École Polytechnique

Rapport de stage

Analyse de Fourier et applications à l'équation de Keller-Segel

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Contents

1 Littlewood-Paley theory

1.1 Hölder and convolution inequalities

In this section we present some classical inequalities that will be of constant use though the rest of the paper.

Theorem 1.1.1 (Hölder) *Let (X, μ) be a measure space and $(p, q, r) \in [1, \infty]^3$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

If (f, g) belongs to $L^p(X, \mu) \times L^q(X, \mu)$ then fg belongs to $L^r(X, \mu)$ and

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

The following lemma states that Hölder's inequality is in some sense optimal.

Lemma 1.1.1 *Let (X, μ) be a measure space and $p \in [1, \infty]$. Let f be a measurable function. If*

$$\sup_{\|g\|_{L^{p'}} \leq 1} \int_x |f(x)g(x)|\, d\mu(x) < \infty$$

then f belongs to L^p and[1]

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}} \leq 1} \int_X f(x)g(x)d\mu(x).$$

The convolution between two functions makes sense for real or complex valued measurable functions defined on some locally compact topological group G endowed with a left invariant Haar measure μ[2]. Then the formal definition of convolution between two such functions reads:

$$f \star g(x) = \int_G f(y)g(y^{-1}x)d\mu(y).$$

One can now state Young's inequality for the convolution of two functions.

Theorem 1.1.2 (Young) *Let G be a locally compact topological group endowed with a left invariant Haar measure μ that satisfies in addition the following:*

$$\mu(A^{-1}) = \mu(A)$$

for all borelian sets A. Then if $(p, q, r) \in [1, \infty]^3$ such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

[1] We agree that in p' denotes the conjugate exponent of p.

[2] That means that μ is a borelian measure on G such that for any borelian set A and for any element a of g, we have $\mu(aA) = \mu A$

and for any couple $(f,g) \in L^p(G,\mu) \times L^q(G,\mu)$ *we have that* $f \star g \in L^r(G,\mu)$ *and*

$$\|f \star g\|_{L^r(G,\mu)} \leq \|f\|_{L^p(G,\mu)} \|g\|_{L^q(G,\mu)} .$$

We will often apply Young's inequality for $\mathbb{Z}$ equipped with the counting measure. In this case we will deal with the convolution of sequences $(a_k)_{k\in\mathbb{Z}}$, $(b_k)_{k\in\mathbb{Z}}$ which will be defined as:

$$(a_k)_{k\in\mathbb{Z}} \star (b_k)_{k\in\mathbb{Z}} (j) = \sum_{j'\in\mathbb{Z}} a_{j-j'} b_{j'}$$

for all $j \in \mathbb{Z}$.

1.2 Functions with compactly supported Fourier transform

In all that follows we shall call an annulus any set of the type:

$$\left\{\xi \in \mathbb{R}^d : 0 < r_1 \leq \xi \leq r_2\right\}.$$

Lemma 1.2.1 *Let* $\mathcal{C}$ *be an annulus* $m \in \mathbb{R}$ *and* $k = 2[1 + d/2]$*. Let* σ *be a* k*-times differentiable function on* $\mathbb{R}^d \backslash \{0\}$ *such that for any* $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \leq k$*, there exists a constant* C_α *such that for all* $\xi \in \mathbb{R}^d$ *we have:*

$$|\partial^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{m-|\alpha|} .$$

Then there exists a constant C *depending only on the constants* C_α *such that for any* $p \in [1,\infty]$*, any* $\lambda > 0$ *and for any function* $u \in L^p$ *with Fourier transform supported in* $\mathcal{C}$*, we have:*

$$\|\sigma(D)u\|_{L^p} \leq C\lambda^m \|u\|_{L^p} .$$

Proof. Let us consider $\phi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ such that it takes value 1 on $\mathcal{C}$. It is clear that we have:

$$\sigma(D)u = \lambda^d K_\lambda(\lambda\cdot) \star u;$$
$$K_\lambda(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} \phi(\xi)\sigma(\lambda\xi) d\xi.$$

Let $M = [1 + d/2]$. We have:

$$\begin{aligned}\left(1+|x|^2\right)^M K_\lambda(x) &= \int_{\mathbb{R}^d} (\mathbb{I}-\Delta_\xi)^M e^{i\langle\xi,x\rangle}\phi(\xi)\sigma(\lambda\xi)d\xi \\ &= \int_{\mathbb{R}^d} e^{i\langle\xi,x\rangle} (\mathbb{I}-\Delta_\xi)^M \phi(\xi)\sigma(\lambda\xi)d\xi \\ &= \sum_{|\alpha|+|\beta|\leq 2M} c_{\alpha,\beta}\lambda^\beta \int_{\mathbb{R}^d} e^{i\langle\xi,x\rangle}\partial^\alpha\phi(\xi)\partial^\beta\sigma(\lambda\xi)d\xi,\end{aligned}$$

where $c_{\alpha,\beta}$ are integers. Of course, the integration may be restricted to $Supp(\phi)$ thus, in virtue of the hypothesis, we obtain:

$$\left(1+|x|^2\right)^M |K_\lambda(x)| \leq C_M \lambda^m.$$

Because $2M > d$ we have that the integral in the left hand side of the bellow inequality is well-defined:

$$\|K_\lambda\|_{L^1} \leq \int\limits_{\mathbb{R}^d} \frac{dx}{\left(1+|x|^2\right)^M} \left\| \left(1+|\cdot|^2\right)^M |K_\lambda(\cdot)| \right\|_{L^\infty}$$

and Young's inequality yields the desired result. ∎

Lemma 1.2.2 *Let $\mathcal{C}$ be an annulus and $\mathcal{B}$ a ball. Then a constant $C > 0$ exists so that for any positive integer k, any couple $(p,q) \in [1,\infty]^2$ with $q \geq p \geq 1$ and any function of L^p, we have:*

$$Supp(\hat{u}) \subset \lambda\mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{L^p};$$

$$Supp(\hat{u}) \subset \lambda\mathcal{C} \Rightarrow C^{-k-1}\lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1}\lambda^k \|u\|_{L^p}.$$

Proof. Let $\lambda = 1$. Let us consider $\phi \in \mathcal{D}(\mathbb{R}^d)$ such that ϕ takes the value 1 near $\mathcal{B}$ and let $\psi = \mathcal{F}^{-1}\phi$. From the support condition we gather that:

$$u = \psi \star u \Rightarrow \partial^\alpha u = (\partial^\alpha \psi) \star u.$$

Thus Young' inequality assures us that:

$$\|(\partial^\alpha\psi) \star u\|_{L^q} \leq \|\partial^\alpha \psi\|_{L^r} \|u\|_{L^p}$$

where

$$\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}.$$

Next, we have:

$$\begin{aligned}\|\partial^\alpha\psi\|_{L^r} &\leq \|\partial^\alpha\psi\|_{L^\infty}^{\frac{r-1}{r}} \|\partial^\alpha\psi\|_{L^1}^{\frac{1}{r}} \leq \frac{r-1}{r}\|\partial^\alpha\psi\|_{L^\infty} + \frac{1}{r}\|\partial^\alpha\psi\|_{L^1} \\ &\leq \|\partial^\alpha\psi\|_{L^\infty} + \|\partial^\alpha\psi\|_{L^1}.\end{aligned}$$

Because

$$\forall x \in \mathbb{R}^d: \ |\partial^\alpha\psi(x)| \leq \left(1+|x|^2\right)^d |\partial^\alpha\psi(x)|,$$

$$\|\partial^\alpha\psi\|_{L^1} \leq \int_{\mathbb{R}^d} \frac{dx}{\left(1+|x|^2\right)^d} \left\|(1+|\cdot|)^d \partial^\alpha\psi\right\|_{L^\infty},$$

we get that:

$$
\begin{aligned}
\|\partial^\alpha \psi\|_{L^r} &\le C \left\| (1+|\cdot|)^d \, \partial^\alpha \psi \right\|_{L^\infty} \\
&\le C \left\| \mathcal{F}^{-1}\left((1+|\cdot|)^d \, \partial^\alpha \psi \right) \right\|_{L^1} \\
&\le C \left\| (\mathbb{I}-\Delta)^d \left((\cdot)^\alpha \, \phi \right) \right\|_{L^1} \\
&\le C^{k+1}.
\end{aligned}
$$

Thus we have proved the desired inequality for $\lambda = 1$. Let us consider $u \in L^p$ with $Supp(\hat{u}) \subset \lambda\mathcal{B}$; applying what we proved for $u_\lambda = \frac{1}{\lambda^d} u(\lambda \cdot)$ we find the general result. Let us move towards the second inequality. Due to an argument similar to that one of above, we shall prove the statement only in the case $\lambda = 1$.

Let us consider a function $\tilde{\phi} \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ taking value 1 near $\mathcal{C}$. Let us observe that for all $\xi \in \mathbb{R}^d \backslash \{0\}$ we have the following algebraic identity:

$$
\begin{aligned}
1 = \frac{|\xi|^{2k}}{|\xi|^{2k}} &= \frac{1}{|\xi|^{2k}} \left(\sum_{i=1}^d \xi_i^2 \right)^k = \frac{1}{|\xi|^{2k}} \sum_{|\alpha|=k} A_\alpha \xi^{2\alpha} \\
&= \frac{1}{|\xi|^{2k}} \sum_{|\alpha|=k} A_\alpha (i\xi)^\alpha (-i\xi)^\alpha.
\end{aligned}
$$

Thus we get:

$$
u = \sum_{|\alpha|=k} g_\alpha \star \partial^\alpha u, \tag{1.2.1}
$$

with g_α defined as

$$
g_\alpha = A_\alpha \mathcal{F}^{-1}\left(\frac{(-i\xi)^\alpha}{|\xi|^{2k}} \phi \right).
$$

Thus, applying the triangle inequality along with Young's inequality we have

$$
\|u\|_{L^p} \le \left(\sum_{|\alpha|=k} \|g_\alpha\|_{L^1} \right) \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p}
$$

But in view of lemma (1.2.1) we have that:

$$
\begin{aligned}
\|g_\alpha\|_{L^1} = A_\alpha \left\| \mathcal{F}^{-1}\left(\frac{(-i\xi)^\alpha}{|\xi|^{2k}} \phi \right) \right\|_{L^1} \le A_\alpha C \left\| \mathcal{F}^{-1}\phi \right\|_{L^1} \\
\sum_{|\alpha|=k} \|g_\alpha\|_{L^1} \le C \sum_{|\alpha|=k} A_\alpha = C d^k \le C^{k+1}.
\end{aligned}
$$

This last inequality obviously ends the proof. ■

Lemma 1.2.3 *Let $\mathcal{C}$ be an annulus. Then there exist two positive constants c, C such that for any $p \in [1, \infty]$ and any couple (t, λ) of positive real numbers we have:*

$$Supp(\hat{u}) \subset \lambda\mathcal{C} \Rightarrow \left\|e^{t\Delta}u\right\|_{L^p} \leq Ce^{-ct\lambda^2}\left\|u\right\|_{L^p}.$$

Proof. Let us consider a function $\phi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ which takes value 1 near the annulus $\mathcal{C}$. As above we shall prove only for $\lambda = 1$. Then we have that:

$$\begin{aligned} e^{t\Delta}u &= \mathcal{F}^{-1}\left(e^{-t|\xi|^2}\hat{u}\right) = \mathcal{F}^{-1}\left(e^{-t|\xi|^2}\phi\hat{u}\right) \\ &= g(t,\cdot) \star u, \end{aligned} \tag{1.2.2}$$

where g is defined by:

$$g(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\phi e^{-t|\xi|^2}d\xi.$$

We shall prove that we can find two constants c and C such that for all $t > 0$ we have:

$$\left\|g(t,\cdot)\right\|_{L^1} \leq Ce^{-ct}. \tag{1.2.3}$$

Obviously once (1.2.3) is proved the lemma follows from Young's inequality applied to relation (1.2.2).

$$\begin{aligned} g(t,x) &= \frac{1}{\left(1+|x|^2\right)^d}\int_{\mathbb{R}^d}\left(1+|x|^2\right)^d e^{i\langle x,\xi\rangle}\phi(\xi)e^{-t|\xi|^2}d\xi \\ &= \frac{1}{\left(1+|x|^2\right)^d}\int_{\mathbb{R}^d}\left(\mathbb{I}-\Delta_\xi\right)^d e^{i\langle x,\xi\rangle}\phi(\xi)e^{-t|\xi|^2}d\xi \\ &= \frac{1}{\left(1+|x|^2\right)^d}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\left(\mathbb{I}-\Delta_\xi\right)^d\left(\phi(\xi)e^{-t|\xi|^2}\right)d\xi. \end{aligned}$$

From Leibniz formula we obtain:

$$\left(\mathbb{I}-\Delta_\xi\right)^d\left(e^{-t|\cdot|^2}\phi\right) = \sum_{\substack{\beta\leq\alpha \\ |\alpha|\leq 2d}} C_{\alpha,\beta}\partial^{(\alpha-\beta)}\phi\partial^\beta e^{-t|\cdot|^2}.$$

According to Faà-di-Bruno's formula we have that:

$$e^{t|\xi|^2}\partial^\beta\left(e^{-t|\xi|^2}\right) = \sum_{\substack{\gamma_1+\ldots+\gamma_m=\beta \\ |\gamma_j|\geq 1}} (-t)^m\prod_{j=1}^m \partial^{\gamma_j}(|\xi|^2),$$

Next, for any $\xi \in Supp(\phi)$ we have that:

$$\begin{aligned} \left|\partial^{(\alpha-\beta)}\phi(\xi)\partial^\beta e^{-t|\xi|^2}\right| &\leq C(1+t)^{|\beta|}e^{-t|\xi|^2} \\ &\leq C(1+t)^{|\beta|}e^{-ct}. \end{aligned}$$

Thus putting together all the above relations we get (1.2.3). ■

From now on, we agree that if I is an interval of $\mathbb{R}$, X is a Banach space and $p \in [1, \infty]$, then $L^p_I(X)$ stands for the set of Lebesgue measurable functions $u : I \to X$ such that $t \mapsto \|u(t)\|_X$ belongs to $L^p(I)$. If $I = [0, T]$ or $I = \mathbb{R}^+$ we shall use the notations $L^p_T(X)$ respectively $L^p(X)$.

Corollary 1.2.1 *Let $\mathcal{C}$ be an annulus and λ a positive real number. Let u_0 and $f = f(t, x)$ having the property that $Supp(u_0) \subset \lambda\mathcal{C}$ and for all $t > 0$, $Supp(f(t)) \subset \lambda\mathcal{C}$. Let us consider u the solution of*

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0, \end{cases}$$

and v a solution of

$$\begin{cases} \partial_t v - \Delta v = f \\ v_{|t=0} = 0. \end{cases}$$

Then there exist two positive constants c and C such that for any $1 \le a \le b \le \infty$ and $1 \le p \le q \le \infty$, we have:

$$\|u\|_{L^q_T(L^b)} \le C\lambda^{-\frac{2}{q}+d(\frac{1}{a}-\frac{1}{b})} \|u_0\|_{L^a},$$
$$\|v\|_{L^q_T(L^b)} \le C\lambda^{-1+\left(\frac{1}{p}-\frac{1}{q}\right)+d\left(\frac{1}{a}-\frac{1}{b}\right)} \|f\|_{L^p_T(L^a)}.$$

Proof. The desired inequalities come from the fact that

$$u(t) = e^{\Delta t} u_0$$

and

$$v(t) = \int_0^t e^{(t-t')\Delta} f(t') dt'.$$

■

1.3 Dyadic partition of unity

Proposition 1.3.1 *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^d : 3/4 \le |\xi| \le 8/3\}$. There exist two radial functions $\chi \in \mathcal{D}(B(0, 4/3))$ and $\varphi \in \mathcal{D}(\mathcal{C})$ valued in the interval $[0, 1]$ and such that:*

$$\forall \xi \in \mathbb{R}^d, \ \ \chi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1, \tag{1.3.1}$$

$$\forall \xi \in \mathbb{R}^d \backslash \{0\}, \ \ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{1.3.2}$$

$$2 \le |j - j'| \Rightarrow Supp(\varphi(2^{-j}\cdot)) \cap Supp(\varphi(2^{-j'}\cdot)) = \emptyset \tag{1.3.3}$$

$$j \ge 1 \Rightarrow Supp(\chi) \cap Supp(\varphi(2^{-j}\cdot)) = \varnothing \tag{1.3.4}$$

the set $\tilde{\mathcal{C}} = \mathcal{B}(0, 2/3) + \mathcal{C}$ *is an annulus and we have*

$$|j - j'| \geq 5 \Rightarrow 2^j \mathcal{C} \cap 2^{j'} \tilde{\mathcal{C}} = \varnothing. \tag{1.3.5}$$

Also the following inequalities hold true:

$$\forall \xi \in \mathbb{R}^d, \;\; \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1, \tag{1.3.6}$$

$$\forall \xi \in \mathbb{R}^d \backslash \{0\}, \;\; \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{1.3.7}$$

Proof. Let $\alpha \in (1, 4/3)$ and denote by $\mathcal{C}'$ the annulus of small radius α^{-1} and big radius 2α. Let us choose a radial function θ with values in $[0,1]$ supported in $\mathcal{C}$ and with value 1 on a neghborhood of $\mathcal{C}'$. Let us observe that for any couple of integers (j, j') we have that:

$$|j - j'| \geq 2 \Rightarrow 2^j \mathcal{C} \cap 2^{j'} \mathcal{C} = \varnothing. \tag{1.3.8}$$

Indeed, let us suppose that $j' \geq j$ and that the set $2^j\mathcal{C} \cap 2^{j'}\mathcal{C}$ is non-empty; then $2^{j'}\frac{3}{4} \leq 2^j \frac{4}{3}$ which implies that $j' - j \leq 1$. Let us set

$$S(\xi) = \sum_{j \in \mathbb{Z}} \theta(2^{-j}\xi).$$

Because of (1.3.8) the sum is locally finite on the set $\mathbb{R}^d \backslash \{0\}$. Thus the function S is smooth on $\mathbb{R}^d \backslash \{0\}$. As $\alpha > 1$ we get that:

$$\bigcup_{j \in \mathbb{Z}} 2^j \mathcal{C}' = \mathbb{R}^d \backslash \{0\}.$$

As the function θ is nonnegative and has value 1 near $\mathcal{C}'$, it follows from the above property that the function S is positive. Let us set:

$$\varphi = \frac{\theta}{S}.$$

It is clear that we have $\varphi \in \mathcal{D}(\mathcal{C})$ and that the function $1 - \sum_{j \geq 0} \varphi(2^{-j}\cdot)$ is smooth and because $Supp(\theta) \subset \mathcal{C}$ we get that:

$$|\xi| \geq \frac{4}{3} \Rightarrow \sum_{j \geq 0} \varphi(2^{-j}\cdot) = 1. \tag{1.3.9}$$

Thus setting

$$\chi(\xi) = 1 - \sum_{j \geq 0} \varphi(2^{-j}\cdot),$$

we get the identities (1.3.1) and (1.3.3). Identity (1.3.4) is an obvious consequence of (1.3.8) and of (1.3.9) Now let us prove (1.3.5). It is clear that $\tilde{\mathcal{C}}$ the annulus of center 0 of small radius 1/12 and of big radius 10/3. Then we have that:

$$2^k\tilde{\mathcal{C}} \cap 2^j\mathcal{C} \neq \varnothing \Rightarrow 2^j\frac{3}{4} \leq 2^k\frac{10}{3} \text{ or } 2^k\frac{1}{12} \leq 2^j\frac{8}{3}.$$

Now let us prove (1.3.6): as χ and φ have their values in $[0,1]$ it is clear from (1.3.1) that

$$\chi^2(\xi) + \sum_{j\geq 0}\varphi^2(2^{-j}\xi) \leq 1.$$

Let us bound from below the sum:

$$1 = \left(\chi(\xi) + \sum_{j\geq 0 \text{ odd}}\varphi(2^{-j}\xi) + \sum_{j\geq 0 \text{ even}}\varphi(2^{-j}\xi)\right)^2.$$

It is clear that:

$$\begin{aligned} 1 &\leq 2\left(\chi(\xi) + \sum_{j \text{ odd}}\varphi(2^{-j}\xi)\right)^2 + \left(\sum_{j \text{ even}}\varphi(2^{-j}\xi)\right)^2 \\ &= \chi^2(\xi) + \sum_{j\geq 0}\varphi^2(2^{-j}\xi). \end{aligned}$$

The last equality comes from the support conditions. Proving (1.3.7) is similar. ■

From now on we fix two functions χ and φ satisfying the assertions of the above proposition. Let us denote by $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low frequency cut-off operators $\dot{S}_j$ are defined as:

$$\begin{aligned} \dot{\Delta}_j u &= \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} h(2^{jd}y)u(x-y)dy, \\ \dot{S}_j u &= \chi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} \tilde{h}(2^{jd}y)u(x-y)dy. \end{aligned}$$

Remark 1.3.1 *The above operators map L^p into L^p with norms independent of j and p.*

Note that at least formally we can write the Littlewood-Paley decomposition:

$$\mathbb{I} = \sum_{j\in\mathbb{Z}}\dot{\Delta}_j. \tag{1.3.10}$$

The above decomposition fails to hold for all tempered distributions: it fails for nonzero polynomials. Thus we shall work in a more restrictive framework in order for (1.3.10) to make sense. This is the purpose of the following definition:

Definition 1.3.1 *We denote by S'_h the space of tempered distributions u such that:*

$$\lim_{\lambda\to\infty} \|\theta(\lambda D)\, u\|_{L^\infty} = 0,$$

for any θ in $\mathcal{D}(\mathbb{R}^d)$.

Remark 1.3.2 *It is not hard to check that $u \in S'_h$ if and only if one can find a smooth compactly supported function θ satisfying the above equality and with the property that it takes the value* 1 *near* 0*.*

Let us give some examples of distributions belonging to S'_h:

- If a tempered distribution u is such that its Fourier transform $\hat{u}$ is locally integrable near 0, then u belongs to S'_h. In particular, the space $\mathcal{E}$ of compactly supported distributions is included in S'_h.
- If u is a tempered distribution such that $\theta(D)u \in L^p$ for some $p \in [1,\infty)$ and some $\theta \in \mathcal{D}(\mathbb{R}^d)$ with $\theta(0) \neq 0$ then $u \in S'_h$.
- A nonzero polynomial P does not belong to S'_h because for any $\theta \in \mathcal{D}(\mathbb{R}^d)$ with value 1 near the origin and any $\lambda > 0$, one may write $\theta(\lambda D)P = P$. However if $\eta \in \mathbb{R}^d \backslash \{0\}$ then $e^{i\langle\cdot,\eta\rangle}P$ belongs to S'_h because the support of its Fourier transform is $\{\eta\}$. This example implies that S'_h is not a closed subspace of S' for the topology of weak convergence.

Proposition 1.3.2 *Let $u \in S'(\mathbb{R}^d)$. Then we have*

$$u = \lim_{j\to\infty} \dot{S}_j u \text{ in } S'(\mathbb{R}^d).$$

Proof. Let us observe that $\left\langle u - \dot{S}_j u,\ f\right\rangle = \left\langle u,\ f - \dot{S}_j f\right\rangle$ for all $f \in S(\mathbb{R}^d)$ so we have to prove that $f = \lim_{j\to\infty} \dot{S}_j f$ in the $S(\mathbb{R}^d)$. Because the Fourier transform acts like an automorphism on $S(\mathbb{R}^d)$ it is enough to prove that $\chi(2^j\cdot)\hat{f} \to \hat{f}$ in S when $j \to \infty$ which is obvious. ■

Let us observe that if $u \in S'_h$ then we get that (1.3.10) holds true as $\dot{S}_j u \to 0$ when $j \to -\infty$. We finish this paragraph with two results of convergence of series of functions which are spectrally localized.

Proposition 1.3.3 *Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of bounded functions such that the Fourier transform of u_j is supported in $2^j\mathcal{C}$ where $\mathcal{C}$ is a given annulus. Let us assume that for some integer N, the sequence $\left(2^{-jN}\|u_j\|_{L^\infty}\right)_{j\in\mathbb{N}}$ is bounded. Then the series $\sum_{j\in\mathbb{N}} u_j$ converges in S'.*

Proof. We observe that by rescaling and using (1.2.1) we obtain that for all integers j and k we have

$$u_j = 2^{-jk} \sum_{|\alpha|=k} 2^{jd} g_\alpha(2^j\cdot) \star \partial^\alpha u_j.$$

Then for any $\phi \in S$ we have:

$$\langle u_j, \phi \rangle = 2^{-jk} \sum_{|\alpha|=k} \left\langle u_j, 2^{jd} g_\alpha(-2^j \cdot) \star (-\partial)^\alpha \phi \right\rangle$$

and thus we get:

$$|\langle u_j, \phi \rangle| \leq C 2^{-jk} \sum 2^{jN} \|\partial^\alpha \phi\|_{L^1}.$$

Let us chose $k > N$. Then $\sum_j \langle u_j, \phi \rangle$ is a convergent series, the sum of which is less than $C \|\phi\|_{M,S}$ for some integer M. Thus the formula

$$\langle u, \phi \rangle = \lim_{j \to \infty} \sum_{j' \leq j} \langle u_{j'}, \phi \rangle,$$

defines a tempered distribution. ■

Proposition 1.3.4 *Let $(u_j)_{j\in\mathbb{Z}}$ be a sequence of bounded functions such that the support of $\hat{u}_j$ is included in $2^j \mathcal{C}$ where $\mathcal{C}$ is a given annulus. Let us assume that for some integer N, the sequence $\left(2^{-jN} \|u_j\|_{L^\infty}\right)_{j\in\mathbb{N}}$ is bounded and that the series $\sum_{j<0} u_j$ converges in L^∞. Then the series $\sum_{j\in\mathbb{Z}} u_j$ converges to some $u \in S'$ and $u \in S'_h$.*

Proof. Thanks to the above proposition we know that the series $\sum_{j\in\mathbb{Z}} u_j$ converges to some u in S'. The only thing left to prove is that $u \in S'_h$. For some integer N_0 we have that

$$\left\|\dot{S}_j u\right\|_{L^\infty} \leq \left\|\dot{S}_j \sum_{j' \leq j+N_0} u_{j'}\right\|_{L^\infty} \leq C \left\|\sum_{j' \leq j+N_0} u_{j'}\right\|_{L^\infty}.$$

The fact that the series $\sum_{j<0} u_j$ converges in L^∞ concludes the proof. ■

1.4 Homogeneous Besov spaces

In this paragraph we shall define and establish some properties of homogeneous Besov spaces which will be used in the second part of this paper when proving existence and uniqueness results for the Keller-Segel equation.

Definition 1.4.1 *Let s be a real number, $(p,r) \in [1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is the subset of distributions $u \in S'_h$ such that:*

$$\|u\|_{\dot{B}^s_{p,r}} = \left\| \left(2^{js} \left\|\dot{\Delta}_j u\right\|_{L^p}\right)_{j\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

Proposition 1.4.1 *The space $\dot{B}^s_{p,r}$ endowed with $\|\cdot\|_{\dot{B}^s_{p,r}}$ is a normed space.*

Proof. It is obvious that $\|\cdot\|_{\dot{B}^s_{p,r}}$ is a semi-norm. Let us assume that for some $u \in S'_h$, we have that $\|u\|_{\dot{B}^s_{p,r}} = 0$. This implies that the support of $\hat{u}$ is included in $\{0\}$ and thus $\dot{S}_j u = u$ for all $j \in \mathbb{Z}$. As $u \in S'_h$ we conclude that $u = 0$. ∎

Remark 1.4.1 *The definition of the Besov space $\dot{B}^s_{p,r}$ is independent of the function φ used for defining the dyadic blocks $\dot{\Delta}_j$ and changing φ yields an equivalent norm. Indeed, if $\tilde{\varphi}$ is another function having the properties as in Proposition (1.3.1) then there exists an integer N_0 such that if $|j - j'| \geq N_0$ then Supp$(2^{-j}\cdot) \cap$ Supp$(2^{-j'}\cdot) = \varnothing$. Thus we get:*

$$\begin{aligned}
2^{js} \left\|\tilde{\varphi}(2^{-j}D)u\right\|_{L^p} &= 2^{js} \left\| \sum_{|j-j'|\leq N_0} \tilde{\varphi}(2^{-j}D)\dot{\Delta}_{j'}u \right\|_{L^p} \\
&\leq C2^{N_0|s|} \sum_{j'} \mathbf{1}_{[-N_0,N_0]}(j-j')2^{j's} \left\|\dot{\Delta}_{j'}u\right\|_{L^p} \\
&= C2^{N_0|s|} \left(\mathbf{1}_{[-N_0,N_0]}(k)\right)_{k\in\mathbb{Z}} \star \left(2^{ks} \left\|\dot{\Delta}_k u\right\|_{L^p}\right)_{k\in\mathbb{Z}}.
\end{aligned}$$

Young's inequality implies the result.

Remark 1.4.2 *A distribution $u \in S'_h$ belongs to $\dot{B}^s_{p,r}$ if and only if there exists some constant C and a sequence with positive terms $(c_j)_{j\in\mathbb{Z}}$ such that $\left\|(c_j)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} = 1$ and for all $j \in \mathbb{Z}$ we have:*

$$\left\|\dot{\Delta}_j u\right\|_{L^p} \leq Cc_j 2^{-js}.$$

Homogeneous Besov spaces have nice scaling properties. Indeed if u is a tempered distribution then let us consider the tempered distribution defined by $u_N = u(2^N\cdot)$. Then we have the following:

Proposition 1.4.2 *Let us consider an integer N and a distribution $u \in S'_h$. Then $\|u\|_{\dot{B}^s_{p,r}}$ is finite if and only if $\|u_N\|_{\dot{B}^s_{p,r}}$ is finite. Moreover we have that:*

$$\|u_N\|_{\dot{B}^s_{p,r}} = 2^{N(s-\frac{d}{p})} \|u\|_{\dot{B}^s_{p,r}}.$$

Proof. By definition of $\dot{\Delta}_j$ and by the change of variable $z = 2^N y$, we get that

$$\begin{aligned}
\dot{\Delta}_j u_N(x) &= 2^{jd} \int_{\mathbb{R}^d} h(2^j(x-y))u(2^N y)dy \\
&= 2^{(j-N)d} \int_{\mathbb{R}^d} h(2^{j-N}(2^N x - z))u(z)dz \\
&= \dot{\Delta}_{j-N}u(2^N x).
\end{aligned}$$

Thus we get the following:

$$\left\|\dot{\Delta}_j u_N\right\|_{L^p} = 2^{-N\frac{d}{p}} \left\|\dot{\Delta}_{j-N} u\right\|_{L^p},$$
$$2^{js}\left\|\dot{\Delta}_j u_N\right\|_{L^p} = 2^{N(s-\frac{d}{p})} 2^{(j-N)s} \left\|\dot{\Delta}_{j-N} u\right\|_{L^p}.$$

The conclusion follows immediately by summation. ■

Remark 1.4.3 *More generally, there exists a constant C depending only on s such that for all positive λ we have that:*

$$C^{-1}\lambda^{s-\frac{d}{p}} \|u\|_{\dot{B}^s_{p,r}} \leq \|u_\lambda\|_{\dot{B}^s_{p,r}} \leq C\lambda^{s-\frac{d}{p}} \|u\|_{\dot{B}^s_{p,r}}.$$

where $u_\lambda = u(\lambda\cdot)$.

Proposition 1.4.3 *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then, for any real number s the space $\dot{B}^s_{p_1,r_1}$ is continuously embedded in $\dot{B}^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$.*

Proof. Lemma 1.2.2 yields

$$\left\|\dot{\Delta}_j u\right\|_{L^{p_2}} \leq C 2^{jd\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \left\|\dot{\Delta}_j u\right\|_{L^{p_1}}$$

so because $\ell^{r_1}(\mathbb{Z})$ is continuously embedded in $\ell^{r_2}(\mathbb{Z})$, the proposition is proved. ■

A important feature of the Besov spaces in contrast with the other more classical functional spaces is that it contains nontrivial homogeneous functions.

Proposition 1.4.4 *Let σ be in $(0,d)$. For any $p \in [1,\infty]$ the function $|\cdot|^{-\sigma}$ belongs to $\dot{B}^{\frac{d}{p}-\sigma}_{p,\infty}$.*

Proof. Using the previous proposition it is enough to prove that $\rho_\sigma = |\cdot|^{-\sigma}$ belongs to $\dot{B}^{d-\sigma}_{1,\infty}$. In order to achieve this let us consider a smooth compactly supported function χ which is identically equal to 1 near the unit ball an let us write

$$\rho_\sigma = \rho_0 + \rho_1$$

with ρ_0 and ρ_1 defined as:

$$\rho_0 = \chi(x)\rho_\sigma, \ \rho_1 = (1-\chi(x))\,\rho_\sigma.$$

A simple computation leads us to the fact that $\rho_0 \in L^1$ and $\rho_1 \in L^q$ for all $q > d/\sigma$. This implies that $\rho_\sigma \in S'_h$ and because of the homogeneity of ρ_σ we have:

$$\begin{aligned}\dot{\Delta}_j \rho_\sigma &= 2^{jd}\rho_\sigma \star h(2^j\cdot) = 2^{j(d+\sigma)}\rho_\sigma(2^j\cdot) \star h(2^j\cdot)\\ &= 2^{j\sigma}\dot{\Delta}_0\rho_\sigma(2^j\cdot).\end{aligned}$$

Therefore we get that $\left\|\dot{\Delta}_j \rho_\sigma\right\|_{L^1} = 2^{j(\sigma-d)} \left\|\dot{\Delta}_0 \rho_\sigma\right\|_{L^1}$ which reduces the problem to proving that the function $\dot{\Delta}_0 \rho_\sigma \in L^1$. Because $\rho_0 \in L^1$ then $\dot{\Delta}_0 \rho_0 \in L^1$. Using lemma 1.2.2 we get that:

$$\left\|\dot{\Delta}_0 \rho_1\right\|_{L^1} \leq C \sup_{|\alpha|=k} \left\|\partial^\alpha \dot{\Delta}_0 \rho_1\right\|_{L^1} \leq C \sup_{|\alpha|=k} \left\|\partial^\alpha \rho_1\right\|_{L^1}.$$

By the Leibniz formula, we get that $\sup_{|\alpha|=k} \rho_1 - (1-\chi) \sup_{|\alpha|=k} \rho_\sigma$ is a smooth compactly supported function. Then, we conclude the theorem by choosing k such that $k > d - \sigma$. ■

The following lemma provides a useful criterion to determinate whether the sum of a series belongs to some homogeneous Besov space.

Lemma 1.4.1 *Let $\mathcal{C}$ be an annulus and $(u_j)_{k\in\mathbb{Z}}$ be a sequence of functions such that the support of the Fourier transform of u_j is contained in $2^j\mathcal{C}$ and*

$$\left\|\left(2^{js} \|u_j\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} < \infty.$$

If the series $\sum_{j\in\mathbb{Z}}$ converges in S' to some $u \in S'_h$ then $u \in \dot{B}^s_{p,r}$ and there exists a constant depending only on s such that:

$$\|u\|_{\dot{B}^s_{p,r}} \leq C_s \left\|\left(2^{js} \|u_j\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}.$$

Proof. It is clear that there exists a integer N such that $\dot{\Delta}_{j'} u_j = 0$ if $|j - j'| \geq N$. Therefore we obtain

$$\left\|\dot{\Delta}'_j u\right\|_{L^p} \leq C \sum_{|j-j'|\leq N} \left\|\dot{\Delta}_{j'} u_j\right\|_{L^p}$$

$$2^{j's} \left\|\dot{\Delta}'_j u\right\|_{L^p} \leq C \sum_{|j-j'|\leq N} 2^{js} \|u_j\|_{L^p}$$

$$2^{j's} \left\|\dot{\Delta}'_j u\right\|_{L^p} \leq \left(1_{[-N,N]}(j)\right)_{k\in\mathbb{Z}} \star \left(2^{ks} \|u_k\|\right)_{k\in\mathbb{Z}} (j')$$

so that applying Young's inequality and taking the $\ell^r(\mathbb{Z})$ norm yields the desired result. ■

Remark 1.4.4 *Let us point out that if (s, p, r) satisfy the following condition:*

$$s < \frac{d}{p} \text{ or } s = \frac{d}{p} \text{ and } r = 1 \tag{1.4.1}$$

then owing to lemma 1.2.2 we get that

$$\lim_{j\to-\infty} \sum_{j'<j} u_{j'} = 0$$

in L^∞ *and thus* $\sum_{j\in\mathbb{Z}} u_j$ *converges to some* $u \in S'$ *and* $\dot{S}_j u$ *tends to* 0 *when* j *goes to* $-\infty$*. In particular we do have that* $u \in S'_h$*.*

Theorem 1.4.1 *Let* $(s_1, s_2) \in \mathbb{R}^2$ *and* $p_1, p_2, r_1, r_2 \in [1, \infty]$*. Let us assume that* (s_1, p_1, r_1) *satisfies* $s_1 < d/p_1$*. Then the space* $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ *endowed with the norm* $\|\cdot\|_{\dot{B}^{s_1}_{p_1,r_1}} + \|\cdot\|_{\dot{B}^{s_2}_{p_2,r_2}}$ *is complete and satisfies the Fatou property: if* $(u_n)_{n\in\mathbb{N}}$ *is a bounded sequence of* $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ *then an element* $u \in \dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ *and a subsequence* $u_{\psi(n)}$ *exist such that*

$$\lim_{n\to\infty} u_{\psi(n)} = u$$

in S' *and also for* $k = 1, 2$ *we have that:*

$$\|u\|_{\dot{B}^{s_k}_{p_k,r_k}} \le \lim_{n\to\infty} \inf \left\|u_{\psi(n)}\right\|_{\dot{B}^{s_k}_{p_k,r_k}}.$$

Proof. We first prove the Fatou property. According to lemma 1.2.2 for any $j \in \mathbb{Z}$ the sequence $\left(\dot{\Delta}_j u_n\right)_{n\in\mathbb{N}}$ is bounded in $L^{\min(p_1,p_2)} \cap L^\infty$. Using Cantor's diagonal process gives rise to a sequence $\left(u_{\psi(n)}\right)_{n\in\mathbb{N}}$ and a sequence of C^∞ functions $(\tilde{u}_j)_{j\in\mathbb{Z}}$ with Fourier transform supported in $2^j\mathcal{C}$ such that for any $j \in \mathbb{Z}$ and $k \in 1, 2$ we have

$$\lim_{n\to\infty} \left\langle \dot{\Delta}_j u_{\psi(n)}, \phi \right\rangle = \langle \tilde{u}_j, \phi \rangle$$

and also that

$$\|\tilde{u}_j\|_{L^{p_k}} \le \lim_{n\to\infty} \inf \left\|\dot{\Delta}_j u\right\|_{L^{p_k}}.$$

Now since the sequence $\left(2^{js_k} \left\|\dot{\Delta}_j u_{\psi(n)}\right\|_{L^{p_k}}\right)_{j\in\mathbb{Z}}$ is bounded in ℓ^{r_k} there exists an element $\left(\tilde{c}^k_j\right)_{j\in\mathbb{Z}}$ of ℓ^{r_k} such that up to an omitted extraction we have for any sequence $(d_j)_{j\in\mathbb{Z}}$ of nonnegative real numbers different from 0 only for a finite number of indice j,

$$\lim_{n\to\infty} \sum_{j\in\mathbb{Z}} 2^{js_k} \left\|\dot{\Delta}_j u_{\psi(n)}\right\|_{L^{p_k}} d_j = \sum_{j\in\mathbb{Z}} \tilde{c}^k_j d_j$$

and that

$$\left\|\left(\tilde{c}^k_j\right)_{j\in\mathbb{Z}}\right\|_{\ell^{r_k}} \le \lim_{n\to\infty} \inf \left\|u_{\psi(n)}\right\|_{\dot{B}^{s_k}_{p_k,r_k}}.$$

Passing to the limit in the sum and using lemma 1.1.1 gives us that $\left(2^{js_k} \|\tilde{u}_j\|_{L^{p_k}}\right)_{j\in\mathbb{Z}}$ belongs to ℓ^{r_k}. From the definition of $\tilde{u}_j$ we easily gather that $\mathcal{F}\tilde{u}_j$ is supported in the annulus $2^j\mathcal{C}$. Because (s_1, p_1, r_1) satisfies (1.4.1) then the previous lemma assures us that the series $\sum_{j\in\mathbb{Z}} \tilde{u}_j$ converges to some $u \in S'_h$. For all $M < N$ and $\phi \in S$ we have that

$$\left\langle \sum_{j=M}^{N} \dot{\Delta}_j u, \phi \right\rangle = \left\langle \sum_{j=M}^{N} \sum_{|j-j'|\le 1} \dot{\Delta}_j \tilde{u}_{j'}, \phi \right\rangle.$$

Hence owing to the definition of $\tilde{u}_j$ we get that

$$\sum_{j=M}^{N} \dot{\Delta}_j u = \lim_{n\to\infty} \sum_{j=M}^{N} \dot{\Delta}_j u_{\psi(n)}$$

in S'. Since condition (1.4.1) is satisfied by (s_1, p_1, r_1) and $\left(u_{\psi(n)}\right)_{n\in\mathbb{N}}$ is bounded in $\dot{B}^{s_1}_{p_1,r_1}$, lemma 1.2.2 assures us $\dot{S}_M u_{\psi(n)}$ tends uniformly to 0 when M goes to $-\infty$. Similarly $\left(\mathbb{I} - \dot{S}_N\right) u_{\psi(n)}$ tends to 0 in $\dot{B}^{s_2}_{p_2,r_2}$ such that u is the limit of $\left(u_{\psi(n)}\right)_{n\in\mathbb{N}}$ which completes the proof of the Fatou property. Finally let us consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$. This sequence is of course bounded thus there exists u in $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ and a subsequence $\left(u_{\psi(n)}\right)_{n\in\mathbb{N}}$ such that it converges to u in S'. For every positive ε there exists an integer n_ε such that

$$n \geq m \geq n_\varepsilon \Rightarrow \left\|u_{\psi(m)} - u_{\psi(n)}\right\|_{\dot{B}^{s_1}_{p_1,r_1}} + \left\|u_{\psi(m)} - u_{\psi(n)}\right\|_{\dot{B}^{s_2}_{p_2,r_2}} < \varepsilon.$$

The Fatou property for the $\left(u_{\psi(m)} - u_{\psi(n)}\right)_{n\in\mathbb{N}}$ ensures that

$$\forall m \geq n_\varepsilon, \ \left\|u_{\psi(m)} - u\right\|_{\dot{B}^{s_1}_{p_1,r_1}} + \left\|u_{\psi(m)} - u\right\|_{\dot{B}^{s_2}_{p_2,r_2}} < C\varepsilon.$$

This obvious concludes the proof of the whole theorem. ■

Remark 1.4.5 *We can observe that because of Proposition 1.4.3 we deduce that $\dot{B}^s_{p,r}$ are Banach space in the case when $s < d/p$. It can be proved that this is also the case with $\dot{B}^{\frac{d}{p}}_{p,1}$. In the case when $s > d/p$ or $s = d/p$ and $r > 1$, $\dot{B}^s_{p,r}$ is no longer a Banach space. This is due to a defect of convergence for low frequencies.*

Proposition 1.4.5 *If p, r are finite then the space $S_0(\mathbb{R}^d)$ of functions of $S(\mathbb{R}^d)$ the Fourier transform of which is supported away from 0, is dense in $\dot{B}^s_{p,r}$.*

Proof. Let $u \in \dot{B}^s_{p,r}$. Because $r < \infty$ for all $\varepsilon > 0$ one can fin some integer N such that

$$\|u - u_N\|_{\dot{B}^s_{p,r}} \leq \varepsilon/2$$

with

$$u_N = \sum_{|j|\leq N} \dot{\Delta}_j u.$$

Let us fix θ smooth supported in $B(0,2)$ with value 1 on $B(0,1)$. For $R > 0$ we set $\theta_R = \theta(\cdot/R)$. Next we fix an integer $M > N$ and we define

$$u^R_{N,M} = \left(\mathbb{I} - \dot{S}_{-M}\right)(\theta_R u_N).$$

Because $M > N$ we get that:

$$u^R_{N,M} - u_N = \left(\mathbb{I} - \dot{S}_{-M}\right)((\theta_R - 1)u_N).$$

According to lemma 1.2.2 we have that for all $j \in \mathbb{N}$ and $k = \max(0, [s] + 2)$,

$$\begin{aligned} 2^{js} \left\| \dot{\Delta}_j (u^R_{N,M} - u_N) \right\|_{L^p} &\leq 2^{-j} 2^{jk} \left\| \dot{\Delta}_j \left(\left(\mathbb{I} - \dot{S}_{-M} \right) ((\theta_R - 1) u_N) \right) \right\|_{L^p} \\ &\leq C_s 2^{-j} \left\| D^k ((\theta_R - 1)\, u_N) \right\|_{L^p}. \end{aligned}$$

If $-M - 1 < j < -1$ one may write that:

$$2^{js} \left\| \dot{\Delta}_j (u^R_{N,M} - u_N) \right\|_{L^p} \leq C 2^{js} \left\| (\theta_R - 1) u_N \right\|_{L^p}$$

and if $j < -M - 2$ we have that $\dot{\Delta}_j (u^R_{N,M} - u_N) = 0$. So finally we get that:

$$\left\| u^R_{N,M} - u_N \right\|_{\dot{B}^s_{p,r}} \leq C \left(\left\| D^k ((\theta_R - 1)\, u_N) \right\|_{L^p} + \sum_{j=-M-1}^{-1} 2^{js} \left\| (\theta_R - 1)\, u_N \right\|_{L^p} \right).$$

Now by virtue of Leibniz formula and Lebesgue's dominated convergence theorem, the right hand side of the above inequality tends to 0 when R goes to infinity. So a positive real number exists such that

$$\left\| u^R_{N,M} - u_N \right\|_{\dot{B}^s_{p,r}} \leq \varepsilon/2.$$

As $u^R_{N,M}$ is a function of S_0 this completes the proof of the proposition. ■

Remark 1.4.6 *The same arguments show that when $r = \infty$, the closure of S_0 for the Besov norm is the set of S'_h such that:*

$$\lim_{|j| \to \infty} 2^{js} \left\| \dot{\Delta}_j u \right\|_{L^p} = 0.$$

Let us introduce a class of space which will be used in the second part of this paper:

Definition 1.4.2 *For any $T > 0$, $s \in \mathbb{R}$, $1 \leq r, \rho \leq \infty$ the space $\tilde{L}^\rho_T(\dot{B}^s_{p,r})$ is the set of tempered distributions u over $(0, T) \times \mathbb{R}^d$ such that*

$$\lim_{j \to -\infty} \dot{S}_j u = 0$$

in $L^\rho\left([0, T]; L^\infty(\mathbb{R}^d)\right)$ and

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} := \left\| \left(2^{js} \left\| \dot{\Delta}_j u \right\|_{L^\rho_T(L^p)} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

This spaces may be linked with the more classical space $L^\rho_T(\dot{B}^s_{p,r})$ by means of the Minkowski inequality. Thus we obtain that:

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} \leq \|u\|_{L^\rho_T(\dot{B}^s_{p,r})}$$

when $r > \rho$ and

$$\|u\|_{L^\rho_T(\dot{B}^s_{p,r})} \leq \|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})}$$

when $r < \rho$.

2 The Keller-Segel model for chemotaxis

2.1 Introduction

In this second part of the paper we will show how one can obtain existence and uniqueness results for a simplified form of the Keller-Segel model for chemotaxis[3]. We consider the following:

$$\begin{cases} \partial_t \rho - \Delta\rho = -\chi \operatorname{div}(\rho \nabla S) \\ \Delta S = -\rho \\ \rho_{|t=0} = \rho_0. \end{cases} \tag{$\mathcal{KS}$}$$

These equations describe the evolution of the density ρ of a biological population submitted to the influence of a chemical agent with concentration S.

From now on, for each $s \in \mathbb{R}$ we consider the operator Δ^{-1} defined as:

$$\Delta^{-1} u = \mathcal{F}\Big(-|\xi|^{-2}\,\mathcal{F}(u)\Big).$$

Let us consider the bilinear operator $Q(u,v)$ defined as:

$$\begin{aligned} Q(u,v) &= \operatorname{div}\operatorname{div}\left(\nabla\Delta^{-1}u \otimes \nabla\Delta^{-1}v\right) - \Delta\left\langle \nabla\Delta^{-1}u, \nabla\Delta^{-1}v\right\rangle \\ &= \sum_{k,l=1}^{d} \partial^2_{kl}\left(\partial_k\Delta^{-1}u\,\partial_l\Delta^{-1}v\right) - \sum_{k=1}^{d}\partial^2_{kk}\left(\partial_k\Delta^{-1}u\,\partial_k\Delta^{-1}v\right). \end{aligned} \tag{2.1.1}$$

We observe that $\mathcal{KS}$ can be written as:

$$\begin{cases} \partial_t\rho - \Delta\rho = \chi Q(\rho,\rho) \\ \rho_{|t=0} = \rho_0. \end{cases}$$

Let us consider $B(u,v)$ defined as:

$$\begin{cases} \partial_t B(u,v) - \Delta B(u,v) = Q(u,v) \\ B(u,v)_{|t=0} = 0. \end{cases} \tag{2.1.2}$$

Let us rigorously precise what shall be understood by solution for ($\mathcal{KS}$).

Definition 2.1.1 *Let us consider E a Banach space such that it is continuously embedded in $S'(\mathbb{R}^d)$. A function $\rho : [0,T) \to E$ is called a mild solution for ($\mathcal{KS}$) on $[0,T)$ if the following two conditions are verified:*

$$\rho \in \mathcal{C}^\star\left([0,T), E\right), \tag{2.1.3}$$

$$\rho = e^{t\Delta}\rho_0 + B(\rho,\rho). \tag{2.1.4}$$

[3]Chemotaxis is the phenomen whereby somatic cells, bacteria, and other single-cell or multicellular organisms direct their movements according to certain chemicals in their environment.

Remark 2.1.1 *If instead of (2.1.3) we have the stronger condition:*

$$\rho \in \mathcal{C}\left([0,T),E\right), \tag{2.1.5}$$

we shall call ρ a strong mild solution on $[0,T)$.

Remark 2.1.2 *Of course whenever $T=\infty$ we shall refer to the solution as being global.*

The above definition is well adapted for applying a fix point argument. More precisely we shall look for mild solutions for $(\mathcal{KS})$ as a fix point for the operator:

$$\rho \mapsto e^{t\Delta}\rho_0 + B(\rho,\rho).$$

The following theorem will assure us existence in appropriate Banach spaces:

Theorem 2.1.1 *Let E be a Banach space and let us consider $L: E \to E$ and $B: E\times E \to E$ a linear operator respectively a bilinear operator such that*

$$\|L\| = \sup_{\|x\|\le 1}\|Lx\| < 1/2,$$
$$\|B\| = \sup_{\|x\|,\|y\|\le 1}\|B(x,y)\| < \infty.$$

Let us chose a constant $a < \frac{1-2\|L\|}{4\|B\|}$. Then for any $\alpha \in B_E(0,a)$ we have that the equation:

$$x = \alpha + Lx + B(x,x), \tag{2.1.6}$$

has a unique solution in $B_E(0,2a)$.

Proof. Let us define the sequence:

$$x_{n+1} = \alpha + Lx_n + B(x_n,x_n), \tag{2.1.7}$$

with $x_0=\alpha$. Let us first show that it remains in $B_E(0,2a)$. Indeed because of the triangle inequality and a inductive argument we have:

$$\begin{aligned}\|x_{n+1}\| &\le a + \|Lx_n\| + \|B(x_n,x_n)\|\\ &\le a + 2\|L\|a + 4\|B\|a^2\\ &\le 2a.\end{aligned}$$

Secondly, we have that:

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \|L\|\,\|x_n - x_{n-1}\| + \|B\|\,\|x_n-x_{n-1}\|\,\|x_n+x_{n-1}\|\\ &\le (\|L\| + 4\|B\|a)\,\|x_n-x_{n-1}\|,\end{aligned}$$

thus because $\|L\|+4\|B\|a<1$ we get that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence thus it converges to some x. Passing to the limit in (2.1.7) yields the desired existence result. Finally let us chose y another solution for (2.1.6) in $B_E(0,2a)$; by taking the norm of the difference we get that:

$$\begin{aligned}\|x-y\| &\le (\|L\| + 4a\|B\|)\,\|x-y\|\\ &< \|x-y\|,\end{aligned}$$

and thus that $x=y$. ■

2.2 A global existence result

We begin by observing that if $\rho_0 \in \dot{B}^{\frac{d}{p}-2}_{p,r}$ and $q \in [1,\infty]$ we have the following relations:

$$\left\|\dot{\Delta}_j e^{t\Delta}\rho_0\right\|_{L^q(L^p)} \leq C2^{-\frac{2j}{q}} \left\|\dot{\Delta}_j \rho_0\right\|_{L^p}$$
$$2^{j\left(\frac{d}{p}+\frac{2}{q}-2\right)} \left\|\dot{\Delta}_j e^{t\Delta}\rho_0\right\|_{L^q(L^p)} \leq C2^{j\left(\frac{d}{p}-2\right)} \left\|\dot{\Delta}_j \rho_0\right\|_{L^p} \tag{2.2.1}$$
$$\left\|e^{t\Delta}\rho_0\right\|_{\tilde{L}^q(\dot{B}^{\frac{d}{p}+\frac{2}{q}-2}_{p,r})} \leq C \left\|\rho_0\right\|_{\dot{B}^{\frac{d}{p}-2}_{p,r}}. \tag{2.2.2}$$

This observation motivates us to consider E_T the closure in the space of tempered distributions of functions $u \in \mathcal{C}^\infty_c([0,T)\times\mathbb{R}^d)$ by the norm:

$$\|u\|_{E_T} = \|u\|_{\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-2}_{p,r})} + \|u\|_{\tilde{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,r})} < \infty.$$

Thus because of (2.2.2) we have that:

$$\left\|e^{t\Delta}\rho_0\right\|_{E_T} \leq C \left\|\rho_0\right\|_{\dot{B}^{\frac{d}{p}-2}_{p,r}}. \tag{2.2.3}$$

Let us now state the desired existence result:

Theorem 2.2.1 *Let $1 \leq p < \infty$, $r \in [1,\infty]$. There exists a universal constant $C > 0$ such that for any $\rho_0 \in \dot{B}^{\frac{d}{p}-2}_{p,r}$ with $\|\rho_0\|_{\dot{B}^{\frac{d}{p}-2}_{p,r}} < C$ there exists a global mild solution ρ in $\dot{B}^{\frac{d}{p}-2}_{p,r}$ for $(\mathcal{KS})$. Moreover the solution ρ is in E_∞ and it is the unique solution in the ball centred at the origin and of radius $2C$ of the space E_∞.*

The above theorem is of course a consequence of (2.2.3) and the following Lemma:

Lemma 2.2.1 *Let $B(u,v)$ be defined as in (2.1.2). Then for all $T > 0$ there exists a constant $C > 0$ such that:*

$$\|B(u,v)\|_{E_T} \leq C \|u\|_{E_T} \|v\|_{E_T}. \tag{2.2.4}$$

Proof. According to corollary 1.2.1, for any $q \in [1,\infty]$ we have that:

$$2^{\frac{2j}{q}} \left\|\dot{\Delta}_j B(u,v)\right\|_{L^q_T(L^p)} \leq \left\|\dot{\Delta}_{j'} Q(u,v)\right\|_{L^1_T(L^p)}. \tag{2.2.5}$$

From (2.1.1) we get that:

$$\begin{aligned}\left\|\dot{\Delta}_{j'} Q(u,v)\right\|_{L^p} &\leq C \sup_{k,l} \left\|\partial^2_{kl}\left[\dot{\Delta}_j\left(\partial_l\Delta^{-1}u\partial_k\Delta^{-1}v\right)\right]\right\|_{L^p}\\ &\leq 2^{2j} C \sup_{k,l} \left\|\dot{\Delta}_j\left(\partial_l\Delta^{-1}u\partial_k\Delta^{-1}v\right)\right\|_{L^p}.\end{aligned} \tag{2.2.6}$$

Let us point out the following Bony decomposition of the product:

$$\partial_l \Delta^{-1} u \partial_k \Delta^{-1} v = \sum_{j'} \dot{S}_{j'} \partial_l \Delta^{-1} u \dot{\Delta}_{j'} \partial_k \Delta^{-1} v + \dot{S}_{j'+1} \partial_k \Delta^{-1} v \dot{\Delta}_{j'} \partial_l \Delta^{-1} u$$

Because the supports of the Fourier transforms of $\dot{S}_{j'} \partial_l \Delta^{-1} u \dot{\Delta}_{j'} \partial_k \Delta^{-1} v$ and $\dot{S}_{j'+1} \partial_k \Delta^{-1} v \dot{\Delta}_{j'} \partial_l \Delta^{-1} u$ are contained in $2^{j'}\mathcal{B}_1$ and $2^{j'}\mathcal{B}_2$ with $\mathcal{B}_1$ and $\mathcal{B}_2$ being two Euclidian balls, we get the existence of an integer N such that if $j - j' > N$ we have:

$$\dot{\Delta}_j (\dot{S}_{j'} \partial_l \Delta^{-1} u \dot{\Delta}_{j'} \partial_k \Delta^{-1} v) = 0$$

and

$$\dot{\Delta}_j (\dot{S}_{j'+1} \partial_k \Delta^{-1} v \dot{\Delta}_{j'} \partial_l \Delta^{-1} u) = 0.$$

Thus we can write:

$$\begin{gathered} \dot{\Delta}_j \partial_l \Delta^{-1} u \partial_k \Delta^{-1} v = \\ = \sum_{j' \geq j-N} \dot{\Delta}_j (\dot{S}_{j'} \partial_l \Delta^{-1} u \dot{\Delta}_{j'} \partial_k \Delta^{-1} v) + \sum_{j' \geq j-N} \dot{\Delta}_j (\dot{S}_{j'+1} \partial_k \Delta^{-1} v \dot{\Delta}_{j'} \partial_l \Delta^{-1} u) \end{gathered}$$

and we get:

$$\begin{aligned} \left\| \dot{\Delta}_j \sum \dot{S}_{j'} \partial_l \Delta^{-1} u \dot{\Delta}_{j'} \partial_k \Delta^{-1} v \right\|_{L^p} &\leq \sum_{j' \geq j-N} \| \dot{\Delta}_j (\dot{S}_{j'} \partial_l \Delta^{-1} u \dot{\Delta}_{j'} \partial_k \Delta^{-1} v) \|_{L^p} \\ &\leq \sum_{j' \geq j-N} \left\| \dot{S}_{j'} \partial_l \Delta^{-1} u \dot{\Delta}_{j'} \partial_k \Delta^{-1} v \right\|_{L^p} \\ &\leq \sum_{j' \geq j-N} 2^{-j'} \left\| \dot{\Delta}_{j'} v \right\|_{L^p} \left\| \dot{S}_{j'} \partial_l \Delta^{-1} u \right\|_{L^\infty} . \end{aligned} \tag{2.2.7}$$

A similar relation holds for the other member so by summation and (2.2.6) we get that[4]:

$$\left\| \dot{\Delta}_{j'} Q(u,v) \right\|_{L^p} \leq 2^{2j} \sum^{sym} \sum_{j' \geq j-N} 2^{-j'} \left\| \dot{\Delta}_{j'} v \right\|_{L^p} \left\| \dot{S}_{j'} \partial_l \Delta^{-1} u \right\|_{L^\infty} \tag{2.2.8}$$

[4]We agree that $\sum^{sym}$ means the sum of the term after the sign added up with term obtained from the previous by changing u with v

Now according to lemma 1.2.1 and lemma 1.2.2 we have that:

$$\begin{aligned}
2^{-j'}\left\|\dot{S}_{j'}\partial_l\Delta^{-1}u\right\|_{L^\infty} &\le 2^{-j'}\sum_{j''\le j'-1}\left\|\dot{\Delta}_{j'}\partial_l\Delta^{-1}u\right\|_{L^\infty}\\
&= \sum_{j''\le j'-1} 2^{-j'}\left\|\partial_l\Delta^{-1}\dot{\Delta}_{j"}u\right\|_{L^\infty}\\
&\le 2^{-j'}\sum_{j''\le j'-1} 2^{(\frac{d}{p}-1)j''}\left\|\dot{\Delta}_{j"}u\right\|_{L^p}\\
&\le \sum_{j"}\mathbf{1}_{[1,\infty)}(j'-j")2^{-(j'-j")}2^{(\frac{d}{p}-2)j''}\left\|\dot{\Delta}_{j"}u\right\|_{L^\infty_T(L^p)}\\
&= \left(\mathbf{1}_{[1,\infty)}(k)2^{-k}\right)_{k\in\mathbb{Z}}\star\left(2^{(\frac{d}{p}-2)k}\left\|\dot{\Delta}_k u\right\|_{L^\infty_T(L^p)}\right)_{k\in\mathbb{Z}}(j')
\end{aligned}$$

Because of Young's inequality we get that

$$\begin{aligned}
&\left(\mathbf{1}_{[1,\infty)}(k)2^{-k}\right)_{k\in\mathbb{Z}}\star\left(2^{(\frac{d}{p}-2)k}\left\|\dot{\Delta}_k u\right\|_{L^\infty_T(L^p)}\right)_{k\in\mathbb{Z}}(j')\\
&\le \left\|\left(\mathbf{1}_{[1,\infty)}(k)2^{-k}\right)_{k\in\mathbb{Z}}\star\left(2^{(\frac{d}{p}-2)k}\left\|\dot{\Delta}_k u\right\|_{L^\infty_T(L^p)}\right)_{k\in\mathbb{Z}}\right\|_{\ell^\infty(\mathbb{Z})}\\
&\le \left\|\left(\mathbf{1}_{[1,\infty)}(k)2^{-k}\right)_{k\in\mathbb{Z}}\right\|_{\ell^{r'}(\mathbb{Z})}\left\|\left(2^{(\frac{d}{p}-2)k}\left\|\dot{\Delta}_k u\right\|_{L^\infty_T(L^p)}\right)_{k\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}
\end{aligned}$$

thus we have

$$\left(\mathbf{1}_{[1,\infty)}(k)2^{-k}\right)_{k\in\mathbb{Z}}\star\left(2^{(\frac{d}{p}-2)k}\left\|\dot{\Delta}_k u\right\|_{L^\infty_T(L^p)}\right)_{k\in\mathbb{Z}}(j')\le\|u\|_{\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-2}_{p,r})}\tag{2.2.9}$$

By putting together relations (2.2.9) and (2.2.7) we get that:

$$\left\|\dot{\Delta}_j\sum\dot{S}_{j'}\partial_l\Delta^{-1}u\dot{\Delta}_{j'}\partial_k\Delta^{-1}v\right\|_{L^p}\le C\|u\|_{\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-2}_{p,r})}\sum_{j'\ge j-N}\left\|\dot{\Delta}_{j'}v\right\|_{L^p}.$$

Of course by a similar manner we get that:

$$\left\|\dot{\Delta}_j\sum\dot{S}_{j'+1}\partial_k\Delta^{-1}v\dot{\Delta}_{j'}\partial_l\Delta^{-1}u\right\|_{L^p}\le C\|v\|_{\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-2}_{p,r})}\sum_{j'\ge j-N}\left\|\dot{\Delta}_{j'}u\right\|_{L^p}.$$

Thus we have the following estimation:

$$\left\|\dot{\Delta}_j(\partial_l\Delta^{-1}u\partial_k\Delta^{-1}v)\right\|_{L^p}\le C\sum^{sym}\|u\|_{\tilde{L}^\infty(\dot{B}^{\frac{d}{p}-2}_{p,r})}\sum_{j'\ge j-N}\left\|\dot{\Delta}_{j'}v\right\|_{L^p}.$$

and by virtue of (2.2.6), time integration, (2.2.5) and finally by multiplying with

$2^{j(\frac{d}{p}-2)}$ we successively get that:

$$2^{\frac{2j}{q}}\left\|\dot{\Delta}_j B(u,v)\right\|_{L^q_T(L^p)} \leq 2^{2j}\sum^{sym}\|u\|_{\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-2}_{p,r})}\sum_{j'\geq j-N}\left\|\dot{\Delta}_{j'}v\right\|_{L^1_T(L^p)}$$

$$2^{\left(\frac{d}{p}+\frac{2}{q}-2\right)j}\left\|\dot{\Delta}_j B(u,v)\right\|_{L^q_T(L^p)} \leq \sum^{sym}\|u\|_{\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-2}_{p,r})}\sum_{j'\geq j-N}2^{(j-j')\frac{d}{p}}2^{\frac{d}{p}j'}\left\|\dot{\Delta}_{j'}v\right\|_{L^1_T(L^p)}.$$

Once observed that:

$$\sum_{j'\geq j-N}2^{(j-j')\frac{d}{p}}2^{\frac{d}{p}j'}\left\|\dot{\Delta}_{j'}v\right\|_{L^1_T(L^p)} = \left(\mathbf{1}_{[-\infty,N]}(k)2^{k\frac{d}{p}}\right)_{k\in\mathbb{Z}}\star\left(2^{\frac{d}{p}k}\left\|\dot{\Delta}_k v\right\|_{L^1_T(L^p)}\right)_{k\in\mathbb{Z}}(j')$$

we can apply Young's inequality[5] thus obtaining:

$$\begin{aligned}\|B(u,v)\|_{\tilde{L}^q_T(\dot{B}^{\frac{d}{p}+\frac{2}{q}-2}_{p,r})} &\leq C\sum^{sym}\|u\|_{\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-2}_{p,r})}\|v\|_{\tilde{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,r})} \\ &\leq C\|u\|_{E_T}\|v\|_{E_T}.\end{aligned} \tag{2.2.10}$$

Finally taking in the above inequality $q=1$ respectively $q=\infty$ and by adding them up we get the desired result (2.2.4). ■

In order to establish further properties we will derive some slightly different results from the above lemma namely:

Corollary 2.2.1 *Let Q and B be defined as in (2.1.1) respectively as in (2.1.2). Then for all $q\in[1,\infty]$, $1\leq p<\infty$ and for all $0\leq s<t$ the following relation holds true:*

$$\|B(u,v)\|_{\tilde{L}^q_{[s,t]}(\dot{B}^{\frac{d}{p}+\frac{2}{q}-2}_{p,r})} \leq C\|Q(u,v)\|_{\tilde{L}^1_{[s,t]}(\dot{B}^{\frac{d}{p}-2}_{p,r})} \leq \tilde{C}\|u\|_{\tilde{L}^2_{[s,t]}(\dot{B}^{\frac{d}{p}-1}_{p,r})}\|v\|_{\tilde{L}^2_{[s,t]}(\dot{B}^{\frac{d}{p}-1}_{p,r})}.$$

Moreover if $u\in\tilde{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,r})\cap\tilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-2}_{p,r})$ and $r<\infty$ then for $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that the following relation holds:

$$|t-s|<\delta(\varepsilon)\Rightarrow\|u\|_{\tilde{L}^2_{[s,t]}(\dot{B}^{\frac{d}{p}-1}_{p,r})}<\varepsilon.$$

Proof. The left inequality follows from integration of (2.2.5). For the second

[5] Note that preciesly here one needs the fact that $p<\infty$.

part let us observe that according to relation (2.2.8) we have that:

$$\begin{aligned}\left\|\dot{\Delta}_{j'}Q(u,v)\right\|_{L^p} &\leq 2^{2j}\sum^{sym}\sum_{j'\geq j-N}2^{-j'}\left\|\dot{\Delta}_{j'}v\right\|_{L^p}\left\|\partial_l\Delta^{-1}\dot{S}_{j'}u\right\|_{L^\infty}\\ &\leq 2^{2j}\sum^{sym}\sum_{j'\geq j-N}2^{-2j'}\left\|\dot{\Delta}_{j'}v\right\|_{L^p}\left\|\dot{S}_{j'}u\right\|_{L^\infty}\\ &\leq 2^{2j}\sum^{sym}\sum_{j'\geq j-N}2^{-2j'}\left\|\dot{\Delta}_{j'}v\right\|_{L^p}\sum_{j''\leq j'-1}\left\|\dot{\Delta}_{j''}u\right\|_{L^\infty}\\ &\leq 2^{2j}\sum^{sym}\sum_{j'\geq j-N}2^{-2j'}\left\|\dot{\Delta}_{j'}v\right\|_{L^p}\sum_{j''\leq j'-1}2^{j''\frac{d}{p}}\left\|\dot{\Delta}_{j''}u\right\|_{L^p}\end{aligned}$$

By time integration and Hölder inequality we get that:

$$\int_s^t\left\|\dot{\Delta}_{j'}Q(u,v)\right\|_{L^p}dt' \leq 2^{2j}\sum^{sym}\sum_{\substack{j'\geq j-N\\ j''\leq j'-1}}2^{-2j'}2^{j''\frac{d}{p}}\int_s^t\left\|\dot{\Delta}_{j'}v\right\|_{L^p}\left\|\dot{\Delta}_{j''}u\right\|_{L^\infty}dt'$$

$$\leq 2^{2j}\sum^{sym}\sum_{j'\geq j-N}2^{-2j'}\left\|\dot{\Delta}_{j'}v\right\|_{L^2_{[s,t]}(L^p)}\sum_{j''\leq j'-1}2^{j''\frac{d}{p}}\left\|\dot{\Delta}_{j''}u\right\|_{L^2_{[s,t]}(L^p)}$$

$$= 2^{2j}\sum^{sym}\sum_{j'\geq j-N}2^{-j'}\left\|\dot{\Delta}_{j'}v\right\|_{L^2_{[s,t]}(L^p)}\sum_{j''\leq j'-1}2^{-(j'-j'')}2^{j''(\frac{d}{p}-1)}\left\|\dot{\Delta}_{j''}u\right\|_{L^2_{[s,t]}(L^p)}.$$

From now on the rest of the proof goes exactly like in the previous lemma and thus we omit it. In order to prove the remaining relation let us observe that for any $q\in(0,\infty)$ and we have that $u\in\tilde{L}^q(\dot{B}^{\frac{d}{p}+\frac{2}{q}-2}_{p,r})$ and

$$\|u\|_{\tilde{L}^q(\dot{B}^{\frac{d}{p}+\frac{2}{q}-2}_{p,r})}\leq\frac{1}{q}\|u\|_{\tilde{L}^1(\dot{B}^{\frac{d}{p}}_{p,r})}+\left(1-\frac{1}{q}\right)\|u\|_{\tilde{L}^\infty(\dot{B}^{\frac{d}{p}-2}_{p,r})}.$$

The proof of this result being rather basic we omit it. Let us consider j_0 such that

$$\sum_{j>j_0}2^{j(\frac{d}{p}-1)r}\left\|\dot{\Delta}_ju\right\|^r_{L^2_T(L^p)}<\left(\frac{\varepsilon}{2}\right)^r. \tag{2.2.11}$$

For all $j\in\mathbb{Z}$ we have that

$$\begin{aligned}2^{j(\frac{d}{p}-1)}\int_s^t\left\|\dot{\Delta}_ju\right\|^2dt' &\leq 2^{j(\frac{d}{p}-1)}|t-s|^{\frac{1}{2}}\left\|\dot{\Delta}_ju\right\|_{L^4_T(L^p)}\\ &\leq\left(2^{j_0}|t-s|\right)^{1/2}2^{j(\frac{d}{p}-\frac{3}{2})}\left\|\dot{\Delta}_ju\right\|_{L^4_T(L^p)}.\end{aligned} \tag{2.2.12}$$

Thus by setting

$$|t-s| \leq \frac{\varepsilon^2}{2^{j_0+2}\,\|u\|^2_{L^4_T(\dot B^{\frac{d}{p}-\frac{3}{2}}_{p,r})}}, \tag{2.2.13}$$

taking the r-th power of (2.2.12) and taking the sum for all $j \in \mathbb{Z}$ up to j_0 we get that:

$$\sum_{j\leq j_0} 2^{j(\frac{d}{p}-1)r} \left\|\dot\Delta_j u\right\|^r_{L^2_T(L^p)} < \left(\frac{\varepsilon}{2}\right)^r. \tag{2.2.14}$$

The conclusion comes from (2.2.11) and (2.2.14). ■

2.3 The continuity in time of the solution

In this paragraph we shall prove that in the case when $r<\infty$ we get that the solution given by (2.2.1) is actually a strong mild solution. In order to do that we shall prove that the operators $e^{\Delta(\cdot)}\rho_0$ and $B(\rho,\rho)$ are continuous in the time variable. For the first operator let us observe that:

$$\begin{aligned}
\left\|\dot\Delta_j(e^{\Delta t}\rho_0 - e^{\Delta s}\rho_0)\right\|_{L^p} &= \left\|\mathcal{F}^{-1}\left(\left(e^{-t|\xi|^2}-e^{-s|\xi|^2}\right)\mathcal{F}\left(\dot\Delta_j\rho_0\right)\right)\right\|_{L^p}\\
&= \left\|\left(\frac{e^{-\frac{|\cdot|^2}{4t}}}{[4\pi t]^{d/2}} - \frac{e^{-\frac{|\cdot|^2}{4s}}}{[4\pi s]^{d/2}}\right)\star\dot\Delta_j\rho_0\right\|_{L^p}\\
&\leq \left\|\frac{e^{-\frac{|\cdot|^2}{4t}}}{[4\pi t]^{d/2}} - \frac{e^{-\frac{|\cdot|^2}{4s}}}{[4\pi s]^{d/2}}\right\|_{L^1}\left\|\dot\Delta_j\rho_0\right\|_{L^p}
\end{aligned} \tag{2.3.1}$$

By the dominated convergence theorem we get that

$$\lim_{s\to t}\left\|\frac{e^{-\frac{|\cdot|^2}{4t}}}{[4\pi t]^{d/2}} - \frac{e^{-\frac{|\cdot|^2}{4s}}}{[4\pi s]^{d/2}}\right\|_{L^1} = 0 \tag{2.3.2}$$

Arguing by density it is sufficient to prove the result in the case where $\rho_0 \in S_{0,c}$ where $S_{0,c}$ is the set of functions from the Schwartz class which have compactly supported Fourier transform and their support does not contain 0.Thus we get the existence of a $j_{\rho_0}\in\mathbb{Z}$ such that $|j|\geq j_{\rho_0}$ implies that

$$\dot\Delta_j\rho_0 = 0.$$

Next, for all $|j|\leq j_{\rho_0}$ because of (2.3.1) we get that

$$\sum_j 2^{rj(\frac{d}{p}-2)}\left\|\dot\Delta_j(e^{\Delta t}\rho_0 - e^{\Delta s}\rho_0)\right\|^r_{L^p} \leq \sum_j \left\|\frac{e^{-\frac{|\cdot|^2}{4t}}}{[4\pi t]^{d/2}} - \frac{e^{-\frac{|\cdot|^2}{4s}}}{[4\pi s]^{d/2}}\right\|_{L^1}\left\|\dot\Delta_j\rho_0\right\|_{L^p}$$

which is the same with

$$\left\|e^{t\Delta}\rho_0 - e^{s\Delta}\rho_0\right\|_{\dot B^{\frac{d}{p}-2}_{p,r}} \leq \left\|\frac{e^{-\frac{|\cdot|^2}{4t}}}{[4\pi t]^{d/2}} - \frac{e^{-\frac{|\cdot|^2}{4s}}}{[4\pi s]^{d/2}}\right\|_{L^1}\|\rho_0\|_{\dot B^{\frac{d}{p}-2}_{p,r}}$$

Because of (2.3.2) we get that:

$$\lim_{s\to t}\left\|e^{t\Delta}\rho_0 - e^{s\Delta}\rho_0\right\|_{\dot{B}^{\frac{d}{p}-2}_{p,r}} = 0. \tag{2.3.3}$$

Let us move now to prove that $B(\rho,\rho)$ is continuous in time. We shall split it into two parts namely let us consider $t > s$ and let us write that:

$$B(\rho,\rho)(t) - B(\rho,\rho)(s) =$$
$$= \int_s^t e^{(t-t')\Delta}Q(\rho,\rho)(t')dt' + \int_0^s [e^{(t-t')\Delta} - e^{(s-t')\Delta}]Q(\rho,\rho)(t')dt'. \tag{2.3.4}$$

We us take a look at the firs term of the above sum:

$$\begin{aligned} I_j &:= \int_s^t e^{(t-t')\Delta}\left\|\dot{\Delta}_j Q(\rho,\rho)(t')\right\|_{L^p} dt' \le C\int_s^t e^{-c(t-t')2^{2j}}\left\|\dot{\Delta}_j Q(\rho,\rho)(t')\right\|_{L^p} dt' \\ &\le \int_s^t \left\|\dot{\Delta}_j Q(\rho,\rho)(t')\right\|_{L^p} dt' \\ &\le 2^{2j}\sum_{j'\ge j-N_0}^{sym} 2^{-j'}\left\|\dot{\Delta}_{j'}\rho\right\|_{L^2_{[s,t]}(L^p)}\sum_{j''\le j'-1} 2^{(\frac{d}{p}-1)}\left\|\dot{\Delta}_{j''}\rho\right\|_{L^2_{[t,s]}(L^p)}. \end{aligned}$$

Thus multiplying by $2^{j(\frac{d}{p}-2)}$ and taking the $\ell^r(\mathbb{Z})$ norm we get that we get that:

$$\left\|\left(2^{j(\frac{d}{p}-2)}I_j\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} \le C\left\|\rho\right\|^2_{\tilde{L}^2_{[s,t]}(\dot{B}^{\frac{d}{p}-1}_{p,r})}. \tag{2.3.5}$$

which in view of Corollary (2.2.1) tends to 0 as s tends to t. Let us look now at the second member of (2.3.4). For all $j \in \mathbb{Z}$ we get that:

$$2^{j(\frac{d}{p}-2)r}\left(\int_0^s \left\|e^{(t-t')\Delta} - e^{(s-t')\Delta}\dot{\Delta}_j Q(\rho,\rho)\right\|_{L^p}\right)^r \le 2^r\left(\left\|\dot{\Delta}_j Q(\rho,\rho)\right\|_{L^1(L^p)}\right)^r$$

and thus in order to conclude that the second member of (2.3.4) tends to 0 as s tends to t we should prove that for every j we have:

$$\lim_{s\to t} 2^{jr(\frac{d}{p}-2)}\left(\int_0^s \left\|e^{(t-t')\Delta} - e^{(s-t')\Delta}\dot{\Delta}_j Q(\rho,\rho)\right\|_{L^p}\right)^r = 0. \tag{2.3.6}$$

In order to do so let us observe that:

$$\int_0^s \left\| e^{(t-t')\Delta} - e^{(s-t')\Delta} \dot{\Delta}_j Q(\rho,\rho) \right\|_{L^p} =$$

$$= \int_0^s \left\| \left(\frac{e^{-\frac{|\cdot|^2}{4(t-t')}}}{[4\pi(t-t')]^{d/2}} - \frac{e^{-\frac{|\cdot|^2}{4(s-t')}}}{[4\pi(s-t')]^{d/2}} \right) \star \dot{\Delta}_j Q(\rho,\rho) \right\|_{L^p} dt'$$

$$\leq \int_0^s \left\| \frac{e^{-\frac{|\cdot|^2}{4(t-t')}}}{[4\pi(t-t')]^{d/2}} - \frac{e^{-\frac{|\cdot|^2}{4(s-t')}}}{[4\pi(s-t')]^{d/2}} \right\|_{L^1(\mathbb{R}^d)} \left\| \dot{\Delta}_j Q(\rho,\rho) \right\|_{L^p} dt'$$

$$\leq 2 \left\| \dot{\Delta}_j Q(\rho,\rho) \right\|_{L^1(L^p)}.$$

Recall that according to relation (2.3.2) we have that for all $t' \in [0,s)$ we have that:

$$\lim_{s\to t} \left\| \frac{e^{-\frac{|\cdot|^2}{4(t-t')}}}{[4\pi(t-t')]^{d/2}} - \frac{e^{-\frac{|\cdot|^2}{4(s-t')}}}{[4\pi(s-t')]^{d/2}} \right\|_{L^1(\mathbb{R}^d)} = 0$$

so according to the dominated convergence theorem we have that (2.3.6) is true. Thus we get that

$$\lim_{s\to t} \left\| \left(2^{j(\frac{d}{p}-2)} \int_0^s \left\| e^{(t-t')\Delta} - e^{(s-t')\Delta} \dot{\Delta}_j Q(\rho,\rho) \right\|_{L^p} \right) \right\|_{\ell^r(\mathbb{Z})} = 0 \qquad (2.3.7)$$

Of course, in view of (2.3.3), (2.3.5) and (2.3.7) we get the desired continuity result.

2.4 The local existence and uniqueness

As we saw in the previous paragraph taking ρ_0 sufficiently small in $\dot{B}_{p,r}^{\frac{d}{p}-2}$ we get the existence of a global mild solution for $(\mathcal{KS})$. In this paragraph we shall show how to obtain local results for arbitrarily data in $\dot{B}_{p,r}^{\frac{d}{p}-2}$ when $r < \infty$. Let us proceed in the following manner: take $e^{t\Delta}\rho_0$ the free heat solution and let us consider the operator:

$$\rho \mapsto B(e^{t\Delta}\rho_0, e^{t\Delta}\rho_0) + 2B(e^{t\Delta}\rho_0, \rho) + B(\rho,\rho).$$

It is obvious that a fix point for the above operator gives us a mild solution for $(\mathcal{KS})$ namely $e^{t\Delta}\rho_0 + \rho$. Thus in order to apply Theorem 2.1.1 one should obtain that $B(e^{t\Delta}\rho_0, e^{t\Delta}\rho_0)$ along with the norm of the linear operator $\rho \to 2B(e^{t\Delta}\rho_0, \rho)$ can be made small for appropriate T. Thus the local well posedness becomes a consequence of the following:

Corollary 2.4.1 *Let $u_0 \in \dot{B}^{\frac{d}{p}-2}_{p,r}$. Then the following relations hold true:*

$$\lim_{T\to 0} \left\| e^{\Delta t} u_0 \right\|_{\tilde{L}^q_T(\dot{B}^{\frac{d}{p}+\frac{2}{q}-2}_{p,r})} = 0, \quad q \in [1,\infty),$$
$$\lim_{T\to 0} \left\| B(e^{\Delta t} u_0, e^{t\Delta} u_0) \right\|_{E_T} = 0.$$

Proof. This is a immediate consequence of (2.2.2) and Corollary 2.2.1 gives us the first relation. The second one is deduced from the first one and from the fact that according to Corollary 2.2.1 we have:

$$\left\| B(e^{\Delta t} u_0, e^{t\Delta} u_0) \right\|_{E_T} \le C \left\| e^{\Delta t} u_0 \right\|^2_{\tilde{L}^2_T(\dot{B}^{\frac{d}{p}-1}_{p,r})}. \tag{2.4.1}$$

■

Let us emphasize that Theorem 2.1.1 does not assure us the uniqueness in the whole space but rather in a ball. Nevertheless when $r < \infty$ the uniqueness follows from the arguments presented bellow. Let us suppose that for a initial data ρ_0 we have that two mild solutions ρ and $\tilde{\rho}$ on an interval $[0,T]$. Then we would have that:

$$\rho - \tilde{\rho} = B(\rho + \tilde{\rho}, \rho - \tilde{\rho}),$$

which in view of Corollary 2.2.1 implies that:

$$\begin{aligned}\|\rho - \tilde{\rho}\|_{E_T} &\le \|B(\rho + \tilde{\rho}, \rho - \tilde{\rho})\|_{E_T} \le C \|\rho + \tilde{\rho}\|_{L^2_T(\dot{B}^{\frac{d}{p}-1}_{p,r})} \|\rho - \tilde{\rho}\|_{L^2_T(\dot{B}^{\frac{d}{p}-1}_{p,r})} \\ &\le C \|\rho + \tilde{\rho}\|_{L^2_T(\dot{B}^{\frac{d}{p}-1}_{p,r})} \|\rho - \tilde{\rho}\|_{E_T}.\end{aligned}$$

Also according to the second part of the corollary invoqued above one can choose a $T^\star > 0$ depending only on $\|\rho + \tilde{\rho}\|_{L^4_T(\dot{B}^{\frac{d}{p}-\frac{3}{2}}_{p,r})}$ such that $\|\rho + \tilde{\rho}\|_{L^2_T(\dot{B}^{\frac{d}{p}-1}_{p,r})} < C^{-1}$. Thus ρ and $\tilde{\rho}$ coincide on every interval of length $T^\star$ included in $[0,T]$ and thus they must coincide on the whole $[0,T]$.

2.5 The case $r = \infty$

The above results of local existence and the fact that the solution is strong mild have been proven in the case when $r < \infty$ and clearly the arguments fail to hold in the case $r = \infty$. Nevertheless we are still able to provide a local existence result for a subspace of $\dot{B}^{\frac{d}{p}-2}_{p,\infty}$ namely the closure in $S'(\mathbb{R}^d)$ of S_0 by the norm $\|\cdot\|_{\dot{B}^{\frac{d}{p}-2}_{p,\infty}}$. We denote this space as $\underline{\dot{B}}^{\frac{d}{p}-2}_{p,\infty}$. According to Lemma we know that this space is the set of distribution $u \in S'_h$ such that$\|u\|_{\dot{B}^{\frac{d}{p}-2}_{p,\infty}} < \infty$ and:

$$\lim_{|j|\to\infty} 2^{j(\frac{d}{p}-2)} \left\| \dot{\Delta}_j u \right\|_{L^p} = 0. \tag{2.5.1}$$

All the results involving the local existence and the fact that the solution will be continuous in time with values in $\dot{B}^{\frac{d}{p}-2}_{p,\infty}$ that have been proven for $r<\infty$ remain true if we take $\rho_0 \in \underline{\dot{B}}^{\frac{d}{p}-2}_{p,\infty}$. This a consequence of the fact that the following implication holds true:

$$\rho_0 \in \underline{\dot{B}}^{\frac{d}{p}-2}_{p,\infty} \Rightarrow \lim_{T\to 0} \left\| e^{\Delta t}\rho_0 \right\|_{L^q_T(\dot{B}^{\frac{d}{p}+\frac{2}{q}-2}_{p,\infty})} = 0, \quad q\in[1,\infty). \tag{2.5.2}$$

In order to prove (2.5.2) let us emphasize that for any $\varepsilon>0$ we can choose a j_0 such that

$$|j|>j_0 \Rightarrow 2^{j(\frac{d}{p}-2)} \left\| \dot{\Delta}_j \rho_0 \right\|_{L^p} \leq \varepsilon$$

and thus in view of (2.2.1) leads to

$$|j|>j_0 \Rightarrow 2^{j\left(\frac{d}{p}+\frac{2}{q}-2\right)} \left\| \dot{\Delta}_j e^{t\Delta}\rho_0 \right\|_{L^q_T(L^p)} \leq C\varepsilon.$$

Now in view of Lemma 1.2.3 because for all $j\in\mathbb{Z}$:

$$2^{j\left(\frac{d}{p}+\frac{2}{q}-2\right)} \left\| \dot{\Delta}_j e^{t\Delta}\rho_0 \right\|_{L^q_T(L^p)} < C\left(1-e^{-c2^{2j}T}\right) \left\| \dot{\Delta}_j e^{t\Delta}\rho_0 \right\|_{L^q_T(L^p)},$$

we get the existence of a $T=T(\varepsilon)$ such as for all $|j|<j_0$:

$$2^{j\left(\frac{d}{p}+\frac{2}{q}-2\right)} \left\| \dot{\Delta}_j e^{t\Delta}\rho_0 \right\|_{L^q_T(L^p)} \leq C\varepsilon.$$

Thus combining the last two relations we find that:

$$\left\| e^{\Delta t}\rho_0 \right\|_{L^q_T(\dot{B}^{\frac{d}{p}+\frac{2}{q}-2}_{p,\infty})} \leq C\varepsilon.$$

The fact that this solution is a strong mild solution is proved by a similar technique.

References

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