Covers and propagation in symplectic groups

Corinne Blondel

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The purpose of these notes is to give a simplified overview of the theory of types and covers, due to Bushnell and Kutzko, and explain what is the counterpart, in the study of parabolically induced complex representations of reductive p-adic groups, of Tadić's philosophy in terms of covers. As an illustration, we describe a setting in which we can 'propagate' covers in symplectic groups and detail an example in $Sp(12, F)$.

1 Tadić's philosophy and covers

Let F be a local non archimedean field, with ring of integers $\mathfrak{o}_F$, maximal ideal $\mathfrak{p}_F$, residue field k_F of cardinality q_F, and residual characteristic $p = \operatorname{char} k_F$. We will occasionnally use a uniformizing element ϖ_F of $\mathfrak{p}_F$ and a character $\psi : F \to \mathbb{C}^\times$ trivial on $\mathfrak{p}_F$ and non trivial on $\mathfrak{o}_F$. Analogous notations will be used for field extensions of F.

Let $G = \mathbf{G}(F)$ be the group of F-rational points of a connected reductive algebraic group $\mathbf{G}$ defined over F. Let $\mathcal{R}(G)$ be the category of smooth complex representations of G. (Recall that a vector in a representation of G is smooth if its fixator is an open subgroup of G; a representation is smooth if all vectors in its space are smooth.)

There are two basic tools in the study of those representations: the functor of parabolic induction and Jacquet's restriction functor. Let P be a parabolic subgroup of G, let N be its unipotent radical and let M be a Levi factor of P. We need the modular character $\Delta_P : P \longrightarrow \mathbb{R}^{\times +}$ defined by: $\Delta_P(a) = \dfrac{\operatorname{vol}(aKa^{-1})}{\operatorname{vol}(K)}$, $a \in P$, where vol denotes the volume of an open compact subgroup of P with respect to some right Haar measure on P.

The (normalized) **parabolic induction functor** $\operatorname{Ind}_P^G : \mathcal{R}(M) \longrightarrow \mathcal{R}(G)$ is defined as follows. Let $(\sigma, V) \in \mathcal{R}(M)$. Then $\operatorname{Ind}_P^G(\sigma)$ is the representation of G by right translations in the space of functions

$$\operatorname{Ind}_P^G(V) = \{f\colon G \to V \ / \text{ for } m \in M, n \in N, g \in G : f(mng) = \Delta_P(m)^{1/2}\sigma(m)f(g), \text{ and } f \text{ is a smooth vector for the action of } G\}.$$

The **Jacquet restriction functor** $r_N : \mathcal{R}(G) \longrightarrow \mathcal{R}(M)$ is defined as follows. Let $(\pi, V) \in \mathcal{R}(G)$ and let $V(N) = \operatorname{Span}_{\mathbb{C}}\{\pi(n)v - v \ / \ v \in V, n \in N\}$. Then $r_N(\pi)$ is the natural quotient action of M in the space $r_N(V) = V_N = V/V(N)$. The normalized restriction functor r_P^G, defined as the twist of r_N by the character $\Delta_P^{-1/2}$, is left-adjoint to Ind_P^G.

We are using normalized induction here because it is common use; however, from the point of view of types, unnormalized and normalized induction or restriction make no difference: they differ by an unramified character.

The **supercuspidal** representations of G are those smooth representations (π, V) of G which satisfy, for any proper parabolic subgroup $P = MN$ of G: $r_N(\pi) = 0$ (or equivalently, for complex representations, $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) = 0$ for any $\sigma \in \mathcal{R}(M)$).

Let σ be an irreducible supercuspidal representation of M. The **inertial class** $[M, \sigma]_G$ is the equivalence class of the pair (M, σ) under the equivalence relation defined by conjugacy in G and twisting of σ by unramified characters of M – that is, one-dimensional representations of M that are trivial on every compact subgroup. We denote by $\mathcal{R}^{[M,\sigma]}(G)$ the subcategory of smooth representations π of G with the following property: any irreducible subquotient of π is a subquotient of $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ for some unramified character χ of M. Bernstein's decomposition of $\mathcal{R}(G)$ is the decomposition of $\mathcal{R}(G)$ as the direct sum, over inertial classes in G, of the subcategories $\mathcal{R}^{[M,\sigma]}(G)$. We will come back to this later: it is indeed the goal of the theory of types to describe each piece in this decomposition with a type.

The study of $\mathcal{R}(G)$ has for long followed two distinct paths: the study of supercuspidal representations of G, and the study of the parabolically induced representations $\mathrm{Ind}_P^G(\sigma)$ as above. As for the second problem, worked upon by many, Tadić's approach has been to study such induced representations using all possible functors $r_{N'} : \mathcal{R}(G) \longrightarrow \mathcal{R}(M')$, where $M'N'$ is a parabolic subgroup of G with unipotent radical N'. The philosophy of this approach is that *having more parabolic subgroups gives more possibilities to compare informations coming from the Jacquet modules of various parabolic subgroups* ([9], Introduction). The method is indeed very efficient and has produced a lot of results, by Tadić and others.

We will here explain a completely different approach that belongs to a line of thought initialized more than thirty years ago: via compact open subgroups. Let us start with supercuspidal representations; it has been an open question, for at least that amount of time, whether supercuspidal representations were (compactly) induced, that is:

1.1. *Given an irreducible supercuspidal representation π of G, does there exist an open subgroup K of G, compact modulo the center of G, and an irreducible smooth representation κ of K, such that $\pi \simeq \text{c-Ind}_K^G\, \kappa$?*

Here, if V is the space of κ, $\text{c-Ind}_K^G\, \kappa$ is the representation of G by right translations in the space

$$\text{c-Ind}_K^G(V) = \{f\colon G \to V \ / \text{ for } k \in K, g \in G : f(kg) = \kappa(k) f(g), \text{ and } \\ f \text{ is compactly supported mod the center of } G\}.$$

It is a basic fact that if the representation $\text{c-Ind}_K^G\, \kappa$ is irreducible, then it is supercuspidal (supercuspidal representations are also characterized by the fact that their coefficients are compactly supported mod the center). Furthermore its irreducibility is equivalent to the fact that the **intertwining** of κ, that is, the set of $g \in G$ such that $\mathrm{Hom}_{K \cap K^g}(\kappa, \kappa^g) \neq \{0\}$, is equal to K.

There have been too many works on question 1.1 to even try to cite them all: thirty years of efforts led eventually to a positive answer for $\mathbf{G} = GL(N)$ ([5], 1993) and many other reductive groups have followed. We only wish here to give an idea of types, starting with types for supercuspidal representations of $GL(N, F)$, and of covers, following Bushnell and Kutzko's formalism ([6]).

1.1 Types for supercuspidal representations of $GL_N(F)$

We first define the notion of *maximal simple type* $(J(\beta, \mathfrak{A}_0), \lambda(\beta, \mathfrak{A}_0))$ in $G = GL_N(F)$, following [5]. We will skip the technical definitions, to be found in *loc. cit.*, but will give an easy example instead.

We start with a principal $\mathfrak{o}_F$-order $\mathfrak{A}_0$ in $M_N(F)$ (e.g. $M_N(\mathfrak{o}_F)$) and an element β in $M_N(F)$ generating a field extension $E = F[\beta]$ of F. We let $\hat{E}$ be the commutant algebra of E in $M_N(F)$. We assume that:

- $E^\times$ normalises $\mathfrak{A}_0$;
- $n_0 = -\mathrm{val}_E(\beta)$ satisfies $n_0 > 0$;
- $\mathfrak{B}_0 = \mathfrak{A}_0 \cap \hat{E}$ is a *maximal* $\mathfrak{o}_E$-order in $\hat{E}$;
- $[E : F]$ is minimal among the degrees of field extensions possibly generated by elements of $\beta + \mathfrak{A}_0$.

Then we call $[\mathfrak{A}_0, n_0, 0, \beta]$ a *maximal simple stratum* in $M_N(F)$. The simplest object attached to this stratum is a function ψ_β on G defined by $\psi_\beta(g) = \psi \circ \mathrm{tr}\,(\beta(g-1))$, $g \in G$. This function restricts to a character on suitable subgroups.

With these data $\mathfrak{o}_F$-lattices in $M_N(F)$ are built:

$$\mathfrak{H}^1(\beta, \mathfrak{A}_0) \subset \mathfrak{J}^1(\beta, \mathfrak{A}_0) \subset \mathfrak{J}^0(\beta, \mathfrak{A}_0),$$

and compact open subgroups of $GL_N(F)$:

$$H^1(\beta, \mathfrak{A}_0) \subset J^1(\beta, \mathfrak{A}_0) \subset J(\beta, \mathfrak{A}_0),$$

with $H^1 = 1 + \mathfrak{H}^1$, $J^1 = 1 + \mathfrak{J}^1$, $J = \mathfrak{J}^{0\times}$.

A simple type will be constructed by stages up this tower of subgroups. First, the crucial notion of simple character: it is a difficult generalization of an easy construction given in the example below. A *simple character* is a rather special character of $H^1(\beta, \mathfrak{A}_0)$ (one-dimensional representation), among its properties are the following:

- the restriction of a simple character to a suitable subgroup of H^1 coincides with ψ_β;
- the intertwining of a simple character is $J\hat{E}^\times J$.

Let θ_0 be a simple character. Next, Heisenberg construction provides η_0, unique irreducible representation of $J^1(\beta, \mathfrak{A}_0)$ containing θ_0; as such, η_0 has the same intertwining as θ_0.

The third step is the β-*extension* step: extending the representation η_0 to $J(\beta, \mathfrak{A}_0)$ without shrinking its intertwining. It is possible, if difficult. We thus pick κ_0, a β-*extension* of η_0 to $J(\beta, \mathfrak{A}_0)$; the intertwining of κ_0 is the full intertwining of θ_0.

Let $f = N/[E:F]$. The last ingredient in our simple type is σ_0, a cuspidal representation of $GL(f, k_E)$. We inflate it to $J(\beta, \mathfrak{A}_0)$ through:

$$J(\beta, \mathfrak{A}_0)/J^1(\beta, \mathfrak{A}_0) \simeq GL(f, k_E).$$

Our maximal simple type is now defined by $\lambda(\beta, \mathfrak{A}_0) = \kappa_0 \otimes \sigma_0$.

For a complete account of maximal simple types, we also must allow the so-called 'level 0 case': where $\mathfrak{A}_0$ is a maximal order and we take $\beta = 0$, $E = F$, $\lambda(\beta, \mathfrak{A}_0) = \sigma_0$.

Theorem 1.2 (Bushnell-Kutzko [5])**.** *For any extension $\lambda^\star$ of $\lambda(\beta, \mathfrak{A}_0)$ to $J^\star = E^\times J(\beta, \mathfrak{A}_0)$, the induced representation* c-$\mathrm{Ind}_{J^\star}^{GL_N(F)} \lambda^\star$ *is irreducible and supercuspidal.*

Every supercuspidal representation of $GL_N(F)$ has this form for a suitable choice of β and $\mathfrak{A}_0$.

1.2 An example in $GL(4, F)$

Assume the residual characteristic p is not 2. Let $\nu \in \mathfrak{o}_F^\times$ be an element that is not a square and let E_0 be the unramified quadratic extension of F, viewed as $E_0 = F[\left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)]$ in $M_2(F)$. Let $a \geq 1$ be an integer and let $u \in \mathfrak{o}_{E_0}^\times$ satisfy $u^2 \notin \mathfrak{o}_F^\times(1 + \mathfrak{p}_{E_0})$.

Define $\beta = \left(\begin{smallmatrix} 0 & u\varpi_F^{-a} \\ u\varpi_F^{1-a} & 0 \end{smallmatrix}\right)$ in $M_2(E_0) \subset M_4(F)$. Then $E = F[\beta]$ is a ramified quadratic extension of E_0, viewed inside $M_4(F)$, in which β has odd valuation $-n_0 = 1 - 2a$.

Now $\mathfrak{A}_0 = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \end{pmatrix}$ is a principal $\mathfrak{o}_F$-order with radical $\mathfrak{P}_0 = \begin{pmatrix} \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$, normalized by $E^\times$, and $[\mathfrak{A}_0, n_0, 0, \beta]$ is a maximal simple stratum in $M_4(F)$.

This is deliberately an elementary situation, in which definitions in [5] immediately give:

$$H^1(\beta, \mathfrak{A}_0) = J^1(\beta, \mathfrak{A}_0) = (1 + \mathfrak{p}_E)\ (1 + \mathfrak{P}_0^{\frac{n_0+1}{2}})\ ; \quad J(\beta, \mathfrak{A}_0) = \mathfrak{o}_E^\times\ (1 + \mathfrak{P}_0^{\frac{n_0+1}{2}}).$$

Simple characters are obtained as follows. The function ψ_β is a character on $1 + \mathfrak{P}_0^{\frac{n_0+1}{2}}$. Let θ_0 be some character of $(1 + \mathfrak{p}_E)$ agreeing with ψ_β on the intersection $(1 + \mathfrak{p}_E) \cap (1 + \mathfrak{P}_0^{\frac{n_0+1}{2}})$. Then $\theta_0 \psi_\beta$ is a simple character of $H^1(\beta, \mathfrak{A}_0)$, still denoted by θ_0.

Our maximal simple type is here $\lambda(\beta, \mathfrak{A}_0) = \theta^\star \psi_\beta$ for some character $\theta^\star$ of $\mathfrak{o}_E^\times$ extending θ_0.

1.3 Types for non-supercuspidal representations

We come back to the general setting of the beginning of this paragraph and give the definition of a type for a subcategory $\mathcal{R}^{[M,\sigma]}(G)$, or equivalently for an inertial class $[M, \sigma]_G$:

Definition 1.3 ([6])**.** *A pair (J, λ), J an open compact subgroup of G, λ a smooth irreducible representation of J, is a type for $[M, \sigma]_G$ if, for any smooth irreducible representation π of G, the following conditions are equivalent:*

- $\pi|_J$ *contains* λ*;*

- π *is a subquotient of* $\mathrm{Ind}_P^G \sigma \otimes \chi$ *for some unramified character* χ *of* M*.*

(In the situation of Theorem 1.2, the pair $(J(\beta, \mathfrak{A}_0), \lambda(\beta, \mathfrak{A}_0))$ is a type for the inertial class of $\text{c-Ind}_{J^\star}^{GL_N(F)} \lambda^\star$ in ${}_{GL_N(F)}$.)

To explain the interest of this definition, we immediately need to define the **Hecke algebra** of a pair (J, λ) as above; it is the intertwining algebra of the representation $\text{c-Ind}_J^G \lambda$, which can also be described as the following convolution algebra, where W_λ denotes the space of λ (and a Haar measure is fixed on G):

$$\mathcal{H}(G, \lambda) = \{ f : G \longrightarrow \mathrm{End}_{\mathbb{C}}(W_\lambda) \ / \ f \ \text{ compactly supported} \\ \text{and } \ f(ugv) = \lambda(u) f(g) \lambda(v) \ \text{ for } \ u, v \in J, \ g \in G \}.$$

Note that the support of an element of $\mathcal{H}(G, \lambda)$ is a finite union of J-double cosets.

Now the whole point of the definition is to replace the study of induced representations of the form $\mathrm{Ind}_P^G \sigma \otimes \chi$ as above, by the study of corresponding right modules over the algebra $\mathcal{H}(G, \lambda)$, via the following functor, which is an **equivalence of categories** whenever (J, λ) is a type for $[M, \sigma]_G$ (see [6]):

$$\begin{array}{ccc} \mathcal{R}^{[M,\sigma]}(G) & \overset{\mathcal{M}_\lambda}{\underset{\sim}{\longrightarrow}} & \text{Mod-}\mathcal{H}(G, \lambda) \\ \pi & \longmapsto & \mathrm{Hom}_J(\lambda, \pi) \end{array}$$

The right action of $f \in \mathcal{H}(G, \lambda)$ on $\phi \in \mathrm{Hom}_J(\lambda, \pi)$ is given by

$$\phi.f(w) = \int_G \pi(g^{-1}) \, \phi(f(g) w) \, dg \quad (w \in W_\lambda).$$

1.4 Covers

Types are not known yet to exist in all cases; for supercuspidal representations, existence is very closely related to question 1.1. Anyhow, the best way to construct types in the non-supercuspidal case, say $[M, \sigma]_G$ with M a proper Levi subgroup, is to try to start with a type (J_M, λ_M) for σ in M and to build a G-**cover** or *induced type* of (J_M, λ_M) in G: the shape of type best adapted to parabolic induction.

Let as above $P = MN$ be a parabolic subgroup of G of Levi M and unipotent radical N and let $P^- = MN^-$ be the parabolic subgroup opposite to P relative to M.

Definition 1.4 ([6]). *A pair* (J, λ)*,* J *an open compact subgroup of* G*,* λ *a smooth irreducible representation of* J*, is a* G-**cover** *of an analogous pair* (J_M, λ_M) *in* M *if*

- $J = (J \cap N^-) \ (J \cap M) \ (J \cap N)$ *and* $J \cap M = J_M$*,*

- λ *is trivial on* $J \cap N^-$ *and* $J \cap N$ *and* $\lambda_{|J \cap M} = \lambda_M$*;*

- *for any smooth irreducible representation* (π, V) *of* G*, the Jacquet functor* r_N *is injective on the isotypic component of* $\pi_{|J}$ *of type* λ*:*

$$V^\lambda \overset{r_N}{\underset{\sim}{\longrightarrow}} V_N^{\lambda_M}.$$

The two first conditions, dealing with the Iwahori decomposition of the group **and** the representation, are relatively easy to fulfill; pairs (J, λ) satisfying those two conditions are said to be **decomposed** above (J_M, λ_M) with respect to P – note that the representation λ is then entirely determined by λ_M. Given the first two, the third condition is a very strong one and it has indeed very strong consequences:

Theorem 1.5 (Bushnell-Kutzko [6])**.** *A G-cover of a type is a type. With the notation above, if (J, λ) is a G-cover of (J_M, λ_M) and (J_M, λ_M) is a type for $[L, \sigma]_M$ in M, then (J, λ) is a type for $[L, \sigma]_G$ in G.*

Yet there is more to covers than this theorem. Indeed, if we start as before with a decomposed pair (J, λ) above (J_M, λ_M), we can define an injective homomorphism of vector spaces $T : \mathcal{H}(M, \lambda_M) \hookrightarrow \mathcal{H}(G, \lambda)$, by :

$$f \in \mathcal{H}(M, \lambda_M) \text{ and } \operatorname{Supp} f = J_M m J_M \quad \Rightarrow \quad \operatorname{Supp} T(f) = JmJ \ \text{ and } T(f)(m) = f(m).$$

Further, there is a subalgebra $\mathcal{H}(M, \lambda_M)^+$ of $\mathcal{H}(M, \lambda_M)$ on which T restricts to an homomorphism of *algebras*

$$T^+ : \mathcal{H}(M, \lambda_M)^+ \hookrightarrow \mathcal{H}(G, \lambda). \tag{1.6}$$

It is the subalgebra of functions supported on (P, J)*-positive* elements of M, namely those $m \in M$ satisfying $m(J \cap N)m^{-1} \subset J \cap N$ and $m(J \cap N^-)m^{-1} \supset J \cap N^-$.

We twist our morphism by the character $\Delta_P^{1/2}$, which is trivial on compact subgroups of M, letting $T_P^+(f) = T^+(\Delta_P^{1/2} f)$, $f \in \mathcal{H}(M, \lambda_M)^+$. The remarkable feature of covers is the following:

Theorem 1.7 (Bushnell-Kutzko [6])**.** *Let (J, λ) be a decomposed pair above (J_M, λ_M) with respect to P. Then (J, λ) is a G-cover of (J_M, λ_M) if and only if the morphism T_P^+ extends to a homomorphism of algebras*

$$t_P : \mathcal{H}(M, \lambda_M) \hookrightarrow \mathcal{H}(G, \lambda).$$

This extension is unique, injective, and gives rise to the commutative diagram :

$$\begin{array}{ccc} \mathcal{R}^{[L,\sigma]}(G) & \xrightarrow[\sim]{\mathcal{M}_\lambda} & \text{Mod-}\mathcal{H}(G, \lambda) \\ {\scriptstyle \operatorname{Ind}_{P^-}^G} \uparrow & & \uparrow {\scriptstyle (t_P)_*} \\ \mathcal{R}^{[L,\sigma]}(M) & \xrightarrow[\sim]{\mathcal{M}_{\lambda_M}} & \text{Mod-}\mathcal{H}(M, \lambda_M) \end{array} \tag{1.8}$$

Here $(t_P)_*(Y) = \operatorname{Hom}_{\mathcal{H}(M,\lambda_M)}(\mathcal{H}(G, \lambda), Y)$, the structure of right $\mathcal{H}(M, \lambda_M)$-module on $\mathcal{H}(G, \lambda)$ being given by t_P.

In this context, a counterpart to Tadić's philosophy is given by the crucial homomorphism of algebras t_P, plus the transitivity of covers ([6] again):

1.9. *Let $P' = M'N'$ be another parabolic subgroup of G of Levi M' and unipotent radical N'. Assume $P \subset P'$ and $M \subset M'$. Then (J, λ) is a G-cover of $(J \cap M, \lambda_{|J\cap M})$ if and only if (J, λ) is a G-cover of $(J \cap M', \lambda_{|J\cap M'})$ and $(J \cap M', \lambda_{|J\cap M'})$ is a M'-cover of $(J \cap M, \lambda_{|J\cap M})$.*

Indeed, the point of types is to replace the study of parabolic induction by the study of modules over the relevant Hecke algebras. The point of covers is to do so using the commutative diagram 1.8. You do need information on the Hecke algebra $\mathcal{H}(G,\lambda)$ and a crucial tool is the homomorphism of algebras defined in Theorem 1.7. Now, thanks to transitivity, each intermediate Levi subgroup will give rise to such an algebra homomorphism so *the more intermediate parabolic subgroups we have, the more information we get on the Hecke algebra.* We describe in the next paragraph an instance of this technique; more details on the use of these homomorphisms will be found in section 2.5.

2 Propagation of types in symplectic groups

¿From now on we assume the residual characteristic p is different from 2. We look at the following situation: we fix integers $N \geq 1$ and $k \geq 0$ and let an integer $t \geq 1$ vary. We put

$$G_t = Sp\,(2tN + 2k, F)$$

and consider:

- the standard Levi subgroup $M_t = GL_N(F) \times \cdots \times GL_N(F) \times Sp_{2k}(F)$ of G_t
- the standard parabolic subgroup P_t of block-upper-triangular matrices with Levi component M_t and unipotent radical U_t.

We pick an irreducible supercuspidal representation π of $GL_N(F)$ and an irreducible supercuspidal representation ρ of $Sp_{2k}(F)$ and form, for complex numbers $a_1, \ldots, a_t$, the representation of G_t:

$$I(\pi, t, \rho) = \mathrm{Ind}_{P_t}^{G_t}\ \pi|\det|^{a_1} \otimes \cdots \otimes \pi|\det|^{a_t} \otimes \rho.$$

Tadić has proved in [12] that the knowledge of reducibility points (that is, the values of $a_1, \ldots, a_t$ such that the representation is reducible) and composition series of $I(\pi,1,\rho)$ implies the knowledge of reducibility points and composition series of $I(\pi,t,\rho)$ for $t \geq 1$. This is of course in accordance with the philosophy stated in the previous paragraph: the case of maximal parabolic subgroup is the most difficult, if t is greater than 1 there are more parabolics to work with and one can obtain results through the use of Jacquet functors.

What could be the counterpart of this result in terms of types and covers ? To express it, we need first to pick (Γ,γ), a $[GL_N(F),\pi]$- type in $GL_N(F)$, and (Δ,δ), an $[Sp_{2k}(F),\rho]$- type in $Sp_{2k}(F)$. Knowledge of reducibilities for $I(\pi,1,\rho)$ can be replaced by the knowledge of a G_1-cover (Ω_1,ω_1) of $(\Gamma\times\Delta, \gamma\otimes\delta)$ and of its Hecke algebra $\mathcal{H}(G_1,\omega_1)$, through the equivalence of categories:

$$\mathcal{R}^{[M_1,\ \pi\otimes\rho]}(G_1) \ \overset{\mathcal{M}_{\omega_1}}{\underset{\sim}{\longrightarrow}}\ \text{Mod-}\mathcal{H}(G_1,\omega_1).$$

In this setting, a result similar to Tadić's would be the following:

There exists a G_t-cover (Ω_t, ω_t) of $(\Gamma^{\times t} \times \Delta, \gamma^{\otimes t} \otimes \delta)$ with Hecke algebra $\mathcal{H}(G_t, \omega_t)$ admitting a presentation by generators and relations ***entirely determined*** *by t and $\mathcal{H}(G_1, \omega_1)$.*

Indeed, the reducibilities for $I(\pi, t, \rho)$ would then be known through the equivalence of categories:

$$\mathcal{R}^{[M_t,\ \pi^{\otimes t} \otimes \rho]}(G_t) \overset{\mathcal{M}_{\omega_t}}{\underset{\sim}{\longrightarrow}} \text{Mod-}\mathcal{H}(G_t, \omega_t).$$

This is the question we will examine in this section, in the case where the representation π is **self-dual**, that is, equivalent to the contragredient representation $\check{\pi}$ defined as the dual action of $GL_N(F)$ in the space of smooth vectors in the dual of the space of π. This is the most interesting case: if no representation $\pi|\det|^a$ is self-dual, then it follows from the work of Tadić that reducibilities for $I(\pi, t, \rho)$ all come from reducibilities for the parabolically induced representation of $\pi|\det|^{a_1} \cdots \otimes \pi|\det|^{a_t}$ in $GL_{tN}(F)$, which are well known.

2.1 The principle of propagation

We now need to get more technical and fix some more notations. We let w_i be the anti-diagonal matrix $w_i = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ and define $Sp\,(2i, F)$ as the symplectic group of F^{2i} with respect to the symplectic form of matrix $\begin{pmatrix} 0 & w_i \\ -w_i & 0 \end{pmatrix}$. For any matrix x we write ${}^{\tau}x$ for the transpose of x with respect to the anti-diagonal; in particular the identification of $GL_N(F) \times Sp_{2k}(F)$ with M_1 reads:

$$(x, g) \longmapsto \begin{pmatrix} x & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & {}^{\tau}x^{-1} \end{pmatrix}.$$

We start with a G_1-cover (Ω_1, ω_1) of $(\Gamma \times \Delta, \gamma \otimes \delta)$ and think of it in terms of blocks: from the definition, the pair (Ω_1, ω_1) is in particular decomposed above $(\Gamma \times \Delta, \gamma \otimes \delta)$ so is the product of its intersections with U_1^-, M_1 and U_1. We visualize as follows:

$$\Omega_1 = \begin{pmatrix} \Gamma & \mathcal{B}_{12} & \mathcal{B}_{13} \\ \mathcal{B}_{21} & \Delta & \mathcal{B}_{23} \\ \mathcal{B}_{31} & \mathcal{B}_{32} & {}^{\tau}\Gamma \end{pmatrix},$$

this schematization meaning of course that the off-diagonal blocks $\mathcal{B}_{ij}$ are suitable lattices in the relevant matrix space and that Ω_1 is the intersection with $Sp\,(2N + 2k, F)$ of the set of matrices whose block entries belong to the corresponding subgroup or lattice.

We want to produce a G_t-cover (Ω_t, ω_t) of $(\Gamma^{\times t} \times \Delta, \gamma^{\otimes t} \otimes \delta)$; with the same convention of visualization, this would look like

$$\Omega_t = \begin{pmatrix} \Gamma & \mathcal{C}_{12} & \cdots & \mathcal{C}_{1,t+1} & & \cdots & \mathcal{C}_{1,2t+1} \\ \mathcal{C}_{21} & \Gamma & & \mathcal{C}_{2,t+1} & & & \vdots \\ & \vdots & \ddots & \vdots & & & \\ \mathcal{C}_{t+1,1} & \mathcal{C}_{t+1,2} & \cdots & \Delta & \cdots & \cdots & \mathcal{C}_{t+1,2t+1} \\ & & & \vdots & \ddots & & \\ \vdots & & & \mathcal{C}_{2t,t+1} & & \daleth & \mathcal{C}_{2t,2t+1} \\ \mathcal{C}_{2t+1,1} & \cdots & & \mathcal{C}_{2t+1,t+1} & & \cdots & \daleth \end{pmatrix} = \begin{pmatrix} \Gamma_t & \mathcal{D}_{12} & \mathcal{D}_{13} \\ \mathcal{D}_{21} & \Delta & \mathcal{D}_{23} \\ \mathcal{D}_{31} & \mathcal{D}_{32} & \daleth_t \end{pmatrix}.$$

The first shape shows what we mean by propagation: we would expect a relationship between the blocks $\mathcal{C}_{i,t+1}$ or $\mathcal{C}_{t+1,i}$ and the blocks $\mathcal{B}_{j2}$ or $\mathcal{B}_{2j}$ in Ω_1, at best equality for instance, and a relationship between the off-$Sp\,(2k, F)$-part $\begin{pmatrix} \Gamma & \mathcal{B}_{13} \\ \mathcal{B}_{31} & \daleth \end{pmatrix}$ and what is obtained from Ω_t by removing the $t+1$-th row and column, namely $\begin{pmatrix} \Gamma_t & \mathcal{D}_{13} \\ \mathcal{D}_{31} & \daleth_t \end{pmatrix}$ in the second shape.

It seems hopeless to try such a 'propagation' without any specific knowledge of the cover (Ω_1, ω_1). Besides, the best hint we have is the following consequence of the transitivity of covers: the block Γ_t should hold a $GL_{tN}(F)$-cover of $(\Gamma^{\times t}, \gamma^{\otimes t})$. We will thus rely strongly on Bushnell-Kutzko types in $GL_{iN}(F)$.

2.2 A family of $GL_{2tN}(F)$-covers of $(\Gamma \times \cdots \times \Gamma, \gamma \otimes \cdots \otimes \gamma)$

We now pick (Γ, γ), our $[GL_N(F), \pi]$-type in $GL_N(F)$, as a Bushnell-Kutzko maximal simple type $(\Gamma, \gamma) = (J(\beta, \mathfrak{A}_0), \lambda(\beta, \mathfrak{A}_0))$. We use the notations of section 1.1 and will shorten $\mathfrak{H}^1 = \mathfrak{H}^1(\beta, \mathfrak{A}_0)$, $\mathfrak{J}^0 = \mathfrak{J}^0(\beta, \mathfrak{A}_0)$... Recall $E = F[\beta]$.

In the book [5], Bushnell and Kutzko do produce a $GL_{2tN}(F)$-cover of $(\Gamma^{\times 2t}, \gamma^{\otimes 2t})$. Starting with this cover, and with some additional work, one obtains a family $(\Gamma(2t, r), \gamma(2t, r))$ of $GL_{2tN}(F)$-covers of $(\Gamma^{\times 2t}, \gamma^{\otimes 2t})$, indexed by an integer r, $t \leq r \leq 2t$; the cover given in [5] corresponds to $r = t$ while the cover given by $r = 2t$ is the best suited to propagation. In block-matrix form, we have:

$$\Gamma(2t,r)=\overset{\longleftarrow \quad r \quad \longrightarrow}{\underset{\longleftarrow \quad 2t-r \quad \longrightarrow}{\begin{pmatrix} \Gamma & \mathfrak{J}^0 & \dots & \mathfrak{J}^0 & \varpi_E^{-1}\mathfrak{H}^1 & \dots & \varpi_E^{-1}\mathfrak{H}^1 \\ \mathfrak{H}^1 & \Gamma & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & \varpi_E^{-1}\mathfrak{H}^1 \\ \mathfrak{H}^1 & & & & & & \mathfrak{J}^0 \\ \varpi_E\mathfrak{J}^0 & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \Gamma & \mathfrak{J}^0 \\ \varpi_E\mathfrak{J}^0 & \dots & \varpi_E\mathfrak{J}^0 & \mathfrak{H}^1 & \dots & \mathfrak{H}^1 & \Gamma \end{pmatrix}}}$$

and the representation $\gamma(2t,r)$ is trivial on upper or lower triangular block matrices and equal to $\gamma\otimes\cdots\otimes\gamma$ on block diagonal matrices. (Actually we have such a family of covers $(\Gamma(i,r),\gamma(i,r))$, $[\frac{i+1}{2}]\leq r\leq i$, for any integer $i\geq 1$.)

It is convenient to think of this subgroup as a 2×2 block-matrix subgroup and to introduce a notation for the lattices in $M_{tN}(F)$ corresponding to the upper and lower unipotent parts; we write:

$$\Gamma(2t,r)=\begin{pmatrix} \Gamma\ (t,t) & \Gamma^+(t,r) \\ \Gamma^-(t,r) & \Gamma\ (t,t) \end{pmatrix}, \quad t\leq r\leq 2t.$$

Note that $\Gamma(t,t)$ is the most obvious 'propagated type' from $\Gamma(2,2)=\begin{pmatrix} \Gamma & \mathfrak{J}^0 \\ \mathfrak{H}^1 & \Gamma \end{pmatrix}$ and that $\mathfrak{J}^0$ and $\mathfrak{H}^1$ are both rings. Subdiagonals in $\Gamma^+(t,r)$ have either all entries in $\mathfrak{J}^0$ or all entries in $\varpi_E^{-1}\mathfrak{H}^1$ and accordingly for $\Gamma^-(t,r)$, with $\varpi_E\mathfrak{J}^0$ and $\mathfrak{H}^1$.

We are almost ready to produce propagated types. The last ingredient we need is the specific properties that the type of a self-dual supercuspidal representation can be assumed to satisfy, useful to obtain decomposed pairs in symplectic groups from the ones we have in linear groups.

2.3 Type of a self-dual supercuspidal representation

Heuristically, we want to produce decomposed pairs – hopefully covers – in $Sp(2tN,F)$ with the subgroups $\Gamma(2t,r)$ in $GL_{2tN}(F)$. Intersecting with $Sp(2tN,F)$ will give interesting subgroups only if $\Gamma(2t,r)$ is stable under the involution defining the symplectic group. Up to some conjugacy, we can achieve this condition when π is self-dual, provided that we find a type for π that is itself self-dual in some sense. This is the motivation of the following theorem:

Theorem 2.1 (Blondel [2]). *Let π be a self-dual irreducible supercuspidal representation of $GL_N(F)$. One can choose a maximal simple type $(\Gamma,\gamma) = (J(\beta,\mathfrak{A}_0),\ \lambda(\beta,\mathfrak{A}_0))$ for π satisfying the following properties:*

1. *$\mathfrak{A}_0$ is a τ-stable principal $\mathfrak{o}_F$-order in $M_N(F)$.*
2. *If $\beta \neq 0$, $E = F[\beta]$ is a quadratic extension of $F[\beta^2]$. We let $x \mapsto \bar{x}$ denote the non trivial element of* $\mathrm{Gal}(F[\beta]/F[\beta^2])$.
3. *There is an element σ in $U(\mathfrak{A}_0)$ such that $\sigma^{-1}x\sigma = {}^\tau\bar{x}$ for all $x \in E$.*
4. *$\mathfrak{H}^1 = \mathfrak{H}^1(\beta,\mathfrak{A}_0)$ and $\mathfrak{J}^0 = \mathfrak{J}^0(\beta,\mathfrak{A}_0)$ are stable under $x \mapsto \sigma\, {}^\tau x \sigma^{-1}$.*
5. *The pairs (H^1,θ_0), (J^1,η_0) and (Γ,γ) are stable under $x \mapsto \sigma\, {}^\tau x^{-1}\sigma^{-1}$ (the group is stable and the representation transformed into an equivalent representation).*

Example. In our previous example in $GL_4(F)$ (1.2), the extension E is τ-stable and conditions (i) to (iv) hold with $\sigma = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$. The fifth condition holds if and only if we choose a character $\theta^\star$ such that $\theta^\star(x\bar{x}) = 1$ for all $x \in \mathfrak{o}_E^\times$.

2.4 The propagation theorem

We fix a maximal simple type (Γ,γ) for our self-dual representation π having the properties in Theorem 2.1 and use the corresponding notations. It follows that the conjugate of $\Gamma(2t,r)$ by $\Sigma = \mathrm{diag}(I_N,\cdots,I_N,\sigma,\cdots,\sigma)$ (t blocks I_N, t blocks σ), namely

$$\Gamma(2t,r)^\Sigma = \begin{pmatrix} \Gamma\ (t,t) & \Gamma^+(t,r)\sigma \\ \sigma^{-1}\Gamma^-(t,r) & {}^\tau\Gamma\ (t,t) \end{pmatrix},\ t \leq r \leq 2t,$$

is stable by the involution defining the symplectic group. We also fix a type (Δ,δ) for the supercuspidal representation ρ of $Sp(2k,F)$ such that $\rho = \text{c-Ind}_\Delta^{Sp(2k,F)}\delta$; the existence of such a type follows from recent work of Stevens ([11]).

We first state the hypotheses needed on the G_1-cover of $(\Gamma\times\Delta,\gamma\otimes\delta)$. The main point is that we actually need two such covers, related to the twin groups $\Gamma(2,1)$ and $\Gamma(2,2)$ in $GL_{2N}(F)$, or rather a little less: two decomposed pairs that are 'almost' covers. (As a consequence of the theorem they will turn out to be covers.)

The first hypothesis thus concerns the existence of two decomposed pairs of a special shape: there must exist $\mathfrak{o}_F$-lattices $\Delta^+ \subset M_{N,2k}(F)$ and $\Delta^- \subset M_{2k,N}(F)$ such that

$$\Omega(1,1) = \begin{pmatrix} \Gamma & \Delta^+ & \varpi_E^{-1}\mathfrak{H}^1\sigma \\ \Delta^- & \Delta & \alpha\,{}^\tau\Delta^+ \\ \sigma^{-1}\mathfrak{J}^0\varpi_E & {}^\tau\Delta^-\alpha^{-1} & {}^\tau\Gamma \end{pmatrix} \text{ (intersected with } G_1)$$

$$\text{and } \Omega(1,2) = \begin{pmatrix} \Gamma & \Delta^+ & \mathfrak{J}^0\sigma \\ \Delta^- & \Delta & \alpha\,{}^\tau\Delta^+ \\ \sigma^{-1}\mathfrak{H}^1 & {}^\tau\Delta^-\alpha^{-1} & {}^\tau\Gamma \end{pmatrix} \text{ (intersected with } G_1),\ \left(\alpha = \begin{pmatrix} -I_k & 0 \\ 0 & I_k \end{pmatrix}\right),$$

are subgroups of G_1 holding representations $\omega(1,1)$ and $\omega(1,2)$ such that $(\Omega(1,1),\omega(1,1))$ and $(\Omega(1,2),\omega(1,2))$ are decomposed pairs above $(\Gamma\times\Delta,\gamma\otimes\delta)$.

To explain the second hypothesis, we have to come back to the definition of covers: the third condition in definition 1.4 is equivalent to the inversibility, in the Hecke algebra, of an element supported on a special double coset; the exact statement is rather technical, hence omitted. Suffice to say that in practice, showing that a given decomposed pair is a cover is very much related to finding invertible generators for the Hecke algebra, that make it a convolution algebra on an affine (or extended affine) Weyl group (possibly twisted).

Now the elements s and q defined below are the generators of the affine Weyl group of type $\tilde{C}_2$ adapted to the situation in the maximal case. The assumptions below imply that the elements e_s and e_q are invertible (b_q and b_s are non zero); if they did belong to the same Hecke algebra, both their invertibilities would imply that the corresponding decomposed pair is a cover. Heuristically, 2.2 and 2.3 say that the two pairs are 'halfway' to being covers, and 'complementarily' so.

We thus let $s=\begin{pmatrix}0&0&\sigma\\0&I_{2k}&0\\-{}^{\tau}\sigma^{-1}&0&0\end{pmatrix}$, $q=\begin{pmatrix}0&0&-{}^{\tau}\sigma\,{}^{\tau}\varpi_E^{-1}\\0&I_{2k}&0\\\sigma^{-1}\varpi_E&0&0\end{pmatrix}$, and assume we have elements

- $e_q\in\mathcal{H}(G_1,\omega(1,1))$, supported on $\Omega(1,1)\;q\;\Omega(1,1)$, such that:

$$e_q^2=a_q\,e_q+b_q\,\mathcal{I}\qquad(a_q\in\mathbb{C},b_q\in\mathbb{C}^\times);\tag{2.2}$$

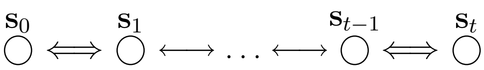
- $e_s\in\mathcal{H}(G_1,\omega(1,2))$, supported on $\Omega(1,2)\;s\;\Omega(1,2)$, such that:

$$e_s^2=a_s\,e_s+b_s\,\mathcal{I}\qquad(a_s\in\mathbb{C},b_s\in\mathbb{C}^\times).\tag{2.3}$$

With those assumptions, propagation holds. We can 'propagate' the subgroups $\Omega(1,1)$ and $\Omega(1,2)$ into the following family in G_t:

$$\Omega(t,r)=\begin{pmatrix}\Gamma(t,t)&M_{t,1}(\Delta^+)&\Gamma^+(t,r)\sigma\\M_{1,t}(\Delta^-)&\Delta&M_{1,t}(\alpha\,{}^{\tau}\Delta^+)\\\sigma^{-1}\Gamma^-(t,r)&M_{t,1}({}^{\tau}\Delta^-\alpha^{-1})&{}^{\tau}\Gamma(t,t)\end{pmatrix}\cap G_t\quad(t\geq1\ \text{ and } t\leq r\leq 2t).$$

Theorem 2.4 (Blondel [3]). *For $t\geq1$ and $t\leq r\leq 2t$, $\Omega(t,r)$ is a subgroup of G_t and holds a (unique) representation $\omega(t,r)$ such that $(\Omega(t,r),\omega(t,r))$ is a G_t-cover of $(\Gamma^{\times t}\times\Delta,\gamma^{\otimes t}\otimes\delta)$. The Hecke algebra $\mathcal{H}(G_t,\omega(t,r))$ is a convolution algebra on an affine Weyl group of type $\tilde{C}_t$, with parameters:*



$$(a_{\mathbf{s}_0},b_{\mathbf{s}_0})=(a_s,b_s),$$
$$(a_{\mathbf{s}_i},b_{\mathbf{s}_i})=(q_E^f-1,q_E^f)\quad\textit{for } 1\leq i\leq t-1,$$
$$(a_{\mathbf{s}_t},b_{\mathbf{s}_t})=(a_q,b_q).$$

(In the next subsection we give precise information on the generators of this Hecke algebra.)

Given the assumptions, this theorem fulfills the requirements in the beginning of section 2: the groups $\Omega(t,r)$ are entirely built from the groups $\Omega(1,1)$ and $\Omega(1,2)$, using propagation and the structure of $GL_{2tN}(F)$-covers; the Hecke algebra $\mathcal{H}(G_t,\omega(t,r))$ is completely determined by the $t=1$ case. Note that the parameters (q_E^f-1,q_E^f) only depend on the (inertial class of the) representation π: they are parameters of the Hecke algebra $\mathcal{H}(GL_{tN}(F),\gamma(t,t))$ (see [5]).

At this point of course, we are left with two difficulties: first, producing covers with the required properties for $t=1$, second, compute the corresponding parameters. So far, the cases in which this theorem applies – that is, the assumptions are fulfilled – are: the level zero case, due to Morris [8]; the $k=0$ case [2]; the case of $Sp(4,F)$ $(2N+2k=4)$ [1]; and a special case with $k=N$ [3] of which we detail an example for $N=4$ below.

2.5 Some elements in the proof

Past the preliminary checking (technical and boring) that $(\Omega(t,r),\omega(t,r))$, $t\le r\le 2t$, are indeed decomposed pairs, the proof that they are covers proceeds along two main lines.

One is the use of a *family* of decomposed pairs and not just one. Technically, the proof works alternatively with $\Omega(t,t)$ and $\Omega(t,2t)$: we work at each step with the most convenient of the two. A crucial intermediate result ([3], Proposition 5) states that the Hecke algebras $\mathcal{H}(G_t,\omega(t,r))$ are all isomorphic in a very precise way. To simplify matters we now shorten: $(\Omega,\omega)=(\Omega(t,r),\omega(t,r))$, $\mathcal{H}=\mathcal{H}(G_t,\omega(t,r))$.

The other is the most related to Tadić's philosophy: it is the one we will roughly explain here. Indeed, transitivity of covers (1.9) and a similar property of transitivity for decomposed pairs allows to move back and forth from one Levi subgroup to the full group and then down to another Levi subgroup.

First, to show that (Ω,ω) is a cover, we produce specific invertible elements in $\mathcal{H}$ as the images, by homomorphisms of the form $T^+:\ \mathcal{H}(M,\omega_M)^+\hookrightarrow\mathcal{H}(G_t,\omega)$ (see 1.6) for well chosen Levi subgroups M, of invertible elements in the subalgebra $\mathcal{H}(M,\omega_M)^+$ itself.

Next, to obtain a full description of the Hecke algebra $\mathcal{H}$ by generators and relations, we use the fact that our covers are covers of their intersection with any intermediate Levi subgroup: the images of the corresponding Hecke algebras by the homomorphisms as defined in Theorem 1.7 will span the algebra $\mathcal{H}$. Precisely, we define elements s_0, s_i $(1\le i\le t-1)$, s_t of G_t, together with Levi subgroups particularly relevant to those elements.

$$s_0=\begin{pmatrix} I_{(t-1)N} & & & & \\ & 0 & 0 & \sigma & \\ & 0 & I_{2k} & 0 & \\ & -{}^{\tau}\sigma^{-1} & 0 & 0 & \\ & & & & I_{(t-1)N}\end{pmatrix},\quad M^0=\begin{pmatrix}\ddots & & & & \\ & GL_N(F) & & & \\ & & G_1 & & \\ & & & GL_N(F) & \\ & & & & \ddots\end{pmatrix};$$

$$s_i=\begin{pmatrix} I_{(t-i-1)N} & & & & & & \\ & 0 & I_N & & & & \\ & I_N & 0 & & & & \\ & & & I_{2k+2(i-1)N} & & & \\ & & & & 0 & I_N & \\ & & & & I_N & 0 & \\ & & & & & & I_{(t-i-1)N}\end{pmatrix},\quad M^i=\begin{pmatrix} GL_{tN}(F) & & \\ & Sp_{2k}(F) & \\ & & GL_{tN}(F)\end{pmatrix};$$

$$s_t = \begin{pmatrix} 0 & 0 & -{}^\tau\sigma\, {}^\tau\varpi_E^{-1} \\ 0 & I_{2(t-1)N+2k} & 0 \\ \sigma^{-1}\varpi_E & 0 & 0 \end{pmatrix}, \quad M^t = \begin{pmatrix} \star & 0 & \star & 0 & \star \\ 0 & GL_{(t-1)N}(F) & 0 & 0 & 0 \\ \star & 0 & \star & 0 & \star \\ 0 & 0 & 0 & GL_{(t-1)N}(F) & 0 \\ \star & 0 & \star & 0 & \star \end{pmatrix} \simeq GL_{(t-1)N}(F) \times G_1.$$

The generator e_{s_j} of $\mathcal{H} = \mathcal{H}(G_t, \omega(t,r))$ attached to the Coxeter generator $\mathbf{s}_j$, $0 \le j \le t$, in Theorem 2.4, is an element of $\mathcal{H}$ with support $\Omega\ s_j\ \Omega$; indeed, easy representation-theoretic considerations imply that the double coset $\Omega\ s_j\ \Omega$ supports a one-dimensional subspace of $\mathcal{H}$. We need to produce such an element e_{s_j} that satisfies the quadratic relation $e_{s_j}^2 = a_{\mathbf{s}_j} e_{s_j} + b_{\mathbf{s}_j}\mathcal{I}$. Now, with a good choice of r ($r = t$ for s_t, $r = 2t$ for s_i, $i \le t-1$), the element s_j is Ω-positive in M^j (1.6) and there is an element ϵ_j in $\mathcal{H}(M^j, \omega_{|\Omega \cap M^j})$, with support $(\Omega \cap M^j) s_j (\Omega \cap M^j)$, that satisfies this quadratic relation. Indeed:

- If $1 \le j \le t-1$: $\mathcal{H}(M^j, \omega_{|\Omega\cap M^j}) \simeq \mathcal{H}(GL_{tN}(F), \Gamma(t,t)) \otimes \mathcal{H}(Sp_{2k}(F), \delta)$; there is an element $u_j \in \mathcal{H}(GL_{tN}(F), \Gamma(t,t))$, with support $\Gamma(t,t) \begin{pmatrix} I_{(t-j-1)N} & & & \\ & 0 & I_N & \\ & I_N & 0 & \\ & & & I_{(j-1)N} \end{pmatrix} \Gamma(t,t)$, such that $u_j^2 = (q_E^f - 1)u_j + q_E^f$ ([5]).
- If $j = 0$: $\mathcal{H}(M^0, \omega_{|\Omega\cap M^0}) \simeq \mathcal{H}(GL_N(F), \Gamma)^{t-1} \otimes \mathcal{H}(G_1, \omega(1,2))$; we use assumption 2.3.
- If $j = t$: $\mathcal{H}(M^t, \omega_{|\Omega\cap M^t}) \simeq \mathcal{H}(GL_{(t-1)N}(F), \Gamma(t-1,t-1)) \otimes \mathcal{H}(G_1, \omega(1,1))$; we use assumption 2.2.

Using the homomorphism $T^+ : \mathcal{H}(M^j, \omega_{|\Omega\cap M^j}) \hookrightarrow \mathcal{H}$ of 1.6 gives the result for $e_{s_j} = T^+(\epsilon_j)$.

Finally, to prove that the family $(e_{s_j})_{0\le j\le t}$ generates $\mathcal{H}$, we use the 'Bernstein presentation' given by Bushnell and Kutzko in [6] and [7].

3 An example in $Sp(12, F)$

We come back to our example in section 1.2 and notice that E/F is a biquadratic extension – that is, its Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and its norm subgroup is $N_{E/F}(E^\times) = F^{\times 2}$. We have a maximal simple stratum $[\mathfrak{A}_0, n_0, 0, \beta]$ in $M_4(F)$ and a simple character θ_0 of $H^1(\beta, \mathfrak{A}_0)$. Skipping to the notations of §2.3, we let $\sigma = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ and make the additional assumption that $\theta_0(x\bar{x}) = 1$ for all $x \in 1 + \mathfrak{p}_E$.

The involution defining the symplectic group is actually $x \mapsto \sigma\, {}^\tau x \sigma^{-1}$. Our stratum is thus a *skew* stratum and our simple character θ_0 is a *skew* simple character ([10]): the groups $H^1(\beta, \mathfrak{A}_0) = J^1(\beta, \mathfrak{A}_0)$ and $J(\beta, \mathfrak{A}_0)$ are stable under this involution, as is the character θ_0.

We let $G = GL(4, F)$, $\bar{G} = Sp(4, F)$, and for any subgroup K of G we let $\bar{K} = K \cap \bar{G}$. According to [10], Theorem 5.2, the restriction to $\bar{H}^1 = \bar{J}^1 = \left((1 + \mathfrak{p}_E) \cap \bar{G}\right)\left((1 + \mathfrak{P}_0^{\frac{n_0+1}{2}}) \cap \bar{G}\right)$ of the skew simple character θ_0 underlies supercuspidal representations of $Sp(4, F)$: for any extension δ of θ_0 to $\Delta = \bar{J}(\beta, \mathfrak{A}_0)$, the representation $\rho = \text{c-Ind}_\Delta^{Sp(4,F)} \delta$ is irreducible supercuspidal and the pair (Δ, δ) satisfies the assumptions in Theorem 2.4. We have here $\bar{\Delta} = \mathfrak{o}_E^\times \bar{H}^1$ so there are two such extensions δ.

On the other hand the same skew simple character θ_0 underlies self-dual supercuspidal representations of G – but it turns out that the character we must use here is not θ_0 but its square θ_0^2, attached to the skew simple stratum $[\mathfrak{A}_0, n_0, 0, 2\beta]$ (see [2], Lemma 4.3.1). Anyway, θ_0^2 has two extensions to a self-dual character γ of $\Gamma = J(\beta, \mathfrak{A}_0) = \mathfrak{o}_E^\times H^1$, i.e. satisfying $\gamma(x\bar{x}) = 1$ for all $x \in \mathfrak{o}_E^\times$; note that γ is either trivial or quadratic on $\mathfrak{o}_{E_0^\times}$. Now (Γ, γ) is a maximal simple type contained in a self-dual supercuspidal representation π of G and satisfying the properties in Theorem 2.1.

By arguments similar to [2] §3.3, one can show that

$$\Omega(1,1) = \begin{pmatrix} \Gamma & \mathfrak{J}^0 & \varpi_E^{-1}\mathfrak{H}^1\sigma \\ \mathfrak{H}^1 & \Delta & \mathfrak{J}^0\sigma \\ \sigma^{-1}\mathfrak{J}^0\varpi_E & \sigma^{-1}\mathfrak{H}^1 & {}^\tau\Gamma \end{pmatrix} \cap Sp(12, F)$$

$$\text{and} \quad \Omega(1,2) = \begin{pmatrix} \Gamma & \mathfrak{J}^0 & \mathfrak{J}^0\sigma \\ \mathfrak{H}^1 & \Delta & \mathfrak{J}^0\sigma \\ \sigma^{-1}\mathfrak{H}^1 & \sigma^{-1}\mathfrak{H}^1 & {}^\tau\Gamma \end{pmatrix} \cap Sp(12, F)$$

are subgroups holding decomposed pairs above $(\Gamma\times\Delta, \gamma\otimes\delta)$ with respect to P_1 (remember that $\Delta = \bar{\Gamma}$) and that assumptions 2.2 and 2.3 hold. The propagation theorem 2.4 thus applies, giving covers in $Sp\,(4(2t+1), F)$ attached to the inertial class $[GL(4, F)^{\times t} \times Sp(4, F), \pi^{\otimes t} \otimes \rho]_{Sp(4(2t+1),\,F)}$ and the structure of their Hecke algebra, provided we know the parameters of the $t = 1$ case.

In the present situation, we can compute the parameters (a_s, b_s) and (a_q, b_q), following the recipe in [1], §1.d; in particular we use the Haar measure on $Sp(12, F)$ giving $\Omega(1,1)$ and $\Omega(1,2)$ volume 1 and normalise e_s and e_q by $e_s(s) = 1$, $e_q(q) = 1$. Let L be the quadratic character of $\mathfrak{o}_{E_0}^\times$ and let G be the Gauss sum

$$G(x) = \sum_{z \in \mathfrak{o}_{E_0}^\times/1+\mathfrak{p}_{E_0}} L(z)\ \psi \circ \mathrm{tr}_{E_0/F}(zx), \quad x \in \mathfrak{o}_{E_0}^\times.$$

The residual field of E_0 has cardinality q_F^2 so -1 is a square in $k_{E_0}^\times$ and $G(x)^2 = q_F^2$; we write $G(x) = \epsilon(x) q_F$ with $\epsilon(x) = \pm 1$. We find:

$$b_s = q_F^2, \quad a_s = \delta(-1)(q_F^2 - 1), \quad b_q = q_F^8, \quad a_q = \begin{cases} 0 & \text{if } \gamma \text{ is trivial on } \mathfrak{o}_{E_0^\times}, \\ (-1)^{\frac{q+1}{2}}\, \epsilon(u)\, q_F^3\, (q_F^2 - 1) & \text{otherwise.} \end{cases} \tag{3.1}$$

The interest of this example lies in the fact that, according to the class of u mod $(\mathfrak{o}_{E_0}^\times)^2$, the representation π is either generic or non generic ([4]). When it is not, reducibilities for $I(\pi, t, \rho)$ are not attained by the usual methods; the construction of a cover above plus the computation of the Hecke algebra do, however, give those reducibilities. One can notice that, in the case studied above, reducibilities will be the same whether the representation is generic or not.

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Corinne Blondel
C.N.R.S. - Théorie des Groupes - Case 7012
Institut de Mathématiques de Jussieu
Université Paris 7
F-75251 Paris Cedex 05
France
blondel@math.jussieu.fr