$, ϵ_ρ can be obtained in a finitely many steps (starting from appropriate alternated $Jord_\rho^+$).

15. Unitary groups

In this section we shall explain necessary modifications which one needs to make that the classifications obtained for symplectic and odd-orthogonal groups in former sections, holds also for the unitary groups.

Fix a series S_n of unitary groups (see the first section). Note that S_n are connected reductive groups over F. Further, the formula (1-1) holds (and therefore (1-2) also holds). This follows in a similar way as in the case of non-split odd-orthogonal groups (the reduced root system is either of type B or C; in both cases the Weyl group is the same as in the symplectic and odd-orthogonal cases).

The Casselman's square integrability criterion has the same form as in the case of symplectic and odd-orthogonal groups.

In the definition of an admissible triple, we need only to specify which L-function we need to take in the definition of the parity of a for $(\rho,a)\in Jord$. The L-function is determined by the representation. We shall recall of the representation introduced in [M2] which enteres the definition of the parity.

Let S_n be the unitary group of the unitary space V_n from the Witt tower and denote by $n^* = \dim_{F'}(V_n)$ (note that $n^*=2n$ if $\dim_{F'}(V_n)$ is even, and $n^*=2n+1$ otherwise; recall that S_n is the unitary group $U(n^*,F'/F)$).

Take an irreducible F'/F-selfdual cuspidal representation ρ of a general linear group $GL(d_\rho,F')$. The L-group of F-group $GL(d_\rho,F')$ is isomorphic to a semidirect product

$$\big(GL(d_\rho,\mathbb{C})\times GL(d_\rho,\mathbb{C})\big)\leftthreetimes Gal(F'/F),$$

where (the non-trivial element of) $Gal(F'/F)$ acts on the normal subgroup $GL(d_\rho,\mathbb{C})\times GL(d_\rho,\mathbb{C})$ by

$$\theta(g_1,g_2,1)\theta^{-1} = ({}^tg_2^{-1},{}^tg_1^{-1},1)$$

(here ${}^t g$ denotes the transposed matrix of g).

For $\eta \in \{\pm 1\}$, denote by $R^{(\eta)}_{d_\rho}$ the representation of the above L-group of $GL(d_\rho, F')$ on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{d_\rho})$ given by:

$$(g_1, g_2, 1)u = g_1\, u\, {}^t g_2 \qquad \text{and} \qquad (1, 1, \theta)u = \eta\, {}^t u.$$

Suppose now that S_n is a series of groups such that dimensions $\dim_{F'}(V_n)$ are even. Then we shall denote by

$$R_{d_\rho}$$

the representation $R^{(1)}_{d_\rho}$. Otherwise, in the odd case, R_{d_ρ} will denote $R^{(-1)}_{d_\rho}$.

With R_{d_ρ} defined in this way for unitary groups, we have the same definition of the parity as in the second section.

The degree

$$\sum_{(\rho,a)\in Jord(\pi)} a\, d_\rho$$

needs to be n^*, for an irreducible square integrable representation π of S_n.

These are required modifications in the unitary case.

16. Even-orthogonal groups

Fix a series S_n of even-orthogonal groups (see the first section). First we need to describe L-functions which enter the definition of Jordan blocks. For an irreducible cuspidal selfdual representation ρ of $GL(d_\rho, F)$, we denote by R_{d_ρ} the representation of $GL(d_\rho, \mathbb{C})$ on $\wedge^2 \mathbb{C}^{d_\rho}$.

The degree $\sum_{(\rho,a)\in Jord(\pi)} a\, d_\rho$ needs to be here $2n$, for an irreducible square integrable representation π of S_n.

Denote the subgroup of elements in S_n of determinant one by S'_n (S'_n has index two in S_n).

Now we shall comment the case of non-split even-orthogonal groups S_n. Then the Weyl group of S'_n is the same as in the case of symplectic and odd-orthogonal groups (the root system is of type B). Therefore, we can apply the calculations done in the section 4. of [T5] to the groups S'_n. Further, one can easily see that the analogue of Lemma 5.1 holds here (recall that the unipotent radicals in S_n are already contained in S'_n). From this one gets that the formula (1-1) holds also for groups S_n.

Further, since we have the same root system as in the case of odd-orthogonal groups, the Casselman's square integrability criterion hold for groups S'_n (and S_n) in the same form as in the case of symplectic and odd-orthogonal groups.

These are the only comments that we need in the case of non-split even-orthogonal groups.

Suppose now that S_n consists of split even-orthogonal groups. Now the formula (1-1) holds by [B] (the strategy of proving (1-1) requires modification in this case, since the root systems of the groups S'_n are of type D, and the Weyl group is slightly different from the previous cases; there are also other ways to prove (1-1), different from proof in [B]).

A possible differences with the case of the groups that we have studied before appears if π_{cusp} is a representation of $S_0 = \{1\}$. In this case we have two standard "Siegel parabolic subgroups" in S_n. They are also parabolic subgroups in S_n, but they are conjugated in S_n. Therefore, we can proceed in this situation in the same way as in the cases of groups that we have considered before.

For $n \neq 1$, a (finite length) representation π_{cusp} of S_n is cuspidal if $\pi_{cusp}|S'_n$ is cuspidal representation of S'_n. The group S_1 does not have cuspidal representations.

Further, if $n \neq 1$, then a representation π of S_n is square integrable if and only if $\pi|S'_n$ is square integrable representation of S'_n. One directly sees that S_1 does not have square integrable representations (neither it has essentially square integrable representations).

A comment regarding the Casselman's square integrability criterion is necessary, since the root systems of S'_n are different from the previous ones. First we shall say few words about parabolic subgroups.

Denote by $s \in S_n$ a quasi-diagonal matrix

$$\text{q-diag}\Big(\underbrace{1,\dots,1}_{n-1 \text{ times}}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \underbrace{1,\dots,1}_{n-1 \text{ times}}\Big).$$

Recall that standard parabolic subgroups in $SO(2n,F)$ are parameterized by partitions $\beta = n_1 + \cdots + n_k$ of $0 \leq m \leq n$, $m \neq n-1$ into a sum of positive integers. Besides these standard parabolic subgroups, the remaining ones are parabolic subgroups $sP_\beta s$, when $n = m$ and $n_k \geq 2$ (see [B]).

Let (π, V) be an irreducible representation of S_n, whose partial support is π_{cusp}. Denote $V(N) = \text{span}_{\mathbb{C}}\{\pi(n)v - v; n \in N, v \in V\}$ and let $V_N = V/V(N)$ be the normalized Jacquet module.

For checking square integrability of π, we need to check the Casselman's square integrability criterion for parabolic subgroups of type P_γ or $sP_\gamma s$, for which Jacquet modules are cuspidal.

Now the Levi factor of a parabolic subgroup M_γ of P_γ is naturally isomorphic to $GL(n_1,F) \times \cdots \times GL(n_k,F) \times SO(n-m,F)$. In the case of $sP_\gamma s$, it is also naturally isomorphic to $GL(n_1,F) \times \cdots \times GL(n_k,F) \times SO(n-m,F)$, by the conjugation of this subgroup with s (in the first case, we shall say that we are in the non-conjugate situation, while in the other case we shall say that we are in the conjugate situation). Let $\rho = \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ be an irreducible cuspidal subquotient of the Jacquet module, or its conjugate, if we are in the conjugated situation. Define

$$e_*(\rho) = (\underbrace{e(\rho_1),\dots,e(\rho_1)}_{n_1 \text{ times}}, \dots, \underbrace{e(\rho_k),\dots,e(\rho_k)}_{n_k \text{ times}}, \underbrace{e(\sigma),\dots,e(\sigma)}_{n-m \text{ times}}),$$

where we take $e(\sigma) = 0$ if $n - m \geq 2$.

First consider the case $n - m \geq 2$. The Casselman's criterion for square integrability tells in this case that the square integrability is equivalent to

$$\sum_{i=1}^{j} e(\rho_i) n_i > 0, \qquad j = 1, \dots, k. \tag{16-1}$$

These are our usual relations for square integrability.

Let $n = m$. Suppose $n_k = 1$. Then the last two simple roots of the root system D_n are not in the roots that define the parabolic subgroup. The square integrability criterion now is equivalent to the following relations

$$\sum_{i=1}^{j} e(\rho_i)n_i > 0, \qquad j = 1, \dots, k-2, \tag{16-2}$$

$$\left(\sum_{i=1}^{k-1} e(\rho_i)n_i\right) - e(\rho_k)n_k > 0, \tag{16-3}$$

$$\sum_{i=1}^{k} e(\rho_i)n_i > 0. \tag{16-4}$$

Summing (16-3) and (16-4) we get

$$\sum_{i=1}^{k-1} e(\rho_i)n_i > 0. \tag{16-5}$$

Thus

$$\sum_{i=1}^{j} e(\rho_i)n_i > 0, \qquad j = 1, \dots, k. \tag{16-6}$$

Note that (16-6) represents our usual relations for square integrability. We need to see that they are enough for square integrability, i.e. that they also imply (16-3) (recall that π is a representations of $O(2n, F)$).

Suppose that (16-6) hold for each irreducible cuspidal subquotient ρ of the Jacquet module. We shall now see that (16-3) holds. We know that there is an epimorphism $V \to V_{N_\gamma} \to \rho = \rho_1 \otimes \cdots \otimes \rho_k \otimes 1$ of $P_\gamma-$representations (the unipotent radical is assumed to act trivially in the last representation). Now conjugating this epimorphism by s, we get that there is P_γ epimorphism onto $\rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes \tilde{\rho}_k \otimes 1$. Now (16-4) applied to the last subquotient of the Jacquet module, implies that (16-3) holds.

It remains to consider the case $n = m$ and $n_k \geq 2$. Then $\sigma = \pi_{cusp}$ is the trivial representation (of $O(0, F)$). Now the square integrability criterion gives relations (16-6). Note that these relations need to hold in the non-conjugate situations, as well as in conjugate ones.

Suppose that relations (16-6) hold for non-conjugate situations only. Let ρ be a subquotient of the Jacquet module for $sP_\gamma s$ (note $N_\gamma = sN_\gamma s$). Then we have an epimorphism $V \to V_{N_\gamma} \to s\rho s = s(\rho_1 \otimes \cdots \otimes \rho_k \otimes 1)s \cong \rho = \rho_1 \otimes \cdots \otimes s(\rho_k \otimes 1)s$ of $sP_\gamma s-$representations (the unipotent radical is assumed to act trivially in the last representation, as in the previous case). Conjugating this epimorphism by s, we obtain an epimorphism from V onto

$\rho_1 \otimes \cdots \otimes \rho_k \otimes 1$ of P_γ representations. Now relations (16-6) applied to the last subquotient of the Jacquet module (in the non-conjugate situation), imply that (16-6) also hold in the conjugated situation.

At different places, we have use Harish-Chandra's results on the Plancherel measure, specialy in **2.2**; this has only be written in [W1] for connected group but we have extend what we need for even orthogonal groups in the appendix of [M2]. The proof of **2.2** is already in [M2].

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