Introduction to the Calculus of Variations

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Abstract

These are notes for the M2 course "Calcul des variations" given at Université Paris Cité in the period November-December 2025.

The aim of theese notes is to introduce the reader to the calculus of variations, namely techniques that allow to study the existence of extrema of functions defined on a Banach space. These notes are largely inspired from classical refferences such as those signed by Bernard Dacorogna, Filip Rindler or Filippo Santambrogio or the "polycopie" of J.-F. Babadjian. The more "analysis"-oriented chapters are inspired by the books of H. Brezis, L. Grafakos, G. Leoni and H. Royden.

1 Introduction

In the following we present a few exemples arising in physics that can be formalized as minimization problems.

1.1 Exemples

1.1.1 The branchistochrone problem.

The problem credited to be the "birthcertificate of the calculus of variations" is the so called branchistochrone problem. We consider two points A, B in the plane. Let a curve $\mathcal{C}$ conect the two points and assume that a material point on which only the gravity acts starts rolling from A to B. The time that the material point takes to arrive from A to B clearly depends on the curve $\mathcal{C}$. The branchistochrone problem asks to find the curve for which this time is minimal.

Assume that $A = (0,0)$, $B = (\bar{x}, \bar{y})$ with $\bar{x} > 0$, $\bar{y} < 0$ and that $u = u(x)$ is a generic curve with endpoints A and B. Then

$$dt = \frac{1}{\frac{ds}{dt}} ds = \frac{1}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{du}{dt}\right)^2}} \sqrt{(dx)^2 + (du)^2} = \frac{\sqrt{(dx)^2 + (du)^2}}{\sqrt{-2gu}} = \sqrt{\frac{(dx)^2 + (du)^2}{-2gu}}$$

from which

$$\frac{dt}{dx} = \sqrt{\frac{1 + \left(\frac{du}{dx}\right)^2}{-2gu}}$$

such that the time is

$$T[u] = \int_0^{\bar{x}} \sqrt{\frac{1 + \left(\frac{du}{dx}\right)^2}{-2gu}} \mathrm{d}x.$$

The above formula is an expression for the time that the material point takes to arrive from A to B along the curve $\mathcal{C}$. Finding the curve which realizes the minimal time is thus casted into a problem of finding the minimum of the above application which is a functional.

1.1.2 The isoperimetric problem.

Given $L > 0$, the isoperimetric problem asks to enclose the maximal area with curve that has legth L. A variant is the following find $u \geq 0$, $u(0) = u(1) = 0$ which realizes the minimum of

$$A[u] = \int_0^1 u(x)\,\mathrm{d}x$$

under the constraint

$$\sqrt{1 + \left(\frac{du}{dx}\right)^2} dx = L.$$

1.1.3 Elasticity

We begin with the following simple exemple. A plane elastic membrane is glued at the level of its boundary to a plane support. We denote $\Omega \subset \mathbb{R}^2$ the domain occupied by the membrane. We apply a force $f(x_1, x_2)\,\mathrm{d}x_1\mathrm{d}x_2$ on each surface element dx_1dx_2 and we denote the vertical displacement by u i.e. $u : \Omega \to \mathbb{R}$. For small forces it can be established empirically that u is found as the minimum of the functional

$$\int_\Omega (\frac{1}{2}|Du(x)|^2 - f(x)\,u(x))\mathrm{d}x.$$

Moreover, $u = 0$ on $\partial\Omega$ since the membrane is glued at the level of its boundary.

More generaly, consider an elastic body occupying a bounded connected domain $\Omega \subset \mathbb{R}^3$. If some force field $\overrightarrow{f} = \overrightarrow{f}(x) \in \mathbb{R}^3$ acts on the body, each point $x = (x_1, x_2, x_3) \in \Omega$ will be sent to a new position $y = y(x_1, x_2, x_3) \in \mathbb{R}^3$. We introduce the displacement vector

$$\overrightarrow{u} = y - x.$$

In order to measure the local deformation the so-called strain tensor is introduced witch is given by

$$\mathbb{D}\overrightarrow{u} = \frac{1}{2}\left(D\overrightarrow{u} + D^T\overrightarrow{u}\right).$$

In general, in elasticity one postulates that the deformed configuration is obtained by minimizing a functional of the type

$$\int_\Omega \left(W(\mathbb{D}\overrightarrow{u}(x)) - \overrightarrow{u}(x) \cdot \overrightarrow{f}(x)\right)\mathrm{d}x$$

Some considerations of physical nature show that

1. Undeformed state does not costs energy : $W(O_3) = 0$.
2. Frame indiference : $W(QA) = W(A)$ for any $A \in \mathbb{R}^{3\times3}$ and orthogonal Q.
3. Infinite compression costs infinite energy $\lim_{\det A\to0} W(A) = +\infty$.
4. Infinite stretching costs infinite energy $\lim_{|A|\to\infty} W(A) = +\infty$.

1.1.4 Fluid mechanics

A fluid wich initially occupies a bounded region of the space $\Omega \subset \mathbb{R}^3$ can be described by a familly of functions

$$X_t : \Omega \to \mathbb{R}^3$$

which for each $t > 0$ are C^1-difeomorphims on their images. The phisical interpretation is that $X_t(x)$ is the position at time $t > 0$ of the particle that was at $t = 0$ in position $x \in \Omega$. The Euler equations for a inviscid barotropic compressible fluid are the equations verified by the critical point of the following functional

$$\mathcal{A} = \int_{t_0}^{t_1} \int_{\Omega} \rho_0(x) \left(\frac{1}{2} \left| \dot{X}_t(x) \right|^2 - F\left(\frac{\rho_0(x)}{\det DX_t(x)} \right) \right) \mathrm{d}x.$$

where t_0, t_1 are given.

1.2 The Euler-Lagrange equations

The above problems and many more arrising in physics can be casted as a minimisation problem (or the question of finding critical points) of some abstract functional defined on a Banach space. Of course the nature of this space depends on the nature of the problem. Let us recall how minimisation works for functions of real variables.

1.2.1 Necessary and Sufficient conditions for functions of a real variable

First recall that in finite dimension, bounded sequences are pre-compact:

Theorem 1.1 *Let $a, b \in \mathbb{R}$, $a < b$ and $(x_n)_{n\in\mathbb{N}} \subset [a, b]$. Then $(x_n)_{n\in\mathbb{N}}$ admits a convergent subsequence.*

This property is crucial in order to obtain

Theorem 1.2 *Consider $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ continious. Then f is bounded and attains its sup/inf bounds.*

Variant : $f : \mathbb{R} \to \mathbb{R}$ continious such that $\lim_{|x|\to\infty} f(x) = +\infty$. Then f is bounded by below and attains its inf bound.

If f is differentiable then we can deduce an equation for the points where the min/max are atteined. We begin we the well known first order condition know as Fermat's theorem.

Theorem 1.3 *Let $f :]a, b[\to \mathbb{R}$ be a differentiable function on $]a, b[$ and $x_0 \in]a, b[$ a point where f attains a local minimum/maximum. Then,*

$$f'(x_0) = 0.$$

Example 1.1 *Of course not every critical point (i;e. where the derivative vanishes) is a point of min/max as the exemple $f(x) = x^3$ shows.*

If f is assumed to be two times differentiably, it turns out that f'' verifies an inequality in a local min/max point.

Theorem 1.4 *Let $f :]a, b[\to \mathbb{R}$ be a twice differentiable function on $]a, b[$ and $x_0 \in]a, b[$. If f attains a local minimum in x_0 (resp. a local maximum in x_0) then $f''(x_0) \geq 0$ (resp. $f''(x_0) \leq 0$).*

The second derivative also allows us to obtain a sufficient condition for min/max.

Theorem 1.5 *Let $f\ :]a,b[\to \mathbb{R}$ be a twice differentiable function on $]a,b[$ and $x_0 \in]a,b[$ such that $f'(x_0) = 0$. Then if $f''(x_0) > 0$ then x_0 is a local minimum (resp. if $f''(x_0) < 0$ then x_0 is a local maximum).*

There is also an important class of functions where there is only one extremum point : strctly convex/strictly concave functions. We recall that a function is called strictly convex if for all x, y from its domain of definition and any $\lambda \in (0,1)$ it holds true that

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y).$$

A function is strictly concave if $-f$ is strictly convex.

Theorem 1.6 *Assume that $f : \mathbb{R} \to \mathbb{R}$ is such that $\lim\limits_{|x|\to\infty} f(x) = +\infty$ and that it is strictly convex. Then there exists an unique point $x_0 \in \mathbb{R}$ such that $f(x_0) = \inf\limits_{x\in\mathbb{R}} f(x)$.*

1.2.2 The infinite dimensional case

Notations In the following, we will always reserve $d \geq 1$ for the space dimension and we reserve the notation $\Omega \subset \mathbb{R}^d$ *for a bounded open set in* $\mathbb{R}^d$. Sometimes we will assume some smoothness on the boundary for instance C^2 which implies that there exists a C^1-normal field $n : \partial\Omega \to \mathbb{R}^d$. The space variable will be denoted

$$x \in \mathbb{R}^d \text{ with } x = (x_1, x_2, ..., x_d).$$

For $m \geq 1$ and $k \in \mathbb{N} \cup \{\infty\}$ we let

$$C^k(\Omega; \mathbb{R}^m) = \{u : \Omega \to \mathbb{R}^m \ : \ u \text{ is } k\text{-times cont.-differentiable}\}.$$

If $m = 1$ we will simply use the notation $C^k(\Omega)$ insteand of $C^k(\Omega; \mathbb{R})$. We will denote by

$$C_c^k(\Omega; \mathbb{R}^m) = \left\{ u : \Omega \to \mathbb{R}^m \ : \ \begin{array}{c} u \text{ is } k\text{-times cont.-differentiable,} \\ u \text{ has compact support in } \Omega. \end{array} \right\}$$

Recall that the support of u is

$$\operatorname{Supp} u = \overline{\{x \in \Omega : u \neq 0\}}.$$

Also, we denote by

$$C^k\left(\overline{\Omega}; \mathbb{R}^m\right) = \left\{ u_{|\Omega} \, : \, u \in C_c^k\left(\mathbb{R}^d; \mathbb{R}^m\right) \right\}.$$

If E is a set we will use the notation 2^E for the power set i.e. the set of all subsets of E.

We will work with functions of the form $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ will belong to a certain class of admissible (to be defined later) functions. In the following we denote by

$$f = f(x, u, \xi)$$

$u = (u_1, u_2, \cdots, u_m) \in \mathbb{R}^m$, while $\xi \in \mathbb{R}^{m\times d}$ i.e.

$$\xi = (\xi_{ij})_{i\in\overline{1,m},\ j\in\overline{1,d}}.$$

Such a function f will be called Lagrangian.

We will use the following notations for the partial derivatives of f with respect to the group of variables denoted by u :

$$f_{u_i} \text{ or } \frac{\partial f}{\partial u_i},\ i \in \overline{1,m}$$

$$f_u \in \mathbb{R}^m,\ f_u = \left(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}, \cdots, \frac{\partial f}{\partial u_m}\right)$$

respectively for the group of variables denoted by ξ :

$$f_{\xi_{ij}} \text{ or } \frac{\partial f}{\partial \xi_{ij}},\ i \in \overline{1,m},\ j \in \overline{1,d},$$

$$f_\xi \in \mathbb{R}^{m\times d},\ f_\xi = \left(\frac{\partial f}{\partial \xi_{ij}}\right)_{i\in\overline{1,m},\ j\in\overline{1,d}}.$$

For a function

$$u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^m,$$
$$u = (u_1, u_2, ..., u_m)$$

we let

$$\begin{cases} Du : \Omega \subset \mathbb{R}^d \to \mathbb{R}^{m\times d}, \\ (Du)_{ij} = \partial_j u_i,\ i \in \overline{1,m},\ j \in \overline{1,d}. \end{cases}$$

The case $m = 1$ will be reffered to as the scalar case while the case $m \geq 2$ will be reffered to as the vectorial case.

This course will be devoted to the study of extrama for functionals of the form

$$J[u] = \int_\Omega f(x, u(x), Du(x))\, dx. \tag{1}$$

For the moment, we will be interested in understanding in a somehow informal manner, first order conditions which are known as the Euler-Lagrange equations. These are a system of PDEs that an extremum of J should satisfy. Let us gather in the following lines a list with very basic definitions regarding functionals defined on normed spaces.

Definition 1.1 *Consider* $(E, \|\cdot\|)$ *a normed vector space and let* $J : E \to (-\infty, \infty]$. *We denote by* $\mathrm{Dom}(J) \in 2^E$ *the domain of* J *:* $\mathrm{Dom}(J) = \{u \in E : J[u] \leq \infty\}$.

1. *We say that* J *is lower semi-continious (abv. l.s.c.) in* $\bar{u} \in E$ *if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $u \in E$

$$\|u - \bar{u}\|_E \leq \delta \Rightarrow J[\bar{u}] \leq \varepsilon + J[u].$$

This is equivalent to sequential lower semi-continuity : J *is l.s.c. in* $\bar{u} \in E$ *if for all* $(u_n)_n$ *converging to* $\bar{u} \in E$,

$$J[u] \leq \liminf J[u_n].$$

We say that $J : E \to [-\infty, \infty)$ *is upper semi-continious in* $\bar{u} \in X$ *if* $-J$ *is l.s.c. in* $\bar{u} \in E$. *We say that* $J : E \to \mathbb{R}$ *is continious in* $\bar{u} \in Y$ *if* J *and* $-J$ *are l.s.c. in* $\bar{u} \in E$.

2. *Let* $J : E \to (-\infty, \infty]$ *be as above. We say that* $\bar{u} \in E$ *is* <u>*a local minimizer on* $D \subset E$</u> *if there exists some* $\delta > 0$ *such that for all* $u \in D$

$$\|u - \bar{u}\|_E \leq \delta \Rightarrow J[\bar{u}] \leq J[u].$$

Assume that $(E, \|\cdot\|_E)$ is a normed vector space of functions defined on Ω taking values in $\mathbb{R}^m$ and that E contains smooth functions $C_c^\infty(\Omega; \mathbb{R}^m)$. Let

$$J : E \to]-\infty, \infty]$$

be defined by (1) with the Lagrangian, $f \in C^2$ and that

(H1) $\bar{u} \in C^2(\Omega; \mathbb{R}^m) \in D \subset E$ is a smooth local minimizer for J on D,

(H2) $D \subset E$ has the property that $\forall h \in C_c^\infty(\Omega; \mathbb{R}^m)$ there exists $\varepsilon_h \in (-1, 1)$ such that

$$\forall t \in (-\varepsilon_h, \varepsilon_h)\ ,\ u_t = \bar{u} + th \in D.$$

Then, fix $h \in C_c^\infty(\Omega; \mathbb{R}^m)$ and using u_t introduced above, using that $\bar{u}$ is a local minimum on D we obtain that

$$\begin{aligned}
0 &= \frac{d}{dt}\{t \to J[u_t]\}_{|t=0} \\
&= \frac{d}{dt}\int_\Omega f(x, \bar{u}(x) + th(x), D\bar{u}(x) + tDh(x))\,\mathrm{d}x\Big|_{t=0} \\
&= \int_\Omega \frac{\partial f}{\partial u_i}(x, \bar{u}(x), D\bar{u}(x))h_i(x) + \frac{\partial f}{\partial \xi_{ij}}(x, \bar{u}(x), D\bar{u}(x))\partial_j h_i(x)\,\mathrm{d}x \\
&= \int_\Omega \left\{\frac{\partial f}{\partial u_i}(x, \bar{u}(x), D\bar{u}(x))h_i(x) - \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial \xi_{ij}}(x, \bar{u}(x), D\bar{u}(x))\right)\right\} h_i(x)\,\mathrm{d}x \\
&+ \int_{\partial\Omega} \frac{\partial f}{\partial \xi_{ij}}(x, \bar{u}(x), D\bar{u}(x))h_i(x)\, n_j(x)\,\mathrm{d}\sigma(x).
\end{aligned}$$

Since $h = 0$ on $\partial\Omega$ and $h \in C_c^\infty(\Omega; \mathbb{R}^m)$ was fixed arbitrarely, we obtain the following system of partial differential equations that should be verified by a smooth extremum of the functional J : $\bar{u}$ has to be a solution for

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial \xi_{ij}}(x, u(x), Du(x))\right) = \frac{\partial f}{\partial u_i}(x, u(x), Du(x)),\ \text{for all } i \in \overline{1, m}. \tag{2}$$

In the scalar case, i.e. $m = 1$ the Euler-Lagrange equation writes

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial \xi_j}(x, u(x), Du(x))\right) = \frac{\partial f}{\partial u}(x, u(x), Du(x)). \tag{3}$$

In the multi dimensional case, the Euler-Lagrange equations are a system of PDEs. The PDEs (2) or (3) for the scalar case, are supplemented with the boundary condition

$$u = u_{\partial\Omega} \text{ on } \partial\Omega \tag{4}$$

with $u_{\partial\Omega} : \partial\Omega \to \mathbb{R}^m$ given. The problem of finding a solution for (2) togheater with the boundary conditions (4) is the classical (or strong) formulation of a *Dirichlet boundary value problem*. We could (and we later will) restrict to the search of functions verifying the integral indentities obtaining when diferentiating $\varepsilon \to J[u_\varepsilon]$. We obtain then the :

Weak formulation for the Dirichlet bvp :

$$\left|\begin{array}{l}\text{find } u \text{ such that } u = u_{\partial\Omega} \text{ on } \partial\Omega \text{ and for all } h \in C_c^\infty(\Omega; \mathbb{R}^m), \\ \int_\Omega \frac{\partial f}{\partial u_i}(x, u(x), Du(x))h_i(x) + \frac{\partial f}{\partial \xi_{ij}}(x, u(x), Du(x))\partial_j h_i(x)\, dx = 0.\end{array}\right.$$

If we impose a stronger hypothesis on D, namely that

(H2') $D \subset E$ has the properthy that $\forall h \in C^{\infty}\left(\overline{\Omega};\mathbb{R}^m\right)$ there exists $\varepsilon_h \in (-1,1)$ such that

$$\forall t \in (-\varepsilon_h, \varepsilon_h)\ ,\ u_t = \bar{u} + th \in D,$$

then, besides the equations (2) we obtain also that for all $i \in \overline{1,m}$

$$\frac{\partial f}{\partial \xi_{ij}}(x, u(x), Du(x))n_j\,(x) = 0 \text{ on } \partial\Omega, \tag{5}$$

or

$$f_\xi \cdot n = \frac{\partial f}{\partial \xi_j}(x, u(x), Du(x))n_j\,(x) = 0 \text{ on } \partial\Omega, \tag{6}$$

in the scalar case. Remark that the condition (H2') is so strong that it will impose the boundary behaviour (5) to the minimizers. The problem of finding a solution for (2) togheater with the boundary conditions (5) is the classical formulation of a *Neumann boundary value problem.*

Weak formulation for the Neumann bvp :

$$\left|\begin{array}{l}\text{find } u \text{ such that for all } h \in C_c^{\infty}\left(\mathbb{R}^d;\mathbb{R}^m\right), \\ \int_\Omega \frac{\partial f}{\partial u_i}(x,u(x),\overline{Du(x))h_i\,(x) + \frac{\partial f}{\partial \xi_{ij}}}(x,u(x),Du(x))\partial_j h_i\,(x)\ dx = 0.\end{array}\right.$$

In a rather informal manner, it should be understood that the boundary conditions are implied in the set of test functions.

1.3 Exemples of Euler-Lagrange equations

1.3.1 The case of ODEs

Here $d = 1$, $m = 1$, $\Omega = (a,b) \subset \mathbb{R}$, and let

$$J[u] = \int_a^b f\left(x, u\left(x\right), u'\left(x\right)\right) \mathrm{d}x.$$

Minimizers of J over subsets of , say $E = C^2(a,b)$ verify the Euler-Lagrange equations which in strong form are given by

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \xi}\left(x, u\left(x\right), u'\left(x\right)\right)\right) = \frac{\partial f}{\partial u}\left(x, u\left(x\right), u'\left(x\right)\right) \text{ for all } x \in (a,b)\,.$$

The Neumann conditions in this case read

$$\frac{\partial f}{\partial \xi}\left(a, u\left(a\right), u'\left(a\right)\right) = \frac{\partial f}{\partial \xi}\left(b, u\left(b\right), u'\left(b\right)\right) = 0$$

at the boundary.

The Euler-Lagrange equations associated to the minimization of

$$J[u] = \int_0^1 \left(\frac{m(u')^2}{2} - f\left(u\right)\right) \mathrm{d}t$$

with $m > 0$ constant and $f \in C^1(\mathbb{R})$ are

$$mu'' = -f'\left(u\right)$$

where we recognise Newton's equation. If the minimisation takes place in, say $D = \left\{u \in C^1[0,1] : u(0) = 0, u(1) = 1\right\}$ then local minimizers on D are solutions of

$$\begin{cases} mu'' = -f'(u), \\ u(0) = 0, u(1) = 1. \end{cases}$$

Of course, D verifies (H2) but it does not verify (H2'). If instead $D = C^1[0,1]$ then (H2') is verified and a local minimizer on D verify

$$\begin{cases} mu'' = -f'(u), \\ u'(0) = u'(1) = 0. \end{cases}$$

In the case of the branchistochrone problem

$$J[u] = \int_0^{\bar{x}} \sqrt{\frac{1 + (u')^2}{-2gu}} dx$$

we have (do not forget that in that setting $u(x) < 0$ for $x \in (0, \bar{x})$) the following Euler-Lagrange equations

$$\left(\frac{u'}{\sqrt{u\left(1 + (u')^2\right)}} \right)' = g \frac{\sqrt{1 + (u')^2}}{(u)^{\frac{3}{2}}}.$$

1.3.2 The Poisson equation

Consider the functional defined by

$$J[u] = \int_\Omega (\frac{1}{2} |Du(x)|^2 - g(x) u(x)) dx$$

where $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ (thus $m = 1$, it is a scalar problem) and

$$|Du(x)|^2 = |\partial_1 u(x)|^2 + |\partial_2 u(x)|^2 + \cdots + |\partial_d u(x)|^2.$$

The associated Euler-Lagrange equation is

$$-\Delta u = g$$

which is called the Poisson equation.

The classical formulation for the Dirichlet problem for the Poisson equation is to find a function $u \in C^2(\Omega)$ such that

$$\begin{cases} -\Delta u = g, \text{ in } \Omega, \\ u = u_0, \text{ on } \partial\Omega. \end{cases}$$

The weak formulation for the Dirichlet bvp in this case is to find u such that

$$\int_\Omega Du(x) \cdot Dh(x) \, \mathrm{d}x = \int_\Omega g(x) h(x) \, \mathrm{d}x \ \underline{\forall h \in C_c^\infty(\Omega)}.$$

The classical formulation for the Neumann problem for the Poisson equation is

$$\begin{cases} -\Delta u = g, \text{ in } \Omega, \\ Du \cdot n = 0, \text{ on } \partial\Omega, \end{cases}$$

while the weak formulation of the Neumann problem for the Poisson equation is just to find u such that

$$\int_\Omega Du(x) \cdot Dh(x) \, \mathrm{d}x = \int_\Omega g(x) h(x) \, \mathrm{d}x \ \underline{\forall h \in C_c^\infty\left(\mathbb{R}^d\right)}.$$

1.3.3 The p-Poisson equation

The p-Poisson equation is obtained considering

$$J[u] = \int_\Omega (\frac{1}{p} |Du(x)|^p - g(x) u(x)) \mathrm{d}x$$

which has the corresponding Euler-Lagrange equation:

$$-\operatorname{div}\left(|Du(x)|^{p-2} Du(x)\right) = g(x).$$

The above equation is nonlinear. The operator

$$\Delta_p u := \operatorname{div}\left(|Du|^{p-2} Du\right)$$

is called p-Laplacian.

1.3.4 Lamé's system

The next exemple comes from linear elasticity

$$J[u] = \int_\Omega (2\mu |\mathbb{D}u|^2 + \frac{\lambda}{2}(\operatorname{div} u)^2 - g(x) u(x)) dx$$

where $u : \Omega \to \mathbb{R}^d$ (vectorial case $m = d$) and

$$\mathbb{D}u = \frac{1}{2}(Du(x) + (Du(x))^T).$$

The coresponding Euler-Lagrange equations are

$$-\mu\Delta u - (\mu + \lambda) D \operatorname{div} u = g.$$

2 The direct method in an abstract setting

2.1 Informal description of the direct method

The indirect method, which had some succes in the 1D case, consists in solving the EL equations and prove that the solution is a minimiser. This is much more harder to implement in the multidimensional case since the EL equations are PDEs. The direct method of the calculus of variations consists of showing that a functional has a local extremum/critical point without using the associated EL equations. It turns out that showing the existence of C^1-minimizers has a drawback from the functional analysis point of view : the space is not reflexive and showing that a sequence is compact is not easy. The strategy to avoid this problem is to first show the existence of extrema/critical points in a larger space. In a second time, one can focus on recovering extra regularity properties. For instance, for the Poisson problem, the functional

$$J[u] = \int_\Omega (\frac{1}{2} |Du(x)|^2 - g(x) u(x)) dx$$

is naturally defined on the Sobolev space $W^{1,2}(\Omega)$ which consists of $L^2(\Omega)$ functions having generalized derivatives in $L^2(\Omega)$. In the chapters that follow, we will present the analysis tools will be needed in order to show the existence of minima for such a functional.

The Direct Method can be informally described as the search for a Banach space X such that

1. Boundness : J extends naturaly to this space and J is lower bounded $\inf J[u] > -\infty$: the boundness implies the existence of a minimizing sequence i.e. $(u_n) \subset X$ s.t. $\lim_n J[u_n] = \inf J[u]$.

2. Coercivity : we wish to deduce that any minimizing sequence is bounded for this we would need to establish an inequality of the type

$$cte + J[u] \geq \|u\|_X .$$

3. Weak-Compactness : we should be able to extract a subsequence from $(u_n)_n$ that converges to some candidate for minimum. Since we are working in infinite dimensional Banach space, we have to abandon the norm-topology : we will work in the so-called weak-topology.

4. Weak lower semi-continuity : this is exactly the property that ensures that by passing to the limit J does not increse.

5. Investigate uniqueness using convexity of the integrand.

In the next section we introduce and present the main properties of the weak topology on a Banach space.

2.2 Weak topology on a Banach space

This chapter is largely inspired from Chapter 3 of Brezis's "Functional Analysis". The reader is strongly encoureged to see this chapter of the aforementioned book.

The question at the heart of weak topologies is the following : if we are given a set E, a priori we do not ask for this set to be endowed with any particular structure, a topological space Y and a collection of function $\{\varphi\}_{i\in I}$, $\varphi : E \to Y$, what is the coarsest topology which renders all the functions $\{\varphi\}_{i\in I}$ continious? One can show that this topology corresponds to the class of sets obtained as arbitrary reunions of sets that are finite intersections of $\left\{\varphi_i^{-1}(D) : i \in I \text{ and } D \text{ open in } Y\right\}$. There is more practical charcterisation of convergent sequences w.r.t. this topology namely, if $(u_n)_{n\in\mathbb{N}} \subset E$, $u \in E$ then u_n converges to u w.r.t. to the weak topology if

$$\forall i \in I : \varphi_i(u_n) \to \varphi_i(u) \text{ in } Y.$$

In this lecture we will introduce the weak topology induced by continious functions on Banach spaces.

Definition 2.1 *Consider a set $Y \neq \emptyset$ and $\mathcal{T}_1, \mathcal{T}_2$ two topologies on Y such that $\mathcal{T}_1 \subset \mathcal{T}_2$. We say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ or that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.*

To put things in perspective, the reader can solve

Exercise 2.1 *Consider $Y \neq \emptyset$ and $\mathcal{T}_1, \mathcal{T}_2$ two topologies on Y such that $\mathcal{T}_1 \subset \mathcal{T}_2$.*

1. *If K is $\mathcal{T}_2-$quasi-compact then it is $\mathcal{T}_1-$quasi-compact.*

2. *If $(Z, \mathcal{T}_Z)$ is another topological space then if $J : Y \to Z$ is $\mathcal{T}_1-$continious then it is $\mathcal{T}_2-$continious.*

Thus in an informal manner, weaker topologies contain more compact sets but less continious functions. Keep in mind that lower semi-continuity is just continuity when $Z =]-\infty, \infty]$ and $\mathcal{T}_Z = \{(a, \infty], a \in \mathbb{R}\} \subset 2^{]-\infty,\infty]}$.

We consider a Banach space $(E, \|\cdot\|_E)$ and E' its dual space which is endowed with

$$\|f\|_{E'} = \sup_{\|u\|_E \leq 1} \langle f, u \rangle .$$

Recall that, $(E', \|\cdot\|_{E'})$ is a Banach space.

Definition 2.2 *The weak topology $\sigma(E, E') \subset 2^E$ is the coarsest topology with respect to which all $f \in E'$ are continious. In particular the weak topology is weaker than the topology induced by $\|\cdot\|_E$ which will be reffered to as the strong topology.*

The next proposition gives a basis of neighborhoods at a given point $u \in E$.

Proposition 2.1 *Let $u_0 \in E$; given $\varepsilon > 0$ and a finite set $\{f_1, f_2, \ldots, f_k\}$ in E' consider*

$$V = V(f_1, f_2, \ldots, f_k; \varepsilon) = \{u \in E;\ |\langle f_i, u - u_0\rangle| < \varepsilon\ \forall i = 1, 2, \ldots, k\}.$$

Then V is a neighborhood of u_0 for the topology $\sigma(E, E')$. Moreover, we obtain a basis of neighborhoods of u_0 for $\sigma(E, E')$ by varying ε, k, and the f_i's in E'.

As a consequence of the Hahn-Banach theorem we obtain that

Proposition 2.2 *The weak topoloogy $\sigma(E, E')$ is Hausdorff.*

In the quasi-totality of this course we will be working with sequences.

Definition 2.3 *If a sequence $(u_n)_n$ in E converges to u in the weak topology $\sigma(E, E')$ we shall write*

$$u_n \rightharpoonup u \text{ weakly in } \sigma(E, E')$$

or when it will be clear from the context we will also write simply that

$$u_n \rightharpoonup u.$$

In order to emphasize strong convergence we will say $u_n \to u$ strongly, which of course means that $\|u_n - u\|_E \to 0$.

We gather in the following proposition some basic properties of convergence w.r.t. the weak topology.

Proposition 2.3 *Let $(u_n)_n$ be a sequence in E. Then*

1. $[u_n \rightharpoonup u$ *weakly in* $\sigma(E, E')] \iff [\forall f \in E' : \langle f, u_n\rangle \to \langle f, u\rangle\ ]$.
2. *If $u_n \to u$ strongly, then $u_n \rightharpoonup u$ weakly in $\sigma(E, E')$.*
3. *If $u_n \rightharpoonup u$ weakly in $\sigma(E, E')$, then $(u_n)_n$ is bounded and $\|u\| \leq \liminf \|u_n\|$.*
4. *If $u_n \rightharpoonup u$ weakly in $\sigma(E, E')$ and if $f_n \to f$ strongly in E' (i.e., $\|f_n - f\|_{E'} \to 0$), then $\langle f_n, u_n\rangle \to \langle f, u\rangle$.*

The first property is a consequence of the Banach-Steinhauss Theorem. The third property is a bitt more delicate and we will prove it latter.

When E is finite-dimensional, the weak topology $\sigma(E, E')$ and the norm topology are the same. In particular, a sequence (u_n) converges weakly if and only if it converges strongly.

Open (resp. closed) sets in the weak topology $\sigma(E, E')$ are always open (resp. closed) in the strong topology. In any infinite-dimensional space, the weak topology is strictly coarser than the strong topology; i.e., there exist open (resp. closed) sets in the strong topology that are not open (resp. closed) in the weak topology. Here are two examples:

Exercise 2.2 *The unit sphere $S = \{u \in E; \|u\| = 1\}$, with E infinite-dimensional, is never closed in the weak topology $\sigma(E, E')$. More precisely, we have*

$$S_{\sigma(E,E')} = \overline{B}_E,$$

where $S_{\sigma(E,E')}$ denotes the closure of S in the topology $\sigma(E, E')$.

A sufficient condition that a strongly closed set to be weakly closed is the following

Proposition 2.4 *Let C be a convex subset of E. Then C is closed in the weak topology $\sigma(E, E')$ if and only if it is closed in the strong topology.*

As a consuqeunce we have the following property which is very helpfull in application

Theorem 2.1 (Mazur) *Assume $(u_n)_n$ converges weakly to u. Then there exists a sequence $(v_n)_n$ made up of convex combinations of the u_n's that converges strongly to u.*

We have the following important result:

Theorem 2.2 *Assume that $\varphi : E \to (-\infty, +\infty]$ is convex and lower semicontinuous (l.s.c.) in the strong topology. Then φ is l.s.c. in the weak topology $\sigma(E, E')$.*

Definition 2.4 *A Banach space is called separable if it contains a countable dense subset.*

Definition 2.5 *Let E be a Banach space and let $j : E \to E'' = (E')'$ be the canonical injection from E into E'' :*

$$\begin{cases} j(u) : E' \to \mathbb{R}, \\ \langle j(u), f \rangle_{E'',E'} :=_{E'} \langle f, u \rangle_{E',E}. \end{cases}$$

The space E is said to be reflexive if j is surjective, i.e., $j(E) = E''$. When E is reflexive, E'' is usually identified with E.

Banach space with separable duals have the property that the unit ball is metrisable. This is important in applications since it allows us to work with sequences instead of the more general topological concepts.

Theorem 2.3 *Assume that E is a reflexive and separable Banach space. Then the closed unit ball in E*

$$B_E = \{u \in E \mid \|u\|_E \leq 1\}$$

endowed with the weak topology $\sigma(E, E')$ is compact.

Proof.

In a first step we will show that B_E with the topolgy $\sigma(E, E')$ is metrizable. First of all, E is reflexive and separable if and only if E' reflexive and separable. Assume that $(f_n)_n \subset E'$ is dense in the unit ball of E'. Then, for all $u \in E$ we introduce

$$[u] = \sum_{i=0}^{+\infty} \frac{|\langle f_i, u\rangle|}{2^i}$$

which induces a norm on E. Remark that if for all $i \in \mathbb{N}$ $\langle f_i, u\rangle = 0$ then owing to the density we obtain that $\langle f, u\rangle = 0$ for all $f \in E'$ and $\|f\|_{E'} \leq 1$ which in turn implies that $u = 0$.

Consider $u_0 \in B_E$ and $V = V(g_1, g_2, \dots, g_k; \varepsilon) = \{u \in E;\ |\langle g_i, u - u_0\rangle| < \eta\ \forall i = 1, 2, \dots, k\}$. Let us show that there exists $r > 0$ such that

$$\{x \in B_E,\ [u - u_0] \leq r\} \subset V.$$

Let $\varepsilon > 0$ and $f_{\varphi(i)}$ such that for all $i \in 1, 2 \cdots, k$:

$$\left\|f_{\varphi(i)} - g_i\right\|_{E'} \leq \varepsilon.$$

Then if $u \in B_E$, $[u - u_0] \leq r$ it implies in particular that

$$|\langle f_i, u - u_0\rangle| \leq 2^i r.$$

We see that for all $i \in 1, 2 \cdots, k$:

$$|\langle g_i, u - u_0\rangle| \leq |\langle g_i - f_{\varphi(i)}, u - u_0\rangle| + \left|\left\langle f_{\varphi(i)}, u - u_0\right\rangle\right| \leq 2\varepsilon + 2^{\varphi(i)} r.$$

We take r such that the RHS of the previous inequality to be smaller than η.

Now consider

$$B_r = \{u \in B_E,\ [u - u_0] \leq r\}$$

and let us show that it contains a $\sigma(E, E')$-open set containg u_0. Consider N large enough such that

$$\sum_{i=N+1}^{+\infty} \frac{1}{2^i} \leq \frac{r}{4}$$

and let

$$V = \left\{u \in B_E : |\langle f_i, u - u_0\rangle| \leq \frac{r}{4} \text{ for } i \in \overline{1, N}\right\}.$$

For any $u \in V$ we have that

$$\sum_{i=0}^{+\infty} \frac{|\langle f_i, u - u_0\rangle|}{2^i} = \sum_{i=0}^{N} \frac{|\langle f_i, u - u_0\rangle|}{2^i} + \sum_{i=N+1}^{+\infty} \frac{|\langle f_i, u - u_0\rangle|}{2^i} \leq \frac{r}{4} \sum_{i=0}^{N} \frac{1}{2^i} + \frac{r}{2} < r$$

which implies that $x \in B_r$. Observe that in this part we did not use the reflexivity.

Since the unit ball is metrizable and separable, sequential compactness implies compactness. Let us consider a sequence $(u_n)_n \subset B_E$. We show that we can extract from it a convergent subsequence. By a Cantor diagonal argument along with the density of $(f_i)_{i\in\mathbb{N}}$ we can extract a subsequence such that for all $f \in E'$

$$\lim_{n\to\infty} \left\langle f, u_{\varphi(n)}\right\rangle \overset{not.}{:=} \langle j, f\rangle.$$

By the uniform boundeness principle (Banach Steinhauss theorem) it results that $j \in E''$ and using reflexivity we obtain the existence of a point u such that for all $f \in E'$

$$\lim_{n \to +\infty} \left\langle f, u_{\varphi(n)} \right\rangle = \langle f, u \rangle .$$

Thus, $(B_E, \sigma(E, E'))$ is sequentially compact.

We obtain two imediate consequences.

Corollary 2.1 *Assume that E is a reflexive Banach space with separable dual and let $(u_n)_n$ be a bounded sequence in E. Then there exists a subsequence $(u_{\varphi(n)})_n$ that is convergent w.r.t. the weak topology $\sigma(E, E')$.*

Corollary 2.2 *Let E be a reflexive Banach space with separable dual. Let $K \subset E$ be a bounded, closed, and convex subset of E. Then K is compact in the topology $\sigma(E, E')$.*

Corrollary 2.1 is of great importance when treating problems from calculus of variations or PDEs : indeed, when considering a minimizing sequence the only information we have is that this sequence is bounded in some Sobolev space. It is here that the weak-compactness plays an important role : it provides the natural candidate for the solution of the minimization problem. We illustrate this principle in the following result witch will be reffered to as the Direct Method for proving the existence of minimizers for problems in the calculus of variations:

Theorem 2.4 (Direct Method) *Let E be a reflexive Banach space and let $J : E \to (-\infty, +\infty]$ be a weakly lower semicontinuous function, A a $\sigma(E, E')$-closed subset of E such that $J \not\equiv +\infty$ on A and*

$$\lim_{\substack{\|u\| \to \infty, \\ u \in A}} J[u] = +\infty.$$

Then J achieves its minimum on A, i.e., there exists some $\bar{u} \in A$ such that

$$J[\bar{u}] = \inf_{u \in A} J[u] > -\infty.$$

In practice, weakly lower semicontinuity is not easy to verify. The following corollary gives a first result that can be used for convex problems.

Corollary 2.3 *Let E be a reflexive Banach space and let $A \subset E$ be a nonempty, closed, convex subset of E. Let $J : E \to (-\infty, +\infty]$ be a convex lower semicontinuous function such that $J \not\equiv +\infty$ on A and*

$$\lim_{\|x\| \to \infty,\, x \in A} J(x) = +\infty. \tag{7}$$

Then J achieves its minimum on A, i.e., there exists some $\bar{u} \in A$ such that

$$J[\bar{u}] = \inf_{u \in A} J[u] > -\infty.$$