Basic representation theory of reductive p-adic groups

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The following notes are aimed at presenting basic notions of the representation theory of reductive p-adic groups in an elementary manner, suitable for beginners. To keep them short we have made drastic choices, oriented by two guidelines. The first is to keep the representation theory quite general as long as possible, in that representations are taken in vector spaces over a (commutative) field R about which assumptions are made only when necessary; admittedly, from chapter 2 on, the characteristic of R is not equal to p. The second is to fix a reachable yet major goal: prove that smooth irreducible representations of a reductive p-adic group over a field of characteristic not p are admissible, a cornerstone in the theory.

The main references are the book by C.J. Bushnell and G. Henniart [6] which, although explicitly dealing with GL(2), actually explains the main ideas in the subject of complex representation theory of reductive p-adic groups; the book by D. Renard [11] which gives the full arguments in that subject; and the book by M.-F. Vignéras [16] for questions depending on the base field of the representations. The never-published-but-ever-cited notes by W. Casselman are such a common background of everyone in the subject that they cannot be left aside [8]. Other references are occasional. Unpublished lecture notes made available to us by B. Lemaire and by S. Stevens have also been very useful, as well as the remarks of the anonymous referee which encouraged us to clarify some proofs.

1 Smooth representations of locally profinite groups

1.1 Locally profinite groups

A topological group G is *profinite* if it is *compact* and *totally disconnected* (that is, there is no connected subset of G with more than one element). If G is profinite the quotient maps combine into a topological isomorphism

$$G \quad \cong \quad \varprojlim G/U$$

where U runs through all open, normal subgroups of G, hence G is the projective limit of a system of finite groups. Conversely such a projective limit is a compact and totally disconnected group, hence profinite [9]. Of course finite groups themselves, with the discrete topology, are profinite. If the quotients G/U above are all p-groups, then G is a pro-p-group. Example: the additive group of the ring of p-adic integers is the pro-p-group $\mathbb{Z}_p = \varprojlim \mathbb{Z} / p^n\mathbb{Z}$ ($n \geq 1$).

Definition 1.1. *A topological group G is* locally profinite *if it satisfies one of the equivalent following conditions [9]:*

(i) *every neighbourhood of the identity in G contains a compact open subgroup;*

(ii) *G is locally compact and totally disconnected.*

Compact locally profinite groups G are profinite.

Example: the additive group of the field $\mathbb{Q}_p$ of p-adic numbers is a locally profinite group, with a fundamental system of neighbourhoods of 0 given by $p^n\mathbb{Z}_p$, $n \in \mathbb{Z}$. The same indeed holds for the additive group of a local non-archimedean field F. We denote by $\mathfrak{o}_F$ the ring of integers of F, by $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$, by k_F the residual field, obtained as the finite quotient $\mathfrak{o}_F/\mathfrak{p}_F$, of characteristic p (the *residual characteristic* of F) and cardinality q. Then $\mathfrak{o}_F \cong \varprojlim \mathfrak{o}_F / \mathfrak{p}_F^n$ ($n \geq 1$) is a pro-p-group and the fractional ideals $\mathfrak{p}_F^n$, $n \in \mathbb{Z}$, make up a family of compact open subgroups such that $F = \cup_{n\in\mathbb{Z}}\mathfrak{p}_F^n$.
Let $a \in F$, $a \neq 0$. The *valuation* of a, denoted by $\mathrm{val}_F(a)$, is the maximum integer n such that $a \in \mathfrak{p}_F^n$; one puts $\mathrm{val}_F(0) = \infty$. The topology of F is actually that of a metric space, defined by the absolute value of F: $|a|_F = q^{-\mathrm{val}_F(a)}$ for $a \in F^\times$, $|0|_F = 0$.
The multiplicative group $F^\times$ is locally profinite as well, with a fundamental system of neighbourhoods of 1 given by the compact open subgroups $1+\mathfrak{p}_F^n$, $n \geq 1$, contained in the unique maximal compact subgroup $\mathfrak{o}_F^\times$. Observe that $1 + \mathfrak{p}_F$ is a pro-p-group but $\mathfrak{o}_F^\times$ is not.
Finite-dimensional vector spaces over F can be identified with a product of copies of F, by choosing a basis, and endowed with the product topology: they are thus locally profinite groups as well – one must of course check that the topology does not depend on the choice of basis. This applies to the algebra $M_n(F)$ of n by n matrices over F, of which the group $\mathrm{GL}_n(F)$ is an open subset. We give it the induced topology. One can check that product and inverse mappings are continuous, making $\mathrm{GL}_n(F)$ a topological group. The subgroups $K = \mathrm{GL}_n(\mathfrak{o}_F)$ and $K_t = 1 + \mathfrak{p}_F^t M_n(\mathfrak{o}_F)$, $t \geq 1$, are compact open and give a fundamental system of neighbourhoods of 1 in $\mathrm{GL}_n(F)$: this is a locally profinite group. Here again K is profinite and K_1 is a pro-p-group.

We will now work in this totally disconnected world. It is important to keep in mind some fundamental properties of the locally compact totally disconnected topology: in a locally profinite group G, closed subgroups are locally profinite, as well as quotient groups by closed normal subgroups, and any compact subgroup is contained in an open compact subgroup. Points are closed. The centraliser of any element $x \in G$ is closed (as the kernel of the continuous map $g \mapsto gxg^{-1}x^{-1}$). The centre of G is closed, as the intersection of those centralisers.

We will see that calculations oftentimes reduce to finite computations. This is the case for instance whenever we work with the quotient of a compact group by an open subgroup: such a quotient is discrete as the quotient *by* an open subgroup, it is compact as the quotient *of* a compact subgroup, whence it is *finite*.

1.2 Basic representation theory

From now on R will be a (commutative) field, with characteristic l, possibly positive. Whenever necessary we will assume that R is algebraically closed, or add an assumption on the characteristic.

[In [16] representations over a commutative ring R with a unit are considered.]

A *representation* (π, V) of a group G (in the vector space V, over the field R) is a homomorphism π from G into the group of linear automorphisms of an R-vector space V:

$$\pi \ : \ G \longrightarrow \mathrm{Aut}_R(V).$$

A *morphism* between two representations (π, V) and (π', V') of G is a linear homomorphism ϕ from V to V' such that, for any $g \in G$, $\phi \circ \pi(g) = \pi'(g) \circ \phi$. The set of those morphisms is denoted by $\mathrm{Hom}_G(\pi, \pi')$. The two representations are *isomorphic* if this set contains a linear isomorphism. A *subrepresentation* of (π, V) is a pair $(\pi_{|W}, W)$ for a G-stable subspace W of V (G-stable: stable under the automorphisms in $\pi(G)$). If $(\pi_{|W}, W)$ is a subrepresentation of (π, V), then the operators $\pi(g)$, $g \in G$, define automorphisms of the quotient vector space V/W, hence a *quotient representation* $(\bar{\pi}, V/W)$ of (π, V). If $(\pi_{|W}, W)$ and $(\pi_{|W'}, W')$ are subrepresentations of (π, V) with W contained in W', the quotient representation $(\bar{\pi}_{|W'}, W'/W)$ is called a *subquotient* (representation) of (π, V).

A representation (π, V) of G is *indecomposable* if V cannot be decomposed as a direct sum of proper G-stable subspaces. It is *irreducible* if it is non-zero and no proper non-zero subspace of V is stable under the automorphisms in $\pi(G)$.

A *composition series* for a representation (π, V) of G is a finite strictly increasing sequence $\{0\} = V_0 \subset V_1 \subset \cdots V_{n-1} \subset V_n = V$ of G-stable subspaces of V such that the quotients V_i/V_{i-1} are irreducible for $1 \leq i \leq n$; its length is the integer n. The Jordan-Hölder Theorem says that if (π, V) admits two composition series then they have the same length n and the corresponding unordered n-tuples of irreducible subquotients are the same up to isomorphisms. A representation of G is said to have *finite length* (n) if it admits a composition series (of length n).

A representation of G is *finitely generated* or *of finite type* if there are vectors v_i, $1 \le i \le n$, in V such that V is spanned over R by the $\pi(g)v_i$ for $g \in G$ and $1 \le i \le n$.

Proposition 1.2. *Let (π, V) be a non-zero representation of G.*

(i) *(π, V) has an irreducible subquotient.*

(ii) *If (π, V) is finitely generated, it has an irreducible quotient.*

(iii) *If (π, V) has finite length, it has an irreducible subrepresentation.*

Proof. Certainly (iii) is obvious from the definition and (i) follows from (ii) applied to the finitely generated subrepresentation spanned by $\pi(G)v$ for some non-zero $v \in V$. Now (ii) is a classical result based on *Zorn's Lemma :*

Consider a non-empty ordered set in which every non-empty totally ordered subset has an upper bound. Then the set has a maximal element.

If V is not irreducible, we apply Zorn's Lemma to the set of proper G-subspaces of V, ordered by inclusion. Let $\{W_i/\ i \in I\}$ be a totally ordered subset and consider $W = \cup_{i \in I} W_i$. This is a G-subspace, we need to show that it is again proper. Otherwise $W = V$ so the generators $v_1, \cdots, v_n$, say, of V all belong to W, hence for $1 \le k \le n$ there is $i_k \in I$ such that v_k belongs to W_{i_k}. One of the subspaces W_{i_k}, $1 \le k \le n$, contains the others so eventually contains V, a contradiction.

Hence V contains a maximal proper G-subspace W and the quotient V/W is irreducible. ■

Finally we have the following:

Proposition 1.3. *Let (π, V) be a representation of G. The following conditions are equivalent:*

(i) *(π, V) is the direct sum of a family of irreducible G-subspaces;*

(ii) *(π, V) is the sum of its irreducible G-subspaces;*

(iii) *any G-subspace has a G-complement in V.*

The representation (π, V) is completely reducible *if it satisfies those conditions.*

Proof. To show that (ii) implies (i), one fixes a family $\{U_i, i \in I\}$ of irreducible G-subspaces of V such that V is the sum of the U_i, $i \in I$, and one applies Zorn's Lemma to the set of subsets J of I such that the sum $\oplus_{j \in J} U_j$ is direct.

Now, assume (i) holds and write $V = \oplus_{i \in I} U_i$ for a family $\{U_i, i \in I\}$ of irreducible G-subspaces of V. Let W be a G-subspace of V. Let K be a maximal element (Zorn's Lemma again) of the set X of subsets J of I such that $W \cap \sum_{j\in J} U_j = \{0\}$. Then the sum of W and $\sum_{j\in K} U_j$ is direct and its intersection with any U_i, $i \in I$, is either U_i or $\{0\}$. It must be U_i, otherwise the set $K \cup \{i\}$ would belong to X. Hence $\oplus_{k\in K} U_k$ is a G-complement of W in V.

Finally, assume (iii) holds and let W be a complement in V to the sum V_0 of all irreducible G-subspaces of V. Assume W is non-zero. It has a non-zero irreducible subquotient, say W_1/W_0 with $W_0 \subset W_1 \subseteq W$. By (iii) again we have $V = V_0 \oplus W_0 \oplus U$ for some G-subspace U of V, hence a G-projection from V onto U under which the image of W_1 is irreducible, a contradiction since U cannot contain an irreducible G-subspace. ■

1.3 Smooth representations

We now come back to a locally profinite group G and add conditions on the representations according to this extra structure we have on G.

Here, a fruitful notion is that of a *smooth* representation, which takes into account the topology of G: we want the map $(g, v) \mapsto \pi(g)v$ from $G \times V$ to V to be continuous when V is regarded as a discrete space. In other words:

Definition 1.4. *The representation (π, V) of G is* smooth *if for any $v \in V$, the stabilizer $G_v = \{g \in G/\pi(g)(v) = v\}$ of v in G is an* open *subgroup of G.*

Smoothness is certainly preserved by surjective morphisms and by the operations in the previous section: taking subrepresentations, subquotients, direct sums. We write $\mathfrak{R}(G)$ for the category of smooth representations of G; it is an abelian category (see [11, §A.6] or [10, §IV.8]). Let us now look at some fundamental examples.

We first look at an irreducible smooth representation (π, V) of a profinite group H. Pick a non-zero vector v in V. Then H_v is an open subgroup of H, the coset space H/H_v is finite and V is spanned by $\{\pi(g)v/\, g \in H/H_v\}$ hence *finite dimensional.* Actually the representation factors through the *finite* quotient group of G by the intersection $K = \cap_{g\in H/H_v} gH_v g^{-1}$ of the stabilizers of those vectors. In this setting, let us recall the classical argument showing that

(cr) *if G is finite and the characteristic of R is 0 or prime to the order of G, then any finite dimensional representation (π, V) of G is completely reducible.*

Indeed let (π, W) be a subrepresentation of (π, V) and let T be a projector from V onto W. Then

$$T^G = \frac{1}{|G|} \sum_{g\in G} \pi(g) T \pi(g)^{-1}$$

is a G-invariant projector in V, its kernel is a G-invariant complement to W in V.

We now turn to right translations acting on spaces of functions on G with various properties. The right translate of some function f on G by $g \in G$ is the function $x \mapsto f(xg)$, $x \in G$. (One can also consider the action by left translations: $g.f(x) = f(g^{-1}x)$.) This is the basic method of constructing representations, however, the right translate of a smooth function need not be smooth. One has to appeal to the general process of "*taking the smooth part*". In any representation (π, V) of G, the subspace of smooth vectors, namely the vectors having an open stabilizer, provides a smooth representation of G denoted by (π^∞, V^∞). We have

$$V^\infty = \bigcup_K V^K$$

where K ranges over the family of open compact subgroups of G and V^K is the subspace of K*-invariant vectors*, i.e.

$$V^K \ = \ \{v \in V \ / \ \forall g \in K \quad \pi(g)v = v\}.$$

Let S be an R-vector space. A function $f : G \to S$ is *locally constant* if

$$\forall x \in G \quad \exists H_x \text{ open subgroup of } G \quad \forall g \in H_x \quad f(xg) = f(x).$$

(Note that, since $xg = (xgx^{-1})x$, one also has $f(gx) = f(x)$ for all $g \in xH_x x^{-1}$. In other words the notions of left locally constant and right locally constant coincide.)

The space $\mathcal{C}(G, S)$ of locally constant functions from G to S provides a representation of G by right translations. Its smooth part is the space $\mathcal{C}^\infty(G, S)$ of *smooth functions*, that is, functions $f : G \to S$ such that

$$\exists H \text{ open subgroup of } G \quad \forall x \in G \quad \forall g \in H \quad f(xg) = f(x).$$

Recall that the *support* $\mathrm{Supp}(f)$ of a function f on a topological group G with values in some vector space is the closure of the set $\{x \in G / \ f(x) \neq 0\}$. For a locally constant function f on a locally profinite group G, this set is closed. *Locally constant functions with compact support* from G to S are actually smooth functions. Indeed, let $f \in \mathcal{C}(G, S)$ have compact support; the open subgroups H_x defined above provide an open cover $(xH_x)_{x \in \mathrm{Supp}\, f}$ of the compact subset $\mathrm{Supp}\, f$, from which a finite subcover can be found, whence the smoothness.

We write $\mathcal{C}_c^\infty(G, S)$ for the space of locally constant functions with compact support from G to S. It provides smooth representations of G by right translations and by left translations. If K fixes $f \in \mathcal{C}_c^\infty(G, S)$ on the right and K' fixes f on the left, then $K \cap K'$ will fix f on both sides. Hence

$$\mathcal{C}_c^\infty(G, S) \ = \ \cup_K \mathcal{C}_c(G//K, S)$$

where K ranges over the family of compact open, or small enough compact open, subgroups of G, and $\mathcal{C}_c(G//K, S)$ is the space of bi-K-invariant compactly supported functions from

G to S. The space $\mathcal{C}_c(G//K,R)$ in particular has a natural basis given by the characteristic functions I_{KgK} of the double cosets KgK in $K\backslash G/K$.

Finally let (π,V) be a smooth representation of G. The dual space $V^*=\mathrm{Hom}_R(V,R)$ of V is equipped with the dual action of G given by

$$<\pi^*(g)v^*,v>=<v^*,\pi(g^{-1})v> \qquad \text{for } v\in V, v^*\in V^*, g\in G.$$

This representation of G in V^* is not smooth in general. The *contragredient representation* of (π,V) is the smooth representation $(\tilde{\pi},\tilde{V})$ provided by the smooth part of (π^*,V^*), i.e.

$$(\tilde{\pi},\tilde{V})=((\pi^*)^\infty,(V^*)^\infty).$$

1.4 Induced representations

We come back to the generic example of smooth representation: the action of G by right tranlations in some space of smooth functions, and give a construction that is largely used in the theory.

Let H be a closed subgroup of G and let (σ,W) be a smooth representation of H. Consider the space $\mathrm{Ind}_H^G W$ of functions $f\colon G\to W$ which satisfy:

(i) for all $h\in H$ and $g\in G$, $f(hg)=\sigma(h)f(g)$;

(ii) there exists an open subgroup K_f of G such that for $k\in K_f$ and $g\in G$: $f(gk)=f(g)$.

This space is indeed stable under right translations by elements of G and the second condition is the required smoothness, hence a smooth representation $(\mathrm{Ind}_H^G\sigma,\mathrm{Ind}_H^G W)$ of G called the *representation (smoothly) induced by σ.* The subspace c-$\mathrm{Ind}_H^G W$ of functions with compact support modulo H (that is, the support is contained in some $H\Omega$ where Ω is a compact set in G) is G-stable and provides a subrepresentation $(c\text{-}\mathrm{Ind}_H^G\sigma, c\text{-}\mathrm{Ind}_H^G W)$ of G called *the representation compactly induced by σ.* The two representations coincide of course when the quotient space G/H is compact.

The following properties of smooth induction and compact induction are easily checked:

- There is a canonical H-homomorphism: $\mathrm{Ind}_H^G W\longrightarrow W$ given by $f\mapsto f(1)$.
- If H is open in G, there is a canonical H-homomorphism: $W\longrightarrow c\text{-}\mathrm{Ind}_H^G W$ given by $w\mapsto f_w$, where $f_w(h)=\sigma(h)(w)$ for $h\in H$ and f_w is null outside H. The image of this homorphism generates c-$\mathrm{Ind}_H^G W$ as a G-representation.

  In particular, if H is open then c-Ind_H^G respects finite type.
- *Functoriality.* Smooth and compact induction provide functors: $\mathfrak{R}(H)\to\mathfrak{R}(G)$. [To the morphism $\varphi: W_1\to W_2$ corresponds $\mathrm{Ind}\varphi$ given, for $f: G\to W_1$, by $\mathrm{Ind}\varphi(f)=\varphi\circ f$.]

- *Transitivity.* Both smooth induction and compact induction are transitive, that is, if $H \subset H_1 \subset G$ are closed subgroups, then $\mathrm{Ind}_H^G = \mathrm{Ind}_{H_1}^G \mathrm{Ind}_H^{H_1}$ and similarly for c-Ind.

- *Frobenius reciprocity.* Ind_H^G is right adjoint to Res_H^G, the restriction functor from G to H: for $(\pi, V) \in \mathfrak{R}(G)$, $(\sigma, W) \in \mathfrak{R}(H)$, there is a canonical isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) & \xrightarrow{\sim} & \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma) \\ \varphi & \longmapsto & [v \mapsto \varphi(v)(1)] \end{array}$$

[The inverse map sends $\psi \in \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma)$ to $v \mapsto (x \mapsto \psi(\pi(x)v))$.]

- *Frobenius reciprocity – open subgroup case.* If H is open in G, c-Ind_H^G is left adjoint to Res_H^G: for $(\pi, V) \in \mathfrak{R}(G)$, $(\sigma, W) \in \mathfrak{R}(H)$, there is a canonical isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_H^G \sigma, \pi) & \xrightarrow{\sim} & \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \pi) \\ \varphi & \longmapsto & [w \mapsto \varphi(f_w)] \end{array}$$

[The inverse map sends $\psi \in \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \pi)$ to $f \mapsto \sum_{H\backslash G} \pi(g^{-1})\psi(f(g))$.]

- The functors Ind_H^G and c-Ind_H^G are left exact.

We end this section with a technical statement. Let K be a compact open subgroup of G and let f be a K-invariant vector in the compactly induced representation c-$\mathrm{Ind}_H^G \sigma$. Then the support of f is a finite union of double cosets HgK and the value $f(g) \in W$ is fixed by the compact open subgroup $H \cap gKg^{-1}$ of H. Conversely an element w of $W^{H \cap gKg^{-1}}$ determines a unique function in c-$\mathrm{Ind}_H^G \sigma$ with support HgK and value w at g. Thus [6, §3.5], [11, III.2.2]:

Proposition 1.5. *Let K be a compact open subgroup of G and let Λ be a set of coset representatives for the double cosets $H\backslash G/K$. Then the map*

$$\begin{array}{rcl} \left(c\text{-}\mathrm{Ind}_H^G \sigma\right)^K & \longrightarrow & \bigoplus_{g \in \Lambda} W^{H \cap gKg^{-1}} \\ f & \longmapsto & (f(g))_{g \in \Lambda} \end{array}$$

is an isomorphism.

For the proof we send the reader to the relevant books. A more general statement, *Mackey's decomposition*, is given in [16, §I.5].

2 Admissible representations of locally profinite groups

2.1 Admissible representations.

Definition 2.6. *A smooth representation (π, V) of G is called* admissible *if V^K is finite dimensional for any compact open subgroup K of G.*

We remark that this is equivalent to asking that V^K be finite dimensional for all *small enough* compact open subgroups K of G. Indeed, if $K \subset K'$ then $V^{K'} \subset V^K$. For an admissible representation the space V is a union $\cup_K V^K$ of finite dimensional subspaces.

To obtain consistent results associated with this notion, we know have to make an assumption on the field R. From now on, we assume:

($\star$) **There exists a compact open subgroup K in G such that the index of any open subgroup of K is invertible in R.**

For a reductive p-adic group, this is equivalent to: *p invertible in R.*

Of course if such a K exists, all compact open subgroups contained in K satisfy the same requirement, which holds then for any small enough compact open subgroup of G. Saying that the field R satisfies ($\star$) with respect to G amounts to saying that:

the set $\mathcal{K}^\star(G)$ of compact open subgroups K of G such that the index of any open subgroup of K is invertible in R is a fundamental system of neighbourhoods of 1 in G.

Proposition 2.7. *Assume that R satisfies ($\star$) and let $K \in \mathcal{K}^\star(G)$.*

(i) *The restriction to K of any smooth representation (π, V) of G is completely reducible (i.e. π is K-semi-simple). In particular the subspace V^K has a K-complement in V, given by*
$$V(K) =< \pi(k)v - v / k \in K, v \in V > .$$

(ii) *We have $(\tilde{V})^K \simeq (V^K)^*$ as R-vector spaces.*

(iii) *The functor $V \mapsto V^K$, from $\mathfrak{R}(G)$ into the category of R-vector spaces, is exact.*

(iv) *The functor $\mathfrak{R}(G) \to \mathfrak{R}(G)$ assigning to (π, V) the contragredient representation $(\tilde{\pi}, \tilde{V})$ is contravariant and exact.*

Proof. (i) For any $v \in V$, the representation ξ of K in the subspace spanned by the vectors $\pi(k)v$, $k \in K$, factors through a finite quotient of K and the assumption ensures that it is completely reducible (see (cr) in §1.3). Summing over $v \in V$ we get the first assertion through Proposition 1.3. Any K-homomorphism from V to V^K is nul on $V(K)$ hence $V(K)$ is contained in any K-complement of V^K, kernel of a K-projection onto V^K. On the other hand any K-complement of V^K in V is a sum of non-trivial irreducible representations of K. Let U be one of them; by irreducibility $U(K) = U$ is contained in $V(K)$.

(ii) From the definition of the contragredient representation we have for $v \in V$, $v^* \in \tilde{V}$, $g \in K$: $< \tilde{\pi}(g)v^*, v >=< v^*, \pi(g^{-1})v >$. We see that $v^* \in \tilde{V}$ is K-invariant if and only if it annihilates the space $V(K)$. By restriction v^* thus defines an element of the dual space of V^K, hence an injective morphism $(\tilde{V})^K \hookrightarrow (V^K)^*$. Conversely, if $\phi \in (V^K)^*$ then, extending it trivially on $V(K)$ gives an injection $(V^K)^* \to \tilde{V}^K$.

(iii) Left as an exercise using the complete reducibility of the restrictions of smooth representations to K (see [6, §2.3]).

(iv) Suppose $V_1 \xrightarrow{\varphi} V_2 \xrightarrow{\psi} V_3$ is exact. The morphism $\psi^* : V_3^* \longrightarrow V_2^*$ defined by $\langle \psi^*(v_3^*), v_2 \rangle = \langle v_3^*, \psi(v_2) \rangle$ maps smooth vectors to smooth vectors, as well as $\phi^* : V_2^* \longrightarrow V_1^*$. We get a sequence $\widetilde{V}_3 \xrightarrow{\psi^*} \widetilde{V}_2 \xrightarrow{\varphi^*} \widetilde{V}_1$ with $\varphi^* \circ \psi^* = 0$. Then $V_1^K \to V_2^K \to V_3^K$ is exact so $(V_3^K)^* \to (V_2^K)^* \to (V_1^K)^*$ is exact, i.e. $\widetilde{V}_3^K \to \widetilde{V}_2^K \to \widetilde{V}_1^K$ is exact. This holds for any $K \in \mathcal{K}^\star(G)$ hence for any small enough K. Any $v \in \widetilde{V}_2$ is fixed by such a K hence if $\varphi^*(v) = 0$ we have $v = \psi^*(w)$ for some $w \in \widetilde{V}_3^K$ so $\widetilde{V}_3 \to \widetilde{V}_2 \to \widetilde{V}_1$ is exact. ∎

Note that (iii) implies that any subquotient of an admissible representation is admissible. We are now ready to demonstrate the importance of the notion of admissibility.

Corollary 2.8. *Assume R satisfies $(\star)$. The following conditions on a smooth representation (π, V) of G are equivalent:*

- *the representation (π, V) is admissible;*
- *the contragredient representation $(\widetilde{\pi}, \widetilde{V})$ is admissible;*
- *the natural G-embedding from (π, V) to $(\widetilde{\widetilde{\pi}}, \widetilde{\widetilde{V}})$ given by*
$$v \mapsto \delta(v) \,:\, \widetilde{V} \to R,\ v^* \mapsto \langle v^*, v \rangle.$$
is an isomorphism.

Let (π, V) be admissible. Then (π, V) is irreducible if and only if $(\widetilde{\pi}, \widetilde{V})$ is irreducible.

Proof. The equivalence π admissible $\iff$ $\widetilde{\pi}$ admissible follows from (ii) above. The map δ induces a map $\delta^K : V^K \to \widetilde{\widetilde{V}}^K$ for each $K \in \mathcal{K}^\star(G)$, and δ is surjective if and only if δ^K is surjective for each such K. From (ii) above $\widetilde{\widetilde{V}}^K = (V^K)^{**}$ is isomorphic to V^K if and only if V^K is finite dimensional.

Now let (π, V) be admissible and let U be a proper G-stable subspace of V. Then from the exactness of $\pi \mapsto \widetilde{\pi}$ the space $U^\perp = \{\widetilde{v} \in \widetilde{V} : \langle \widetilde{v}, U \rangle = 0\}$ is a proper subspace of $\widetilde{V}$. The converse is true since $\pi \simeq \widetilde{\widetilde{\pi}}$. ∎

Induced representations provide us with some examples. Indeed the technical Proposition 1.5 has the following consequences:

Proposition 2.9. *If G/H is compact, the functor $\mathrm{Ind}_H^G = c\text{-}\mathrm{Ind}_H^G$ preserves admissibility.*

Proof. Indeed, for any open compact subgroup K the number of double cosets $H\backslash G/K$ is finite and if σ is admissible each space $W^{H\cap gKg^{-1}}$ is finite dimensional. ■

Proposition 2.10. *If R satisfies $(\star)$, the functor $c\text{-}\mathrm{Ind}_H^G$ is exact.*

Proof. We already have left-exactness, we only need to show that if $0 \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$ is an exact sequence of smooth representations of H then $c\text{-}\mathrm{Ind}\psi : c\text{-}\mathrm{Ind}_H^G V \to c\text{-}\mathrm{Ind}_H^G W$ is surjective. So we take some $K \in \mathcal{K}^\star(G)$ and some $f \in (c\text{-}\mathrm{Ind}_H^G W)^K$. We may assume that the support of f is only one double class HgK, so f is determined by $f(g) \in W^{H\cap gKg^{-1}}$. We need to find $f' : G \to V$ such that, for all $x \in G$, $\psi(f'(x)) = f(x)$. From Proposition 2.7 (iii) we know that there is some $v \in V^{H\cap gKg^{-1}}$ such that $\psi(v) = f(g)$. Hence v determines a unique $f' \in V^K$ with support HgK, which answers the question. ■

2.2 Haar measure

We will see again that $(\star)$ is the exact assumption that will allow us to define very useful additional tools. The first is a Haar measure on G, the second is the structure of an algebra on the space $\mathcal{C}_c^\infty(G,R)$ defined in §1.3.

Definition 2.11. *A measure on G with values in R is a linear form on $\mathcal{C}_c^\infty(G,R)$. A* (left) Haar measure *on G is a non-zero measure μ which is invariant under left translation by G.*

Suppose μ is a left Haar measure on G. For X an open compact subset of G, we let I_X be the characteristic function of X and we put $\mu(X) = \mu(I_X)$. We have $\mu(gX) = \mu(X)$ for all $g \in G$, by left-invariance. Now take f in $\mathcal{C}_c(G//K,R)$ (§1.3). The support of f is a finite union of K-cosets and $f = \sum_{gK\in\mathrm{Supp}(f)/K} f(g) I_{gK}$ whence

$$\mu(f) = \sum_{gK\in\mathrm{Supp}(f)/K} f(g)\mu(K). \tag{2.12}$$

Since μ is non-zero and the characteristic functions I_{gK}, for $g \in G$ and K a compact open subgroup of G, span $\mathcal{C}_c^\infty(G,R)$, there is such a K with $\mu(K) = \mu(gK) \neq 0$. Then, for any K' compact open subgroup of G, we have

$$\mu(K') = [K' : K'\cap K]\,\mu(K'\cap K) \quad \text{and} \quad \mu(K) = [K : K'\cap K]\,\mu(K'\cap K)$$

with finite indices $[K' : K' \cap K]$ and $[K : K' \cap K]$. We note in passing that the quotient

$$(K' : K) = [K' : K' \cap K]/[K : K' \cap K]$$

is called the *generalised index* of K in K'.
This discussion and formula 2.12 easily imply the following proposition [16, §I.2.4].

Proposition 2.13. *There exists a (left) Haar measure on G with values in R if and only if R satisfies $(\star)$. If so, compact open subgroups with non-zero volume are exactly the elements of $\mathcal{K}^\star(G)$ and a Haar measure μ on G is uniquely determined by the non-zero volume $\mu(K_0)$ of some $K_0 \in \mathcal{K}^\star(G)$.*

For a function $f \in \mathcal{C}_c^\infty(G, R)$, we will often use the notation

$$\mu(f) = \int_G f(g) d\mu(g).$$

The group G is said to be *unimodular* if the *modulus character* δ_G from G to $\mathbb{Q}^\times$ defined by $\delta_G(g) = (gKg^{-1} : K)$ for some (for any – exercise !) compact open subgroup K of G is trivial. One can show (exercise or [16, §I.2.7]) that $f \mapsto \mu(f\delta_G)$ is a right Haar measure on G. Hence, if G is unimodular, any left Haar measure on G is also a right Haar measure.
Example. Let $G = \{\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} / a \in F^\times, x \in F\}$, a subgroup of $\mathrm{GL}(2, F)$. Check that

$$\delta_G(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) = |a|_F.$$

As a first application we give the following proposition and refer to [6] or [16] for a proof. Note that our convention for the modulus character is the one in [16]; in [6] the inverse character is used.

Proposition 2.14. *[6, §3.5] [16, §5.11] Assume that R satisfies $(\star)$. Let H be a closed subgroup of G, let σ be a smooth representation of H and let $\tilde{\sigma}$ be the contragredient representation. Then the contragredient representation of $c\text{-}\mathrm{Ind}_H^G \sigma$ is isomorphic to $\mathrm{Ind}_H^G \delta_G^{-1}\delta_H\tilde{\sigma}$.*

2.3 Hecke algebra of a locally profinite group

We continue to assume that R satisfies $(\star)$. We fix a left Haar measure μ on G and use it to define a product $*$, called *convolution*, on $\mathcal{C}_c^\infty(G, R)$. For $f_1, f_2 \in \mathcal{C}_c^\infty(G, R)$ and $x \in G$ we put

$$f_1 * f_2(x) = \int_G f_1(g)\, f_2(g^{-1}x) d\mu(g).$$

This product gives $\mathcal{C}_c^\infty(G,R)$ the structure of an R-algebra, called *the Hecke algebra of G over R* and denoted $\mathcal{H}_R(G)$. (Check associativity as an exercise.) This algebra has no unit element unless G is discrete, but it has a lot of idempotent elements (that is, $e * e = e$). Indeed for each $K \in \mathcal{K}^\star(G)$ the function

$$e_K = \mu(K)^{-1} I_K$$

satisfies $e_K * e_K = e_K$ (another exercise).

This algebra is a powerful tool in the study of smooth representations of G, which pass through to $\mathcal{H}_R(G)$-modules. Indeed, let (π, V) be a smooth representation of G and let $f \in \mathcal{C}_c^\infty(G,R)$. For each $v \in V$ we can find an open compact subgroup K of G such that f is K-invariant on the right and v is K-invariant, hence the finite linear combination

$$\sum_{gK \in \mathrm{Supp}(f)/K} f(g)\pi(g)v$$

is meaningful. It does depend on K though (for a $K' \subset K$ the index $[K : K']$ turns up) but for $K \in \mathcal{K}^\star(G)$ this dependence is just a volume. Thanks to the Haar measure we have a well-defined linear operator on V given by

$$\pi(f) = \int_G f(g)\pi(g)d\mu(g).$$

For K open compact subgroup and $v \in V^K$ we have

$$\pi(I_{gK})v = \int_K \pi(gk)vdk = \mu(K)\pi(g)v, \quad \text{for all } g \in G. \tag{2.15}$$

In particular, if $K \in \mathcal{K}^\star(G)$ then the idempotent element e_K acts through $\pi(e_K)$, which is zero on $V(K)$ and the identity on V^K, hence a projection onto V^K. This projection is extremely useful as we will see.

We also leave as an exercise the fact that the action is an algebra action, i.e. for f and f' in $\mathcal{H}_R(G)$, we have $\pi(f * f') = \pi(f)\pi(f')$. (Hint: either use Fubini's Theorem, which says that we can reverse the order of a double integral, or apply to a $v \in V$ and reduce to the case of characteristic functions of K-cosets with K fixing v.)

Hence, for (π, V) a smooth representation of G, we have given V the structure of a left $\mathcal{H}_R(G)$-module. Since $V = \cup_K V^K$ where K ranges over $\mathcal{K}^\star(G)$, and $\pi(e_K)V^K = V^K$, this module satisfies $\pi(\mathcal{H}_R(G))V = V$, namely it is *non-degenerate* (or *smooth*, or *unital*). This defines a functor M from $\mathfrak{R}(G)$ to the category $\mathcal{H}_R(G)$-Mod of non-degenerate left $\mathcal{H}_R(G)$-modules, which takes (π, V) to V and $\varphi \in \mathrm{Hom}_G(V_1, V_2)$ to $\varphi \in \mathrm{Hom}_{\mathcal{H}_R(G)}(V_1, V_2)$ (note that φ really is a morphism of $\mathcal{H}_R(G)$-modules by (2.15) and linearity). The functor is clearly injective, surjectivity is Proposition 2 in [6, §4.2]. Summing up:

Proposition 2.16. *The functor $M : \mathfrak{R}(G) \to \mathcal{H}_R(G)$-Mod is an equivalence of categories.*

Let $K \in \mathcal{K}^\star(G)$ as before and define $\mathcal{H}_R(G, K)$ as the sub-algebra of $\mathcal{H}_R(G)$ consisting of bi-K-invariant functions. It is easy to check that $\mathcal{H}_R(G, K) = e_K * \mathcal{H}_R(G) * e_K$, and since e_K is a projection onto V^K for any smooth representation (π, V) of G, the space V^K inherits the structure of an $\mathcal{H}_R(G, K)$-module. At some point we will need the following fact from [6, Proposition 4.3 (1)]:

Lemma 2.17. *Let $K \in \mathcal{K}^\star(G)$. If (π, V) is a smooth irreducible representation of G, then V^K is either zero or an irreducible $\mathcal{H}_R(G, K)$-module.*

Proof. Assume that V^K is non-zero and let M be a non-zero $\mathcal{H}_R(G, K)$-submodule of V^K. By irreducibility we have $V = \pi\left(\mathcal{H}_R(G)\right) M$ hence

$$V^K = \pi(e_K)V = \pi(e_K)\pi\left(\mathcal{H}_R(G)\right) M = \pi\left(\mathcal{H}_R(G, K)\right) M = M.$$

■

2.4 Coinvariants

We have mentioned earlier the notion of *invariants:* let (π, V) be a smooth representation of G and H a closed subgroup, then V^H is the biggest subspace of V on which H acts trivially. One can also consider the *coinvariants*, that is, the biggest quotient of V on which H acts trivially, namely the quotient:

$$V_H = V/V(H), \quad V(H) = \langle \pi(h)v - v \;/\; h \in H, v \in V \rangle.$$

We obtain a functor $V \mapsto V_H$ from $\mathfrak{R}(G)$ into the category of R-vector spaces, which is easily seen to be *right exact.* [Indeed, let $0 \to U \overset{\varphi}{\to} V \overset{\psi}{\to} W \to 0$ be an exact sequence of smooth representations of H. The definition gives $\phi(U(H)) \subset V(H)$ and $\psi(V(H)) \subset W(H)$ hence a sequence $U_H \overset{\varphi_H}{\to} V_H \overset{\psi_H}{\to} W_H \to 0$. This sequence is exact because the kernel of the composed map $V \overset{\psi}{\to} W \to W_H$ is equal to $\ker \psi + V(H)$.]
Note that if H belongs to $\mathcal{K}^\star(G)$, then the quotient map identifies V^H and V_H by Proposition 2.7 (i). Also note that G plays no part in the definition, which makes sense for any smooth representation of the locally profinite group H. However, it is specially interesting when applied to the restriction to H of a representation of a bigger group G, since the coinvariants V_H then provide a representation of the G-normalizer of H.

Proposition 2.18. *Suppose H is a union $\bigcup_{t \in \mathbb{N}} K_t$ of compact open subgroups belonging to $\mathcal{K}^\star(H)$ and such that $K_t \subset K_{t+1}$ for all $t \in \mathbb{N}$. Then for any smooth representation (π, V) of H we have $V(H) = \bigcup_{t \in \mathbb{N}} V(K_t)$ and the functor $V \to V_H$ is exact.*

Proof. Let $v \in V(H)$, $v = \sum_{i=1}^{r} (\pi(h_i)v_i - v_i)$. There exists $t \in \mathbb{N}$ such that $h_i \in K_t$ for all $i = 1, \dots, r$, so $v \in V(K_t)$. The converse is clear.

We already have right-exactness, so we only need to show that if $0 \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$ is exact then $\varphi_H : U_H \to V_H$ is injective, i.e. the inverse image of $\operatorname{Im}\varphi \cap V(H)$ is contained in $U(H)$. Now from Proposition 2.7 (i) again, we have: $V(K_t) = \ker \pi_V(e_{K_t})$ so

$$\operatorname{Im}\varphi \cap V(H) = \bigcup_{t\in\mathbb{N}} \operatorname{Im}\varphi \cap \ker \pi_V(e_{K_t}) = \bigcup_{t\in\mathbb{N}} \varphi\left(\ker \pi_U(e_{K_t})\right) = \varphi\left(U(H)\right).$$

(Indeed, for $u \in U$ we have $\pi_V(e_{K_t})\varphi(u) = \varphi(\pi_U(e_{K_t})u)$, and φ is injective.) ■

For F a non-archimedean local field, the additive group of $F = \cup_{i\in\mathbb{Z}}\mathfrak{p}_F^i$ satisfies the hypothesis of Proposition 2.18 if p is invertible in R. The same holds for the group N of upper triangular unipotent matrices in $GL(n, F)$ since we have

$$N = \begin{pmatrix} 1 & F & F & \cdots & F \\ 0 & 1 & F & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & F \\ \vdots & \ddots & \ddots & 1 & F \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} = \bigcup_{i\in\mathbb{Z}} \begin{pmatrix} 1 & \mathfrak{p}_F^i & \mathfrak{p}_F^{2i} & \cdots & \mathfrak{p}_F^{(n-1)i} \\ 0 & 1 & \mathfrak{p}_F^i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathfrak{p}_F^{2i} \\ \vdots & \ddots & \ddots & 1 & \mathfrak{p}_F^i \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

3 Schur's Lemma and Z-compact representations

3.1 Characters

A *character* of a locally profinite group G is a smooth representation of G on a one-dimensional vector space. So by definition, a character is just a homomorphism

$$\chi : G \longrightarrow R^\times$$

with open kernel. We denote by $\hat{G}$ the group of characters of G into $R^\times$.

We mentioned in §2.2 the *modulus character* δ_G from G to $\mathbb{Q}^\times$, defined by means of the generalised index as $\delta_G(g) = (gKg^{-1} : K)$ for some (for any) compact open subgroup K of G. The vocabulary is consistent for it is indeed a character: it is trivial on all compact subgroups. In particular, a profinite group is unimodular.

[The vocabulary is dangerous. Within the representation theory of finite groups, character means the trace of a finite dimensional representation. There is a risk of confusion here. Furthermore, even in the context of smooth representations of p-adic groups, some definitions may differ slightly: character may mean complex character of modulus 1 while characters in the above sense are called quasi-characters.]

Whatever the group topology on $R^\times$, characters are continuous. Now, if $R = \mathbb{C}$, characters are exactly the continuous homomorphisms from G into $\mathbb{C}^\times$; furthermore, if G is the union of its compact open subgroups, their image is contained in the unit circle of complex numbers of modulus 1. Indeed a small enough compact neighbourhood of 1 in $\mathbb{C}^\times$ contains no non-trivial subgroup and any compact subgroup of $\mathbb{C}^\times$ is contained in the unit circle [6, 1.6]. This applies in particular to F, union of its compact subgroups $\mathfrak{p}_F^n$, $n \in \mathbb{Z}$.

[Recall the Pontryagin duality for locally compact abelian groups: in that setting characters are continuous morphisms from G into complex numbers of modulus 1, the group of characters is called the dual group of G and the bidual group of G is canonically isomorphic to G. If G is compact, its dual group is discrete and vice versa.]

Let us say a word on characters of a finite abelian group G. First remark $|\hat{G}| \leq |G|$. Indeed G is a product of cyclic groups and if C_n is a cyclic group of order n, the choice of a generator identifies $\hat{C}_n$ with the group of roots of unity of order dividing n in the field R.

Now let H be a subgroup of G. Then *if R is algebraically closed, every R-character of H extends to a character of G.* The proof goes by induction on the index $[G : H]$. If the index is 1 there is nothing to prove. Otherwise, let $x \in G$, $x \notin H$ and let n be the least integer such that x^n belongs to H. For ψ a character of H, one can choose an n-th root ξ of $\psi(x^n)$ in $R^\times$ and extend ψ to a character $\bar{\psi}$ of the subgroup generated by H and x by letting $\bar{\psi}(hx^t) = \psi(h)\xi^t$ for $h \in H$ and $t \in \mathbb{Z}$.

Characters of F (additive group) play an important part in the (complex or l-modular) representation theory of reductive groups defined over F, they deserve a word here.

We start with a character $\chi : k_F \to R^\times$ of the residual field. Since k_F has characteristic p, the image of χ is contained in the p-th roots of unity in R. Moreover k_F is a separable extension of the prime field $\mathbb{F}_p$ with a trace map $\mathrm{tr} : k_F \to \mathbb{F}_p$, so if R has a non-trivial p-th root of 1, say ω, the assignment $1 \mapsto \omega$ defines a non-trivial character χ_0 of $\mathbb{F}_p$ and $\chi_0 \circ \mathrm{tr}$ is a non-trivial character of k_F.

So assume R has a non-trivial p-th root of 1 and let χ be a non-trivial character of k_F. For any $a \in k_F$, the map $a\chi : x \mapsto \chi(ax)$ is a character of k_F, non-trivial if and only if $a \neq 0$, hence an injective morphism $a \mapsto a\chi$ from k_F to its group of characters $\hat{k}_F$, which must be onto since $|\hat{k}_F| \leq |k_F|$. Hence *if R has a non-trivial p-th root of 1, any choice of a non-trivial character of k_F provides a group isomorphism $\hat{k}_F \simeq k_F$.*

Now, let $\psi : F \to R^\times$ be a non-trivial character of F (if any). From the topology of F, there is a least integer d, called the *level* of ψ, such that $\mathfrak{p}_F^d \subset \ker \psi$. For any $a \in F$, the map $a\psi : x \mapsto \psi(ax)$ is a character of F, of level $d - \mathrm{val}_F(a)$ if $a \neq 0$, hence again an injective morphism $a \mapsto a\psi$ from F to its group of characters. This morphism is also *onto*. For let θ be another non-trivial character. Using the action of a suitable element of $F^\times$ we may as well assume that it has level d. We compare ψ and θ on $\mathfrak{p}_F^{d-1}/\mathfrak{p}_F^d$, isomorphic to the additive group of the residual field k_F. From the above, R necessarily has a non-trivial p-th root of

1 and there is a unit $u_1 \in \mathfrak{o}_F^\times$ such that θ and $u_1\psi$ agree on $\mathfrak{p}_F^{d-1}$. Pursuing this process provides a Cauchy sequence u_n in $\mathfrak{o}_F^\times$ that converges to $u \in \mathfrak{o}_F^\times$ such that $\theta = u\psi$ [6, 1.7]. We conclude:

Lemma 3.19. *If F has a non-trivial character ψ, then R has a non-trivial p-th root of 1 and $a \mapsto a\psi$, $a \in F$, is a group isomorphism from F to $\hat{F}$.*

All non-trivial characters of F have the same image, contained in the group of roots of unity of order a power of p in $R^\times$. In particular F has no non-trivial character if R has characteristic p. Actually, the only irreducible smooth representation of a pro-p-group over a field of characteristic p is the trivial representation, due to the following generalization to pro-p-groups of Proposition 26 in [13, §8.3]:

Lemma 3.20. *[1, Lemma 3] Let (π, V) be a non-zero smooth representation of a pro-p-group G over a field R of characteristic p. Then there is a non-zero vector $v \in V$ that is fixed under G.*

On the other hand if the characteristic of R is not p and R has primitive p^n-th roots of unity for any integer n, then F has non-trivial additive characters. We refer to the paper by A. Robert [12] for complements; that paper is actually quite a precursor in the subject of l-modular representations and is a worthwhile read.

Let $G = \mathrm{GL}(n, F)$. The kernel of a character ζ of G must contain the commutator subgroup of G, equal to $\mathrm{SL}(n, F)$ (for any field except $\mathbb{F}_2$). Hence ζ factors as $\chi \circ \det$ for some homomorphism $\chi : F^\times \to R^\times$. The determinant map is continuous and open, so χ is a character of $F^\times$ [6, 9.2] that may be studied using the filtration $1 + \mathfrak{p}_F^t$, $t \geq 1$, of $\mathfrak{o}_F^\times$. If we choose a uniformising element ϖ_F of F, we have $F^\times = \varpi_F^{\mathbb{Z}} \mathfrak{o}_F^\times$, hence a character is given by the image of ϖ_F and the restriction to the maximal compact subgroup $\mathfrak{o}_F^\times$. The easiest example of a character of $F^\times$ is $x \mapsto |x|_F$. It takes its values in the subgroup $|k_F|^{\mathbb{Z}}$ of $\mathbb{Q}_+^\times$, contained in $p^{\mathbb{Z}}$, so it can be defined over any field R of characteristic not p.

3.2 Schur's Lemma and central character

Lemma 3.21. Schur's Lemma - General Version. *Let (π, V) and (π', V') be two smooth irreducible representations of the locally profinite group G.*

(i) $\mathrm{Hom}_G(\pi, \pi')$ *is non-zero if and only if (π, V) and (π', V') are isomorphic.*

$\mathrm{End}_G(\pi)$ *is a division algebra over R.*

(ii) *If R is algebraically closed and there exists K open compact subgroup of G such that $0 < \dim_R V^K < |R|$, then $\mathrm{End}_G(\pi) = R$.*

Proof. (i) is hardly more than a remark, though a crucial one. Indeed, the kernel and the image of a G-morphism from V to V' are subrepresentations of V and V' respectively.

To go further we now assume that R is algebraically closed. Let $v \in V^K$ be non-zero. The map $\phi \mapsto \phi(v)$ is an injective linear map from $\mathrm{End}_G(\pi)$ to V^K so

$$\dim_R \mathrm{End}_G(\pi) \leq \dim_R V^K.$$

Assume for a contradiction that $\mathrm{End}_G(\pi)$ contains an element ϕ not in R. Since R is algebraically closed, the mapping $R(X) \to \mathrm{End}_G(\pi)$ which associates to a rational fraction $Q(X)$ the element $Q(\phi)$ is an injection of fields, hence the dimension of $\mathrm{End}_G(\pi)$ over R is at least that of the field of rational fractions $R(X)$. But in $R(X)$ the set $\{(X-c)^{-1}/c \in R\}$ is linearly independent over R, hence

$$\dim_R \mathrm{End}_G(\pi) \geq |R|,$$

a contradiction since $\dim_R V^K < |R|$. ■

When the assumptions are fulfilled the lemma says that any G-endomorphism of π is *scalar*. It thus plays a part in representation theory which is crucial enough to put forward its most usual versions.

Lemma 3.22. Schur's Lemma - *Assume R is algebraically closed. Let (π, V) be an irreducible smooth representation of G. If either one of the following conditions holds :*

(i) *(π, V) is admissible;*

(ii) *R is uncountable and for any compact open subgroup K of G, the set G/K is countable;*

then $\mathrm{End}_G(\pi) = R$.

Proof. (i) is just a rephrasing of the previous statement since $|R|$ cannot be finite. For (ii) we only need to remark that V has countable dimension since, for a choice of a non-zero $v \in V$ and a compact open subgroup K fixing v, V is spanned by $\{\pi(g)v \,/\, g \in G/K\}$. ■

The algebraic closure of a countable field is countable, hence (ii) does not apply to $\bar{\mathbb{Q}}$ nor to $\bar{\mathbb{F}}_l$. It does apply of course to $\mathbb{C}$.

The choice of K in (ii) is irrelevant since, for any other compact open subgroup K' of K, the intersection $K \cap K'$ has finite index in K and K'. Assumption (ii) is satisfied in $\mathrm{GL}(n, F)$ and more generally in any reductive p-adic group, as follows from the Cartan decomposition [16, II.1.3].

Write $Z = Z(G)$ for the centre of G and let (π, V) be a smooth representation of G such that $\mathrm{End}_G(\pi) = R$ (which does *not* imply that π is irreducible, unless it is completely reducible, see the example in [6, §9.10]). Then, for all $z \in Z$, $\pi(z)$ belongs to $\mathrm{End}_G(\pi)$ so it can be written as $\pi(z) = \omega_\pi(z) I_V$ and $\omega_\pi : Z \to R^\times$ is a character, called the *central character* of π. (Observe that if K is a compact open subgroup of G such that V^K is non-zero, then ω_π must be trivial on $Z \cap K$.) Of course, a representation $(\pi, V) \in \mathfrak{R}(G)$ may have a central character even if $\mathrm{End}_G(\pi) \neq R$.

In particular, if G is abelian and assumption (ii) in the previous lemma holds, irreducible smooth representations of G are just characters. One can reach the same conclusion if G is the product of a profinite group by a free abelian group of finite rank, as in [16, I.7.12] (where the proof is different):

Lemma 3.23. *Assume R is algebraically closed. Let G be abelian.*

(i) A finite dimensional smooth representation of G is irreducible if and only if it is one-dimensional.

(ii) If G is the product of an abelian profinite group by a free $\mathbb{Z}$-module of finite rank, then any irreducible smooth representation of G is a character.

Proof. (i) The image of any element of G has an eigenvalue and the corresponding eigenspace is stable under the action of G.

(ii) Write $G = \mathbb{Z}^n \times G_0$ with G_0 profinite and $n \geq 0$. Let (π, V) be an irreducible smooth representation of G. For $v \in V$, $v \neq 0$, the span of $\pi(G_0)v$ is finite-dimensional (smoothness) hence contains a character ω of G_0 by (i) and Proposition 1.2 (iii). The subspace

$$V(\omega, G_0) = \{v \in V \ / \ \forall g \in G_0 \ \ \pi(g)v = \omega(g)v\}$$

is non-zero and stable under $\pi(G)$, hence it is equal to V. In other words G_0 acts by a character on V, so we are reduced to the case $G = \mathbb{Z}^n$ with $n \geq 1$.

The R-linear span E of the operators $\pi(g) \in \mathrm{End}_R(V)$ for $g \in G$, is a field containing R. Indeed, it is commutative and contained in $\mathrm{End}_G(\pi)$ which, by Schur's Lemma, is a division algebra over R. The field E is finitely generated as an R-algebra: if $\{g_1, \cdots, g_n\}$ is a basis of G over $\mathbb{Z}$, then E is generated by $\{\pi(g_i)/ \ i = 1, \cdots, n\} \cup \{\pi(g_i)^{-1}/ \ i = 1, \cdots, n\}$. We now cite the *Algebraic Nullstellensatz* from [5, Proposition 2.3]:

Let K be a field, and let E be an extension field which is finitely generated as a K-algebra. Then E/K is a finite algebraic extension.

Since R is algebraically closed we get $E = R$, q.e.d. ∎

3.3 Z-compact representations

Let (π, V) be a smooth representation of G, let $(\widetilde{\pi}, \widetilde{V})$ be the contragredient representation, let $v \in V$ and $\widetilde{v} \in \widetilde{V}$. The *matrix coefficient* associated to $v, \widetilde{v}$ is the function $\varphi_{v,\widetilde{v}} : G \to R$ given by

$$\varphi_{v,\widetilde{v}}(g) = \langle \widetilde{v}, \pi(g)v \rangle.$$

We let $\mathcal{C}(\pi)$ be the subspace (of the space of smooth functions from G to R) spanned by the matrix coefficients of π. We recall that Z is the centre of G.

Definition 3.24. *A smooth representation (π, V) of G is Z-compact if all the matrix coefficients of π are compactly supported modulo Z.*

Recall that a function is *compactly supported mod Z* if its support is contained in some subset ZE where E is compact. Note also that if (π, V) and $(\widetilde{\pi}, \widetilde{V})$ are both irreducible (see §2.1) then (π, V) is Z-compact if and only if it has one non-zero matrix coefficient compactly supported mod Z.
When R satisfies $(\star)$ one can give an equivalent definition using the Hecke algebra.

Proposition 3.25. *Assume R satisfies $(\star)$. Let (π, V) be a smooth representation of G having a central character.*

(i) *The following are equivalent:*

(a) *(π, V) is Z-compact;*

(b) *for any $K \in \mathcal{K}^\star(G)$ and any $v \in V$, the function $g \mapsto \pi(e_K)\pi(g)v$ on G is compactly supported mod Z.*

(ii) *If (π, V) is Z-compact and finitely generated, then it is admissible.*

Proof. [2, 2.40] (i) First assume (π, V) is Z-compact and pick $v \in V$ and $K \in \mathcal{K}^\star(G)$. We claim that *the subspace E spanned by the $v_g = \pi(e_K)\pi(g)v$, $g \in G$, is finite dimensional.* For otherwise, one could find a sequence g_i, $i \in \mathbb{N}$, such that the v_{g_i} are linearly independent. Completing the set $\{v_{g_i}\ ;\ i \in \mathbb{N}\}$ with a set Λ into a basis of V^K, we could define a linear functional λ on V by $\langle \lambda, w\rangle = \langle \lambda, \pi(e_K)w\rangle$ for all $w \in V$, $\langle \lambda, v_{g_i}\rangle = 1$ for all $i \in \mathbb{N}$, and $\langle \lambda, \Lambda\rangle = 0$. Then λ would belong to $(\widetilde{V})^K$ (Proposition 2.7) but the coefficient $\langle \lambda, \pi(g)v\rangle = \langle \lambda, \pi(e_K)\pi(g)v\rangle$ would not be compactly supported mod Z since the support would contain the infinite number of disjoint classes g_iZK', for some open compact subgroup K' fixing v.
Hence one can find a basis $\{b_1, \cdots, b_t\}$ of the dual space of E made of elements of $(\widetilde{V})^K$ (see Proposition 2.7). A vector $v_g \in E$ is null if and only if $\langle b_i, v_g\rangle = 0$ for $i = 1, \cdots, t$, so the support of $g \mapsto v_g$ is the union of the supports of the coefficients $g \mapsto \langle b_i, \pi(g)v\rangle$ for $1 \le i \le t$, compact mod Z.
For the converse, take any $v \in V$, any $\widetilde{v} \in \widetilde{V}$. Then $\widetilde{v}$ is K-fixed for a small enough compact open subgroup $K \in \mathcal{K}^\star(G)$ hence, for any $g \in G$, we have $\langle \widetilde{v}, \pi(g)v\rangle = \langle \widetilde{\pi}(e_K)\widetilde{v}, \pi(g)v\rangle = \langle \widetilde{v}, \pi(e_K)\pi(g)v\rangle$, compactly supported mod Z.
(ii) is a consequence of the claim at the beginning of the proof. ∎

Many variants can be given, see [16]. In particular, using Schur's Lemma 3.22 we also have:

Corollary 3.26. *Assume that*

- *R is algebraically closed, uncountable and satisfies $(\star)$;*
- *for any compact open subgroup K of G, the set G/K is countable.*

Then any irreducible and Z-compact representation is admissible.

The most useful variant for us will be the following proposition, particularly suitable to the context of reductive p-adic groups as we will see later (Proposition 4.37). To fix ideas, one can check directly that in $G = \mathrm{GL}(2, F)$, the subgroup G_0 defined as the kernel of $g \mapsto |\det(g)|_F$ satisfies the hypothesis below.

Proposition 3.27. *Assume that R satisfies $(\star)$ and that G contains an open normal subgroup G_0 with compact centre such that $G_0 Z$ has finite index in G. Let (π, V) be a smooth representation of G.*

(i) *The following are equivalent:*

  (a) *(π, V) is Z-compact;*

  (b) *for any $K \in \mathcal{K}^\star(G)$ and any $v \in V$, the function $g \mapsto \pi(e_K)\pi(g)v$ on G is compactly supported mod Z;*

  (c) $\pi_{|G_0}$ is compact*, that is, its matrix coefficients are compactly supported.*

(ii) *Assume furthermore that any irreducible smooth representation of Z over R is a character. If (π, V) is Z-compact and irreducible, then it is admissible and has a central character.*

Proof. For (i) we follow [2, 3.21]. The proof of (b) implies (a) in 3.25 remains valid and indeed (a) implies (c): G_0 is open in G hence the matrix coefficients of $\pi_{|G_0}$ are restrictions of those of π. We must prove that (c) implies (b) so we assume that $\pi_{|G_0}$ is compact and pick v, K as in (b). We may assume that K is contained in G_0. The proof of (a) implies (b) above holds for compact representations so the support $S_{G_0}(v, K)$ of $g \mapsto \pi(e_K)\pi(g)v$, $g \in G_0$, is compact in G_0. Let $\{g_1, \cdots, g_k\}$ be a set of coset representatives for G/G_0Z. The support $S_G(v, K)$ of that same function on G is contained in the finite union, for $1 \leq i \leq k$, of the sets $Zg_iS_{G_0}(\pi(g_i)v, K)$, compact mod Z.

For (ii) we use Waldspurger's argument as in [15, Lemme 20]. The restriction of π to ZG_0 is finitely generated, we let X be a finite set of generators. Let $K \in \mathcal{K}^\star(G_0)$ as before; we also assume that V^K is non-zero. The subspace W of V^K generated by the $\pi(e_K)\pi(g)v$, $g \in G_0$, $v \in X$, is finite-dimensional by (b) and smoothness, and since X generates V over ZG_0, we have $V^K = \pi(e_K)V = \pi(Z)W$.

Hence V^K is a finitely generated representation of Z and as such admits an irreducible quotient Y (Proposition 1.2) which is a character χ by assumption. The subspace of χ-invariants of V^K: $V^K(\chi) = \{\pi(z)v - \chi(z)v / v \in V^K, z \in Z\}$ is a $\mathcal{H}_R(G,K)$-submodule of V^K. Its image in Y is 0 so it is not equal to V^K and by irreducibility (Lemma 2.17) $V^K(\chi) = \{0\}$. So Z acts on V^K by the character χ and V^K, finitely generated over Z, is finite dimensional. The space $V(\chi)$ is non-zero and G-stable hence equal to V. ■

The main tool for constructing Z-compact representations is compact induction from open, compact mod centre, subgroups of G, for which the following result is crucial.

Proposition 3.28. *Suppose that R satisfies $(\star)$. Let H be an open, compact modulo Z subgroup of G and let σ be a smooth representation of H. Assume that $\pi = c\text{-}\mathrm{Ind}_H^G \sigma$ is irreducible and admissible. Then π is Z-compact.*

Proof. From §2.1 the contragredient representation $\widetilde{\pi}$ is also irreducible (actually this is why we have assumed admissibility; irreducibility of $\widetilde{\pi}$ may be assumed instead, but in practice, it is the admissibility that will be checked most of the time) hence it is enough to find one non-zero matrix coefficient compactly supported mod Z. If σ is zero there is nothing to prove, otherwise we can find v, a non-zero vector in the space W of σ, and $\widetilde{v}$, a non-zero element in the space of $\widetilde{\sigma}$ such that $<\widetilde{v}, v> \neq 0$. The function f_v on G defined by $f_v(x) = \sigma(x)v$ if $x \in H$, $f_v(x) = 0$ otherwise, belongs to the space of π. The function $f_{\widetilde{v}}$ defined analogously can be regarded as a vector in $\widetilde{\pi}$ since the composed map

$$c\text{-}\mathrm{Ind}_H^G W \to W \to R \quad : \quad f \mapsto f(1) \mapsto <f_{\widetilde{v}}(1), f(1)>$$

is a smooth linear form on $c\text{-}\mathrm{Ind}_H^G W$. It is non-zero since $<f_{\widetilde{v}}(1), f_v(1)> = <\widetilde{v}, v>$ is non-zero. The corresponding matrix coefficient is:

$$\varphi_{f_v, f_{\widetilde{v}}}(g) = <f_{\widetilde{v}}, \pi(g) f_v> = <f_{\widetilde{v}}(1), \pi(g) f_v(1)> = <\widetilde{v}, f_v(g)> .$$

Its support is certainly contained in the support of f_v hence in H, compact mod Z. ■

We remark that the above proof could be completed to show that, under the hypotheses of Proposition 3.28, the contragredient representation of π is actually equivalent to $c\text{-}\mathrm{Ind}_H^G \widetilde{\sigma}$. For a general duality theorem, we refer to [6, §3.5].

This is about as far as we will go for a locally profinite group at large. Further study of those induced representations can be found in [6] (especially §3.5 and §11.4, paying special attention to Remark 1 there) and [16, §I.5].

3.4 An example

Let $M = \{\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \ / \ a \in F^\times, x \in F\}$, a subgroup of $\mathrm{GL}(2, F)$ that we have already encountered. Then M is the semi-direct product of N by S with

$$N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \ / \ x \in F\}, \qquad S = \{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \ / \ a \in F^\times\}.$$

Let ϑ be a *non-trivial* character of N. Form the smooth representations $c\text{-}\mathrm{Ind}_N^M \vartheta$ and $\mathrm{Ind}_N^M \vartheta$. Since any other non-trivial character of N is an S-conjugate of ϑ (Lemma 3.19), those induced representations do not depend, up to isomorphism, on the choice of ϑ.

Proposition 3.29. *[6, §8.2] Assume that R is algebraically closed and that the residual characteristic p of F is invertible in R. The representation $c\text{-}\mathrm{Ind}_N^M \vartheta$ is irreducible and is a proper subrepresentation of $\mathrm{Ind}_N^M \vartheta$.*

Proof. From §3.1 we know that, under the given assumption over R, the group $N \simeq F$ has non-trivial characters. The assumption over R is also used in the proof of irreducibility, for which we refer to [6, §8.2]. This proof relies on the notion of ϑ-coinvariants, similar to the notion of coinvariants in §2.4: for a smooth representation (π, V) of N, the space of ϑ-coinvariants V_ϑ is the biggest quotient of V on which N acts by ϑ. It is the quotient of V by the subspace spanned by the $\pi(n)v - \vartheta(n)v$, $v \in V$, $n \in N$.

We now prove the second part of the statement to give an idea of the shape of the representation. We identify S with $F^\times$ through the map $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mapsto a$, $a \in F^\times$. A function f in $V = \mathrm{Ind}_N^M \vartheta$ is determined by its restriction to $S \simeq F^\times$, which is a smooth function ϕ_f on $F^\times$, hence an injective linear map $f \mapsto \phi_f$ from V to $\mathcal{C}^\infty(F^\times)$. We need to characterize its image.

Let $\phi \in \mathcal{C}^\infty(F^\times)$. We look for a function $f \in V$ extending ϕ. Such a function must satisfy

$$f(an) = \vartheta(ana^{-1})\phi(a) \qquad (\dagger)$$

for any $a \in S$ and $n \in N$. Furthermore f must be smooth: there must exist a compact open subgroup N_f of N such that $f(gu) = f(g)$ for any $u \in N_f$ and any $g \in M$. In particular $u \mapsto \vartheta(aua^{-1})$ must be trivial on N_f for any $a \in S$ such that $\phi(a)$ is non-zero. Observe from

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix} \quad (t \in F^\times,\ x \in F)$$

that aN_fa^{-1} gets bigger as $|a|_F$ gets bigger. Since the character ϑ is non-trivial, $\phi(a)$ must be zero for $|a|_F$ big enough. Conversely it is straightforward to see that the formula $(\dagger)$ does define a function in V whenever ϕ is a smooth function on $F^\times$ that vanishes outside the intersection of $F^\times$ with a compact set of F.

By definition, the image of the subspace $V_c = c\text{-Ind}_N^M \vartheta$ under this map is the subset $\mathcal{C}_c^\infty(F^\times)$ of smooth functions with compact support on $F^\times$, that is, those smooth functions which vanish outside a compact set of $F^\times$, that is, $\phi(a) = 0$ both for $|a|_F$ big enough and small enough: it is a proper subrepresentation. ■

In view of Corollary 2.8, since the contragredient representation of $c\text{-Ind}_N^M \vartheta$ is the non-irreducible representation $\text{Ind}_N^M \vartheta'$ for a non-trivial character ϑ' of N (Proposition 2.14), we conclude:

Corollary 3.30. *Under the same assumption over R, let ϑ be a non-trivial character of N. The smooth irreducible representation $c\text{-Ind}_N^M \vartheta$ of M is not admissible.*

Indeed M is not the group of F-points of a connected reductive algebraic group defined over F. In the next section we will see how the structure of such groups forces admissibility upon any smooth irreducible (complex) representation.

4 Cuspidal representations of reductive p-adic groups

We let now G be the group of F-points of a connected reductive algebraic group $\mathbb{G}$ defined over F: in short and somewhat abusively, a "reductive p-adic group". To simplify matters one can think of $G = \text{GL}(n, F)$. We recall that in such a group, for any compact open subgroup K of G, the set G/K is countable. Furthermore the crucial assumption ($\star$) on R just means that the characteristic of R is not p, the residual characteristic of F..

A central part in this section will be played by *parabolic subgroups* of G, their *unipotent radical* and their *Levi factors*, which are Levi subgroups of G. For $\text{GL}(n, F)$, up to conjugacy, a parabolic subgroup is a subgroup of upper block-triangular matrices, its unipotent radical is made of those matrices whose diagonal blocks are identity matrices, and an obvious Levi factor is the subgroup of block-diagonal matrices, a product of smaller $\text{GL}(n_i, F)$.

We will now explain how this structure allows an inductive approach to the smooth representation theory of G, inducing up from smaller connected reductive algebraic groups that are Levi subgroups of G. Eventually this aproach will culminate in the proof that *every smooth irreducible representation of G is admissible*, when the characteristic of R is not p. This is a deep property that does not hold true if the hypothesis that G be reductive is removed, as we have seen in Corollary 3.30.

4.1 Parabolic induction and restriction

Let P be a parabolic subgroup of G with unipotent radical N and let L be a Levi factor of P, so $P = LN$ and we have a canonical isomorphism $L \simeq P/N$ given by composing the injection $L \hookrightarrow P$ with the quotient map $P \to P/N$. Then P is a closed subgroup of G with a compact quotient G/P and we can induce representations from P to G – in this case there is no difference between smooth induction and compact induction. Actually we are going to induce very special representations of P, indeed those representations that are trivial on N hence uniquely determined by their restriction to L. The *inflation* of a representation ρ of L to P is the unique representation of P which restricts to ρ on L and trivial on N; we will usually denote it by ρ as well. The *parabolic induction* functor is the following composition:

$$i_{L,P}^G : \mathfrak{R}(L) \xrightarrow{\text{inflation}} \mathfrak{R}(P) \xrightarrow{\quad \mathrm{Ind}_P^G \quad} \mathfrak{R}(G).$$

Inflation is an exact functor; using the properties of smooth and compact induction given in §1.4 we get directly that the parabolic induction functor is *left exact*; if the residual characteristic p of F is invertible in R it is *exact* (Proposition 2.10).

If K is a compact open subgroup of G, then $K \cap P$ is a compact open subgroup of P and $K \cap L$ is a compact open subgroup of L. Admissibility is thus preserved by inflation from L to P and by Ind_P^G (Corollary 2.9), so *parabolic induction preserves admissibility.*

As we have seen in §1.4, smooth induction from P to G has a left adjoint, the restriction from G to P. To obtain the left adjoint functor to parabolic induction we have to compose restriction with the N-coinvariants functor, which is left adjoint to inflation. Indeed, since N is normal in P, for any smooth representation (π, V) of P the space $V(N)$ is stable under P and the quotient V_N provides a smooth representation of P trivial on N, hence a smooth representation of L. The *parabolic restriction* functor, or *Jacquet [restriction]* functor, is the composite functor:

$$r_{L,P}^G : \mathfrak{R}(G) \xrightarrow{\mathrm{Res}_P^G} \mathfrak{R}(P) \xrightarrow{N\text{-coinvariants}} \mathfrak{R}(L).$$

It is indeed left adjoint to $i_{L,P}^G$: for $(\pi, V) \in \mathfrak{R}(G)$, $(\sigma, W) \in \mathfrak{R}(L)$, we have

$$\mathrm{Hom}_L(r_{L,P}^G \pi, \sigma) = \mathrm{Hom}_G(\pi, i_{L,P}^G \sigma).$$

Lemma 4.31. *Let $P = LN$ be a parabolic subgroup of G. Then:*

(i) *$r_{L,P}^G$ is right exact. If p is invertible in R then $r_{L,P}^G$ is exact.*

(ii) *$r_{L,P}^G$ preserves finite type.*

Proof. (i) This is a consequence of §2.4, indeed N is a union of pro-p-subgroups.

(ii) Suppose $(\pi, V) \in \mathfrak{R}(G)$ is of finite type: V is spanned by $\pi(G)(\{v_1, \cdots, v_t\})$ for some finite set of vectors $\{v_1, \cdots, v_t\}$. This set is contained in V^K for some open compact subgroup K of G. Let Γ be a set of coset representatives of the finite set $P\backslash G/K$, then $\pi(\Gamma)(\{v_1, \cdots, v_t\})$ is a set of generators for $\mathrm{Res}_P^G V$ and its image in V_N is a finite set of generators of the L-representation $r_{L,P}^G \pi$. ∎

4.2 Parabolic pairs

A parabolic pair in G is a pair (P, L) made of a parabolic subgroup of G and a Levi factor L of P. There is a partial order on the set of parabolic pairs: $(P, L) \preceq (P', L')$ if $P \subseteq P'$ and $L \subseteq L'$. We thus fix in G a minimal parabolic pair (P_0, L_0) for this order and we call *standard parabolic pair* any parabolic pair (P, L) such that $(P_0, L_0) \preceq (P, L)$ and *standard parabolic subgroup* any parabolic subgroup P such that $P_0 \subseteq P$.

In $\mathrm{GL}(n, F)$ we choose for P_0 the subgroup of upper triangular matrices and for L_0 the subgroup of diagonal matrices. Standard parabolic subgroups are just the subgroups of upper-block-triangular matrices.

The setup we need is the following [4]:

Proposition 4.32. (i) *The set of standard parabolic pairs in G is finite. Each conjugacy class of parabolic pairs of G contains exactly one standard parabolic pair.*

(ii) *A parabolic subgroup P containing P_0 has a unique Levi factor L_P containing L_0. Standard parabolic subgroups are in a one-to-one correspondence with standard parabolic pairs.*

(iii) *Let (P, L_P) be a standard parabolic pair. The set of pairs $(Q \cap L_P, L_Q)$ attached to standard parabolic pairs $(Q, L_Q) \preceq (P, L_P)$ is the set of standard parabolic pairs in L_P relative to the minimal pair $(P_0 \cap L_P, L_0)$.*

With this in hand we can state correctly the *transitivity* of both functors of parabolic induction and restriction. The proof is straightforward and left as an exercise (for instance, one can check it directly for parabolic induction and then deduce it for parabolic restriction by adjunction).

Lemma 4.33. *Let $(P, L) \preceq (P', L')$ be standard parabolic pairs of G. Then*

$$i_{L,P}^G = i_{L',P'}^G \circ i_{L,L'\cap P}^{L'} \quad \textit{and} \quad r_{L,P}^G = r_{L,L'\cap P}^{L'} \circ r_{L',P'}^G.$$

4.3 Cuspidal representations

The group G itself is a parabolic subgroup of G, with Levi factor G and unipotent radical $\{1\}$. A parabolic subgroup of G is *proper* if it is not equal to G.

Definition 4.34. *A smooth representation* (π, V) *of* G *is* cuspidal *if it satisfies the equivalent following conditions:*

- *for any proper parabolic subgroup* $P = LN$ *of* G *we have* $r^G_{L,P}\pi = \{0\}$*;*
- *for any proper parabolic subgroup* $P = LN$ *of* G *and any smooth representation* σ *of* L *we have* $\mathrm{Hom}_G(\pi, i^G_{L,P}\sigma) = \{0\}$.

The equivalence of those two conditions is a direct consequence of the adjunction property $\mathrm{Hom}_L(r^G_{L,P}\pi, \sigma) = \mathrm{Hom}_G(\pi, i^G_{L,P}\sigma)$. We observe that for any closed subgroup H of G, for any $g \in G$ and for any $(\pi, V) \in \mathfrak{R}(G)$ we have $V(gHg^{-1}) = \pi(g)V(H)$. It follows that $V_N = \{0\}$ if and only if $V_{gNg^{-1}} = \{0\}$, so one can replace in the definition *proper parabolic subgroup* by *proper* standard *parabolic subgroup*.

Remark. A smooth irreducible representation (π, V) of G is *supercuspidal* if it is not a *subquotient* of a proper parabolically induced representation. A supercuspidal representation is certainly irreducible cuspidal. The converse holds for $R = \mathbb{C}$ – in which case the two words tend to be used indifferently – but does not hold in general, indeed the comparison between the two notions is a difficult problem for which we refer to [16, §II.2, II.3].

We are at last in a position to state and prove the first of the two theorems that are the cornerstones of the theory, first step towards a classification of smooth irreducible representations of G.

Theorem 4.35. *Let* (π, V) *be a smooth irreducible representation of* G*. Then there exist a parabolic subgroup* $P = LN$ *of* G *and an irreducible cuspidal representation* $\sigma \in \mathfrak{R}(L)$ *such that* $\mathrm{Hom}_G(\pi, i^G_{L,P}\sigma) \neq \{0\}$*, i.e.* π *is a subrepresentation of* $i^G_{L,P}\sigma$*.*

Proof. [16, II.2.4] The set of standard parabolic pairs is finite so there exists a standard parabolic pair (P, L) such that $r^G_{L,P}\pi \neq 0$ and $r^G_{L',P'}\pi = 0$ for all standard parabolic pairs $(P', L') \preceq (P, L)$, $(P', L') \neq (P, L)$. Since π is irreducible, the representation $r^G_{L,P}\pi$ is of finite type, hence it has an irreducible quotient σ (Proposition 1.2). By transitivity of $r^G_{L,P}$, the representation $r^G_{L,P}\pi$ must be cuspidal, as well as σ because $r^G_{L,P}$ is right exact. Now

$$\mathrm{Hom}_G(\pi, i^G_{L,P}\sigma) = \mathrm{Hom}_L(r^G_{L,P}\pi, \sigma) \neq \{0\}.$$

We actually proved that there is a *standard* parabolic subgroup satisfying the requested condition. ∎

One can prove further that the equivalence class of the pair (L, σ) given by the theorem, for the equivalence relation combining G-conjugacy of such pairs and equivalence of representations, is uniquely determined by the isomorphism class of (π, V). It is called the *cuspidal support* of (π, V) [16, II.2.4, II.2.20].

A full classification of smooth irreducible representations of G thus requires a classification of the irreducible cuspidal representations of the Levi subgroups of G. This has been achieved so far for complex representations of linear and classical (for $p \neq 2$) p-adic groups : cuspidal irreducible representations are induced from open, compact mod centre, subgroups (see Proposition 3.28 and Theorem 4.38). We refer to Colin Bushnell's notes in the present volume.

4.4 Iwahori decomposition

We fix as before a minimal parabolic pair (P_0, L_0), we let N_0 be the unipotent radical of P_0 and we fix a maximal split torus A_0 in G contained in L_0. The corresponding set Σ^+ of *positive roots* is defined – in short, they are the eigencharacters of the adjoint action of A_0 on the Lie algebra of N_0. We then define

$$A_0^- = \{a \in A_0 / \forall \alpha \in \Sigma^+ \ |\alpha(a)| \leq 1\}.$$

For $G = \mathrm{GL}(n, F)$ both A_0 and L_0 are the subgroup of diagonal matrices and we have

$$A_0^- = \{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & \cdots & 0 & a_n \end{pmatrix} \in A_0; |a_i/a_{i+1}|_F \leq 1 \text{ for } i = 1, \cdots, n-1\}$$

(indeed it is enough to check the inequality $|\alpha(a)| \leq 1$ for *simple* roots).

For a parabolic pair (P, L), we write N for the unipotent radical of P and N^- for the opposite of N relative to L, that is: the unipotent radical of the unique parabolic subgroup with Levi factor L and intersection with P exactly L. In $\mathrm{GL}(n, F)$ and for P standard, N^- is just the transpose of N.

Proposition 4.36. *[8, Proposition 1.4.4][11, V.5.2] Let (P_0, L_0) be the minimal parabolic pair in G fixed above.*

(i) *There exists a maximal compact subgroup K_0 of G satisfying*

  (a) $G = K_0 P_0 = P_0 K_0$,

  (b) (Cartan decomposition) *There is a finite subset Ω of L_0 such that $G = K_0 A_0^- \Omega K_0$.*

(ii) *There exists a fundamental system $(K_m)_{m\in\mathbb{N}}$ of neighbourhoods of the identity in G, made of normal compact open subgroups of K_0, satisfying $K_{m+1} \subset K_m$ for $m \in \mathbb{N}$ and, for any standard parabolic pair (P, L) and any $m \geq 1$:*

(a) Iwahori decomposition with respect to (P, L): *the product map*

$$(K_m \cap N^-) \times (K_m \cap L) \times (K_m \cap N) \longrightarrow K_m$$

is an isomorphism.

(b) *For any $a \in A_0^-$ which is central in L we have*

$$a(K_m \cap N)a^{-1} \subset K_m \cap N \text{ and } a^{-1}(K_m \cap N^-)a \subset K_m \cap N^-.$$

Furthermore, for any $i \geq 1$ there exists $a \in A_0^-$, central in L, such that

$$a(K_m \cap N)a^{-1} \subset K_i \cap N.$$

(c) *$K_m \cap L$ has an Iwahori decomposition with respect to $(L \cap P_0, L_0)$.*

For $G = \mathrm{GL}(n, F)$ we can use the standard filtration of the standard maximal compact subgroup $K_0 = \mathrm{GL}(n, \mathfrak{o}_F)$:

$$K_m = I + \varpi_F^m M_n(\mathfrak{o}_F) = \{g \in K/g \equiv I \bmod \mathfrak{p}_F^m\}.$$

The Iwahori decompositions are a consequence of their obvious additive analogues: $M_n(\mathfrak{p}_F^m)$ is certainly the sum of its intersections with the Lie algebras of N^-, L and N. We observe that $A_0 \cap K_0 = A_{0,0}$ is the subgroup of diagonal matrices with entries in $\mathfrak{o}_F^\times$. A system of representatives for $A_0^-/A_{0,0}$ is given by the semigroup

$$D = \left\{ d = \begin{pmatrix} \varpi_F^{s_1} & & 0 \\ & \ddots & \\ 0 & & \varpi_F^{s_n} \end{pmatrix} : s_1 \geq s_2 \geq \cdots \geq s_n \right\}$$

and the Cartan decomposition for $\mathrm{GL}(n, F)$ actually reads $G = K_0 D K_0$.

To give an idea of the way conjugation by $d \in D$ shrinks N_0, a simple computation in $\mathrm{GL}(3, F)$ should suffice. A transposition shows how conjugation by d^{-1} shrinks N_0^-.

$$\begin{pmatrix} \varpi_F^{s_1} & 0 & 0 \\ 0 & \varpi_F^{s_2} & 0 \\ 0 & 0 & \varpi_F^{s_3} \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi_F^{-s_1} & 0 & 0 \\ 0 & \varpi_F^{-s_2} & 0 \\ 0 & 0 & \varpi_F^{-s_3} \end{pmatrix} = \begin{pmatrix} 1 & \varpi_F^{s_1-s_2} x_{12} & \varpi_F^{s_1-s_3} x_{13} \\ 0 & 1 & \varpi_F^{s_2-s_3} x_{23} \\ 0 & 0 & 1 \end{pmatrix}.$$

If we start with a compact open subgroup N_0^0 of N_0, the resulting subgroup $dN_0^0 d^{-1}$ can be made arbitrarily small by making the differences $s_1 - s_2$, $s_2 - s_3$ big enough.

If (P, L) is attached to the partition $n = n_1 + \cdots + n_t$ the elements of D central in L are those that satisfy $s_1 = \cdots = s_{n_1}$, $\ldots$, $s_{n_1+\cdots+n_{t-1}+1} = \cdots = s_{n_t}$. We let D_L^- be the set of elements in D central in L and such that $s_{n_i} > s_{n_i+1}$ for $1 \leq i \leq t-1$. The second property in (b) above can also be stated as: for any $d \in D_L^-$ there is an integer k such that, for any $s \geq k$, $d^s(K_m \cap N)d^{-s} \subset K_i \cap N$.

4.5 Smooth irreducible representations are admissible

The last but not least result uses the whole specificity of the structure of reductive p-adic groups: it manages to link two properties of some smooth representations that appear different, the compactness mod centre of the coefficients and the nullity of the space of coinvariants under the unipotent radical of any proper parabolic subgroup. The proof shows how this link works; in particular Proposition 4.36 is an indispensable tool. We don't need to limit ourselves to representations having a central character thanks to yet another structure property, allowing us to rely on Proposition 3.27 instead of Proposition 3.25:

Proposition 4.37. *[11, §V.2][14, §0.4]*

(i) *The centre Z of G is the product of a profinite group by a free $\mathbb{Z}$-module of finite rank.*

(ii) *Let G_0 be the intersection of the kernels of the characters $g \mapsto |\chi(g)|_F$, for all rational characters χ of G. Then G_0 is open and normal in G, has a compact centre, and G_0Z has finite index in G. Every compact subgroup of G is contained in G_0.*

For $G = \mathrm{GL}(n, F)$, the subgroup G_0 is just the kernel of $g \mapsto |\det(g)|_F$. The center is isomorphic to $F^\times \simeq \mathfrak{o}_F^\times \times \mathbb{Z}$.

Theorem 4.38. *Assume p is invertible in R. Then a smooth representation of G is cuspidal if and only if it is Z-compact.*

Proof. [16, II.2.7] We use the notation of the previous paragraph : a minimal parabolic pair (P_0, L_0), a maximal split torus A_0 in G contained in L_0 and a sequence $(K_m)_{m\geq 0}$ of compact open subgroups given by Proposition 4.36. Since p is invertible in R, those subgroups belong to $\mathcal{K}^\star(G)$ (§2.1) for m large and we may as well assume that they belong to $\mathcal{K}^\star(G)$ for $m \geq 1$. We then have at our disposal the corresponding sequence of idempotent elements e_{K_m}, $m \geq 1$, in $\mathcal{H}_R(G)$.
Let (π, V) be a smooth representation of G. Recall that the operator $\pi(e_{K_m})$, $m \geq 1$, is a projection onto V^{K_m} (§2.3). We will translate the properties of cuspidality and Z-compactness in terms of those projections.

Lemma 4.39. *(π, V) is Z-compact if and only if, for any $m \geq 1$, for any $v \in V$, the map $a \mapsto \pi(e_{K_m})\pi(a)v$, from A_0^- to V^{K_m}, is compactly supported mod the centre.*

Proof. The analogous statement with G instead of A_0^- holds (Proposition 3.27), the "only if" part is thus clear. For the converse we use Cartan decomposition (Proposition 4.36) to describe the support on G of the map $g \mapsto \pi(e_{K_m})\pi(g)v$. Indeed let $i \geq m$ such that v is K_i-fixed and let $X = \{\pi(x)v \ / \ x \in \Omega K_0\}$. This is a finite set and by assumption, for any w in X the support of $a \mapsto \pi(e_{K_m})\pi(a)w$ is compact mod Z, say H_wZ with H_w compact in

A_0^-. Let g in G and write $g = kax$, $k \in K_0$, $a \in A_0^-$, $x \in \Omega K_0$. Then, writing $w = \pi(x)v$, we have $\pi(e_{K_m})\pi(g)v = \pi(e_{K_m})\pi(k)\pi(a)w = \pi(k)\pi(e_{K_m})\pi(a)w$ since K_m is normal in K_0. For this to be non-zero, a must belong to $H_w Z$ hence the support of the function is contained in the finite union over $x \in \Omega K_0/K_i$ of $K_0 H_w Z x K_i$, and the Z-compactness follows. ∎

Note that the assumption $v \in V$ can be replaced by $v \in V^{K_m}$ since $\pi(e_{K_m}) = \pi(e_{K_m})\pi(e_{K_i})$ for $i \geq m$.

Lemma 4.40. *Let $(P = LN, L)$ be a standard parabolic pair in G. Let H_i, $i \geq 1$, be an increasing sequence of compact open subgroups of N with $N = \cup_{i\geq 1} H_i$. For $v \in V$, the following conditions are equivalent:*

(i) *$v \in V(N)$;*

(ii) *for i large enough: $\pi(e_{H_i})v = 0$;*

(iii) *for any $m \geq 1$, there is an integer k such that, for all $d \in A_0^-$ satisfying $dH_kd^{-1} \subset K_m \cap N$, we have: $\pi(e_{K_m})\pi(d)v = 0$.*

(iv) *for any $m \geq 1$, there is an integer k such that, for all $d \in A_0^-$ central in L and satisfying $dH_kd^{-1} \subset K_m \cap N$, we have: $\pi(e_{K_m})\pi(d)v = 0$.*

Proof. (i) ⇔ (ii) is just Proposition 2.18 applied to N and the restriction of π to N, a smooth representation of N. We have: $\pi(e_{H_i})v = \int_{H_i} \pi(k)v \, d\mu_{H_i}(k)$, where μ_{H_i} is simply the unique Haar measure on H_i giving H_i volume 1 (Proposition 2.13).
Now we prove (ii) ⇒ (iii). Let $m \geq 1$. Assume $\pi(e_{H_k})v = 0$. If $d \in A_0^-$ satisfies $dH_kd^{-1} \subset K_m \cap N$, then

$$\pi(e_{K_m})\pi(d)v = \pi(e_{K_m})\pi(e_{dH_kd^{-1}})\pi(d)v = \pi(e_{K_m})\pi(d)\pi(e_{H_k})v = 0, \quad \text{q.e.d.}$$

Certainly (iii) implies (iv). For (iv) ⇒ (ii) we start with $v \in V$ and $m \geq 1$ such that $v \in V^{K_m}$. We leave it as an exercise for the reader ([6, §3.2 and 7.6], [2, Proposition 1.26, Lemma 3.11]) to check that the Iwahori decomposition in Proposition 4.36 translates into

$$\pi(e_{K_m}) = \pi(e_{K_m\cap N})\pi(e_{K_m\cap L})\pi(e_{K_m\cap N^-}),$$

where again the right-hand side refers to Haar measures on N, L and N^- giving volume 1 to $K_m \cap N$, $K_m \cap L$ and $K_m \cap N^-$ respectively. Let $d \in A_0^-$ be central in L. We have $d^{-1}(K_m \cap L)d \subset K_m \cap L$ and, by Proposition 4.36, $d^{-1}(K_m \cap N^-)d \subset K_m \cap N^-$, so

$$\begin{aligned}
\pi(e_{K_m})\pi(d)v &= \pi(e_{K_m\cap N})\pi(d)\pi(d^{-1})\pi(e_{K_m\cap L})\pi(e_{K_m\cap N^-})\pi(d)v \\
&= \pi(e_{K_m\cap N})\pi(d)\pi(e_{K_m\cap L})\pi(e_{d^{-1}(K_m\cap N^-)d})v \\
&= \pi(e_{K_m\cap N})\pi(d)v \\
&= \pi(d)\pi(e_{d^{-1}(K_m\cap N)d})v,
\end{aligned}$$

hence $\pi(e_{K_m})\pi(d)v = 0$ if and only if $\pi(e_{d^{-1}(K_m \cap N)d})v = 0$. So picking $d \in A_0^-$, central in L and satisfying $dH_kd^{-1} \subset K_m \cap N$ for a k given by (iii), we have $\pi(e_{d^{-1}(K_m \cap N)d})v = 0$, hence $\pi(e_{H_i})v = 0$ for any H_i containing $d^{-1}(K_m \cap N)d$, q.e.d. ■

Confronting those two lemmas will give the theorem. Assume first that (π, V) is not Z-compact: there are $m \geq 1$ and $v \in V^{K_m}$ such that the map $a \mapsto \pi(e_{K_m})\pi(a)v$ on A_0^- is not compactly supported mod the centre. Let Y be its support.

Assume first that $G = \mathrm{GL}(n, F)$. Then there exist indices $i < j$ such that $|a_j/a_i|$ is unbounded above for $a = \mathrm{diag}(a_1, \dots, a_n) \in Y$. In fact, since $|a_s|$ is increasing with s, there exists $i \leq r < j$ such that $|a_{r+1}/a_r|$ is unbounded above. Let P_r be the (maximal) standard parabolic subgroup associated to the partition $(r, n-r)$ and let N_r be its unipotent radical. Let $(H_k)_{k \geq 1}$ be an increasing filtration of N_r as in Lemma 4.40. Then, for any $k \geq 1$, the family $\{aH_ka^{-1}; a \in Y\}$ of compact open subgroups of N_r contains arbitrarily small subgroups. In particular, for any k there is $a \in Y$ such that $aH_ka^{-1} \subset K_m \cap N_r$ and $\pi(e_{K_m})\pi(a)v \neq 0$. Hence v does not belong to $V(N_r)$, so (π, V) is not cuspidal.

For a general G, there are roots $\alpha \in \Sigma^+$ such that the values $|\alpha(a)^{-1}|$ are not bounded above on Y; among them must be a simple root α_0, defining a proper standard parabolic pair $(P_{\alpha_0} = L_{\alpha_0}N_{\alpha_0}, L_{\alpha_0})$ in G. The proof carries on replacing N_r with N_{α_0}.

Conversely, let (π, V) be Z-compact, let $v \in V$ be non-zero and let $m \geq 1$ be such that $v \in V^{K_m}$. The support Y of the map $a \mapsto \pi(e_{K_m})\pi(a)v$ on A_0^- is compact mod centre hence the values $|\alpha(a)^{-1}|$ for $\alpha \in \Sigma^+$ (for $\mathrm{GL}(n, F)$: the quotients $|a_j/a_i|$, $1 \leq i < j \leq n$) are bounded above on Y. In particular, for any proper standard parabolic pair $(P = LN, L)$ in G, the subgroups $a^{-1}(K_m \cap N)a$, for a in Y and central in L, are all strictly contained in some subgroup H_k in a given filtration of N as in Lemma 4.40, that is: $K_m \cap N \subsetneq aH_ka^{-1}$ for all $a \in Y$, central in L. This means that (iii) in Lemma 4.40 holds, so v belongs to $V(N)$. Finally v belongs to $V(N)$ for any proper standard parabolic subgroup so π is cuspidal. ■

We now draw the consequences for admissibility.

Corollary 4.41. *Assume R is algebraically closed of characteristic different from p, the residual characteristic of F. Then any smooth irreducible representation of G over R is admissible.*

Proof. Proposition 3.27, with Lemma 3.23 and Lemma 4.37, ensures that irreducible Z-compact representations are admissible; so are irreducible cuspidal representations by the theorem. By Theorem 4.35, any smooth irreducible representation of G is a subrepresentation of some proper parabolically induced representation $i_{L,P}^G\sigma$ where σ is an irreducible cuspidal representation of the reductive p-adic group L, whence admissible. Parabolic induction preserves admissibility, q.e.d. ■

This corollary is the goal we had fixed for these notes. Yet it turns out that the only assumption over R really needed is that the characteristic is not p. A proof, due to M.-F. Vignéras, can be found in [3, Proposition 2]. For convenience we reproduce it here.

Theorem 4.42. *Assume that the characteristic of R is not p. Then any smooth irreducible representation of G over R is admissible.*

Proof. Let (π, V) be a smooth irreducible representation of G over R and let $\bar{R}$ be an algebraic closure of R. The representation $(\pi_{\bar{R}}, V_{\bar{R}} = \bar{R} \otimes_R V)$ of G over $\bar{R}$ is generated by a single element hence has an irreducible quotient (π', V') (Proposition 1.2) which is realizable on a finite extension E of R in $\bar{R}$ [16, II.4.7], that is: $V' \simeq \bar{R} \otimes_E V'_E$ for a representation (π_E, V_E) of G over E.

Let then f be a non-zero $\bar{R}G$-homomorphism from $\bar{R} \otimes_R V$ to $\bar{R} \otimes_E V'_E$ and let $v \neq 0$, $v \in V$. Since v generates $\bar{R} \otimes_R V$ as a $\bar{R}G$-module, its image $f(v)$ is non-zero and belongs to $K \otimes_E V'_E = V'_K$ for some finite extension K of E in $\bar{R}$. Composing with the natural injection from V to $\bar{R} \otimes_R V$, one gets an injective RG-homomorphism $i : V \hookrightarrow V'_K$.

We thus have, for any open compact subgroup H of G : $V^H \hookrightarrow (V'_K)^H$. But $\bar{R} \otimes_K (V'_K)^H$ is contained in $(V')^H$, finite dimensional over $\bar{R}$, hence $(V'_K)^H$ is finite dimensional over K and over R. ■

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