$Sp(2N)$-covers for self-contragredient supercuspidal representations of $GL(N)$

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November 15, 2003

Let G be the group of F-points of a connected reductive algebraic group defined over F, a local non-archimedean field. The goal of the theory of types is the description of direct summands of the category $\mathcal{R}(G)$ of smooth complex representations of G as categories of modules over Hecke algebras.

More precisely, the Bernstein decomposition of this category states that it is the direct sum, over the set of inertial classes in G, of full subcategories $\mathcal{R}^{[M,\pi]}(G)$ attached to each inertial class. Recall that an inertial class in G is the equivalence class $[M,\pi]_G$ of a pair (M,π) made up of a F-Levi subgroup M of G and an irreducible supercuspidal representation π of M; the equivalence relation includes G-conjugacy and twisting of π by an unramified character of M. The subcategory $\mathcal{R}^{[M,\pi]}(G)$ consists of representations each of whose irreducible subquotients is a subquotient of a representation parabolically induced to G from an unramified twist of π.

Finding a type (J,λ) for this subcategory means finding a compact open subgroup J of G and a smooth irreducible representation λ of J such that the subcategory $\mathcal{R}^{[M,\pi]}(G)$ consists of representations generated by their isotypic component of type λ under J. If (J,λ) is a type for $[M,\pi]_G$, i.e. for $\mathcal{R}^{[M,\pi]}(G)$, this subcategory is then equivalent to the category of non-degenerate modules over the Hecke algebra $\mathcal{H}(G,J,\lambda)$ (for all this see [BK2]).

The problem of finding types in G naturally breaks into two pieces which are very different in nature. One is finding types for the inertial classes of supercuspidal representations of G. The other is finding types for inertial classes $[M,\pi]_G$ where M is a proper Levi subgroup of G. C. J. Bushnell and P. C. Kutzko in [BK2] have developped a method to address this second problem, based on the definition of *covers*.

We say that (J,λ) is a *G-cover of* (J_M,λ_M), an analogous pair in M, if there is an F-parabolic subgroup P of G with unipotent radical U and Levi decomposition $P=MU$ such that :

(i) (J,λ) is a *decomposed pair with respect to* (M,P), i.e.
  – $J=(J\cap U^-)(J\cap M)(J\cap U)$, where U^- is the unipotent radical of the parabolic subgroup P^- opposite of P relative to M, and
  – λ is trivial on $J\cap U^-$ and $J\cap U$;

(ii) $J\cap M=J_M$ and $\lambda|J_M\simeq\lambda_M$;

(iii) for any smooth irreducible representation (σ,V) of G, the restriction to V^λ of the Jacquet functor r_U is injective.

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Note that the definition in [BK2] requires those properties to hold for any such parabolic subgroup P; nonetheless it follows from [Bu2] that one may restrict the definition to just one parabolic subgroup.

C. J. Bushnell and P. C. Kutzko have shown that:

if (J_M, λ_M) *is a type for* $[M, \pi]_M$ *in* M *and if* (J, λ) *is a* G*-cover of* (J_M, λ_M)*, then* (J, λ) *is a type for* $[M, \pi]_G$ *in* G*.*

Let now $\bar{G}$ be $Sp_{2N}(F)$ where F has odd residual characteristic, let $\bar{P}$ be the Siegel parabolic subgroup, and let $\bar{M}$ be the Siegel Levi subgroup, which we identify with $GL_N(F)$ (see §I.1). Let π be an irreducible supercuspidal representation of $GL_N(F)$ and $(J_{\bar{M}}, \lambda_{\bar{M}})$ be a Bushnell-Kutzko type for $[GL_N(F), \pi]_{GL_N(F)}$ in $\bar{M}$. Observe that the non-trivial element s in $N_{\bar{G}}(\bar{M})/\bar{M}$ stabilizes the inertial class $[\bar{M}, \pi]_{\bar{G}}$ if and only if π and its contragredient representation are equivalent up to twisting by an unramified character of $GL_N(F)$ – yet, since any unramified character of $GL_N(F)$ is a square, π and its contragredient representation are in the same inertial class if and only if this class actually contains a self-contragredient representation.

If this is not the case, it should follow from [BK2], Theorem 12.1, that the Hecke algebra $\mathcal{H}(\bar{G}, J, \lambda)$ of a $\bar{G}$-cover (J, λ) of $(J_{\bar{M}}, \lambda_{\bar{M}})$ is commutative, isomorphic to $\mathcal{H}(\bar{M}, J_{\bar{M}}, \lambda_{\bar{M}})$, and the corresponding subcategories are equivalent. In any case, a recent result of A. Roche ([R], Theorem 3.1) states, in our present setting, that parabolic induction from $\mathcal{R}^{[\bar{M},\pi]}(\bar{M})$ to $\mathcal{R}^{[\bar{M},\pi]}(\bar{G})$ is an equivalence of categories if and only if s does not stabilize $[\bar{M}, \pi]_{\bar{G}}$.

Hence, although the question of existence of $\bar{G}$-covers is interesting in itself, the most interesting case is the case when π is *self-contragredient.* Indeed, given a $\bar{G}$-cover (J, λ) of $(J_{\bar{M}}, \lambda_{\bar{M}})$, one expects the description of $\mathcal{H}(\bar{G}, J, \lambda)$ to give insight into reducibility problems for parabolically induced representations in $\mathcal{R}^{[\bar{M},\pi]}(\bar{G})$ (see e.g. [BB2] for details in the case of $Sp_4(F)$; although, for this group, the results concerning reducibility were already known). In particular, obtaining $\bar{G}$-covers and their Hecke algebras for all such representations π should lead to an exact knowledge of the real numbers α such that the parabolically induced representation to $\bar{G}$ of the twisted representation $\pi \otimes |\det|^{\alpha}$ is reducible (those numbers are known to belong to $\{0, \pm 1/2)\}$ if $N > 1$, by the work of Shahidi [Sh1] [Sh2]).

We construct in this paper $\bar{G}$-covers for Bushnell-Kutzko types attached to inertial classes $[GL_N(F), \pi]_{GL_N(F)}$ where π is self-contragredient, which is the first step in the above program. We do not compute the corresponding Hecke algebras. The principle of the construction is to start with a well chosen $GL_{2N}(F)$-cover attached to the inertial class $[GL_N(F) \times GL_N(F), \pi \otimes \pi]_{GL_{2N}(F)}$ and then restrict it to $Sp_{2N}(F)$. In the process we need some strong properties of simple types attached to self-contragredient supercuspidal representations. We prove the following in part II (Corollary II.3):

Theorem 1. *Let* (Γ, γ) *be a maximal simple type (in the sense of* [BK1]*) in*

$GL_N(F)$ such that the corresponding inertial class contains a self-contragredient representation. Then either (Γ,γ) has level zero, or the simple character θ_0 attached to γ can be attached to a simple stratum $[\mathfrak{A}_0,n_0,0,\beta]$ in $M_N(F)$ with the following properties:

1 –The field $F[\beta]$ is a quadratic extension of $F[\beta^2]$ (in particular N is even).

2 –Let ν be an element in $\mathfrak{A}_0^\times$ realizing the Galois conjugation of $F[\beta]$ over $F[\beta^2]$. The character θ_0 satisfies $\theta_0(\nu x\nu^{-1})=\theta_0(x^{-1})$ (x in $H^1(\beta,\mathfrak{A}_0)$).

This property of self-contragredient supercuspidal representations was known in the tame case: such a representation is then attached to an admissible character of a maximal field extension contained in $GL_N(F)$ and Adler [A] proved that this character is trivial on the group of norms relative to a quadratic subextension (or $N=1$ and the character is quadratic). In *loc. cit.* Adler also gives a full description of level zero self-contragredient supercuspidal representations (which exist only if N is even or $N=1$).

Let $g\mapsto {}^{\tau}g$ be the transposition relative to the anti-diagonal. Theorem 1 essentially amounts to saying that, for a suitable order $\mathfrak{A}$ in $M_{2N}(F)$ related to the order $\mathfrak{A}_0$ above, the stratum $\Lambda=[\mathfrak{A},2n_0,0,\begin{pmatrix}\beta & 0\\ 0 & -{}^{\tau}\beta\end{pmatrix}]$ in $M_{2N}(F)$ is simple. Let $G=GL_{2N}(F)$ and P be the maximal parabolic subgroup in G of upper block-diagonal matrices with Levi subgroup M isomorphic to $GL_N(F)\times GL_N(F)$. The process in [BK1], §7.2, provides us (Corollary II.2) with a G-cover (J_P,λ_P) of $(\Gamma\times{}^{\tau}\Gamma,\gamma\otimes\gamma^*)$, with $\gamma^*(x)=\gamma({}^{\tau}x^{-1})$, attached to the stratum Λ. It will lead us (Theorem III.1) to the cover we are looking for:

Theorem 2. *Let (Γ,γ) be as in Theorem 1 and (J_P,λ_P) be as above. The unique representation ω of $\Omega=J_P\cap\bar{G}$ such that (Ω,ω) is a decomposed pair with respect to $(\bar{M},\bar{P})$ with $\Omega\cap\bar{M}=\Gamma$ and $\omega|\Omega\cap\bar{M}=\gamma$ is a $\bar{G}$-cover of (Γ,γ).*

In the case when (Γ,γ) has level zero, the cover given by Theorem 2 has previously been obtained by L. Morris in [M2]. Also recall that Ju-Lee Kim [Ki] has constructed a set of types in classical groups, under the assumption that the characteristic of F is 0 and the residual characteristic is "big enough". The types in her work that correspond to our present setting need not be the same as those above, in particular they may not be $\bar{G}$-covers (see [BB1]).

The main goal of this paper is Theorem 2, while Theorem 1 appears as a necessary tool. In part I, we establish notation and explain the basic mechanism allowing one to build decomposed pairs in $Sp_{2N}(F)$ from the restriction of decomposed pairs in $GL_{2N}(F)$. In part II, we detail the structure of the maximal simple type (Γ,γ) and of suitable $GL_{2N}(F)$-covers of $(\Gamma\times{}^{\tau}\Gamma,\gamma\otimes\gamma^*)$. This leads us to a proof of Theorem 1, as a Corollary of Theorem II.3. In part III, we first build a periodic infinite sequence (Ω_i,ω_i) of decomposed pairs in $\bar{G}$, with $\Omega_i\cap\bar{M}=\Gamma$ and $\omega_i|\Omega_i\cap\bar{M}=\gamma$, then we show that certain sufficient criteria for this sequence to be a sequence of $\bar{G}$-covers are satisfied. Part IV is devoted to the proof of an intertwining property (Proposition IV.1) that has been assumed in part III.

From paragraph II.2 on, we assume that the order $\mathfrak{A}_0$ is standard. Any maximal simple type (Γ, γ) in $GL_N(F)$ is conjugate to a maximal simple type satisfying this property, hence theorems 1 and 2 hold without this restriction (see the remarks after Corollary II.3 and Theorem III.1).

I would like to thank here Bertrand Lemaire for helpful conversations, David Manderscheid and Shaun Stevens for linguistic comments, Shaun Stevens for his careful reading of an earlier draft of the manuscript and pointing out some mistakes, and Guy Henniart for his constant advice.

I - Framework and basic tool

I.1 - Notations

Let F be a non-archimedean local field of residual characteristic p different from 2, let $\mathfrak{o}_F$ or $\mathfrak{o}$ be its ring of integers, $\mathfrak{p}_F$ or $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}_F$, ϖ_F or ϖ a uniformizing element and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue class field, of cardinality q_F. We will be working with the group $G = GL_{2N}(F)$ and its subgroup $\bar{G} = Sp_{2N}(F)$ viewed as the symplectic group of the F-vector space $V = F^{2N}$ equipped with the symplectic form $<\ ,\ >$ with matrix $\begin{pmatrix} 0 & -w_N \\ w_N & 0 \end{pmatrix}$ in the canonical basis $\{e_1, \ldots, e_{2N}\}$, where $w_N = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \ldots & 0 \end{pmatrix}$. Most matrices written below will be 2×2 block matrices with $N \times N$ blocks. Hence:

$$Sp_{2N}(F) = \{g \in GL_{2N}(F)\ ;\ \begin{pmatrix} 0 & -w_N \\ w_N & 0 \end{pmatrix}\ {}^t g^{-1} \begin{pmatrix} 0 & w_N \\ -w_N & 0 \end{pmatrix} = g\}.$$

We will let $X \mapsto {}^T X$ denote the corresponding involution on $M_{2N}(F)$:

$${}^T X = \begin{pmatrix} 0 & -w_N \\ w_N & 0 \end{pmatrix}\ {}^t X \begin{pmatrix} 0 & w_N \\ -w_N & 0 \end{pmatrix}\ ;\ {}^T\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} {}^\tau D & -{}^\tau B \\ -{}^\tau C & {}^\tau A \end{pmatrix},$$

where $g \mapsto {}^\tau g$, $g \in GL_i(F)$, is the transposition relative to the antidiagonal; in other words: ${}^{t\tau} g = {}^{\tau t} g = w_i\ g\ w_i$.

For any subgroup H of G, we put $\bar{H} = H \cap \bar{G}$. Let P be the stabilizer of the subspace $< e_1, \ldots, e_N >$ in F^{2N}, a parabolic subgroup of G. Let U be its unipotent radical and let M be the Levi factor of P consisting of matrices stabilizing $< e_{N+1}, \ldots, e_{2N} >$. We let P^- be the parabolic subgroup of G opposite of P relative to M and we let U^- be its unipotent radical. We have

$$M = \{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}\ ;\ g_1, g_2 \in GL_N(F)\},\ U = \{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\ ;\ X \in M_N(F)\},$$

$$\bar{M} = \{ \begin{pmatrix} g & 0 \\ 0 & {}^\tau g^{-1} \end{pmatrix}\ ;\ g \in GL_N(F)\},\ \bar{U} = \{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\ ;\ X \in M_N(F), X = {}^\tau X\}.$$

We will accordingly identify M with $GL_N(F) \times GL_N(F)$ and $\bar{M}$ with $GL_N(F)$, the latter through the isomorphism i from $GL_N(F)$ to $\bar{M}$ defined by

$$i(g) = \begin{pmatrix} g & 0 \\ 0 & {}^\tau g^{-1} \end{pmatrix}, \quad g \in GL_N(F).$$

If μ is a representation of a subgroup H of $GL_N(F)$, $i(\mu)$ will be the representation of $i(H)$ defined by $i(\mu)(i(g)) = \mu(g)$ $(g \in H)$.

Let H be a compact open subgroup of G and ρ a smooth irreducible representation of H. The G-intertwining of ρ is:

$$I_G(\rho) = I_G(\rho, H) = \{g \in G \ / \ \mathrm{Hom}_{H \cap H^g} \ (\rho, \rho^g) \neq \{0\}\}.$$

For any g in G we define the intertwining space of ρ at g to be

$$I_g(\rho) = I_g(\rho, H) = \mathrm{Hom}_{H \cap H^g} \ (\rho, \rho^g).$$

I.2 - Some decomposed pairs in $Sp_{2N}(F)$

Let π be a smooth irreducible self-contragredient supercuspidal representation of $GL_N(F)$, hence viewed as a representation of $\bar{M}$; likewise $\pi \otimes \pi$ is viewed as a representation of M. We want to find *types* in G and $\bar{G}$ for the inertial classes attached to these representations, and we want those types to be a *G-cover* and a *$\bar{G}$-cover* respectively, of types attached to $\pi \otimes \pi$ in M and to $i(\pi)$ in $\bar{M}$. The situation in G has been settled by Bushnell and Kutzko in [BK1], as we will recall in part II. Indeed we will use the types build in *loc. cit.* to construct the $\bar{G}$-covers we are looking for: the process will involve a suitable conjugation followed by a restriction to $Sp_{2N}(F)$. The basic mechanism is the following:

Proposition. *Let Γ be a compact open subgroup of $GL_N(F)$ and let γ be a smooth finite-dimensional complex representation of Γ. Let γ^* be the representation of ${}^\tau\Gamma$ defined by: $\gamma^*(g) = \gamma({}^\tau g^{-1})$, $g \in {}^\tau\Gamma$.*

If (J, λ) is a decomposed pair in G relative to (M, P) such that $J \cap M = \Gamma \times {}^\tau\Gamma$ and $\lambda_{|J \cap M} \simeq \gamma \otimes \gamma^*$, then $\bar{J} \cap \bar{M} = i(\Gamma)$ and there exists a unique representation $\widehat{\lambda}$ of $\bar{J}$ such that $(\bar{J}, \widehat{\lambda})$ is a decomposed pair in $\bar{G}$ relative to $(\bar{M}, \bar{P})$ with

$$\widehat{\lambda}_{|i(\Gamma)} = i(\gamma).$$

The representation $\bar{\lambda} = \lambda_{|\bar{J}}$ of $\bar{J}$ is isomorphic to $\widehat{\lambda} \otimes \widehat{\lambda}$.

Proof. We recall the following useful fact: let $x \in U^-$, $m \in M$ and $y \in U$ be such that their product xmy belongs to $\bar{G}$; then x, m and y already belong to $\bar{G}$. Indeed the involution defining $Sp_{2N}(F)$ stabilizes U^-, M and U.

Hence taking intersections with $\bar{G}$ provides a decomposed pair $(\bar{J}, \bar{\lambda})$ in $\bar{G}$ relative to $(\bar{M}, \bar{P})$. We have $\bar{J} \cap \bar{M} = \overline{J \cap M} = \overline{\Gamma \times {}^\tau\Gamma} = i(\Gamma)$ and

$$\bar{\lambda}\left(\begin{pmatrix} g & 0 \\ 0 & {}^\tau g^{-1} \end{pmatrix}\right) \simeq \gamma(g) \otimes \gamma^*({}^\tau g^{-1}) = \gamma(g) \otimes \gamma(g), \quad g \in \Gamma.$$

In particular: $(\bar{J} \cap \bar{U})(\bar{J} \cap \bar{U}^-) \subset (\bar{J} \cap \bar{U}^-) \ T \ (\bar{J} \cap \bar{U})$, with

$$T = \{ \begin{pmatrix} g & 0 \\ 0 & {}^\tau g^{-1} \end{pmatrix} \ ; \ g \in \Gamma, \ \gamma(g) \otimes \gamma(g) = I\}.$$

To go from there to the decomposed pair we are looking for, it is enough ([Bl1], Lemme 1) to prove a similar inclusion with T replaced by

$$T' = \{ \begin{pmatrix} g & 0 \\ 0 & {}^{\tau}g^{-1} \end{pmatrix} \ ; \ g \in \Gamma, \ \gamma(g) = I \}.$$

Indeed, the representation $\widehat{\lambda}$ will then be uniquely defined by the condition $\widehat{\lambda}_{|i(\Gamma)} = i(\gamma)$, plus the fact that it is trivial on $\bar{J} \cap \bar{U}$ and $\bar{J} \cap \bar{U}^-$.

Now the subgroup of $\bar{J}$ generated by $\bar{J} \cap \bar{U}$ and $\bar{J} \cap \bar{U}^-$ is a pro-p-group ([Bl3]) and so is its intersection with $\bar{M}$, hence we can replace T in the above inclusion by a suitable pro-p-subgroup of T. All we have to show is :

Lemma. *Let γ be a finite-dimensional smooth complex representation of a pro-p-group H, with p odd. If the representation $\gamma \otimes \gamma$ of H is trivial, so is γ.*

Indeed γ factors through a finite quotient of H, so it is unitarisable. In particular each operator $\gamma(h)$, $h \in H$, is diagonalisable, and the triviality of $\gamma \otimes \gamma$ implies that any product of two eigenvalues of $\gamma(h)$ is equal to 1. Hence $\gamma(h)$ is a scalar operator, namely $\pm I$. Now $-I$ is impossible for p odd. □

If the pair (Γ, γ) in the proposition is a maximal simple type in $GL_N(F)$ and the pair (J, λ) is a G-cover of $(\Gamma \times {}^{\tau}\Gamma, \gamma \otimes \gamma^*)$, one would like to know whether or not, under relevant conditions on (J, λ), the associated pair $(\bar{J}, \widehat{\lambda})$ is a $\bar{G}$-cover of (Γ, γ). We do address this question here in the special case of a pair (Γ, γ) attached to the inertial class of a self-contragredient representation ; the object of part II is to use Bushnell and Kutzko's simple types to produce in this context a G-cover (J, λ) with suitable properties for that purpose.

II - "Self-contragredient" $GL_{2N}(F)$-covers

II.1 - Bushnell and Kutzko's $GL_{2N}(F)$-covers

All references in this paragraph are to [BK1], any undefined notion or notation comes from [BK1].

Let π be an irreducible supercuspidal representation of $GL_N(F)$ and (Γ, γ) a maximal simple type in $GL_N(F)$ attached to the inertial class of π. From Definition (5.5.10) – where we do treat (b) as a special case of (a) – and Theorems (6.2.1), (6.2.2), the pair $(\Gamma, \gamma) = (J(\beta, \mathfrak{A}_0), \lambda(\beta, \mathfrak{A}_0))$ comes equipped with the following data:

(i) A principal $\mathfrak{o}_F$-order $\mathfrak{A}_0$ and a simple stratum $[\mathfrak{A}_0, n_0, 0, \beta]$ in $M_N(F)$; in particular $E = F[\beta]$ is a field extension of F.
We let $\mathfrak{P}_0$ be the radical of $\mathfrak{A}_0$ and B_0 be the commutant of E in $M_N(F)$. Then $\mathfrak{B}_0 = \mathfrak{A}_0 \cap B_0$ is a maximal $\mathfrak{o}_E$-order with radical $\mathfrak{Q}_0 = \mathfrak{P}_0 \cap B_0$.

(ii) A simple character $\theta_0 \in \mathcal{C}(\mathfrak{A}_0, 0, \beta)$ and a β-extension κ_0 to $J(\beta, \mathfrak{A}_0)$ of the unique irreducible representation η_0 of $J^1(\beta, \mathfrak{A}_0)$ which contains θ_0.

(iii) An irreducible cuspidal representation σ_0 of $GL(f, k_E)$ inflated to Γ through:

$$J(\beta, \mathfrak{A}_0)/J^1(\beta, \mathfrak{A}_0) \simeq \mathbf{U}(\mathfrak{B}_0)/\mathbf{U}^1(\mathfrak{B}_0) \simeq GL(f, k_E), \ f = N/[E : F].$$

We now write $M_{2N}(F)$ as 2×2 block matrices with entries in $M_N(F)$. This amounts to a decomposition of the underlying vector space $V = F^{2N}$, written as column matrices, as a direct sum $V = V^{(1)} \oplus V^{(2)}$ with $V^{(1)}$ (resp. $V^{(2)}$) the subspace of column matrices having their first (resp. last) N entries equal to 0.

Let $(\Lambda_{0,i})_{i\in\mathbb{Z}}$ be the lattice chain in F^N associated to the order $\mathfrak{A}_0$. It determines lattice chains $(\Lambda_{0,i}^{(j)})_{i\in\mathbb{Z}}$ in $V^{(j)}$, $j = 1, 2$, under the natural identification of $V^{(j)}$ with F^N. Let $(\Lambda_i)_{i\in\mathbb{Z}}$ be the lattice chain in V defined by:

$$\Lambda_{2i} = \Lambda_{0,i}^{(1)} \oplus \Lambda_{0,i}^{(2)}, \quad \Lambda_{2i+1} = \Lambda_{0,i+1}^{(1)} \oplus \Lambda_{0,i}^{(2)} \quad (i \in \mathbb{Z}).$$

The corresponding principal order in $M_{2N}(F)$ is $\mathfrak{A} = \begin{pmatrix} \mathfrak{A}_0 & \mathfrak{A}_0 \\ \mathfrak{P}_0 & \mathfrak{A}_0 \end{pmatrix}$.

We identify E with its block-diagonal image in $M_{2N}(F)$, hence we also write β for $\begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$. We write B for the commutant of E in $M_{2N}(F)$ and define the $\mathfrak{o}_E$-order $\mathfrak{B} = \mathfrak{A} \cap B$, with radical $\mathfrak{Q} = \mathfrak{P} \cap B$ (where $\mathfrak{P}$ is the radical of $\mathfrak{A}$).

Since the period of $(\Lambda_i)_{i\in\mathbb{Z}}$ is twice the period of $(\Lambda_{0,i})_{i\in\mathbb{Z}}$, we get (see 1.2.11, 1.4.13, 1.2.4):

Fact. $[\mathfrak{A}, 2n_0, 0, \beta]$ *is a simple stratum in* $M_{2N}(F)$ *and all assumptions in (7.1.11), (7.2.1) are satisfied, with* $t = e(\mathfrak{B}|\mathfrak{o}_E) = 2$.

The following proposition can be regarded as obvious: it is a paraphrase of [BK1], §7. We state it to fix notations and make references easy, and give a sketch of proof as a matter of conscientiousness. The groups P, U, M are defined in (7.1.13) or equivalently in §I.1 above.

Proposition. *There exists a unique representation* λ *of* $J = J(\beta, \mathfrak{A})$ *which is a simple type with the following property.*

Let λ_P *denote the natural representation of* $J_P = (J \cap P)H^1(\beta, \mathfrak{A})$ *on the space of* $J^1(\beta, \mathfrak{A}) \cap U$*-fixed vectors in* λ. *The pair* (J_P, λ_P) *is a decomposed pair in* G *relative to* (M, P) *with :*

$$J_P \cap M = J \cap M = \Gamma \times \Gamma \quad \textit{and} \quad (\lambda_P)_{|J\cap M} = \gamma \otimes \gamma.$$

Proof. Indeed this is Theorem (7.2.17) in [BK1], except that we want an actual equality between representations instead of an equivalence.

From (7.1.16): $J \cap M = J(\beta, \mathfrak{A}^{(1)}) \times J(\beta, \mathfrak{A}^{(2)})$; but we have arranged $\mathfrak{A}$ so that $\mathfrak{A}^{(1)} = \mathfrak{A}^{(2)} = \mathfrak{A}_0$, hence $J(\beta, \mathfrak{A}^{(i)}) = \Gamma$.

If β belongs to $\mathfrak{o}_F$ we just note that $J = J_P$ and $J \cap M = \mathbf{U}(\mathfrak{A}_0) \times \mathbf{U}(\mathfrak{A}_0)$; we take the representation $\sigma_0 \otimes \sigma_0$ there.

We now assume that $\beta \notin \mathfrak{o}_F$. From (7.1.19), the restriction to $H^1 = H^1(\beta, \mathfrak{A})$ of any simple type λ is a multiple of a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ and the pair (H^1, θ) is a decomposed pair in G relative to (M, P), satisfying: $H^1 \cap M = H^1(\beta, \mathfrak{A}^{(1)}) \times H^1(\beta, \mathfrak{A}^{(2)})$ and $\theta_{|H^1\cap M} = \theta^{(1)} \otimes \theta^{(2)}$, where $\theta^{(i)} \in \mathcal{C}(\mathfrak{A}^{(i)}, 0, \beta)$ is the image of θ under the bijection $\mathcal{C}(\mathfrak{A}^{(i)},0,\beta) \xrightarrow{\approx} \mathcal{C}(\mathfrak{A},0,\beta)$ given by Theorem (3.6.14). Since the family of bijections given by (3.6.14) is unique and $\mathfrak{A}^{(1)} = \mathfrak{A}^{(2)} = \mathfrak{A}_0$, we must have $\theta^{(1)} = \theta^{(2)}$, and from (3.6.14) there is a unique $\theta \in \mathcal{C}(\mathfrak{A},0,\beta)$ such that $\theta_{|H^1\cap M} = \theta_0 \otimes \theta_0$.

With this choice of θ, the next step towards λ is the choice of a β-extension κ. From (7.2.5), (7.2.15), (7.2.16), it has the form $\kappa = \mathrm{Ind}_{J_P}^{J} \kappa_P$ where again the pair (J_P, κ_P) is decomposed and $(\kappa_P)_{|J\cap M} = \kappa^{(1)} \otimes \kappa^{(2)}$, both being β-extensions of η_0. Since $(\kappa_P)_{|J\cap M}$ is normalized by $\begin{pmatrix} 0 & I_N \\ \varpi_E & 0 \end{pmatrix}$ (7.2.15) and ϖ_E intertwines $\kappa^{(i)}$, we have $\kappa^{(1)} \simeq \kappa^{(2)} \simeq \kappa_0 \otimes \chi \circ \det_{B_0}$ in the notation of (5.2.2). Since $\kappa \otimes \chi^{-1} \circ \det_B$ is another β-extension we may pick κ in the first place so that $(\kappa_P)_{|J\cap M} = \kappa_0 \otimes \kappa_0$.

All we have to do now is to tensor κ with $\sigma_0 \otimes \sigma_0$ as before (7.2.17). $\square$

Corollary. *The pair (J_P, λ_P) is a G-cover of the pair $(\Gamma \times \Gamma, \gamma \otimes \gamma)$ in M. The pair (J_P, λ_P) is a type in G attached to the inertial class $[M, \pi \otimes \pi]_G$.*

The first assertion follows from (7.3.2) and the results in [BK2, §7], the second from Theorem 8.3 in [BK2]. Note that by symmetry – see (7.1.13) – this also holds for the pair (J_{P^-}, λ_{P^-}), with $J_{P^-} = (J \cap P^-)\ H^1$.

Before turning to the case of self-contragredient supercuspidals in the next paragraphs, let us fix some more notations and write down some properties that will be used later on ; they all derive from [BK1, §3.1].

Let $\mathbb{U} = \begin{pmatrix} 0 & M_N(F) \\ 0 & 0 \end{pmatrix}$ and $\mathbb{U}^- = \begin{pmatrix} 0 & 0 \\ M_N(F) & 0 \end{pmatrix}$. We will identify $\mathbb{U}^-$ with $M_N(F)$ through $\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \mapsto X$ and $\mathbb{U}$ with $M_N(F)$ through $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \mapsto X$. We also use the isomorphisms i^+ and i^- from $M_N(F)$ to $U = 1 + \mathbb{U}$ and $U^- = 1 + \mathbb{U}^-$ respectively, defined by

$$i^+(X) = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}, \qquad i^-(X) = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}.$$

Write $J^1 = J^1(\beta,\mathfrak{A})$, $\mathfrak{J}^1 = \mathfrak{J}^1(\beta,\mathfrak{A})$, $\mathfrak{J}^1_- = \mathfrak{J}^1 \cap \mathbb{U}^-$, $\mathfrak{J}^1_+ = \mathfrak{J}^1 \cap \mathbb{U}$ (both viewed as lattices in $M_N(F)$), and similarly for H^1, $\mathfrak{H}^1$. From [BK1, §3.1, 7.1] we have :

$$\begin{aligned} J^1 = 1 + \mathfrak{J}^1 &= i^-(\mathfrak{J}^1_-)\ (J^1 \cap M)\ i^+(\mathfrak{J}^1_+), \\ H^1 = 1 + \mathfrak{H}^1 &= i^-(\mathfrak{H}^1_-)\ (H^1 \cap M)\ i^+(\mathfrak{H}^1_+). \end{aligned}$$

The lattices $\mathfrak{H}^1$ and $\mathfrak{J}^1$ are invariant under conjugation by $\mathfrak{K}(\mathfrak{B})$, hence by $\begin{pmatrix} 0 & I_N \\ \varpi_E & 0 \end{pmatrix}$. Hence

$$\mathfrak{J}^1_- = \varpi_E \mathfrak{J}^1_+ = \mathfrak{J}^1_+ \varpi_E\ ; \quad \mathfrak{H}^1_- = \varpi_E \mathfrak{H}^1_+ = \mathfrak{H}^1_+ \varpi_E.$$

Those lattices in $M_N(F)$ also satisfy :

$$\varpi_E \mathfrak{J}^1_- \subset \mathfrak{H}^1_- \subset \mathfrak{J}^1_- \subset \mathfrak{H}^1_+ \subset \mathfrak{J}^1_+ \subset \varpi_E^{-1} \mathfrak{H}^1_+.$$

II.2 - Intertwining properties in the self-contragredient case

Let $(\Gamma,\gamma) = (J(\beta,\mathfrak{A}_0),\lambda(\beta,\mathfrak{A}_0))$ be a maximal simple type in $GL_N(F)$ as above, attached to the inertial class of π. We want to use the decomposed pair (J_P,λ_P) given by Proposition II.1 to produce a decomposed pair in $\bar{G}$ through the process described in Proposition I.2. This is easy if Γ is equal to ${}^\tau\Gamma$ and γ^* equivalent to γ, which implies that π and its contragredient representation belong to the same inertial class. We actually want to show that the converse is true up to conjugacy.

Proposition. *Let (Γ,γ) be a maximal simple type in $GL_N(F)$ such that the corresponding inertial class of irreducible supercuspidal representations of $GL_N(F)$ contains a self-contragredient representation π. We keep the notation in §II.1 and assume the order $\mathfrak{A}_0$ is standard.*

(i) There exists σ in $\mathbf{U}(\mathfrak{A}_0)$ such that Γ is stable under $\tilde{\sigma} : x \mapsto \sigma\,{}^\tau x^{-1}\sigma^{-1}$, and γ is equivalent to $\gamma\circ\tilde{\sigma}$.

(ii) Such an element σ is unique up to left multiplication by Γ. It satisfies:

(a) $\sigma\,{}^\tau\sigma^{-1} \in \Gamma$ and $\varpi_E^{-1}\sigma\,{}^\tau\varpi_E\,{}^\tau\sigma^{-1} \in \Gamma$.

(b) The map $\tilde{\sigma}$ stabilizes $H^1(\beta,\mathfrak{A}_0)$ and $J^1(\beta,\mathfrak{A}_0)$ and we have: $\theta_0 = \theta_0\circ\tilde{\sigma}$.

(c) The lattices $\mathfrak{J}^1_+$, $\mathfrak{J}^1_-$, $\mathfrak{H}^1_+$ and $\mathfrak{H}^1_-$ in $M_N(F)$ defined in II.1 are stable under $X \mapsto \sigma\,{}^\tau X\sigma^{-1}$.

Example. Assume $N = 2$ and E is a quadratic extension of F generated by an element of matrix form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$. Then we may take $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; here conjugation by σ realizes the conjugation of E over F and σ plays an explicit part in the construction of $\bar{G}$-covers in [BB1], [BB2].

Proof. We call here *standard* an order whose matrix form is given by [BK1], (2.5.1) or [Bu1], (1.9). Note that we can always conjugate (Γ,γ) into a maximal simple type whose associated order is standard. The property we need here is that $\mathfrak{A}_0$, being standard and principal, is stable under the map $x \mapsto {}^\tau x$.

(i) By a theorem of Gelfand and Kazhdan ([GK], Theorem 2), the contragredient representation of π is equivalent to the representation π^* defined by $\pi^*(x) = \pi({}^\tau x^{-1})$. Since $\mathfrak{A}_0$ is stable under τ, the automorphism $x \mapsto {}^\tau x^{-1}$ transforms the pair $(\Gamma,\gamma) = (J(\beta,\mathfrak{A}_0),\lambda(\beta,\mathfrak{A}_0))$ into $(\Gamma^*,\gamma^*) = (J(-{}^\tau\beta,\mathfrak{A}_0),\lambda(-{}^\tau\beta,\mathfrak{A}_0))$, a maximal simple type underlaid by the simple stratum $[\mathfrak{A}_0, n_0, 0, -{}^\tau\beta]$. Since π^* is equivalent to π, those two types, (Γ,γ) and (Γ^*,γ^*), intertwine in $GL_N(F)$ and we may use [BK1], Theorem (5.7.1), to derive that they are conjugate in $GL_N(F)$. Since the two types are associated to the same order $\mathfrak{A}_0$, the proof of *loc. cit.* actually tells us more: indeed it says that there is an element σ in $\mathbf{U}(\mathfrak{A}_0)$ that conjugates the simple character $\theta_0 \in \mathcal{C}(\mathfrak{A}_0,0,\beta)$ into $\theta_0^* = \theta_0\circ(x \mapsto {}^\tau x^{-1}) \in \mathcal{C}(\mathfrak{A}_0,0,-{}^\tau\beta)$, and that, eventually, that same element σ conjugates (Γ,γ) into (Γ^*,γ') where γ' is equivalent to γ^* (if γ has level 0, it says that γ and γ^* are equivalent).

(ii) Let σ_1 be another such element. The automorphism $\tilde{\sigma}_1 \circ \tilde{\sigma}$ normalizes (Γ, γ) and is a conjugation by $\sigma_1 \, {}^\tau\sigma^{-1}$ that must therefore belong to $E^\times \Gamma \cap \mathbf{U}(\mathfrak{A}_0) = \Gamma$ ([BK1], (6.2.2)). Then the first part in (a) follows whence $\sigma_1 \in \Gamma \, {}^\tau\sigma = \Gamma\sigma$.

For the last part in (a), we use the same argument: the element $\sigma_1 = \varpi_E^{-1}\sigma$ also satisfies the conditions in (i) – except that it does not belong to $\mathbf{U}(\mathfrak{A}_0)$ – because ϖ_E normalizes (Γ, γ), and the determinant of $\sigma_1({}^\tau\sigma_1)^{-1}$ is 1, whence the result.

Now, for (b) we only have to note that the element σ produced in the proof of (i) satisfies the required properties.

The first step in the proof of (c) is the description of $\mathfrak{J}^1(\beta, \mathfrak{A})$ and $\mathfrak{H}^1(\beta, \mathfrak{A})$ in terms of the lattices in $GL_N(F)$ attached to $[\mathfrak{A}_0, n_0, 0, \beta]$, as in [BK1], (7.1.12). Since the given lattices in $M_{2N}(F)$ are the direct sums of their intersections with the $\mathrm{Hom}_F(V^{(i)}, V^{(j)})$ we use the corresponding block-matrix notation.

Lemma. *For any non-negative integer k, we have:*

$$\mathfrak{H}^k(\beta, \mathfrak{A}) = \begin{pmatrix} \mathfrak{H}^{[\frac{k+1}{2}]}(\beta, \mathfrak{A}_0) & \mathfrak{J}^{[\frac{k}{2}]}(\beta, \mathfrak{A}_0) \\ \varpi_E \mathfrak{J}^{[\frac{k}{2}]}(\beta, \mathfrak{A}_0) & \mathfrak{H}^{[\frac{k+1}{2}]}(\beta, \mathfrak{A}_0) \end{pmatrix} \quad \textit{and}$$
$$\mathfrak{J}^k(\beta, \mathfrak{A}) = \begin{pmatrix} \mathfrak{J}^{[\frac{k+1}{2}]}(\beta, \mathfrak{A}_0) & \varpi_E^{-1} \mathfrak{H}^{[\frac{k}{2}]+1}(\beta, \mathfrak{A}_0) \\ \mathfrak{H}^{[\frac{k}{2}]+1}(\beta, \mathfrak{A}_0) & \mathfrak{J}^{[\frac{k+1}{2}]}(\beta, \mathfrak{A}_0) \end{pmatrix}.$$

Proof. The diagonal blocks are already described in [BK1] Proposition (7.1.12), (iii). Since the lattices $\mathfrak{H}^k(\beta, \mathfrak{A})$, $\mathfrak{J}^k(\beta, \mathfrak{A})$ are invariant under conjugation by $\begin{pmatrix} 0 & I_N \\ \varpi_E & 0 \end{pmatrix} \in \mathfrak{K}(\mathfrak{B})$, all we have to show is $\mathfrak{H}^k(\beta, \mathfrak{A}) \cap \mathbb{N} = \mathfrak{J}^{[\frac{k}{2}]}(\beta, \mathfrak{A}_0)$ and $\mathfrak{J}^k(\beta, \mathfrak{A}) \cap \mathbb{N}^- = \mathfrak{H}^{[\frac{k}{2}]+1}(\beta, \mathfrak{A}_0)$. The proof of this goes exactly as in *loc. cit.*: the equality is first checked for β a minimal element, then obtained by induction along β. □

Since $\mathfrak{J}^1_+ = \varpi_E^{-1}\mathfrak{J}^1_-$ and $\mathfrak{H}^1_- = \varpi_E \mathfrak{H}^1_+$ it is easy, using property (a), to check that $\mathfrak{J}^1_+$ and $\mathfrak{H}^1_-$ are stable under the map $\Sigma : X \mapsto \sigma \, {}^\tau X \sigma^{-1}$ if $\mathfrak{J}^1_-$ and $\mathfrak{H}^1_+$ are. From the lemma we have $\mathfrak{J}^1_- = \mathfrak{H}^1(\beta, \mathfrak{A}_0)$ and $\mathfrak{H}^1_+ = \mathfrak{J}(\beta, \mathfrak{A}_0)$.

We know from (b) that Σ stabilizes $H^1(\beta, \mathfrak{A}_0) = 1 + \mathfrak{H}^1(\beta, \mathfrak{A}_0)$ hence $\mathfrak{H}^1(\beta, \mathfrak{A}_0)$.

We have $\mathfrak{J}(\beta, \mathfrak{A}_0) = \mathfrak{B}_0 + \mathfrak{J}^1(\beta, \mathfrak{A}_0)$ ([BK1] (3.1.8)) and $\mathfrak{J}^1(\beta, \mathfrak{A}_0)$ is stable under Σ by (b), so we have to show that $\Sigma(\mathfrak{B}_0)$ is contained in $\mathfrak{J}(\beta, \mathfrak{A}_0)$. We know that $\Sigma(\mathfrak{Q}_0) = \Sigma(\mathfrak{B}_0 \cap \mathfrak{P}_0)$ is contained in $\mathfrak{J}^1(\beta, \mathfrak{A}_0)$. Since $J(\beta, \mathfrak{A}_0) = \Gamma$ is invariant under $\tilde{\sigma}$ we also know that $\Sigma(\mathfrak{B}_0^\times)$ is contained in $J(\beta, \mathfrak{A}_0)$. Our claim then follows from the fact that $\mathfrak{B}_0$ is the $\mathfrak{o}_E$-linear span of $\mathfrak{B}_0^\times$: $\mathfrak{B}_0 = \mathfrak{o}_E[\mathfrak{B}_0^\times]$, as asserted in [Bu1] on page 190 (recall $p \geqslant 3$). □

Let $S = \begin{pmatrix} I & 0 \\ 0 & \sigma \end{pmatrix}$; it belongs to $U(\mathfrak{A})$. Define $J^S = S^{-1}JS$ and $\lambda^S(x) = \lambda(SxS^{-1})$, $x \in J^S$, where (J, λ) is the simple type given by Proposition II.1. Recall that the restriction of λ to $H^1 = H^1(\beta, \mathfrak{A})$ is a multiple of a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ such that (H^1, θ) is a decomposed pair in G relative to (M, P) with $H^1 \cap M = H^1(\beta, \mathfrak{A}_0) \times H^1(\beta, \mathfrak{A}_0)$ and $\theta_{|H^1 \cap M} = \theta_0 \otimes \theta_0$.

Corollary. *The representation* λ^S *of* $J^S = J(S^{-1}\beta S, \mathfrak{A})$ *is a simple type with the following properties:*

(i) The pair (J_P^S, λ_P^S) *is a decomposed pair in* G *relative to* (M, P) *with :*

$$J_P^S \cap M = J^S \cap M = \Gamma \times \ {}^{\tau}\Gamma \quad \textit{and} \quad (\lambda_P^S)_{|J^S \cap M} \simeq \gamma \otimes \gamma^*.$$

The same holds for the pair $(J_{P-}^S, \lambda_{P-}^S)$.

(ii) The groups J^S, J_P^S *and* J_{P-}^S *are invariant under the involution* $X \mapsto {}^T X^{-1}$.

(iii) The group $H^1(\beta, \mathfrak{A})^S$ *is invariant under the involution* $X \mapsto {}^T X^{-1}$ *and so is the simple character* θ^S *. We have* $\theta^S_{|H^1{}^S \cap M} = \theta_0 \otimes \theta_0^*$.

Indeed, using Iwahori decompositions of those groups, one checks easily that the invariance of their intersections with M, U and U^- derives from the properties in the proposition.

II.3 - A block-diagonal skew-simple stratum.

The simple character $\theta^S \in \mathcal{C}(\mathfrak{A}, 0, S^{-1}\beta S)$ in the above Corollary is fixed under the involution $X \mapsto {}^T X^{-1}$; it follows from [St1], Theorem 6.3, that this character can be viewed as a simple character attached to a skew simple stratum, namely:

There exists a simple stratum $[\mathfrak{A}, 2n_0, 0, \delta]$ *in* $\mathrm{End}_F(V)$ *satisfying* $\delta = -{}^T\delta$ *such that* θ^S *belongs to* $\mathcal{C}(\mathfrak{A}, 0, \delta)$.

In our situation we want, though, to work with a field extension both stable under the involution and contained in M. We will thus derive a number of properties of the above stratum that will lead us to that goal: we will first study the hermitian structure of V over the field $L = F[\delta]$ and show that we can conjugate δ into a block diagonal element $g^{-1}\delta g$ (lemma 1); then we will use the very strong intertwining properties of simple types (lemma 2) to show that we can pick g in J^S.

If δ is equal to 0, then $L = F$ is already contained in M anyway. We thus assume that $\delta \neq 0$. The involution $X \mapsto {}^T X$ induces on L an automorphism of order 2; let L_0 be the fixed field of this automorphism. Then L is a separable extension of L_0 (recall $p \neq 2$) with

$$\mathrm{Tr}_{L/L_0}(X) = X + {}^T X \quad (X \in L).$$

Let ϕ be any non-zero F-linear form on L_0; then $\phi \circ \mathrm{Tr}_{L/L_0}$ is a non-zero F-linear form on L, invariant under the involution T. We define a non-degenerate L-anti-hermitian form b_ϕ on V through :

$$\forall a \in L,\ \forall x \in V,\ \forall y \in V, \quad < ax, y >= \phi \circ \mathrm{Tr}_{L/L_0}(a\, b_\phi(x, y)).$$

The intersection of $\mathrm{End}_L(V)$ with $Sp(V)$ is the unitary group $U(V, b_\phi)$ relative to that form.

We first want to find a decomposition of the symplectic F-space V as a direct sum of maximal isotropic subspaces stable under L; it amounts to showing that the anisotropic part of the (anti-)hermitian space (V, b_ϕ) is equal to $\{0\}$. To show this we will use lattice duality, and it will actually be easier, as Shaun Stevens pointed out to us, to work with an hermitian form. We then fix an element u in L satisfying $\mathrm{Tr}_{L/L_0}(u) = 0$ and such that $\mathrm{val}_L u = 0$ if L/L_0 is unramified, $\mathrm{val}_L u = 1$ if L/L_0 is ramified, and we define: $d_\phi(x,y) = ub_\phi(x,y)$ $(x, y \in V)$. This is an hermitian form.

Recall ([BK1], (3.5.1)) that the field extensions $E = F[\beta]$ and $L = F[\delta]$ have the same ramification index and residual degree over F (hence $[E : F]$ is even). In particular the self-dual lattice chain $(\Lambda_i)_{i\in\mathbb{Z}}$ attached to $\mathfrak{A}$ (II.1) has period 2 over L (by definition of a stratum, those lattices are $\mathfrak{o}_L$-lattices). For Y an $\mathfrak{o}_L$-lattice in V we put :

$$Y^\sharp = \{v \in V \ / \ < v, Y > \subseteq \mathfrak{o}_F\} \ ; \quad Y^\natural = \{v \in V \ / \ d_\phi(v, Y) \subseteq \mathfrak{o}_L\}.$$

We fix ϕ such that $\phi(\mathfrak{o}_{L_0}) = \mathfrak{o}_F$ and $\phi(\mathfrak{p}_{L_0}^{-1}) = \mathfrak{p}_F^{-1}$; we then have $Y^\natural = Y^\sharp$. Since $\mathfrak{A}_0$ is standard, we can number the lattice chain $(\Lambda_{0,i})_{i\in\mathbb{Z}}$ in such a way that $\Lambda_{0,0} = \begin{pmatrix} \mathfrak{o}_F \\ \vdots \\ \mathfrak{o}_F \end{pmatrix}$. We get the sequence $\Lambda_1^\sharp = \Lambda_{-1} \supset \Lambda_0 = \Lambda_0^\sharp \supset \Lambda_1 = \varpi_L \Lambda_{-1}$, that reads:

$$\Lambda_1^\natural = \Lambda_{-1} \supset \Lambda_0 = \Lambda_0^\natural \supset \Lambda_1 = \varpi_L \Lambda_{-1}.$$

This is the self-dual slice of the lattice chain in the sense of Morris [M1]. Propositions 1.7, 1.10 in [M1] tell us that we can find a decomposition of the hermitian space (V, d_ϕ) into a direct orthogonal sum $V = V_H \oplus V_a$, where V_a is anisotropic and the anisotropic part of V_H is null, such that, for all $i \in \mathbb{Z}$: $\Lambda_i = \Lambda_i \cap V_H \oplus \Lambda_i \cap V_a$. We now use the following fact:

Let (W, b) be an anisotropic hermitian space over L. Assume there is an $\mathfrak{o}_L$-lattice Y in W satisfying:
- *$Y = Y^\natural$ if L/L_0 is unramified;*
- *$Y = \varpi_L Y^\natural$ if L/L_0 is ramified.*

Then W has dimension 0 or 1 over L.

Remark : The first case is a remark in [MVW], 5.I.1). Both cases rely on the classification of anisotropic hermitian spaces (see, e.g., [MVW], 1.I.4, or [M1], 1.8); indeed such a configuration cannot occur in two-dimensional spaces.

We can now conclude, since V has even dimension over L as over E, that the anisotropic part of V is null. Again, from Propositions 1.7, 1.10 in [M1], we can find a decomposition $V = W_1 \oplus W_2$ into a direct sum of maximal b_ϕ-isotropic L-subspaces such that:
- for all $i \in \mathbb{Z}$, $\Lambda_i = \Lambda_i \cap W_1 \oplus \Lambda_i \cap W_2$;
- the induced lattice chains on W_1 and W_2 have period 1 over L;

– $\Lambda_1 = \Lambda_0 \cap W_1 \oplus \varpi_L \Lambda_0 \cap W_2$.

Let $(f_1, \cdots, f_N)$ be an $\mathfrak{o}_F$-basis for $(\Lambda_i \cap W_1)_{i\in\mathbb{Z}}$ (see definition (1.1.7) in [BK1]) such that $\Lambda_0 \cap W_1 = \mathfrak{o}_F f_1 + \cdots + \mathfrak{o}_F f_N$; one checks easily that the basis $(f_{N+1}, \cdots, f_{2N})$ of W_2 defined by $< f_k, f_{2N-k+1} >= -1$ for $1 \leqslant k \leqslant N$ and $< f_j, f_{2N-k+1} >= 0$ for $1 \leqslant k \leqslant N$, $1 \leqslant j \leqslant N$ and $j \neq k$, is an $\mathfrak{o}_F$-basis for $(\Lambda_i \cap W_2)_{i\in\mathbb{Z}}$.

Let g be the element of $Sp(V)$ that sends the canonical basis $(e_1, \cdots, e_{2N})$ on $(f_1, \cdots, f_{2N})$. We started (see II.1) with a decomposition $V = V^{(2)} \oplus V^{(1)}$ having the same properties with respect to E as the above decomposition with respect to L, and $(e_1, \cdots, e_N)$ is an $\mathfrak{o}_F$-basis for $(\Lambda_i \cap V^{(2)})_{i\in\mathbb{Z}}$. We thus have $g(\Lambda_i \cap V^{(2)}) = \Lambda_i \cap W_1$ for all $i \in \mathbb{Z}$, hence, using duals, $g(\Lambda_i) = \Lambda_i$, so g belongs to $\mathfrak{A}^\times$. We sum up what we have just proved :

Lemma 1. *There exists an element g in $Sp_{2N}(F) \cap \mathfrak{A}^\times$ such that $g^{-1}\delta g$ is block diagonal, namely*

$$g^{-1}\delta g = \begin{pmatrix} \delta_0 & 0 \\ 0 & -\,{}^{\tau}\delta_0 \end{pmatrix}.$$

Let us come back now to the simple type (J^S, λ^S) in Corollary II.2, related to the simple character θ^S in $\mathcal{C}(\mathfrak{A}, 0, S^{-1}\beta S) = \mathcal{C}(\mathfrak{A}, 0, \delta)$. Its conjugate (J^{Sg}, λ^{Sg}) is related to the simple character θ^{Sg} in $\mathcal{C}(g^{-1}\mathfrak{A}g, 0, g^{-1}\delta g)$; note that $g^{-1}\mathfrak{A}g = \mathfrak{A}$ since g belongs to $\mathfrak{A}^\times$. Since $g^{-1}\delta g$ is block diagonal, the machinery of [BK1], §7.1 and 7.2 applies: (J^{Sg}, λ^{Sg}) determines equivalent maximal simple types $\rho^{(1)}$ and $\rho^{(2)}$, attached respectively to the strata $[\mathfrak{A}_0, n_0, 0, \delta_0]$ and $[\mathfrak{A}_0, n_0, 0, -\,{}^{\tau}\delta_0]$; from [BK1], Theorem 7.2.17, we have: $(\lambda^{Sg})_U = \rho^{(1)} \otimes \rho^{(2)}$ (see also terminology 7.2.18, (iii)).

We now recall [BK1], Corollary 7.3.12. Let π' be any smooth irreducible representation of G containing λ. Its supercuspidal support consists of unramified twists of an irreducible supercuspidal representation π of $GL_N(F)$ containing γ, the maximal simple type we started with in II.2: $(\Gamma, \gamma) = (J(\beta, \mathfrak{A}_0), \lambda(\beta, \mathfrak{A}_0))$.

But π' contains λ if and only if it contains λ^{Sg}, hence π also contains the maximal simple type $\rho^{(1)}$. Since the maximal simple types $\lambda(\beta, \mathfrak{A}_0)$ and $\rho^{(1)}$ intertwine in $GL_N(F)$ and are associated to the same order $\mathfrak{A}_0$, they are conjugate in $\mathfrak{A}_0^\times$ ([BK1], Theorem 5.7.1 and its proof). We sum up :

Lemma 2. *Let $(J(\delta_0, \mathfrak{A}_0), \rho^{(1)})$ be the maximal simple type associated to (J^{Sg}, λ^{Sg}). There exists an element a in $\mathfrak{A}_0^\times$ such that $J(\delta_0, \mathfrak{A}_0) = a^{-1}J(\beta, \mathfrak{A}_0)a$ and $\rho^{(1)} \simeq [\lambda(\beta, \mathfrak{A}_0)]^a$.*

Now the element a above is related as follows to the element g in lemma 1:

Proposition. *Let $c = \begin{pmatrix} a & 0 \\ 0 & {}^{\tau}a^{-1} \end{pmatrix}$ be the element of $Sp_{2N}(F) \cap \mathfrak{A}^\times$ associated to a. Then g belongs to the coset cJ^{Sg} in $\mathfrak{A}^\times$.*

Proof. To simplify notation here we let $H = J^{Sg} = J(g^{-1}\delta g, \mathfrak{A})$ and $\mu = \lambda^{Sg}$. We use [BK1], Theorem 7.2.17 to produce two decomposed pairs (H_1, μ_1) and (H_2, μ_2) in H such that $\mu = \mathrm{Ind}_{H_1}^{H} \mu_1 = \mathrm{Ind}_{H_2}^{H} \mu_2$.

For the first one we let μ_1 be the natural action of $H_1 = (H \cap P)H^1(g^{-1}\delta g, \mathfrak{A})$ on the space of $(H \cap U)$-fixed vectors in μ; indeed $H_1 = (J^{Sg})_P$, $\mu_1 = (\lambda^{Sg})_P$. We obtain a decomposed pair (H_1, μ_1) relative to (M, P) with $\mu = \mathrm{Ind}_{H_1}^{H} \mu_1$.

For the second one, we let $H_2 = (J_P)^{Sg}$, $\mu_2 = (\lambda_P)^{Sg}$ and obtain a decomposed pair relative to $(g^{-1}Mg, g^{-1}Pg)$ with $\mu = \mathrm{Ind}_{H_2}^{H} \mu_2$.

We now apply Mackey's theorem [Ku] to the irreducible representation μ. The intertwining of μ in H is one-dimensional, hence there exists a unique double coset $H_2 z H_1$ in H such that the restrictions of μ_1 and μ_2^z to their common domain $H_1 \cap z^{-1} H_2 z$ intertwine.

Let us look at the induced representation $\mathrm{Ind}_H^{\mathfrak{A}^\times} \mu$. It is irreducible – indeed the intertwining of μ in $\mathfrak{A}^\times$ is contained in the intersection with $\mathfrak{A}^\times$ of the intertwining of the simple character θ^{Sg}, hence in $(HD^\times H) \cap \mathfrak{A}^\times = H(D \cap \mathfrak{A})^\times H = H$, where D is the commutant algebra of $g^{-1}\delta g$. Applying the same theorem to $\mathrm{Ind}_{H_1}^{\mathfrak{A}^\times} \mu_1 = \mathrm{Ind}_{H_2}^{\mathfrak{A}^\times} \mu_2$ produces a unique double coset $H_2 z' H_1$ in $\mathfrak{A}^\times$ with the previous properties. We must have $H_2 z H_1 = H_2 z' H_1$, hence the proposition will follow from:

Claim. *$g^{-1}c$ intertwines μ_1 and μ_2.*

First note that the pairs (H_1, μ_1) and (H_2, μ_2) are both invariant (up to equivalence of the representations) under the involution $x \mapsto {}^T x^{-1}$: this follows from Corollary II.2, since g belongs to $Sp(V)$.

We have $c^{-1} g H_2 g^{-1} c = c^{-1} J_P^S c$, so the pairs (H_1, μ_1) and $(H_2^{g^{-1}c}, \mu_2^{g^{-1}c})$ are both decomposed with respect to (M, P) and the representations μ_1 and $\mu_2^{g^{-1}c}$ intertwine if and only if their restrictions to $H_1 \cap M$ and $H_2^{g^{-1}c} \cap M$ intertwine.

Now $H_1 \cap M = J(g^{-1}\delta g, \mathfrak{A}) \cap M = J(\delta_0, \mathfrak{A}_0) \times {}^\tau J(\delta_0, \mathfrak{A}_0)$ and the restriction of μ_1 there is $\rho^{(1)} \otimes \rho^{(2)}$, equivalent to $\rho^{(1)} \otimes (\rho^{(1)})^*$. On the other hand:

$$H_2^{g^{-1}c} \cap M = c^{-1}(J_P \cap M)^S c = (a^{-1} J(\beta, \mathfrak{A}_0) a) \times {}^\tau(a^{-1} J(\beta, \mathfrak{A}_0) a)$$

and the restriction of $\mu_2^{g^{-1}c}$ there is isomorphic to $[\lambda(\beta, \mathfrak{A}_0)]^a \otimes ([\lambda(\beta, \mathfrak{A}_0)]^a)^*$. We now conclude with lemma 2. □

Since $c^{-1}g$ belongs to J^{Sg}, then c^{-1} belongs to $g^{-1}J^S$ so we can write $g = hc$ with $h \in J^S$; note that h belongs to $\bar{G}$ since g and c do. Since the elements $g^{-1}\delta g = c^{-1}h^{-1}\delta h c$ and c belong to M, so does $h^{-1}\delta h$. Furthermore, since h belongs to J^S, it stabilizes the simple character θ^S.

We have finally proved that, given any simple stratum $[\mathfrak{A}, 2n_0, 0, \delta]$ in $M_{2N}(F)$ satisfying $\delta = -{}^T\delta$ and $\theta^S \in \mathcal{C}(\mathfrak{A}, 0, \delta)$, there exists an element $h \in J^S \cap \bar{G}$ such that $h^{-1}\delta h$ belongs to M. We conclude:

Theorem. *Let θ^S be the simple character in Corollary II.2. There exists a simple stratum $[\mathfrak{A}, 2n_0, 0, \alpha]$ in $M_{2N}(F)$, satisfying $\alpha \in M$ and $\alpha = -{}^T\alpha$, such that θ^S belongs to $\mathcal{C}(\mathfrak{A}, 0, \alpha)$.*

Now write $\alpha = \begin{pmatrix} \alpha_0 & 0 \\ 0 & -{}^{\tau}\alpha_0 \end{pmatrix}$ and note that such an element generates a field over F if and only if $\alpha_0 = 0$ (case ruled out from the start) or the field $K = F[\alpha_0]$ is a quadratic extension of $K_0 = F[\alpha_0^2]$.

Recall that the pair $(H^1(\alpha, \mathfrak{A}), \theta^S)$ is a decomposed pair above $(H^1(\alpha_0, \mathfrak{A}_0) \times H^1(-{}^{\tau}\alpha_0, \mathfrak{A}_0), \theta_0 \otimes \theta_0^*)$ and is T-stable, i.e.:

$$\begin{cases} H^1(\alpha_0, \mathfrak{A}_0) \times H^1(-{}^{\tau}\alpha_0, \mathfrak{A}_0) = H^1(\alpha_0, \mathfrak{A}_0) \times {}^{\tau}H^1(\alpha_0, \mathfrak{A}_0) \\ \theta_S\left(\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right)\right) = \theta_0(g_1)\theta_0({}^{\tau}g_2^{-1}). \end{cases}$$

From [BK1], Proposition 7.1.19, we conclude that the character $g \mapsto \theta_0({}^{\tau}g^{-1})$ on $H^1(-{}^{\tau}\alpha_0, \mathfrak{A}_0)$ is the image of θ_0 under the canonical transfer of simple characters from [BK1], Theorem 3.6.14:

$$\mathcal{C}(\mathfrak{A}_0, 0, \alpha_0) \xrightarrow{\approx} \mathcal{C}(\mathfrak{A}_0, 0, -{}^{\tau}\alpha_0)$$

It is difficult here to use the original notations to denote the canonical map; indeed $\mathcal{C}(\mathfrak{A}_0, 0, -{}^{\tau}\alpha_0)$ is still a set of simple characters attached to α_0, but we change the action of K on the underlying vector space by composing it with τ and with the Galois conjugacy over K_0, denoted by $x \mapsto \bar{x}$. We now use the following

Fact. *Let ψ_1, ψ_2 be two F-embeddings of K into $M_N(F)$ such that $\psi_1(K^\times)$ and $\psi_2(K^\times)$ both normalise $\mathfrak{A}_0$. There exists u in $U(\mathfrak{A}_0)$ such that, for all x in K, we have $\psi_2(x) = u^{-1}\psi_1(x)u$. The canonical transfer map between the set of simple characters $\mathcal{C}(\mathfrak{A}_0, m, \psi_1(\alpha_0))$ and $\mathcal{C}(\mathfrak{A}_0, m, \psi_2(\alpha_0))$ (m in $\mathbb{N}$) is then given by $\theta \mapsto \theta^u$.*

(The first assertion above is Lemma 1.6 in [BH1]. The second is so tautological that it is implicit in [BK1]. In any case properties 3.6.13 in [BK1] are easily checked.)

We may then choose an element σ in $U(\mathfrak{A}_0)$ such that $\sigma^{-1}x\sigma = {}^{\tau}\bar{x}$ for all x in K. The canonical transfer map from $\mathcal{C}(\mathfrak{A}_0, 0, \alpha_0)$ to $\mathcal{C}(\mathfrak{A}_0, 0, -{}^{\tau}\alpha_0)$ hence transforms a simple character μ into the simple character $x \mapsto \mu(\sigma x \sigma^{-1})$. We get:

Corollary. *Let (Γ, γ) be a maximal simple type in $GL_N(F)$ such that the corresponding inertial class of irreducible supercuspidal representations of $GL_N(F)$ contains a self-contragredient representation π, and assume the corresponding principal order $\mathfrak{A}_0$ is standard. Then either (Γ, γ) has level zero, or the simple character θ_0 underlying it can be attached to a simple stratum $[\mathfrak{A}_0, n_0, 0, \alpha_0]$ in $M_N(F)$ with the following properties.*

1 –The field $F[\alpha_0]$ is a quadratic extension of $F[\alpha_0^2]$ – in particular N is even.

2 –Let $x \mapsto \bar{x}$ denote the Galois conjugation of $F[\alpha_0]$ over $F[\alpha_0^2]$. There is an element σ in $U(\mathfrak{A}_0)$ such that $\sigma^{-1}x\sigma = {}^{\tau}\bar{x}$ for all x in $F[\alpha_0]$. The simple character θ_0 then satisfies:

$$\theta_0(\sigma\, {}^{\tau}x \sigma^{-1}) = \theta_0(x^{-1}) \qquad (x \in H^1(\alpha_0, \mathfrak{A}_0)).$$

Remark 1. The element σ above satisfies all assumptions in Proposition II.2 (see the proof of II.2). It is unique up to left multiplication by $U(\mathfrak{B}_0)$, and $\sigma\,{}^{\tau}\sigma^{-1}$ belongs to $U(\mathfrak{B}_0)$.

Remark 2. We can apply the above fact to the embedding $x \mapsto {}^{\tau}x$ of K into $M_N(F)$, and get u in $U(\mathfrak{A}_0)$ such that $u^{-1}xu = {}^{\tau}x$ for all x in K. The transfer map between $\mathcal{C}(\mathfrak{A}_0, m, \alpha_0)$ and $\mathcal{C}(\mathfrak{A}_0, m, {}^{\tau}\alpha_0)$ is then given by $\theta \mapsto \theta^u$. Since it is also given by $\theta \mapsto \theta \circ \tau$, any simple character θ in $\mathcal{C}(\mathfrak{A}_0, m, \alpha_0)$ satifies $\theta(g) = \theta(u\,{}^{\tau}gu^{-1})$. We let $\nu = \sigma u^{-1}$ and combine this with the above Corollary: we have $\nu^{-1}x\nu = \bar{x}$ for $x \in K$ and $\theta_0(\nu x\nu^{-1}) = \theta_0(x^{-1})$ for x in $H^1(\alpha_0, \mathfrak{A}_0)$. This is the formulation given in the introduction; it is conjugacy-invariant, hence the assumption that $\mathfrak{A}_0$ is standard can be removed there.

Remark 3. One can actually go further along the same lines and show that $f = N/[E:F]$ is either even or equal to 1; hence either $[E:F]$ is equal to N, or N is a multiple of 4.

III - A sequence of $Sp_{2N}(F)$-covers

III.1 - Construction of the sequence

We do not need in this paragraph the results obtained in §II.3; their use would not help. Henceworth we keep the notations and assumptions in §II.2 – in particular (Γ, γ) is a maximal simple type in $GL_N(F)$, attached to the inertial class of a self-contragredient supercuspidal representation π, and $\mathfrak{A}_0$ is standard – and start with a sequence $(J_i, \lambda_i)_{0\leqslant i\leqslant 4}$ of G-covers of the pair $(\Gamma\times\Gamma, \gamma\otimes\gamma)$ as obtained from Proposition II.1:

$$(J_3, \lambda_3) = (J_P, \lambda_P)\ ;\ (J_2, \lambda_2) = (J_{P^-}, \lambda_{P^-})\ ;$$
$$(J_1, \lambda_1) = (J_2^s, \lambda_2^s)\ ;\ (J_0, \lambda_0) = (J_3^s, \lambda_3^s) \quad \text{with } s = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\ ;$$
$$(J_4, \lambda_4) = (J_0^c, \lambda_0^c) \quad \text{with } c = \begin{pmatrix} \varpi_E & 0 \\ 0 & \varpi_E^{-1} \end{pmatrix}.$$

Note that the elements s and c normalize $(\Gamma\times\Gamma, \gamma\otimes\gamma)$. Let us write down the Iwahori decompositions of the J_i's to visualize them :

$$J_0 = i^-(\mathfrak{J}_+^1)\ (J\cap M)\ i^+(\mathfrak{H}_-^1)\ , \quad J_1 = i^-(\mathfrak{H}_+^1)\ (J\cap M)\ i^+(\mathfrak{J}_-^1)\ ,$$
$$J_2 = i^-(\mathfrak{J}_-^1)\ (J\cap M)\ i^+(\mathfrak{H}_+^1)\ , \quad J_3 = i^-(\mathfrak{H}_-^1)\ (J\cap M)\ i^+(\mathfrak{J}_+^1)\ ,$$
$$J_4 = i^-(\mathfrak{J}_-^1\varpi_E)\ (J\cap M)\ i^+(\varpi_E^{-1}\mathfrak{H}_+^1).$$

The process in Proposition I.2, applied to the conjugates J_i^S of Corollary II.2, provides us with a corresponding sequence of decomposed pairs $\left(\overline{J_i^S}, \widehat{\lambda_i^S}\right)$ in $\bar{G}$, that we will denote by (Ω_i, ω_i) to simplify notations; namely:

$$(\Omega_i, \omega_i)_{0\leqslant i\leqslant 4} \text{ with } \Omega_i = J_i^S \cap \bar{G},\ \ \Omega_i \cap \bar{M} = i(\Gamma) \text{ and } (\omega_i)_{|\Omega_i\cap\bar{M}} = i(\gamma).$$

We have: $J_4^S = \begin{pmatrix} \varpi_E^{-1} & 0 \\ 0 & \sigma^{-1}\varpi_E\sigma \end{pmatrix} J_0^S \begin{pmatrix} \varpi_E & 0 \\ 0 & \sigma^{-1}\varpi_E^{-1}\sigma \end{pmatrix}$ with

$$\begin{pmatrix} \varpi_E^{-1} & 0 \\ 0 & \sigma^{-1}\varpi_E\sigma \end{pmatrix} = \begin{pmatrix} \varpi_E^{-1} & 0 \\ 0 & {}^{\tau}\varpi_E \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & {}^{\tau}\varpi_E^{-1}\sigma^{-1}\varpi_E\sigma \end{pmatrix}$$

and we know from Proposition II.2 that ${}^{\tau}\varpi_E^{-1}\sigma^{-1}\varpi_E\sigma$ belongs to $\sigma^{-1}\Gamma\sigma = {}^{\tau}\Gamma$. Hence Ω_4 is equal to $z^{-1}\Omega_0 z$ where $z = \begin{pmatrix} \varpi_E & 0 \\ 0 & {}^{\tau}\varpi_E^{-1} \end{pmatrix}$ belongs to $\bar{G}$. We can thus derive from the above $(\Omega_i, \omega_i)_{0 \leqslant i \leqslant 4}$ an infinite sequence of decomposed pairs in $\bar{G}$ through :

$$\Omega_{i+4j} = z^{-j}\Omega_i z^j, \quad i \in \{0,1,2,3\},\ j \in \mathbb{N}.$$

Theorem. *The pairs (Ω_i, ω_i), $i \in \mathbb{N}$, are $\bar{G}$-covers of the pair $(i(\Gamma), i(\gamma))$.*

Remark. Let (Γ', γ') be another maximal simple type attached to the inertial class of π. From [BK1], (6.2.4), we can find $a \in GL_N(F)$ such that $\Gamma' = \Gamma^a$, $\gamma' = \gamma^a$. The conjugates of the subgroups in Corollary II.2 by the element $A = i(a)$ in $Sp_{2N}(F)$ satisfy analogous properties with respect to (Γ', γ'). The process in Proposition I.2 then gives us decomposed pairs above $(i(\Gamma'), i(\gamma'))$ which are $\bar{G}$-covers of $(i(\Gamma'), i(\gamma'))$, as A-conjugates of the above. Hence Theorem III.1 actually provides us with a construction of a $\bar{G}$-cover of $(i(\Gamma), i(\gamma))$ whether or not $\mathfrak{A}_0$ is standard.

The proof of this theorem will occupy the remainder of this paper; it is organised as follows. From the properties recalled in §II.1 we know that the sequences $\Omega_i \cap \bar{U}$ and $\Omega_i \cap \bar{U}^-$, $i \in \mathbb{N}$, are respectively increasing and decreasing, with $\bigcup_{i \in \mathbb{N}} \Omega_i = \bar{U}$. We can then use [BB1], Theorem I.3.4: to show that the sequence of decomposed pairs (Ω_i, ω_i) is actually a sequence of *covers*, it is enough to show that each couple of consecutive pairs $((\Omega_i, \omega_i), (\Omega_{i+1}, \omega_{i+1}))$, $i \in \mathbb{N}$, satisfies one of three criteria. In the present paragraph we will prove a convenient periodicity lemma, allowing us to reduce this checking of criteria to the cases $i = 0$ to 3. For $i = 0$ or 2, criterion 1 in *loc. cit.* is satisfied (III.2). For $i = 1$ or 3, criterion 2 is used, but the proof in III.3 takes for granted an intertwining property, property $(\star)$. Part IV is then devoted to proving property $(\star)$, or rather Proposition IV.1 which implies the former; for this we will need Theorem II.3, i.e. the stability of the underlying field extension under the involution T.

Since an appropriate power of ϖ_E belongs to $\varpi_F\Gamma$, the sequence (Ω_i, ω_i) is periodic in the sense of [BL2], Lemma 1, with period $4e(E/F)$. Since we would rather restrict the checking of criteria to the smallest possible number of cases, we have to generalize this lemma to the case of our element z, which does not lie in the center of $\bar{M}$. Note that although we state the periodicity lemma below in our present context, it actually holds in the more general situation of [BL2], Lemma 1.

Lemma. *Let z be an element of $\bar{M}$ which normalizes $\Omega_i \cap \bar{M} = i(\Gamma)$ and such that γ and ${}^{z}\gamma : x \mapsto \gamma(z^{-1}xz)$, are equivalent representations of Γ. Let i,*

$k \in \mathbb{N}$ such that $\Omega_{i+k} = z^{-1}\Omega_i z$ and $\Omega_{i+k+1} = z^{-1}\Omega_{i+1} z$. Let (τ, W) be a smooth representation of $\bar{G}$ and define, for $w \in W^{\omega_i}$:

$$\mathfrak{N}_i^\tau\,(w) = \int_{\Omega_i \cap \bar{U}^-} \tau(y)\, dy \int_{\Omega_{i+1}\cap \bar{U}} \tau(n) w\, dn.$$

We have $\mathfrak{N}_{i+k}^\tau = \tau(z^{-1})\,\mathfrak{N}_i^\tau\,\tau(z)$, hence $\mathfrak{N}_{i+k}^\tau$ is injective on $W^{\omega_{i+k}}$ if and only if $\mathfrak{N}_i^\tau$ is injective on W^{ω_i}.

Proof. Since z belongs to $\bar{M}$ we have : $\Omega_{i+k} \cap \bar{U}^- = z^{-1}(\Omega_i \cap \bar{U}^-)z$ and $\Omega_{i+k+1}\cap\bar{U} = z^{-1}(\Omega_{i+1}\cap\bar{U})z$. From a change of variables $(y,n) \mapsto (z^{-1}yz, z^{-1}nz)$ in the integral defining $\mathfrak{N}_{i+k}^\tau$, we get $\mathfrak{N}_{i+k}^\tau = \tau(z^{-1})\,\mathfrak{N}_i^\tau\,\tau(z)$ (indeed the moduli of the action of z on $\bar{U}$ and $\bar{U}^-$ are mutually inverse). The consequence on injectivity relies on the equality $\tau(z)W^{\omega_{i+k}} = W^{\omega_i}$, due to the fact that, since γ and ${}^z\gamma$ are equivalent, the representations ω_i and ${}^z\omega_{i+k}$ of Ω_i are equivalent. □

Since our present element $z = \begin{pmatrix} \varpi_E & 0 \\ 0 & {}^\tau\varpi_E^{-1} \end{pmatrix}$ satisfies the assumptions in the lemma, showing injectivity of the operators $\mathfrak{N}_i^\tau$ for $i \in \mathbb{N}$ amounts to showing it for $i = 0$ to 3. In other words (see [BL2], Proposition 1), we only need to check the criteria in [BB1], Theorem I.3.4, for $i = 0$ to 3.

III.2 - Injectivity of $\mathfrak{N}_i^\tau$ for $i = 0$ or 2

We start with the case $i = 2$ and will prove that criterion 1 is satisfied, namely: *for any y in $\Omega_3 \cap \bar{U}$, $y \notin \Omega_2$, there is a closed subgroup X of $\Omega_2 \cap \bar{U}^-$ such that $y^{-1}Xy$ is contained in Ω_2 and has no non-zero fixed vectors under ω_2.*

We use the groups $H^1 = H^1(\beta, \mathfrak{A})$ and $J^1 = J^1(\beta, \mathfrak{A})$ from §II. From our definitions of J_2 and J_3, both groups contain H^1 and the restrictions of λ_2 and λ_3 to H^1 are a multiple of the simple character θ which satisfies ([BK1], (7.2.3)):

Fact. *For x in $J^1 \cap U^-$ and y in $J^1 \cap U$, the commutator $[x,y] = xyx^{-1}y^{-1}$ belongs to H^1 and the map:*

$$\begin{aligned}(J^1 \cap U^-/H^1 \cap U^-) \times (J^1 \cap U/H^1 \cap U) &\longrightarrow \mathbb{C}^\times \\ (x,y) &\longmapsto \theta\left([x,y]\right)\end{aligned}$$

is a perfect duality between those two groups.

We have by definition $J_2 \cap U^- = J^1 \cap U^-$, $J_3 \cap U^- = H^1 \cap U^-$, $J_3 \cap U = J^1 \cap U$ and $J_2 \cap U = H^1 \cap U$. Conjugating by S then gives us a perfect duality:

$$\begin{aligned}(J_2^S \cap U^-/J_3^S \cap U^-) \times (J_3^S \cap U/J_2^S \cap U) &\longrightarrow \mathbb{C}^\times \\ (x,y) &\longmapsto \theta^S\left([x,y]\right).\end{aligned}$$

Corollary II.2 states that the involution $X \mapsto {}^T X^{-1}$ on G preserves J_2^S, J_3^S and the above duality given by θ^S. From Stevens's remark in [St1, §4], we conclude that by restriction to $Sp_{2N}(F)$ we still have a perfect duality:

$$\begin{aligned}(\Omega_2 \cap \bar{U}^- / \Omega_3 \cap \bar{U}^-) \times (\Omega_3 \cap \bar{U} / \Omega_2 \cap \bar{U}) &\longrightarrow \mathbb{C}^\times \\ (x,y) &\longmapsto \theta^S\left([x,y]\right).\end{aligned}$$

Now for x in $\Omega_2 \cap \bar{U}^-$ and y in $\Omega_3 \cap \bar{U}$, the commutator $[x,y]$ belongs to $\left(H^1 \cap \bar{U}^-\right) i(g) \left(H^1 \cap \bar{U}\right)$ for some g in $H^1(\beta, \mathfrak{A}_0)$, and we have $\theta^S\left([x,y]\right) = \theta_0(g)^2$. On the other hand $\omega_2([x,y])$ is a multiple of $\theta_0(g)$. Since H^1 is a p-group with p odd, the perfect duality above implies that, if $y \notin \Omega_2$, the subgroup $y^{-1}\left(\Omega_2 \cap \bar{U}^-\right) y$ acts in ω_2 through a non-trivial character, q.e.d.

The case $i = 0$ is entirely similar : indeed $J_1 = sJ_2s^{-1}$ and $J_0 = sJ_3s^{-1}$.

III.3 - Injectivity of $\mathfrak{N}_i^\tau$ for $i = 1$ or 3

Those steps are more involved that the previous ones – indeed, in cases when $\mathfrak{J}_\pm^1 = \mathfrak{H}_\pm^1$, we have $\Omega_2 = \Omega_3$. We start with $i = 1$ and want to show that criterion 2 is satisfied, i.e.:

there is a compact subgroup $\bar{K}$ of $\bar{G}$, containing Ω_1, such that the Hecke algebra $\mathcal{H}(\bar{K}, \Omega_1, \omega_1)$ is supported on $\Omega_1 \cup \Omega_1 t \Omega_1$ for some t in $\bar{K}$ satisfying:
$t^{-1} i(\Gamma) t = i(\Gamma)$, $t^{-1}\left(\Omega_1 \cap \bar{U}\right) t = \Omega_2 \cap \bar{U}^-$, $t^{-1}\left(\Omega_1 \cap \bar{U}^-\right) t = \Omega_2 \cap \bar{U}$.

We certainly have $\Omega_1 = t\,\Omega_2 t^{-1}$ with

$$t = \begin{pmatrix} 0 & \sigma \\ -{}^\tau\sigma^{-1} & 0 \end{pmatrix} = S^{-1} \begin{pmatrix} 0 & I \\ -\sigma\,{}^\tau\sigma^{-1} & 0 \end{pmatrix} S \quad (t \in \bar{G}).$$

Note that $\begin{pmatrix} 0 & I \\ -\sigma\,{}^\tau\sigma^{-1} & 0 \end{pmatrix}$ belongs to the coset $s\,(\Gamma \times \Gamma)$ from Proposition II.2(a); hence t normalizes $(\Omega_2 \cap \bar{M}, i(\gamma))$ and intertwines ω_2.

Since $J_1 = sJ_2s^{-1}$ and J_2, s and S are contained in $K = GL_{2N}(\mathfrak{o})$ (recall $\mathfrak{A}_0$ is standard), the subgroup generated by Ω_1 and Ω_2 is contained in the maximal compact subgroup $\bar{K} = Sp_{2N}(\mathfrak{o})$. Note that working with Ω_2 or Ω_1 here amounts to the same since the element t^2 belongs to $i(\Gamma)$. The support of $\mathcal{H}(\bar{G}, \Omega_2, \omega_2)$ is the $\bar{G}$-intertwining of ω_2, criterion 2 hence amounts to showing :

Proposition. *We have : $I_{\bar{G}}(\omega_2) \cap \bar{K} \subset \Omega_2 \cup \Omega_2\, t\, \Omega_2$.*

Proof. From Proposition I.2 we know that $I_{\bar{G}}(\omega_2)$ is contained in $I_{\bar{G}}\left(\overline{\lambda_2^S}\right)$, itself contained in $I_{\bar{G}}\left(\overline{\theta^S}\right)$, since the restriction of λ_2^S to $H' = S^{-1}H^1S$ is a multiple of θ^S.

We must now make an essential use of Shaun Stevens's results in [St1]. Indeed the character θ^S of H' is fixed under the involution $x \mapsto {}^T x^{-1}$. From [St1], Theorem 6.3, it follows that θ^S can be viewed as a simple character attached to a skew simple stratum, hence satisfies the properties shown in [St1], §3. In particular we have by [St1], Theorem 3.7:

$$I_{\bar{G}}\left(\overline{\theta^S}\right) = I_G(\theta^S) \cap \bar{G}.$$

We thus have the following information on the support we are looking for :

$$I_{\bar{G}}(\omega_2) \cap \bar{K} \subset I_G(\theta^S) \cap \bar{K}.$$

From [BK1], (5.1.1) and (5.5.11), the intertwining of θ is equal to $JB^\times J = J\widetilde{W}J$, where $\widetilde{W}$ is the affine Weyl group of $B^\times$ relative to the basis given in *loc. cit.* Assume for a moment that $[E:F]=N$. Since $B^\times$ is isomorphic to $GL_i(E)$ with $i[E:F]=2N$, we are considering in this case the affine Weyl group of $GL_2(E)$, whose intersection with a maximal compact subgroup has at most two elements, so $I_G(\theta)\cap K$ consists of the two double classes J and JsJ and we get:

$$(\star) \qquad I_{\bar{G}}(\omega_2) \cap \bar{K} \subset (J^S \cup J^S t J^S) \cap \bar{K}.$$

We now drop our assumption on $[E:F]$ and get on with our proof *assuming that* $(\star)$ *holds.*

Since $J = i^-(\mathfrak{J}^1_-)\ (J\cap M)\ i^+(\mathfrak{J}^1_+)\ = (J_2\cap U^-)(J_2\cap M)(J_3\cap U) = J_2J_3$, the group $\Omega = J^S\cap\bar{G}$ satisfies $\Omega = (\Omega_2\cap\bar{U}^-)(\Omega_2\cap\bar{M})(\Omega_3\cap\bar{U}) = \Omega_2\Omega_3 = \Omega_2(\Omega_3\cap\bar{U})$. Furthermore we have:

Lemma. $J^S\ t\ J^S \cap \bar{G} = \Omega\, t\, \Omega.$

Proof. Since $t(J^S\cap U^-)t^{-1}$ is contained in $J^S\cap U$, we have

$$J^S\ t\ J^S = (J^S\cap U)\, t\, (J^S\cap M)(J^S\cap U).$$

We apply [St2], Theorem 2.3, to the automorphism $x \mapsto {}^T x^{-1}$ of G, the pro-p-subgroup $J^S\cap U$, and the subgroup $H = (J^S\cap M)\ \cup\ t\ (J^S\cap M)$. Condition (2.1) in [St2] is easily checked, hence

$$J^S\ t\ J^S \cap \bar{G} = (\Omega\cap\bar{U})\, t\, (\Omega\cap\bar{M})(\Omega\cap\bar{U}) = \Omega\, t\, \Omega. \qquad \square$$

At this point we know: $I_{\bar{G}}(\omega_2)\cap\bar{K} \subset \Omega\cup\Omega\, t\,\Omega$. Now note that the above argument can be applied in exactly the same way to the representation ω_3 of Ω_3; indeed J_3^S also contains H'. We thus get $I_{\bar{G}}(\omega_3)\cap\bar{K}\subset\Omega\cup\Omega\, t\,\Omega$, with the pleasant feature that $\Omega\, t\,\Omega = \Omega_3\, t\,\Omega_3$ since $\Omega\cap\bar{U} = \Omega_3\cap\bar{U}$.

We now use [BK1], (4.1.5) : for $i=2,3$, the dimension of the subspace of the Hecke algebra $\mathcal{H}(\bar{G},\Omega_i,\omega_i)$ supported on Ω (resp. $\Omega\, t\,\Omega$) is equal to the dimension of the subspace of the Hecke algebra $\mathcal{H}(\bar{G},\Omega,\mathrm{Ind}_{\Omega_i}^{\Omega}\omega_i)$ supported on Ω (resp. $\Omega\, t\,\Omega$). But the argument in §III.2 shows that the induced representation of ω_i to Ω is irreducible, hence the first dimension – dimension of the subspaces supported on Ω – is equal to 1. Furthermore the two induced representations $\mathrm{Ind}_{\Omega_i}^{\Omega}\omega_i$, $i = 2, 3$, are isomorphic (use for instance Mackey restriction formula, plus the irreducibility and the fact that the representations ω_2 and ω_3 coincide on $\Omega_2 \cap \Omega_3$), so the second dimension – dimension of the subspaces supported on $\Omega\, t\, \Omega$ – is the same for $i = 2$ and $i = 3$. For $i = 3$ it is equal to 1, because any intertwining between ω_3 and ω_3^t must intertwine the irreducible representation $i(\gamma)$, then it is also equal to 1 for $i = 2$, and since t does intertwine ω_2 we get the required property. □

The last case left, $i = 3$, is dealt with exactly in the same manner, after observing that $\Omega_4 = q\,\Omega_3 q^{-1}$ with :

$$q = \begin{pmatrix} 0 & -{}^{\tau}\sigma\, {}^{\tau}\varpi_E^{-1} \\ \sigma^{-1}\varpi_E & 0 \end{pmatrix} = S^{-1} \begin{pmatrix} -{}^{\tau}\sigma\, {}^{\tau}\varpi_E^{-1}\sigma^{-1}\varpi_E & 0 \\ 0 & I \end{pmatrix} wS,\ w = \begin{pmatrix} 0 & \varpi_E^{-1} \\ \varpi_E & 0 \end{pmatrix}.$$

Again we have $J_4 = wJ_3w^{-1}$. Furthermore, let $y = \begin{pmatrix} 0 & I \\ \varpi_E & 0 \end{pmatrix}$; then:

$$yJ_1y^{-1} = J_4,\ yJ_2y^{-1} = J_3,\ yJ_3y^{-1} = J_2 \text{ and } ysy^{-1} = w.$$

So we can repeat the previous argument, and we obtain a formula analogous to $(\star)$, where we replace K by $(K^y)^S$, ω_2 by ω_3, t by q. The lemma becomes $J^S\ q\ J^S \cap \bar{G} = \Omega\, q\, \Omega$, with the same proof except that the roles of U and U^- are exchanged. Since Ω is contained in $(K^y)^S$, the last part of the argument also follows through after exchanging the roles of ω_2 and ω_3. The proof of Theorem III.1 is now complete, provided we prove $(\star)$ in part IV below.

Remark. Along the lines of the above proof we might get on to show that the Hecke algebra of the pair (Ω_2, ω_2) for instance, in $\bar{G}$, is the algebra with generators T_t and T_q (elements with support the double coset respectively of t and q) and relations the quadratic relations satisfied by T_t and T_q (they belong to a two-dimensional subalgebra). We cannot expect though that these quadratic relations be the same as the relations satisfied by T_s and T_w in the Hecke algebra of the pair (J_2, λ_2) in G – and it is not to be expected either ! For instance, for $N = 2$, the $\bar{G}$-covers constructed in [BB1] are, in the self-contragredient case, instances of the above (Ω_2, ω_2). The corresponding Hecke algebras are described in [BB2]; one can check there that the quadratic relations are, in a number of cases, different from the ones in $GL_4(F)$.

IV - Glauberman's correspondence and intertwining

We must in this last part complete the proof of Theorem III.1, that is, establish the property $(\star)$ in III.3, as well as the analogous property needed in the last case ($i = 3$) in III.3. We will first show that these properties follow from a bound on intertwining, namely proposition IV.1 (compare [BK1], Proposition 5.5.11). To prove this proposition, we will detail in IV.2 properties of the representations involved and use the argument in [BK1], Proposition 5.3.2, to reduce the proof to a very precise intertwining assertion: Proposition IV.3. At last we will establish that assertion using Glauberman's correspondence together with arguments from [BK1], §5.1 and 5.2.

IV.1 - Intertwining and Weyl group

We must now use the full content of Theorem II.3, so we change notations in this last part, both to simplify them and to stick to the notations in [BK1]. We call (J, λ), θ, and so on, what we previously called (J^S, λ^S), θ^S and so on (Corollary II.2), call β the element in $M_N(F)$ previously called α_0 and call $\tilde{\beta}$ the element in $M_{2N}(F)$ previously called α in Theorem II.3, i.e. $\tilde{\beta} = \begin{pmatrix} \beta & 0 \\ 0 & -{}^\tau\beta \end{pmatrix}$.

We know that $(J, \lambda) = (J(\tilde{\beta}, \mathfrak{A}), \lambda(\tilde{\beta}, \mathfrak{A}))$ is a simple type attached to the simple stratum $[\mathfrak{A}, 2n_0, 0, \tilde{\beta}]$ and the simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \tilde{\beta})$. Recall that J, λ and θ are stable under the involution $X \mapsto {}^T\!X^{-1}$. We will abbreviate $H^1 = H^1(\tilde{\beta}, \mathfrak{A})$ and the same for, e.g., J^1, when there is no risk of confusion.

We let E be the field $F[\beta]$ in $M_N(F)$ and B_0 be its commutant; we still call E the field embedding $F[\tilde{\beta}]$ in $M_{2N}(F)$ and call B its commutant. The crucial fact is that the embedding $F[\tilde{\beta}]$ of E in $M_{2N}(F)$ is stable under the involution T and $\bar{B}^\times$ is a unitary group (§II.3).

Let W be the affine Weyl group of $\bar{B}^\times$ relative to the subgroup of diagonal matrices. We have $\bar{B}^\times = \bar{U}(\mathfrak{B}) W \bar{U}(\mathfrak{B})$, since $\bar{U}(\mathfrak{B})$ contains a standard Iwahori subgroup of $\bar{B}^\times$. As in [BK1] §5.5, we let $\mathfrak{M}(\mathfrak{B})^\times$ be the intersection with M of $\mathfrak{B}^\times$.

Let $I(\lambda)$ be the representation of $\bar{J}$ defined by $I(\lambda) = \operatorname{Ind}_{\bar{J}_P}^{\bar{J}} \widehat{\lambda}_P$ (notation defined in I.2). We already know (III.3) that its intertwining is contained in $\overline{JB^\times J} = \overline{J^1 B^\times J^1}$ (recall $J = U(\mathfrak{B}) J^1$). Since B is now stable under T we can use fully [St1], Theorem 3.7, to get: $\overline{J^1 B^\times J^1} = \bar{J}^1 \bar{B}^\times \bar{J}^1$. We will prove in the next paragraphs the following proposition:

Proposition. *The intertwining of* $I(\lambda)$ *is contained in* $\bar{J}^1 N_{\bar{B}^\times}(\mathfrak{M}(\mathfrak{B})^\times) \bar{J}^1$, *equal to* $\bar{J} N_W(\mathfrak{M}(\mathfrak{B})^\times) \bar{J}$.

To derive property $(\star)$ we need only note that the normalizer of $\mathfrak{M}(\mathfrak{B})^\times$ in W is equal to $(\bar{U}(\mathfrak{B}) \cap W)\ W_2\ (\bar{U}(\mathfrak{B}) \cap W)$ with

$$W_2 = \left\{ \begin{pmatrix} \varpi_E^i & 0 \\ 0 & {}^\tau\varpi_E^{-i} \end{pmatrix}, i \in \mathbb{Z} \right\} \cup \left\{ \begin{pmatrix} 0 & \varpi_E^i \sigma \\ -{}^\tau\varpi_E^{-i}\, {}^\tau\sigma^{-1} & 0 \end{pmatrix}, i \in \mathbb{Z} \right\},$$

where σ is an element in $U(\mathfrak{A}_0)$ satisfying the property in Corollary II.3 (or $\sigma = I$ in the level zero case).

The intersection of W_2 with any compact subgroup has at most two elements. It follows that the intersection with any compact subgroup of the intertwining of $I(\lambda)$ contains at most two $\bar{J}$-double classes, q.e.d.

IV.2 - A one-dimensional intertwining space

To prove the above proposition we have to collect informations on the representation $I(\lambda)$. We need more notation. We let $J_- = J \cap U^- = J^1 \cap U^-$, $J_+ = J \cap U = J^1 \cap U$, $J_M = J \cap M$, $J^1_M = J^1 \cap M$, $H_- = H^1 \cap U^-$, $H_+ = H^1 \cap U$, $H^1_M = H^1 \cap M$, and so on. We define the auxiliary subgroups $J^1_P = H_- J^1_M J_+$ and $K = H_- H^1_M J_+$. We will move around the following diagram (where arrows mean inclusion):

$$\begin{matrix}
 & & \bar J = \bar J_- \bar J_M \bar J_+ & & I(\kappa) \\
 & \text{induction} \nearrow & & \nwarrow \text{extension} & \\
\widetilde{i(\kappa_0)} & \bar J_P = \bar H_- \bar J_M \bar J_+ & & \bar J^1 = \bar J_- \bar J^1_M \bar J_+ & I(\eta) \\
 & \text{extension} \nwarrow & & \nearrow \text{induction} & \\
 & & \bar J^1_P = \bar H_- \bar J^1_M \bar J_+ & & \widehat{\eta}_P = \widetilde{i(\eta_0)} \\
 & & \uparrow \text{Heisenberg} & & \\
 & & \bar K = \bar H_- \bar H^1_M \bar J_+ & & \widetilde{i(\theta_0)} \\
 & & \uparrow \text{extension} & & \\
 & & \bar H^1 = \bar H_- \bar H^1_M \bar H_+ & & \widetilde{i(\theta_0)}
\end{matrix}$$

Here, for any subgroup N of $\bar G$ admitting an Iwahori decomposition with respect to (M, P), we denote by $\widetilde{i(\mu)}$ the representation of N trivial on $N \cap U^-$ and $N \cap U$ and with restriction $i(\mu)$ to $N \cap M$, whenever it makes sense. We define $I(\kappa) = \mathrm{Ind}_{\bar J_P}^{\bar J} \widetilde{i(\kappa_0)}$ and $I(\eta) = \mathrm{Ind}_{\bar J^1_P}^{\bar J^1} \widetilde{i(\eta_0)}$. We let η_P be the representation $\eta_{|J^1_P}$ of J^1_P in the space of $\bar J_+$-fixed vectors in η, as in [BK1] §7.2. By definition of $\widehat{\eta}_P$ (proposition I.2) we have $\widetilde{i(\eta_0)} = \widehat{\eta}_P$. Furthermore, since the representation $\widetilde{i(\sigma_0)}$ of $\bar J$ is trivial on $\bar J^1$, we have:

$$\widehat{\lambda}_P = i(\widetilde{\kappa_0 \otimes \sigma_0}) = \widetilde{i(\kappa_0)} \otimes \widetilde{i(\sigma_0)}$$

$$\text{hence} \quad I(\lambda) = \mathrm{Ind}_{\bar J_P}^{\bar J} \widehat{\lambda}_P \simeq (\mathrm{Ind}_{\bar J_P}^{\bar J} \widetilde{i(\kappa_0)}) \otimes \widetilde{i(\sigma_0)} = I(\kappa) \otimes \widetilde{i(\sigma_0)}.$$

We need the following properties:

Proposition.

(i) The representations $I(\kappa)$ and $I(\eta)$ are irreducible.

(ii) The restriction of $I(\kappa)$ to $\bar J^1$ is isomorphic to $I(\eta)$.

(iii) The representation $I(\eta)$ is the Heisenberg representation above $(\bar H^1, \widetilde{i(\theta_0)})$.

(iv) The intertwining of $I(\eta)$ is equal to $\bar J^1 \bar B^\times \bar J^1$ and for any g in $\bar J^1 \bar B^\times \bar J^1$ the dimension of the intertwining space $I_g(I(\eta), \bar J_1)$ is equal to 1.

Proof. The irreducibility of $I(\eta)$ is a consequence of the fact in III.2 and (ii) follows from Frobenius reciprocity. For (iii) the argument is in [St1, §4].

(iv) is more intricate but the proof is entirely in [BK1]. We recall the main points. First of all we already know that the intertwining is contained in $\bar J^1 \bar B^\times \bar J^1$

so we may assume that g belongs to $\bar{B}^\times$. Since $I(\eta)$ is a Heisenberg representation, the argument in [BK1], 5.1.8, 5.1.9, reduces us to proving that

$$[\bar{J}^1 : \bar{J}^1 \cap (\bar{J}^1)^g] = [\bar{H}^1 : \bar{H}^1 \cap (\bar{H}^1)^g]$$

namely lemma 5.1.10 in [BK1], but for $\bar{G}$ instead of G. Using the Cayley transform $x \mapsto (1 + x/2)(1 - x/2)^{-1}$, which is defined on $(\mathcal{J}^1)^- = \{X \in \mathcal{J}^1 / {}^TX = -X\}$ and establishes bijections between $(\mathcal{J}^1)^-$ and $\bar{J}^1$, $(\mathcal{H}^1)^-$ and $\bar{H}^1$, and so on (see [St3]), we replace the equality to be proved by

$$[(\mathcal{J}^1)^- : (\mathcal{J}^1)^- \cap ((\mathcal{J}^1)^-)^g] = [(\mathcal{H}^1)^- : (\mathcal{H}^1)^- \cap ((\mathcal{H}^1)^-)^g].$$

(the Cayley transform on $\mathfrak{P}$ is easily seen to preserve subgroup indices).

Now the proof of *loc. cit.* applies mutatis mutandis: all exact sequences there remain exact after replacing each lattice involved, say Z , by $Z^- = \{X \in Z / {}^TX = -X\}$. Indeed, since g belongs to $B^\times$ and satisfies $g = {}^Tg^{-1}$, all the lattices involved are T-invariant; furthermore, the map a_β is easily seen to commute with the involution T and from [St3], Lemma 2.1.1, we may (and must here) choose a corestriction s that also commutes with T. □

We are now in a position to work out the intertwining of $I(\lambda)$. Let g belong to W and intertwine $I(\lambda)$; we have to show that g normalizes $\mathfrak{M}(\mathfrak{B})^\times$. Since $I(\lambda)$ is isomorphic to $I(\kappa) \otimes \widetilde{i(\sigma_0)}$ and the following two facts hold:

- $\widetilde{i(\sigma_0)}$ is trivial on $\bar{J}^1$;
- $\dim I_g(I(\eta), \bar{J}_1) = 1$;

we can imitate the proof of [BK1], Proposition 5.3.2, to get that any non-zero intertwining operator in $I_g(I(\lambda), \bar{J})$ has the form $S \otimes R$ with $S \in I_g(I(\eta), \bar{J}_1)$ and R an endomorphism in the space of $\widetilde{i(\sigma_0)}$.

Let us use Proposition IV.3 below: for any T-stable minimal $\mathfrak{o}_E$-order $\mathfrak{B}_m$ contained in $\mathfrak{B}$, the operator S also intertwines the restriction of $I(\kappa)$ to the subgroup $\bar{U}^1(\mathfrak{B}_m)\bar{J}^1$ (use one-dimensionality for $I(\eta)$). Again as in *loc. cit.*, this implies that R belongs to $I_g(\widetilde{i(\sigma_0)}, \bar{U}^1(\mathfrak{B}_m)\bar{J}^1)$.

Proposition IV.1 now follows from:

Lemma. *Let $g \in W$ intertwine the restriction of $\widetilde{i(\sigma_0)}$ to $\bar{U}^1(\mathfrak{B}_m)\bar{J}^1$ for any T-stable minimal $\mathfrak{o}_E$-order $\mathfrak{B}_m$ contained in $\mathfrak{B}$. Then g normalizes the group $\mathfrak{M}(\mathfrak{B})^\times$.*

Proof. Indeed we almost recognize [BK1], Proposition 5.5.5, that again we will imitate. The sequence of lemmas there holds unchanged, so we assume g does not normalize $\mathfrak{M}(\mathfrak{B})^\times$ and produce an hereditary order $\mathfrak{B}'_0$, with radical $\mathfrak{Q}'_0$, and a parabolic subgroup Q of $GL_N(F)$, with unipotent radical $N = 1 + \mathfrak{N}$, such that:

(i) $\mathfrak{B}'_0 \subset \mathfrak{B}_0$; $\mathfrak{B}'_0 \cap \mathfrak{N} = \mathfrak{Q}'_0 \cap \mathfrak{N} \not\subseteq \mathfrak{Q}_0$; $g^{-1}(\mathfrak{Q}'_0 \cap \mathfrak{N})\, g \subset \mathfrak{Q}_0$;

(ii) the image of $1+\mathfrak{Q}'_0\cap\mathfrak{N}$ in $U(\mathfrak{B}_0)/U^1(\mathfrak{B}_0)$ is the unipotent radical of a proper parabolic subgroup of $U(\mathfrak{B}_0)/U^1(\mathfrak{B}_0)$.

Indeed, in the notations of *loc. cit.*, $\mathfrak{B}'_0$ is $\bar{\mathfrak{B}}$, contained in some B^i, and one can decompose V^i into a direct sum $V^i = W^1\oplus W^2$ of E- vector spaces such that $\bar{L}_1\cap W^1 = \bar{L}_0\cap W^1$ and $\bar{L}_1\cap W^2 = \varpi_E\bar{L}_0\cap W^2$; then $\mathfrak{N} = \mathrm{Hom}_E(W^2,W^1)\subset \mathrm{End}_E V^i$ satisfies the assumptions. Furthermore, since g belongs to $\bar{B}^\times$, we can as well assume here that $B^i = B^1$.

We put $\mathfrak{B}' = \mathfrak{B}'_0\oplus {}^{\tau}\mathfrak{B}'_0$; hence $(\mathfrak{B}')^\times$ is contained in M and equal to $U(\mathfrak{B}'_0)\times {}^{\tau}U(\mathfrak{B}'_0)$. We put $\mathfrak{Q}' = \mathfrak{Q}'_0\oplus {}^{\tau}\mathfrak{Q}'_0$. Since g belongs to $\bar{B}^\times$ we still have

$$g^{-1}\left(1+[\mathfrak{Q}'_0\cap\mathfrak{N}\oplus {}^{\tau}(\mathfrak{Q}'_0\cap\mathfrak{N})]\right)g\subset 1+\mathfrak{Q} \text{ and } 1+\mathfrak{Q}'\subset U(\mathfrak{B}).$$

We now pick a T-stable minimal $\mathfrak{o}_E$-order $\mathfrak{B}_m$ contained in $\mathfrak{B}'+\mathfrak{Q}$. Then $1+\mathfrak{Q}'\subset 1+\mathfrak{Q}_m = U^1(\mathfrak{B}_m)$.

Assume then that g does intertwine the restriction of $\widetilde{i(\sigma_0)}$ to $\bar{U}^1(\mathfrak{B}_m)\bar{J}^1$; then there is a non-zero operator R in the space of the representation such that:

$$\forall x\in(\bar{U}^1(\mathfrak{B}_m)\bar{J}^1)\cap(\bar{U}^1(\mathfrak{B}_m)\bar{J}^1)^g \qquad R\circ\widetilde{i(\sigma_0)}(x) = \widetilde{i(\sigma_0)}(gxg^{-1})\circ R.$$

This relation holds in particular for $x = g^{-1}i(y)g$ with $y\in 1+\mathfrak{Q}'_0\cap\mathfrak{N}$, because x belongs to $\bar{J}^1$ and $i(y)$ belongs to $\bar{U}^1(\mathfrak{B}_m)$. We get $R = \sigma_0(y)\circ R$, which, with (ii), contradicts the cuspidality of σ_0. □

Remark. The above lemma itself is the full proof of Proposition IV.1 in the case of level 0 representations.

IV.3 - Glauberman's correspondence

This last paragraph will be devoted to the proof of the proposition below.

Proposition. *Let $\mathfrak{B}_m$ be a T-stable minimal $\mathfrak{o}_E$-order in B contained in $\mathfrak{B}$. The restriction of $I(\kappa)$ to $\bar{U}^1(\mathfrak{B}_m)\bar{J}^1$ has the same intertwining as $I(\eta)$.*

The property we want to prove is invariant under conjugation by $\bar{\mathfrak{B}}^\times$; hence we may assume (see [BK1], 1.1.9 and 7.1.15) that $\mathfrak{B}_m = (\mathfrak{B}_{0,m}\oplus {}^{\tau}\mathfrak{B}_{0,m})+\mathfrak{Q}$ where $\mathfrak{B}_{0,m}$ is a minimal $\mathfrak{o}_E$-order in B_0 contained in $\mathfrak{B}_0$. We then have: $\bar{U}^1(\mathfrak{B}_m) = i\left(U^1(\mathfrak{B}_{0,m})\right)\bar{U}^1(\mathfrak{B})$ and

$$\bar{U}^1(\mathfrak{B}_m)\bar{J}^1 = i\left(U^1(\mathfrak{B}_{0,m})\right)\bar{J}^1 = \bar{J}_-\left[i\left(U^1(\mathfrak{B}_{0,m})\right)\bar{J}^1_M\right]\bar{J}^+.$$

Looking at the diagram and proposition in IV.2, we find that

$$I(\kappa)_{|\bar{U}^1(\mathfrak{B}_m)\bar{J}^1}\simeq \mathrm{Ind}_{i(U^1(\mathfrak{B}_{0,m}))\bar{J}^1_P}^{i\left(U^1(\mathfrak{B}_{0,m})\right)\bar{J}^1}\widetilde{i(\kappa'_0)} \quad \text{where } \kappa'_0 = \kappa_{0|U^1(\mathfrak{B}_{0,m})J^1(\beta,\mathfrak{A}_0)}.$$

It is enough to show that *the representations $\widetilde{i(\kappa'_0)}$ and $\widetilde{i(\eta_0)}$ have the same intertwining.* Indeed, by [BK1], 4.1.5, we have: $I_{\bar{G}}\left(I(\kappa)_{|\bar{U}^1(\mathfrak{B}_m)\bar{J}^1}\right) = \bar{J}^1 I_{\bar{G}}\left(\widetilde{i(\kappa'_0)}\right)\bar{J}^1$ and $I_{\bar{G}}\left(I(\eta)\right) = \bar{J}^1 I_{\bar{G}}\left(\widetilde{i(\eta_0)}\right)\bar{J}^1$.

Glauberman's correspondence is the tool we need here. We recall briefly what it is in our setting; more general and precise statements can be found in [St1], §2, or [BH2], §A2, as well as the original references.

Let ϵ be the involution $x \mapsto {}^T x^{-1}$ on G, with fixed points $\bar{G}$. For any open compact pro-p-subgroup H of G which is ϵ-stable, Glauberman's correspondence gives us a unique bijection $\mathfrak{g}$: $\rho \mapsto \mathfrak{g}(\rho)$, between the set Irr $(H)^\epsilon$ of ϵ-stable equivalence classes of smooth irreducible representations of H and the set Irr $(\bar{H})$ of equivalence classes of smooth irreducible representations of $\bar{H}$, characterized by the property that $\mathfrak{g}(\rho)$ occurs in $\rho_{|\bar{H}}$ with odd multiplicity.

Let K be a subgroup of H satisfying the same assumptions as H, let $\sigma \in$ Irr $(K)^\epsilon$ and $\rho \in$ Irr $(H)^\epsilon$. Then $\rho \simeq \mathrm{Ind}_K^H \sigma$ implies $\mathfrak{g}(\rho) \simeq \mathrm{Ind}_{\bar{K}}^{\bar{H}} \mathfrak{g}(\sigma)$, and $\rho_{|K} \simeq \sigma$ implies $\mathfrak{g}(\rho)_{|\bar{K}} \simeq \mathfrak{g}(\sigma)$.

A crucial property of this correspondence is the following:

Fact. [Stevens [St1], Lemma 2.4] *Let $\rho \in$ Irr $(H)^\epsilon$, $g \in \bar{G}$. The dimension of $I_g(\rho, H)$ is odd if and only if the dimension of $I_g(\mathfrak{g}(\rho), \bar{H})$ is odd.*

We now proceed to find the inverse images under Glauberman's correspondence of the representations $\widetilde{i(\kappa_0')}$ and $\widetilde{i(\eta_0)}$. We start a series of lemmas. The first one is valid for any simple stratum in $M_N(F)$.

Lemma 1. *(i) Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in $M_N(F)$ and let $\alpha \in \mathfrak{o}_F^\times$. Then, for any defining sequence $[\mathfrak{A}, n, r_i, \gamma_i]$ for $[\mathfrak{A}, n, 0, \beta]$ (see* [BK1]*, 2.4.2), the sequence $[\mathfrak{A}, n, r_i, \alpha\gamma_i]$ is a defining sequence for $[\mathfrak{A}, n, 0, \alpha\beta]$. In particular we have $\mathcal{J}^k(\beta, \mathfrak{A}) = \mathcal{J}^k(\alpha\beta, \mathfrak{A})$ and $\mathcal{H}^k(\beta, \mathfrak{A}) = \mathcal{H}^k(\alpha\beta, \mathfrak{A})$.*

(ii) The map $\theta \mapsto \theta^2$ is a bijection from $\mathcal{C}(\mathfrak{A}, m, \frac{1}{2}\beta)$ onto $\mathcal{C}(\mathfrak{A}, m, \beta)$ ($m \in \mathbb{N}$), which is compatible with the canonical bijections of [BK1]*, §3.6. We will denote the inverse bijection by $\theta \mapsto \theta^{1/2}$.*

Proof. (i) is simple checking. Since the groups $H^{m+1}(\beta, \mathfrak{A})$ are p-groups with p odd, (ii) is easily checked by induction along a defining sequence for β. □

Let again θ be the simple character that underlies our simple type (J, λ) and let $\theta^{1/2}$ in $\mathcal{C}(\mathfrak{A}, m, \frac{1}{2}\tilde{\beta})$ be its inverse image under the square map. To the simple character $\theta^{1/2}$ we attach representations $\eta^{1/2}$, $\kappa^{1/2}$, and $\eta_P^{1/2}$, $\kappa_P^{1/2}$ in the usual way of [BK1], §5 and 7. For instance, $\kappa^{1/2}$ is a representation of $J(\frac{1}{2}\tilde{\beta}, \mathfrak{A}) = J(\tilde{\beta}, \mathfrak{A})$ which is a beta-extension of $\eta^{1/2}$; note that we have to choose one here, while $\eta^{1/2}$ is completely determined by $\theta^{1/2}$.

Lemma 2. *We still write $\theta^{1/2}$ for the extension of $\theta^{1/2}$ to K trivial on J^+. We have $\mathfrak{g}(\theta^{1/2}) \simeq \widetilde{i(\theta_0)}$ (on K, $\bar{K}$) and $\mathfrak{g}(\eta_P^{1/2}) \simeq \widetilde{i(\eta_0)}$ (on J_P^1, $\bar{J}_P^1$).*

Proof. The character $\theta^{1/2}$ on K is trivial on J^+ and H^-; on H_M^1 it is given by $\theta^{1/2}\left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}\right) = \theta_0^{1/2}(g_1)\theta_0^{1/2}({}^\tau g_2^{-1})$ for g_1, ${}^\tau g_2$ in $H^1(\beta, \mathfrak{A}_0)$ ([BK1], 7.1.19, and Corollary II.2). We thus have: $\theta^{1/2}(i(g)) = \theta_0^{1/2}(g)\theta_0^{1/2}(g) = \theta_0(g)$, for $g \in H^1(\beta, \mathfrak{A}_0)$, which proves the first assertion. The restriction of $\eta_P^{1/2}$ to K is a multiple of $\theta^{1/2}$ so its restriction to $\bar{K}$ is a multiple of $\widetilde{i(\theta_0)}$; unicity of Heisenberg representations says that the restriction of $\eta_P^{1/2}$ to $\bar{J}_P^1$ is then a multiple of $\widetilde{i(\eta_0)}$. □

Of course Glauberman's correspondence does not apply to $J(\tilde{\beta}, \mathfrak{A})$ which is not a p-group, nor to J_P. But it does apply to the following group, intermediate between J_P^1 and J_P:

$$\begin{aligned} L &= \left[U^1(\mathfrak{B}_{0,m}) \times {}^{\mathcal{T}}U^1(\mathfrak{B}_{0,m})\right] J_P^1 \\ &= H_- \left[\left(U^1(\mathfrak{B}_{0,m})J^1(\beta, \mathfrak{A}_0)\right) \times {}^{\tau}\!\left(U^1(\mathfrak{B}_{0,m})J^1(\beta, \mathfrak{A}_0)\right)\right] J_+. \end{aligned}$$

The subgroup L is certainly stable under $x \mapsto {}^{T}x^{-1}$; let us check that the restriction to L of the representation $\kappa_P^{1/2}$ is also stable by this involution, up to isomorphism. From [BK1], §7.2, $\kappa_P^{1/2}$ is trivial on L_- and L_+ and its restriction to $L \cap M$ has the form $\kappa_P^{1/2}\left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}\right) \simeq \kappa_1(g_1) \otimes \kappa_2({}^{\tau}g_2^{-1})$ where κ_1 and κ_2 are both beta-extensions of the Heisenberg representation $\eta_0^{1/2}$ of $J^1(\beta, \mathfrak{A}_0)$ attached to $\theta_0^{1/2}$. Hence κ_1 and κ_2 differ from a character $\chi \circ \det_{B_0}$ where χ is a character of $\mathfrak{o}_E^\times/1 + \mathfrak{p}_E$ ([BK1], 5.2.2). This implies that κ_1 and κ_2 agree on $U^1(\mathfrak{B}_{0,m})J^1$ whence the stability of $(\kappa_P^{1/2})_{|L}$ under ϵ.

Lemma 3. *We have* $\mathfrak{g}((\kappa_P^{1/2})_{|L}) \simeq \widetilde{i(\kappa_0')}$ *(on* L*,* $\bar{L}$*).*

Proof. Both representations have trivial restrictions to $\bar{L}_-$ and $\bar{L}_+$ and irreducible restrictions to $\bar{L} \cap M$. So what we have to show is: $\mathfrak{g}((\kappa_P^{1/2})_{|L\cap M}) \simeq i(\kappa_0')$.

Let $\kappa_M^{1/2} = (\kappa_P^{1/2})_{|L\cap M}$; this is a representation of

$$L \cap M = \left(U^1(\mathfrak{B}_{0,m})J^1(\beta, \mathfrak{A}_0)\right) \times {}^{\tau}\!\left(U^1(\mathfrak{B}_{0,m})J^1(\beta, \mathfrak{A}_0)\right).$$

Denote by $\mathfrak{A}_{0,m}$ the unique hereditary $\mathfrak{o}_F$-order in $\mathfrak{A}_0$ stable under conjugation by $E^\times$ such that $\mathfrak{A}_{0,m} \cap B_0 = \mathfrak{B}_{0,m}$. Let $\theta_{0,m}^{1/2}$ be the image of $\theta_0^{1/2}$ under the canonical tranfer map: $\mathcal{C}(\mathfrak{A}_0, 0, \frac{1}{2}\beta) \to \mathcal{C}(\mathfrak{A}_{0,m}, 0, \frac{1}{2}\beta)$, and let $\eta_{0,m}^{1/2}$ be the unique irreducible representation of $J^1(\beta, \mathfrak{A}_{0,m})$ containing $\theta_{0,m}^{1/2}$. Let $\mu_0^{1/2}$ be the unique extension of $\eta_0^{1/2}$ to $L_0 = U^1(\mathfrak{B}_{0,m})J^1(\beta, \mathfrak{A}_0)$ satisfying

$$\mathrm{Ind}_{L_0}^{U^1(\mathfrak{A}_{0,m})} \mu_0^{1/2} \simeq \mathrm{Ind}_{J^1(\beta,\mathfrak{A}_{0,m})}^{U^1(\mathfrak{A}_{0,m})} \eta_{0,m}^{1/2}.$$

From [BK1], 5.2.6 and 5.1.15 (where $\mu_0^{1/2}$ is denoted by $\tilde{\eta}$, or in our context $\widetilde{\eta_0^{1/2}}$), we have $\kappa_M^{1/2} \simeq \mu_0^{1/2} \otimes \mu_0^{1/2^*}$. The induced representations above are irreducible and *loc. cit.* implies:

$$\begin{aligned} \mathrm{Ind}_{L\cap M}^{U^1(\mathfrak{A}_{0,m})\times {}^{\mathcal{T}}U^1(\mathfrak{A}_{0,m})} \kappa_M^{1/2} &\simeq \mathrm{Ind}_{L_0}^{U^1(\mathfrak{A}_{0,m})} \mu_0^{1/2} \otimes [\mathrm{Ind}_{L_0}^{U^1(\mathfrak{A}_{0,m})} \mu_0^{1/2}]^* \\ &\simeq \mathrm{Ind}_{J^1(\beta,\mathfrak{A}_{0,m})}^{U^1(\mathfrak{A}_{0,m})} \eta_{0,m}^{1/2} \otimes [\mathrm{Ind}_{J^1(\beta,\mathfrak{A}_{0,m})}^{U^1(\mathfrak{A}_{0,m})} \eta_{0,m}^{1/2}]^* \\ &\simeq \mathrm{Ind}_{J^1(\beta,\mathfrak{A}_{0,m})\times {}^{\tau}J^1(\beta,\mathfrak{A}_{0,m})}^{U^1(\mathfrak{A}_{0,m})\times {}^{\mathcal{T}}U^1(\mathfrak{A}_{0,m})} \eta_{0,m}^{1/2} \otimes [\eta_{0,m}^{1/2}]^* \end{aligned}$$

Since the representations involved are irreducible, one gets through Glauberman's correspondence an isomorphism:

$$\mathrm{Ind}_{i(L_0)}^{i\left(U^1(\mathfrak{A}_{0,m})\right)}\ \mathfrak{g}\left(\kappa_M^{1/2}\right) \simeq \mathrm{Ind}_{i(J^1(\beta,\mathfrak{A}_{0,m}))}^{i\left(U^1(\mathfrak{A}_{0,m})\right)}\ \mathfrak{g}\left(\eta_{0,m}^{1/2}\otimes[\eta_{0,m}^{1/2}]^*\right).$$

We already know that $\mathfrak{g}(\eta_P^{1/2})$ is isomorphic to $\widetilde{i(\eta_0)}$ and that $i(\kappa_0')$ extends $i(\eta_0)$. Again from *loc. cit.*, the representation $i(\kappa_0')$ is the unique irreducible representation of $i(L_0)$ extending $i(\eta_0)$ and satisfying:

$$\mathrm{Ind}_{i(L_0)}^{i\left(U^1(\mathfrak{A}_{0,m})\right)}\ i(\kappa_0') \simeq \mathrm{Ind}_{i(J^1(\beta,\mathfrak{A}_{0,m}))}^{i\left(U^1(\mathfrak{A}_{0,m})\right)}\ i(\eta_{0,m}).$$

Hence it is enough to show that $\mathfrak{g}\left(\eta_{0,m}^{1/2}\otimes[\eta_{0,m}^{1/2}]^*\right)\simeq i(\eta_{0,m})$. But the restriction of $\eta_{0,m}^{1/2}\otimes[\eta_{0,m}^{1/2}]^*$ to $i\left(H^1(\beta,\mathfrak{A}_{0,m})\right)$ is a multiple of

$$i(g)\mapsto \theta_{0,m}^{1/2}(g)\otimes[\theta_{0,m}^{1/2}]^*({}^{\tau}g^{-1}) = \left[\theta_{0,m}^{1/2}(g)\right]^2 = \theta_{0,m}(g),$$

so its image is the Heisenberg representation above $i\left(\theta_{0,m}\right)$ namely $i(\eta_{0,m})$. □

End of proof of Proposition IV.3. The intertwining of $\widetilde{i(\eta_0)}$ contains the intertwining of $\widetilde{i(\kappa_0')}$ since the second representation restricts to the first. Let us now take g in $I_{\bar G}(\widetilde{i(\eta_0)}$ and show that g belongs to $I_{\bar G}(\widetilde{i(\kappa_0')}$. The above fact about intertwining spaces and Glauberman's correspondence, combined with the lemmas, gives us:

- g intertwines $\eta_P^{1/2}$ (Fact and Lemma 2). Indeed dim $I_g(\widetilde{i(\eta_0)})$ is equal to 1, from Proposition IV.2 and [BK1], 4.1.5.
- g intertwines $\kappa_P^{1/2}$. Indeed, from [BK1], §7.2, g intertwines $\eta^{1/2}$ (induced from $\eta_P^{1/2}$), hence g intertwines $\kappa^{1/2}$ (that has the same intertwining as $\eta^{1/2}$); furthermore $J_P^1gJ_P^1$ is the unique J_P^1-double coset in J^1gJ^1 that intertwines $\eta_P^{1/2}$ ([BK1], 4.1.5 and 5.1.8). Similarly, in JgJ there is a unique J_P-double coset J_PbJ_P that intertwines $\kappa_P^{1/2}$. Since $J=J_PJ_-$ we may assume that b belongs to $J_-\,g\,J_-$, hence to J^1gJ^1. But then, since b also intertwines $\eta_P^{1/2}$, we must have $b\in J_P^1gJ_P^1$; so $b\in J_PgJ_P$ whence the result.
- the dimension of $I_g(\eta_P^{1/2})$ is equal to 1, so is the dimension of $I_g((\kappa_P^{1/2})_{|L})$ (again the second representation restricts to the first, by [BK1], §7.2). Hence g intertwines the image of this representation by the Glauberman correspondence, namely $\widetilde{i(\kappa_0')}$ (Fact and Lemma 3). □

Remark. Of course proposition IV.3 says something about the restriction of $I(\kappa)$ to a suitable subgroup being a β-extension of $I(\eta)$. To prove Theorem III.1, we actually do not need to know whether or not $I(\kappa)$ itself is a β-extension of $I(\eta)$. It should follow from the study of the Hecke algebra of the $\bar G$-cover that the bound on intertwining given by Proposition IV.1 is actually an equality, i.e. the intertwining of $I(\lambda)$ is equal to $\bar J N_W(\mathfrak{M}(\mathfrak{B})^\times)\bar J$ – see the remark at the end of III.3.

References

[A] J. D. ADLER, 'Self-contragredient supercuspidal representations of GL_n', *Proc. Amer. Math. Soc.* (8) 125 (1997) 2471–2479.

[BB1] L. BLASCO et C. BLONDEL, 'Types induits des paraboliques maximaux de $Sp_4(F)$ et $GSp_4(F)$', *Ann. Inst. Fourier* (6) 49 (1999) 1805–1851.

[BB2] L. BLASCO et C. BLONDEL, 'Algèbres de Hecke et séries principales généralisées de $Sp_4(F)$', *Proc. London Math. Soc.* (3) 85 (2002) 659–685.

[BL1] C. BLONDEL, 'Critère d'injectivité pour l'application de Jacquet', *C. R. Acad. Sci. Paris* (325) I (1997), 1149–1152.

[BL2] C. BLONDEL, 'Une méthode de construction de types induits et son application à G_2', *J. of Algebra* 213 (1999), 231–271.

[BL3] C. BLONDEL, Quelques propriétés des paires couvrantes, *Prépublication 2001.*

[BU1] C. J. BUSHNELL, 'Hereditary orders, Gauss sums and supercuspidal representations of GL_N', *J. reine angew. Math.* 375/376 (1987), 184–210.

[BU2] C. J. BUSHNELL, 'Representations of reductive p-adic groups: localization of Hecke algebras and applications', *J. London Math. Soc.* (2) 63 (2001) 364–386.

[BH1] C. J. BUSHNELL and G. HENNIART, *Local tame lifting for $GL(N)$ I: simple characters,* Publ. Math. IHES 83 (1996), p. 105–233.

[BH2] C. J. BUSHNELL and G. HENNIART, *Local tame lifting for $GL(n)$ II: wildly ramified supercuspidals,* Astérisque 254 (SMF, 1999).

[BK1] C. J. BUSHNELL and P. C. KUTZKO, *The admissible dual of GL_n via compact open subgroups,* Annals of Math. Studies 129 (Princeton University Press, 1993).

[BK2] C.J. BUSHNELL and P.C. KUTZKO, 'Smooth representations of reductive p-adic groups: structure theory via types', *Proc. London Math. Soc.* 77 (1998), 582–634.

[GK] I. M. GELFAND and D. A. KAZHDAN, 'Representations of the group $GL(n, K)$ where K is a local field', in *Lie groups and their representations (Proc. Summer School of the Bolya-Janos Math. Soc., Budapest, 1971),* Halsted, New York, 1975.

[KI] J.-L. KIM, 'Hecke algebras of classical groups over p-adic fields and supercuspidal representations', *Amer. J. Math.* 121 (1999), 967–1029.

[KU] P. C. KUTZKO, 'Mackey's theorem for nonunitary representations', *Proc. Am. Math. Soc.* 64 (1977), 173–175.

[MVW] C. MOEGLIN, M.-F. VIGNÉRAS et J.-L. WALDSPURGER, *Correspondances de Howe sur un corps p-adique,* Lecture Notes in Math. 1291 (Springer Verlag, 1987).

[M1] L. MORRIS, 'Tamely ramified supercuspidal representations of classical groups. I. Filtrations', *Ann. scient. Ec. Norm. Sup.* (4) 24 (1991), 705–738.

[M2] L. MORRIS, 'Level zero G-types', *Compositio Math.* 118 (1999) 2, 135–157.

[R] A. ROCHE, 'Parabolic induction and the Bernstein decomposition', *Compositio Math.* 134 (2002), 113–133.

[SH1] F. SHAHIDI, 'A proof of Langlands' conjecture on Plancherel measures : complementary series for p-adic groups', *Annals of Math.* 132 (1990) 273–330.

[SH2] F. SHAHIDI, 'Twisted endoscopy and reducibility of induced representations for p-adic groups', *Duke Math. J.* 66 (1992) 1–41.

[ST1] S. STEVENS, 'Intertwining and supercuspidal types for p-adic classical groups', *Proc. London Math. Soc.* 83 (2001), 120–140.

[ST2] S. STEVENS, 'Double coset decompositions and intertwining',*Manuscripta mathematica* 106 (2001), 349–364.

[ST3] S. STEVENS, 'Types and representations of p-adic symplectic groups',*PhD Thesis, King's College London, 1998.*

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