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M2 Mathématiques appliquées aux sciences du vivant

Quelques remarques sur un système de type Boussinesq

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Introduction

In the present report we study some mathematical properties of a Boussinesq-type system of partial differential equations namely the Benjamin-Bona-Mahony system which was derived by the last three in [2]. This system of PDE's is derived from the Euler system of equations for a perfect, irrotational fluid flowing through a canal and it is used to describe the propagation of small-amplitude, long wavelength gravity waves at the surface of water. The report is divided into four parts where we approach different aspects:

- In the first part we review the derivation of the BBM-BBM system (and more generally, the *abcd*-systems) and we explain the concept of the Boussinesq time scale. Although, one can already find these features in [2] and [4], we felt the need of presenting them here in order to emphasize the further theoretical developments that we carry on in the rest of the report.

- In the second part we generalize the results dealing with local well-posedness in Sobolev spaces for the BBM-BBM system which can be found in [3] for $d = 1$ or those of [4] for the case of $d \geq 1$ to the sharper class of Besov spaces $B^s_{2,r}$.

- Having proved the local existence results we deal with long time existence via an energy method. Actually, as it will become clearer in the following, this is the most challenging aspect as it is intimately linked with the physical relevance of the model. Long time existence is proved in [4] within the context of Sobolev space with integer regularity index s, under the condition that $s > \left[\frac{d}{2}\right] + 2$. The approach presented below, which relies on the Littlewood-Paley partition of the frequency space, enables us to obtain long time existence results like those in [4] under weaker regularity assumption on the initial data such that in the physical relevant case $d = 2$ (which is actually a 3D theory) we gain "almost" one derivative in comparison with [4]. Nevertheless, the level of regularity $s = \frac{d}{2} + 2$ still prevents us from considering physically more relevant data (for example, an initial disturbance of the free surface modeled by a non-derivable function).

- The report ends with an appendix where we present the definition of Besov space along with some Fourier analysis tools upon which we rely all along the first three parts. Proofs along with complementary results and comments can be found in [5].

Personal comments, with the purpose of easing the reading and understanding of the more technical results can be found all along the text and I wish to apologize to the reader if he/she finds some parts a bit too "rigid".

In the end, I would wish to thank professor Raphäel Danchin for proposing me this subject and for all the discussions we had concerning its different aspects.

1 Derivation of the BBM-BBM system

The Benjamin-Bona-Mahony system of equations (BBM-BBM system)[1] is concerned with surface water waves of an ideal fluid under the force of gravity. More precisely, we consider a layer of incompressible, irrotational, perfect fluid flowing through a canal with flat bottom represented by the plane:

$$\{(x, y, z) : z = -h\},$$

$h > 0$. Assume that the free surface resulting from an initial perturbation of the steady state can be described as being the graph of a function $\eta = \eta(t, x, y)$ over the flat bottom. In this scenario, the water-waves problem reads:

$$\begin{cases} \Delta\phi + \partial_z^2\phi = 0 & \text{in} \quad -h \leq z \leq \eta(x, y, t), \\ \partial_z\phi = 0 & \text{at} \quad z = -h, \\ \partial_t\eta + \nabla\phi\nabla\eta - \partial_z\phi = 0 & \text{at} \quad z = \eta(x, y, t), \\ \partial_t\phi + \frac{1}{2}\left(|\nabla\phi|^2 + |\partial_z\phi|^2\right) + gz = 0 & \text{at} \quad z = \eta(x, y, t) \end{cases} \tag{$\mathcal{WW}$}$$

where ϕ stands for the fluid's velocity potential, and g is the acceleration of gravity. The operators Δ and ∇ are taken with respect to (x, y). Of course, in many applications the above system of equation raises a significant number of problems. This is the reason why an important number of approximate models have appeared each dealing with particular physical regimes. The BBM-BBM system of equations deals with the so called Boussinesq regime which we will explain now. We consider the following quantities: $A = \max_{x,y,t} |\eta|$ the maximum amplitude encountered in the wave motion, l is the smallest wavelength for which the flow has significant energy and $c_0 = \sqrt{gh}$ the kinematic wave velocity. The Boussinesq regime is characterized by the parameters:

$$\alpha = \frac{A}{h},\ \beta = \left(\frac{h}{l}\right)^2,\ S = \frac{\alpha}{\beta}, \tag{1.1}$$

which obey the following relations:

$$\alpha \ll 1,\ \beta \ll 1 \text{ and } S \approx 1,$$

which is to say that we consider waves of small amplitude and long wavelength such that the associated nonlinear and dispersive effects are balanced. Particular situations of interest which fit into the Boussinesq regime are the propagation of tsunamis (before hitting the shore) and bore propagation. The derivation of the BBM-BBM system is presented in [2] along with an analysis of the well-posedness of other different related systems. In order for further theoretical developments to be clearer and in order for the present material to be as auto-contained as possible, we choose to present the derivation of the BBM-BBM system of equations (and more generally the $abcd-$system) as it is done in the last cited paper. We start by choosing the following scaling:

$$x = l\tilde{x},\ y = l\tilde{y},\ z = h(\tilde{z} - 1),\ \eta = A\tilde{\eta},\ t = \tilde{t}l/c_0,\ \phi = gAl\frac{\tilde{\phi}}{c_0},$$

[1] This should not be confussed with the Benjamin-Bona-Mahony equation introduced in 1972 by the three as an alternative for the Korteweg-de Vries equation:

$$\partial_t u + \partial_x u + u\partial_x u - \partial_{txx} u = 0.$$

Comparisons between the BBM equation and the BBM-BBM system can be found in [1].

and we see that $(\mathcal{WW})$ becomes (for visual clarity we will actually drop the tildes):

$$\begin{cases} \beta\Delta\phi+\partial_z^2\phi=0 & \text{in} \quad 0\leq z\leq 1+\alpha\eta\left(x,y,t\right), \\ \partial_z\phi=0 & \text{at} \quad z=0, \\ \partial_t\eta+\alpha\nabla\phi\nabla\eta-\frac{1}{\beta}\partial_z\phi=0 & \text{at} \quad z=1+\alpha\eta\left(x,y,t\right), \\ \partial_t\phi+\frac{1}{2}\left(\alpha\left|\nabla\phi\right|^2+\frac{\alpha}{\beta}\left|\partial_z\phi\right|^2\right)+\eta=0 & \text{at} \quad z=1+\alpha\eta\left(x,y,t\right). \end{cases} \tag{1.2}$$

Of course, formally the rescaled quantities η,ϕ are of order one. We assume that:

$$\phi\left(t,x,y,z\right)=\sum_{k=0}^{\infty}f_k\left(t,x,y\right)z^k.$$

The BBM-BBM system is derived via a formal approximation which goes as it follows: we first see that the first equation in (1.2) gives us:

$$-\beta\Delta f_k=\left(k+1\right)\left(k+2\right)f_{k+2}, \tag{1.3}$$

such that for all $k\in\mathbb{N}$ we get:

$$f_{2k}=\frac{\left(-1\right)^k\beta^k}{(2k)!}\Delta^{(k)}f_0.$$

The second equation of (1.2) amounts to $f_1=0$ such that using relation (1.3) gives us:

$$\phi=\sum_{k=0}^{\infty}\frac{\left(-1\right)^k\beta^k}{(2k)!}\Delta^{(k)}f_0z^{2k}. \tag{1.4}$$

Replacing (1.4) in the third and fourth equation of (1.2) and neglecting the terms that are $O\left(\beta^2\right)$[2] we get:

$$\begin{cases} \partial_t\eta+\Delta f_0+\alpha\nabla f_0\cdot\nabla\eta+\alpha\eta\Delta f_0-\frac{\beta}{6}\Delta^{(2)}f_0=O\left(\beta^2\right), \\ \partial_tf_0+\eta-\frac{\beta}{2}\Delta\partial_tf_0+\frac{\alpha}{2}\left|\nabla f_0\right|^2=O\left(\beta^2\right). \end{cases}$$

Differentiating the second relation with respect to the spatial variable and denoting by $V^0=\nabla f_0$ we see that the last system becomes:

$$\begin{cases} \partial_t\eta+\nabla\cdot V^0+\alpha V^0\cdot\nabla\eta+\alpha\eta\nabla\cdot V^0-\frac{\beta}{6}\nabla\cdot\Delta V^0=O\left(\beta^2\right), \\ \partial_tV^0+\nabla\eta-\frac{\beta}{2}\Delta\partial_tV^0+\frac{\alpha}{2}\nabla\left|V^0\right|^2=O\left(\beta^2\right). \end{cases}$$

or in a more compact form:

$$\begin{cases} \partial_t\eta+\nabla\cdot\left((1+\alpha\eta)V^0\right)-\frac{\beta}{6}\nabla\cdot\Delta V^0=O\left(\beta^2\right), \\ \partial_tV^0+\nabla\eta-\frac{\beta}{2}\Delta\partial_tV^0+\frac{\alpha}{2}\nabla\left|V^0\right|^2=O\left(\beta^2\right). \end{cases} \tag{1.5}$$

As it can be better seen from relation (1.4), f_0 is the rescaled velocity potential at the bottom of the canal and consequently V^0 is the rescaled velocity field at the bottom and thus system (1.5) with the left-hand side dropped yields an approximation of the evolution of the free surface in terms of the rescaled velocity field in the horizontal plane $z=-h$. The philosophy leading to the establishment of the BBM-BBM system (and not only) is that we have no apparent

[2]Of course, in view of (1.1) this includes all quantities of order one multyplied by α^2 or $\alpha\beta$, a fact that we will use without mentioning from now on.

reason for which not to consider the velocity profile at another height as the main actor in the evolution of the free surface η[3]. This feature will be established by making use of lower order relations in higher order terms. Let us consider $\theta \in [0,1]$ and V^θ the scaled horizontal velocity corresponding to the $z = -(1-\theta)\, h$ plane. Formally that is:

$$\begin{aligned} V^\theta &= \nabla \phi_{|z=\theta} = \nabla f_0 - \frac{\beta\theta^2}{2} \Delta \nabla f_0 + O\left(\beta^2\right) \\ &= V^0 - \frac{\beta\theta^2}{2} \Delta V^0 + O\left(\beta^2\right). \end{aligned}$$

Let us apply $\frac{\beta\theta^2}{2}\Delta$ in the above relation in order to observe that

$$\frac{\beta\theta^2}{2}\Delta V^\theta = \frac{\beta\theta^2}{2}\Delta V^0 + O\left(\beta^2\right)$$

and consequently:

$$V^0 = V^\theta + \frac{\beta\theta^2}{2}\Delta V^\theta + O\left(\beta^2\right). \tag{1.6}$$

Replacing (1.6) into (1.5) we end up with:

$$\begin{cases} \partial_t \eta + \nabla \cdot \left((1+\alpha\eta)V^\theta\right) + \frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\nabla \cdot \Delta V^\theta = O\left(\beta^2\right), \\ \partial_t V^\theta + \nabla \eta + \frac{\alpha}{2}\nabla \left|V^\theta\right|^2 + \frac{\beta}{2}\left(\theta^2 - 1\right)\Delta \partial_t V^\theta = O\left(\beta^2\right). \end{cases} \tag{1.7}$$

Observe that applying Δ in the second relation of (1.7) we get:

$$\Delta \partial_t V^\theta = -\Delta \nabla \eta + O(\beta)$$

and that taking μ a real parameter we may write:

$$\begin{aligned} \frac{\beta}{2}\left(\theta^2 - 1\right)\Delta \partial_t V^\theta &= \frac{\beta}{2}\left(\theta^2 - 1\right)(1-\mu+\mu)\Delta \partial_t V^\theta \\ &= (1-\mu)\frac{\beta}{2}\left(\theta^2 - 1\right)\Delta \partial_t V^\theta - \mu\frac{\beta}{2}\left(\theta^2 - 1\right)\Delta \nabla \eta + O\left(\beta^2\right). \end{aligned} \tag{1.8}$$

Similarly, applying Δ in the first relation of (1.7) we get:

$$\Delta \partial_t \eta = -\nabla \cdot \Delta V^\theta + O(\beta)$$

and taking λ a real parameter we may write:

$$\begin{aligned} &\frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\nabla \cdot \Delta V^\theta \\ &= \lambda \frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\nabla \cdot \Delta V^\theta - (1-\lambda)\frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\Delta \partial_t \eta. \end{aligned} \tag{1.9}$$

By letting

$$\begin{aligned} a &= \frac{\mu}{2}\left(1-\theta^2\right),\ b = \frac{1-\mu}{2}\left(1-\theta^2\right), \\ c &= \left(\frac{\theta^2}{2} - \frac{1}{6}\right)\lambda,\ d = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)(1-\lambda), \end{aligned}$$

[3]Acctualy as explained in [2], different models arising from choosing different values of the parameters will be more conveinient in certain phisycal situations.

replacing (1.8) and (1.9) in (1.7) and dropping out the $O\left(\beta^2\right)$ terms, we end up with the so called *abcd*-systems:

$$\begin{cases} \partial_t \eta + \nabla \cdot ((1+\alpha\eta)\, V) + \beta\, (c\Delta\nabla\cdot V - d\Delta\partial_t \eta) = 0, \\ \partial_t V + \nabla\eta + \frac{\alpha}{2}\nabla |V|^2 + \beta\, (a\Delta\nabla\eta - b\Delta\partial_t V) = 0. \end{cases}$$

The BBM-BBM Boussinesq system is obtained by choosing $\theta = \sqrt{2/3}$ and $\mu = \lambda = 0$ such that we finally end up with:

$$\begin{cases} \partial_t \eta + \nabla \cdot ((1+\alpha\eta)\, V) - \frac{\beta}{6}\Delta\partial_t \eta = 0, \\ \partial_t V + \nabla\eta + \frac{\alpha}{2}\nabla |V|^2 - \frac{\beta}{6}\Delta\partial_t V = 0. \end{cases} \qquad \mathcal{BBM}\,(\alpha,\beta)$$

Of course, different choices of the parameters lead to different systems with different mathematical properties which as mentioned before can be used according to the particular physical situation. Let us just mention another one of them, namely the Korteweg-de Vries system (KdV-KdV system) corresponding to $\theta = \sqrt{2/3}$, $\mu = \lambda = 1$:

$$\begin{cases} \partial_t \eta + \nabla \cdot ((1+\alpha\eta)\, V) + \frac{\beta}{6}\Delta\nabla\cdot V = 0, \\ \partial_t V + \nabla\eta + \frac{\alpha}{2}\nabla |V|^2 + \frac{\beta}{6}\Delta\nabla\eta = 0. \end{cases} \qquad \mathcal{KV}\,(\alpha,\beta)$$

Let us end this section by explaining the concept of the Boussinesq time scale. Within system ($\mathcal{BBM}\,(\alpha,\beta)$) the scaled variables are all of order one and the error term are quadratic i.e. of order β^2. It is thus expected that the error between the solution of ($\mathcal{BBM}\,(\alpha,\beta)$) and the solution of ($\mathcal{WW}$) will accumulate like $t\beta^2$ as the solution evolves in time. When t becomes of order β^{-1} the total error will be of order β which is still small compared to η and V. Thus, in order to have a satisfactory model from a practical purpose, well posedness results should be able to develop an existence time of order β^{-1}. This time scale, $T \approx \beta^{-1}$ is named Boussinesq time scale. Although the previous considerations are formal, as explained in [4], page 614, they have been given a rigorous justification. This is the reason why in the next sections, all our efforts will be focussed on obtaining an existence theory on this time scale.

2 Local existence theory

We consider the BBM-BBM system in dimension d which, by means of a simple change of variables, can be rewritten in the following form:

$$\begin{cases} (Id - \varepsilon\Delta)\,\partial_t \eta + \nabla\cdot V + \varepsilon\nabla\cdot(\eta V) = 0, \\ (Id - \varepsilon\Delta)\,\partial_t V + \nabla\eta + \varepsilon\nabla\left(\frac{1}{2}|V|^2\right) = 0 \\ \eta\,(0,x) = \eta_0\,(x)\,,\ V\,(0,x) = V_0\,(x)\,. \end{cases} \qquad (BBM_\varepsilon)$$

Of course, because of $\alpha \approx \beta$, the change of variable does not change the formal order of the terms, and we point out that in this case the Boussinesq time scale is a quantity which is $O\left(\varepsilon^{-1}\right)$. The purpose of this section is to prove a local existence result for initial data in Besov spaces. Let us start by defining what we understand by solution of (BBM_ε)

Definition 2.1. *Let there be $(\eta_0, V_0) \in \mathcal{D}'(\mathbb{R}^d) \times [\mathcal{D}'(\mathbb{R}^d)]^d$ and $T > 0$. A pair, $(\eta, V) \in \mathcal{C}\left([0,T]\,, L^2_{loc}(\mathbb{R}^d) \times [L^2_{loc}(\mathbb{R}^d)]^d\right)$ is called a weak solution of (BBM_ε) on $[0,T]$, with initial* *data* (η_0, V_0) *if for any* $(\phi,\psi)\in\mathcal{C}^1\left([0,T],\mathcal{D}(\mathbb{R}^d)\times[\mathcal{D}(\mathbb{R}^d)]^d\right)$ *the following relation holds true:*

$$\begin{aligned}
&\langle\eta(t),(I-\varepsilon\Delta)\phi(t)\rangle_{\mathcal{D}'\times\mathcal{D}}-\langle\eta_0,(I-\varepsilon\Delta)\phi(0)\rangle_{\mathcal{D}'\times\mathcal{D}}\\
&=\int_0^t\langle\eta(s),(I-\varepsilon\Delta)\partial_t\phi(s)\rangle_{\mathcal{D}'\times\mathcal{D}}\,ds+\int_0^t\langle V(s)(1+\varepsilon\eta(s)),\nabla\phi(s)\rangle_{\mathcal{D}'\times\mathcal{D}}\,ds,\\
&\langle V(t),(I-\varepsilon\Delta)\psi(t)\rangle_{\mathcal{D}'\times\mathcal{D}}-\langle V_0,(I-\varepsilon\Delta)\psi(0)\rangle_{\mathcal{D}'\times\mathcal{D}}\\
&=\int_0^t\langle V(s),(I-\varepsilon\Delta)\partial_t\psi(s)\rangle_{\mathcal{D}'\times\mathcal{D}}\,ds+\int_0^t\left\langle\eta(s)+\varepsilon\frac{|V(s)|^2}{2},\nabla\cdot\psi(s)\right\rangle_{\mathcal{D}'\times\mathcal{D}}ds.
\end{aligned}$$

Let us state the following abstract result:

Proposition 2.1. *Let there be X a Banach space, $x_0\in X$ and $L:X\to X$ and $B:X\times X\to X$,*[4] *a continuous linear operator respectively a continuous bilinear application such that:*

$$\|L\|_\infty+2\|B\|_\infty(R+\|x_0\|)\le c<1, \tag{2.1}$$

where R is given by:

$$R^2=\frac{\|L\|_\infty\|x_0\|}{\|B\|_\infty}+\|x_0\|^2.$$

Under the above condition, the equation

$$x=x_0+Lx+B(x,x),$$

has a unique solution in $B_X(x_0,R)$.

Proof. Of course the above proposition is just a direct application of the Banach fixed point theorem. We start by defining the operator $T:X\to X$ as:

$$Tx=x_0+Lx+B(x,x).$$

Let us first prove that $T(B_X(x_0,R))\subset B_X(x_0,R)$. For any $x\in B_X(x_0,R)$, we have:

$$\begin{aligned}
\|Tx-x_0\|&\le\|L\|_\infty\|x\|+\|B\|_\infty\|x\|^2\\
&\le\|L\|_\infty(\|x_0\|+R)+\|B\|_\infty(\|x_0\|+R)^2=\\
&\|L\|_\infty\|x_0\|+\|B\|_\infty\|x_0\|^2+(\|L\|_\infty+2\|B\|_\infty\|x_0\|)R+R^2\|B\|_\infty\\
&\le\|L\|_\infty\|x_0\|+\|B\|_\infty\|x_0\|^2-R^2\|B\|_\infty+\\
&+(\|L\|_\infty\|x_0\|+2\|B\|_\infty\|x_0\|+2\|B\|_\infty R)R\\
&=(\|L\|_\infty+2\|B\|_\infty(R+\|x_0\|))R\\
&\le R.
\end{aligned}$$

In order to conclude, let us observe that for any $x,y\in B_X(x_0,R)$, we have:

$$\begin{aligned}
\|Tx-Ty\|&\le\|L\|_\infty\|x-y\|+\|B\|_\infty\|x-y\|(\|x\|+\|y\|)\\
&\le\|x-y\|(\|L\|_\infty+2\|B\|_\infty(R+\|x_0\|))\\
&\le c\|x-y\|.
\end{aligned}$$

□

[4] Here, we will use the following notation for the operatorial norm of L and B: $\|L\|_\infty=\inf_{x\in X}\frac{\|Lx\|}{\|x\|}$ and similar, $\|B\|_\infty=\inf_{x,y\in X}\frac{\|B(x,y)\|}{\|x\|\|y\|}$.

We claim that local existence of solutions of (BBM_ε) is a consequence of Proposition 2.1. We begin by rewriting (BBM_ε) in the form of:

$$\begin{cases} \eta(t) = \eta_0 - \int_0^t (Id - \varepsilon\Delta)^{-1} \nabla \cdot V - \varepsilon \int_0^t (Id - \varepsilon\Delta)^{-1} \nabla \cdot (\eta V) = 0, \\ V(t) = V_0 - \int_0^t (Id - \varepsilon\Delta)^{-1} \nabla \eta - \varepsilon \int_0^t (Id - \varepsilon\Delta)^{-1} \nabla \left(\frac{1}{2}|V|^2\right) = 0. \end{cases} \tag{2.2}$$

where $(Id - \varepsilon\Delta)^{-1} a := \mathcal{F}^{-1}\left(\frac{1}{1+\varepsilon|\xi|^2}\hat{a}\right)$. In order to make use of Proposition 2.1, we consider $X = \mathcal{C}\left([0,T], B^s_{2,r} \times \left[B^s_{2,r}\right]^d\right)$, for any $U \in X$, we write $U = (U_1, U_{\tilde{1}})$ with $U_1 \in B^s_{2,r}$ and $U_{\tilde{1}} \in \left[B^s_{2,r}\right]^d$, the projections on the first component, respectively on the last d components. The norm is given by the sum of the Besov norms of the components. Let us also define:

$$[LU](t) = \left(\int_0^t (Id - \varepsilon\Delta)^{-1} \nabla \cdot U_{\tilde{1}}, \int_0^t (Id - \varepsilon\Delta)^{-1} \nabla U_1\right), \tag{2.3}$$

$$B'(U,V) = \left(\frac{U_{\tilde{1}} \cdot V_{\tilde{1}}}{2}, U_1 V_{\tilde{1}}\right), \tag{2.4}$$

$$B(U,V) = \varepsilon L \circ B'(U,V). \tag{2.5}$$

The next proposition aims two rabbits at a time: we want to prove the fact that L is "well-behaved" but also to prepare the ground such that the continuity of B will be an immediate consequence of this behavior. Indeed, as we can better see from (2.4) and (2.5), roughly speaking once we know that the product maps $B^s_{2,r} \times B^s_{2,r}$ into a certain Besov space, we only have to see if L maps the later one back in $B^s_{2,r}$. There are three cases to be considered according to the relation between s and d. When dealing with Besov spaces with $2s > d$ or $2s = d$ and $r = 1$ the simple fact that L is a continuous operator on $\mathcal{C}\left([0,T], B^s_{2,r} \times \left[B^s_{2,r}\right]^d\right)$ will suffice. In the other two cases, in order to proceed as we announced we also have to show that the estimate (2.6), presented just below, holds true[5].

Proposition 2.2. *Consider $s > 0$ and $r \in [1, \infty]$. The application L defined by (2.3) is a linear continuous operator on $\mathcal{C}\left([0,T], B^s_{2,r} \times \left[B^s_{2,r}\right]^d\right)$. Moreover, if $s \leq \frac{d}{2} \leq s+1$, then, for all $t \in [0,T]$, we have that:*

$$\|LU(t)\|_{B^s_{2,r}} \lesssim_\varepsilon \int_0^t \|U(s)\|_{B^{2s'-\frac{d}{2}}_{2,r}} ds, \tag{2.6}$$

with

$$s' = \begin{cases} s & \text{if} \quad s < \frac{d}{2} \\ \frac{d}{2} - \alpha & \text{if} \quad s = \frac{d}{2} \end{cases} \tag{2.7}$$

where α is any real number from $(0,1)$.

[5]Recall that when $\frac{d}{2} > s > 0$ the product maps $B^s_{2,r} \times B^s_{2,r}$ into $B^{2s-\frac{d}{2}}_{2,r}$. See Proposition 4.10 from the Appendix.

Proof. Obviously, L is linear. Next, for any $j \geq -1$, for any $U \in \mathcal{C}\left([0,T], B^s_{2,r} \times \left[B^s_{2,r}\right]^d\right)$, we have[6]:

$$
\begin{aligned}
\left\|\Delta_j LU(t)\right\|_{L^2} &= \left\|\Delta_j \int_0^t (Id - \varepsilon\Delta)^{-1} \nabla \cdot U_{\tilde{1}}\right\|_{L^2} + \left\|\Delta_j \int_0^t (Id - \varepsilon\Delta)^{-1} \nabla U_1\right\|_{L^2} \\
&= \left\|\int_0^t (Id - \varepsilon\Delta)^{-1} \nabla \cdot \Delta_j U_{\tilde{1}}\right\|_{L^2} + \left\|\int_0^t (Id - \varepsilon\Delta)^{-1} \nabla \Delta_j U_1\right\|_{L^2} \\
&\leq \int_0^t \left\|(Id - \varepsilon\Delta)^{-1} \nabla \cdot \Delta_j U_{\tilde{1}}\right\|_{L^2} + \int_0^t \left\|(Id - \varepsilon\Delta)^{-1} \nabla \Delta_j U_1\right\|_{L^2} = \\
& C\int_0^t \left\|\frac{i\xi}{1+\varepsilon\left|\xi\right|^2}\mathcal{F}(\Delta_j U_{\tilde{1}})\right\|_{L^2} + C\int_0^t \left\|\frac{i\xi}{1+\varepsilon\left|\xi\right|^2}\mathcal{F}(\Delta_j U_1)\right\|_{L^2} \\
&\leq C\left\|\frac{\left|\xi\right|^2}{\left(1+\varepsilon\left|\xi\right|^2\right)^2}\right\|_{L^\infty}^{1/2} \left(\int_0^t \left(\left\|\mathcal{F}(\Delta_j U_{\tilde{1}})\right\|_{L^2} + \left\|\mathcal{F}(\Delta_j U_1)\right\|_{L^2}\right)\right) \\
&\leq C\varepsilon^{-\frac{1}{2}} \int_0^t \left(\left\|\Delta_j U_{\tilde{1}}\right\|_{L^2} + \left\|\Delta_j U_1\right\|_{L^2}\right).
\end{aligned} \tag{2.8}
$$

We multiply by 2^{js} and we perform a $\ell^r(\mathbb{Z})$-summation. By virtue of Minkowski's inequality we infer that:

$$
\begin{aligned}
\left\|LU(t)\right\|_{B^s_{2,r}} &\lesssim \varepsilon^{-\frac{1}{2}} \int_0^t \left\|U(s)\right\|_{B^s_{2,r}} ds \\
&\lesssim \varepsilon^{-\frac{1}{2}} T \left\|U\right\|_X .
\end{aligned} \tag{2.9}
$$

Taking the supremum in the left hand-side of the above inequality gives us the desired result. In order to prove the second part of the above proposition, let us observe that for any $t \in [0,T]$

[6] Recall that the dyadic operators Δ_j are defined in the Appendix. See Proposition 4.3 and the remarks that follow.

and for any $j \geq 0$, we have the following:

$$
\begin{aligned}
2^{2js}\left\|\frac{i\xi}{1+\varepsilon\left|\xi\right|^2}\mathcal{F}(\Delta_j U_1(t))\right\|_{L^2}^2 &= \int\limits_{\mathbb{R}^d} \frac{2^{2js}\left|\xi\right|^2}{\left(1+\varepsilon\left|\xi\right|^2\right)^2}\left|\mathcal{F}(\Delta_j U_1(t))\right|^2 \\
&= \int\limits_{2^j\mathcal{C}} \frac{2^{2js}\left|\xi\right|^2}{\left(1+\varepsilon\left|\xi\right|^2\right)^2}\left|\mathcal{F}(\Delta_j U_1(t))\right|^2 \\
&\lesssim \int\limits_{2^j\mathcal{C}} \frac{2^{2js}2^{2j}R_2^2}{(1+\varepsilon 2^{2j}R_1^2)^2}\left|\mathcal{F}(\Delta_j U_1(t))\right|^2 \\
&\lesssim \int\limits_{2^j\mathcal{C}} \frac{2^{2j(s+1)}R_2^2}{(1+\varepsilon 2^{2j}R_1^2)^2}\left|\mathcal{F}(\Delta_j U_1(t))\right|^2 \\
&\lesssim \int\limits_{2^j\mathcal{C}} \frac{2^{2j\left(s+1+\frac{d}{2}-2s'\right)}R_2^2}{(1+\varepsilon 2^{2j}R_1^2)^2}2^{2j\left(2s'-\frac{d}{2}\right)}\left|\mathcal{F}(\Delta_j U_1(t))\right|^2, \qquad (2.10)
\end{aligned}
$$

where R_1, R_2 are the radii of the annulus $\mathcal{C}$ and with α fixed such that $0<\alpha<1$ and s' given by (2.7). The need of imposing the condition $s \leq \frac{d}{2} \leq s+1$ can be better seen from relation (2.10). Indeed, we want

$$
\frac{2^{2j\left(s+1+\frac{d}{2}-2s'\right)}R_2^2}{(1+\varepsilon 2^{2j}R_1^2)^2} \approx_\varepsilon 2^{2j\left(s+\frac{d}{2}-2s'-1\right)}
$$

to remain bounded for all $j \geq 0$ and thus we must assure that

$$
s+\frac{d}{2}-2s'-1 \leq 0.
$$

The apperence of s' is motivated by the fact that in the case $s=\frac{d}{2}$ and $r>1$, we do not have a direct result on the product between two functions of $B^{\frac{d}{2}}_{2,r}$ and thus we will have to make use of the continuous embedding $B^s_{2,r} \subset B^{s-\alpha}_{2,r}$, $\alpha > 0$. From the above discussions we conclude that (2.10) implies that:

$$
2^{js}\left\|\frac{i\xi}{1+\varepsilon\left|\xi\right|^2}\mathcal{F}(\Delta_j U_1(t))\right\|_{L^2} \lesssim_\varepsilon 2^{j\left(2s'-\frac{d}{2}\right)}\left\|\Delta_j U_1(t)\right\|_{L^2}
$$

and a similar result holds if we replace $U_1(t)$ with $U_{\tilde{1}}(t)$. Of course, a similar inequality holds for $j=-1$. Next, owing to (2.8), performing a $\ell^r(\mathbb{Z})$-summation, and by means of Minkowski's inequality we get that:

$$
\left\|LU(t)\right\|_{B^s_{2,r}} \lesssim_\varepsilon \int\limits_0^t \left\|U(s)\right\|_{B^{2s'-\frac{d}{2}}_{2,r}} ds. \qquad (2.11)
$$

as announced. □

Let us now examine the bilinear term.

Proposition 2.3. *Let us consider $s>0$ such that $s \geq \frac{d}{2}-1$ and $r \in [1,\infty]$. Then, the application B defined by* (2.3) *is a bilinear continuous operator on $\mathcal{C}\left([0,T], B^s_{2,r} \times \left[B^s_{2,r}\right]^d\right)$.*

Proof. We begin with the case $s > \frac{d}{2}$ or $s = \frac{d}{2}$ and $r = 1$ since it follows directly from the fact that $B^s_{2,r}$ is an algebra (see Proposition 4.9 from Appendix). Indeed because of (2.4) and (2.9) we have:

$$\begin{aligned}\|B(U,V)\|_{B^s_{2,r}} &\lesssim \varepsilon^{\frac{1}{2}} \int_0^t \left\| \left(\frac{U_{\tilde{1}} \cdot V_{\tilde{1}}}{2}, U_1 V_{\tilde{1}} \right) \right\|_{B^s_{2,r}} ds \lesssim \varepsilon^{\frac{1}{2}} \int_0^t \|UV\|_{B^s_{2,r}} \\ &\lesssim \varepsilon^{\frac{1}{2}} \int_0^t \|U\|_{B^s_{2,r}} \|V\|_{B^s_{2,r}} \lesssim \varepsilon^{\frac{1}{2}} T \|U\|_X \|V\|_X .\end{aligned}$$

Taking the supremum in the left hand-side gives us the desired result. In order to complete the assertion of Proposition 2.3, let us now consider the case when $\frac{d}{2} - 1 \leq s \leq \frac{d}{2}$. We use (2.6) and Proposition 4.10 to infer that:

$$\begin{aligned}\|B(U,V)\|_{B^s_{2,r}} &\lesssim_\varepsilon \int_0^t \left\| \left(\frac{U_{\tilde{1}} \cdot V_{\tilde{1}}}{2}, U_1 V_{\tilde{1}} \right) \right\|_{B^{2s'-\frac{d}{2}}_{2,r}} \lesssim_\varepsilon \int_0^t \|UV\|_{B^{2s'-\frac{d}{2}}_{2,r}} \\ &\lesssim_\varepsilon \int_0^t \|U\|_{B^{s'}_{2,r}} \|V\|_{B^{s'}_{2,r}} \\ &\lesssim_\varepsilon \int_0^t \|U\|_{B^s_{2,r}} \|V\|_{B^s_{2,r}} .\end{aligned}$$

□

We are now in the position of stating and proving the main result of this section:

Theorem 2.1. *Let us consider $s > 0$ such that $s \geq \frac{d}{2} - 1$, $r \in [0, \infty]$ and $(\eta_0, V_0) \in B^s_{2,r}(\mathbb{R}^d) \times \left[B^s_{2,r}(\mathbb{R}^d)\right]$. Then there exists a positive $T > 0$, and a weak solution for (BBM_ε) on $[0,T]$ with initial data (η_0, V_0). Moreover, for all $T > 0$, if there exists $(\eta, V) \in \mathcal{C}\left([0,T], B^s_{2,r}(\mathbb{R}^d) \times [B^s_{2,r}(\mathbb{R}^d)]^d\right)$, a weak solution of (BBM_ε) on $[0,T]$ with initial data (η_0, V_0) then it is the only weak solution of (BBM_ε) which is in $\mathcal{C}\left([0,T], B^s_{2,r}(\mathbb{R}^d) \times [B^s_{2,r}(\mathbb{R}^d)]^d\right)$. Finally, if we denote by $T(\eta_0, V_0)$ the supremum of the set of real numbers $T > 0$ for which there exists a weak solution for (BBM_ε) on $[0,T]$, then if $T(\eta_0, V_0) < \infty$ then we have the following explosion criterion:*

$$\lim_{t \to T} \|(\eta(t), V(t))\|_{B^s_{2,r}} = \infty.$$

Remark 2.1. *In dimension $d = 1$, problem (BBM_ε) remains well posed for initial data in $L^2(\mathbb{R})$. See [1].*

Proof. The existence of a $T > 0$ and for a weak solution of (BBM_ε) is a direct consequence of propositions 2.2, 2.1 and 2.3. The only thing that is left in order to conclude is that we can choose $T > 0$ such that the inequality (2.1) is fulfilled. Indeed, according to the above cited propositions we have that:

$$\|L\|_\infty + 2\|B\|_\infty \left(R + \|(\eta_0, V_0)\|_{B^s_{2,r}}\right) \lesssim_\varepsilon T \left(1 + \|(\eta_0, V_0)\|_{B^s_{2,r}}\right)$$

and thus there exists a positive constant $C_\varepsilon < 1$, independent of η_0, V_0 such that choosing $T = \frac{C_\varepsilon}{1+\|(\eta_0,V_0)\|_{B^s_{2,r}}}$ will give us the desired result. The uniqueness of the solution is a consequence of the fact that each weak solution which is in $\mathcal{C}\left([0,T], B^s_{2,r}(\mathbb{R}^d)\times[B^s_{2,r}(\mathbb{R}^d)]^d\right)$ is given locally by a contraction map argument. Finally, the explosion criterion follows from a classical bootstrap argument. □

We will end this section with a few remarks on the energy of the system (BBM_ε). The results presented below are obtained in [1] ,[2] and [3], in the case of $d=1$. Even if the proofs are basically the same, we choose to briefly present them here (in the general $\mathbb{R}^d$-case) for the sake of completeness:

Proposition 2.4. *Consider (BBM_ε) with initial data $(\eta_0, V_0)\in H^s\left(\mathbb{R}^d\right)\times\left[H^s\left(\mathbb{R}^d\right)\right]^2$ with $s\geq \max\left\{\frac{d}{6},\frac{d}{2}-1\right\}$. Then, the following quantity:*

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^d}\eta(t)^2 + (1+\varepsilon\eta(t))\left|V(t)\right|^2$$

is constant i.e. for all $t\in[0,T(\eta_0,V_0))$:

$$E(t) = E(0).$$

Proof. We consider a smooth enough solution such that the following computations make sense:

$$\begin{aligned}\partial_t E(t) &= \int_{\mathbb{R}^d}\partial_t\eta(t)\left(\eta(t)+\varepsilon\frac{|V(t)|^2}{2}\right)+\partial_t V(t)\left(V(t)+\varepsilon\eta(t)V(t)\right)\\ &= \int_{\mathbb{R}^d}(I-\varepsilon\Delta)^{-1}\nabla\cdot\left(V(t)+\varepsilon\eta(t)V(t)\right)\left(\eta(t)+\varepsilon\frac{|V(t)|^2}{2}\right)+\\ &\quad\int_{\mathbb{R}^d}(I-\varepsilon\Delta)^{-1}\nabla\left(\eta(t)+\varepsilon\frac{|V(t)|^2}{2}\right)\left(V(t)+\varepsilon\eta(t)V(t)\right).\end{aligned}$$

Taking in account that $(I-\varepsilon\Delta)^{-1}$ is auto-adjoint on $L^2\left(\mathbb{R}^d\right)$, we get that:

$$\begin{aligned}&\int_{\mathbb{R}^d}(I-\varepsilon\Delta)^{-1}\nabla\cdot\left(V(t)+\varepsilon\eta(t)V(t)\right)\left(\eta(t)+\varepsilon\frac{|V(t)|^2}{2}\right)\\ &= \int_{\mathbb{R}^d}\nabla\cdot\left(V(t)+\varepsilon\eta(t)V(t)\right)(I-\varepsilon\Delta)^{-1}\left(\eta(t)+\varepsilon\frac{|V(t)|^2}{2}\right)\\ &= -\int_{\mathbb{R}^d}\left(V(t)+\varepsilon\eta(t)V(t)\right)\nabla(I-\varepsilon\Delta)^{-1}\left(\eta(t)+\varepsilon\frac{|V(t)|^2}{2}\right)\\ &= -\int_{\mathbb{R}^d}\left(V(t)+\varepsilon\eta(t)V(t)\right)(I-\varepsilon\Delta)^{-1}\nabla\left(\eta(t)+\varepsilon\frac{|V(t)|^2}{2}\right),\end{aligned}$$

and thus, we get that $\partial_t E(t)=0$. Now, let us consider $(\eta(t), V(t))$ a solution in $\mathcal{C}\left([0,T]\,; H^s\left(\mathbb{R}^d\right)\times H^s\left(\mathbb{R}^d\right)^2\right)$ emanating from $(\eta_0,V_0)\in H^s\left(\mathbb{R}^d\right)\times\left[H^s\left(\mathbb{R}^d\right)\right]^2$. Also, consider a sequence of sufficiently regular initial data $(\eta_{0j}, V_{0j})_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}(\eta_{0j},V_{0j}) = (\eta_0,V_0)\quad\text{in}\quad H^s\left(\mathbb{R}^d\right)\times H^s\left(\mathbb{R}^d\right)^2.$$

Problem (BBM_ε) with initial data (η_{0j}, V_{0j}) for all $j \in \mathbb{N}$, gives birth to a sequence of regular solutions $(\eta_j(t), V_j(t))_{j\in\mathbb{N}}$ which approximates $(\eta(t), V(t))$ in $\mathcal{C}\left([0,T]\,; H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)^2\right)$. According to the above computations, we have:

$$\partial_t \left(\frac{1}{2} \int_{\mathbb{R}^d} \eta_j(t)^2 + (1 + \varepsilon \eta_j(t)) |V_j(t)|^2 \right) = 0.$$

In order to pass to the limit in the last relation and conclude that the same holds true for $(\eta(t), V(t))$ we must be able to make use of the continuous embedding $H^s(\mathbb{R}^d) \subset L^3(\mathbb{R}^d)$. Of course, this can be realized as soon as $s \geq \frac{d}{6}$. □

Remark 2.2. *The above defined energy, $E(t)$, along with:*

$$I(t) = \int_{\mathbb{R}^d} \eta(t) V(t) + \varepsilon \nabla V(t) \nabla \eta(t)$$

and the obviously conserved quantities:

$$\int_{\mathbb{R}^d} \eta(t)\,, \int_{\mathbb{R}^d} V(t)\,,$$

are the only known conservation laws of (BBM_ε).

Remark 2.3. *Of course, the main deficit of the conserved energy is it's "lack of sign". From a physical point of view, the quantity $1 + \varepsilon\eta$ represents the total height of the water above the flat bottom and thus any physical relevant data will satisfy $1 + \varepsilon\eta \geq 0$. Actually, it turns out that, roughly speaking, the solution of (BBM_ε) exists at least as long as the canal does not run dry. In order to emphasize this aspect let us state a result which is a multi-dimensional variant of the one that can be found in [1], Theorem 2.1., page 125, (iii). For more results of this type for other abcd-systems see [3].*

Theorem 2.2. *Let $(\eta_0, V_0) \in H^s \times H^s$, with $s \geq \max\left\{\frac{d}{6}, \frac{d}{2} - 1\right\}$ and let us suppose that there exists $\alpha > 0$ and $T \in (0, T(\eta_0, V_0))$ such that for all $t \in [T, T(\eta_0, V_0))$ we have:*

$$1 + \varepsilon\eta(t) > \alpha.$$

Then, the solution emanating from (η_0, V_0) is global, i.e. $T(\eta_0, V_0) = \infty$.

3 Existence on the Boussinesq time scale

In this part we present a method for obtaining an inferior bound on the maximal time of existence of solutions of (BBM_ε) which is $O(\varepsilon^{-1})$. Suppose that (η_0, V_0) are in $B^{s+1}_{2,r} \times (B^{s+1}_{2,r})^d$ with $s > 1 + d/2$ or $s = 1 + d/2$ and $r = 1$. In order to ease the reading let us rewrite below the BBM-BBM system as follows:

$$\begin{cases} (Id - \varepsilon\Delta)\, \partial_t \eta + \nabla \cdot V + \varepsilon \nabla \cdot (\eta V) = 0, \\ (Id - \varepsilon\Delta)\, \partial_t V + \nabla \eta + \varepsilon \nabla \left(\frac{1}{2} |V|^2 \right) = 0 \\ \eta(0, x) = \eta_0(x)\,,\ V(0, x) = V_0(x)\,. \end{cases} \qquad BBM_\varepsilon$$

The results presented in the previous section assure the existence of a unique solution of (BBM_ε) with initial data (η_0, V_0). Let us denote by $T(\eta_0, V_0)$, the maximal time of existence of the solution emanating from the initial condition (η_0, V_0). Moreover, suppose that $T(\eta_0, V_0) < \infty$. The fact that $s > 1+d/2$ or $s = 1+d/2$ and $r = 1$ will be used when dealing with L^∞-norms, more exactly we will use without explicitly mentioning at each step that $L^\infty \subset B^{s-1}_{2,r} \subset B^{s}_{2,r} \subset B^{s+1}_{2,r}$, with all the previous inclusions being continuous. Let us also set the following notations:

$$\begin{aligned}
\eta_j &:= \Delta_j \eta \\
V_j &:= \Delta_j V \\
\|(\eta_j, V_j)\|^2_{L^2} &:= \|\eta_j\|^2_{L^2} + \|V_j\|^2_{L^2}, \\
\|\nabla(\eta_j, V_j)\|^2_{L^2} &:= \|\nabla\eta_j\|^2_{L^2} + \|\nabla V_j\|^2_{L^2}, \\
U_j^2 &:= \|(\eta_j, V_j)\|^2_{L^2} + \varepsilon\|\nabla(\eta_j, V_j)\|^2_{L^2}, \\
U_{s+1}^2 &:= \|(\eta, V)\|^2_{B^s_{2,r}} + \varepsilon\|(\eta, V)\|^2_{B^{s+1}_{2,r}}.
\end{aligned}$$

Notice that we have:

$$\left\|\left(2^{js}U_j\right)_{j\in\mathbb{Z}}\right\|_{\ell^r} \approx U_{s+1} \approx \|(\eta, V)\|_{B^{s+1}_{2,r}},$$

and according to the explosion criterion from the last section we get that:

$$\lim_{t\to T(\eta_0,V_0)} U_{s+1}(t) = \infty.$$

Let us state the formal result that we will prove in the rest of this section:

Theorem 3.1. *Let there be $(s, r) \in [1, \infty]^2$ such that $s > d/2+1$ or $s = d/2+1$ and $r = 1$. For all initial data $\eta_0 \in B^{s+1}_{2,r}$ and $V_0 \in [B^{s+1}_{2,r}]^d$ there exists a constant $C(\eta_0, V_0)$, independent of ε such that the maximal time of existence of the unique solution of (BBM_ε), $T(\eta_0, V_0)$ emanating from (η_0, V_0) satisfies the following:*

$$\frac{C(\eta_0, V_0)}{\varepsilon} \leq T(\eta_0, V_0).$$

The computations that follow are quite wearisome but in an attempt to render them easier to follow we resume the main ideas:

- We first localize the equation in the frequency space, that is, we write the equation satisfied by $\Delta_j\eta$, $\Delta_j V$.
- Naturally, we obtain a superior bound on the time derivative of the L^2-norm of $\Delta_j\eta, \Delta_j V$ and their gradients. Unfortunately if we wish to apply directly a Gronwall type argument, this leads to a loss of the "quality" of the maximal time of existence, that is we obtain an existence time that is of order $O\left(\varepsilon^{-\delta}\right)$ with $\delta \in (0, 1)$. Of course, the explanation is that we can use the regularizing effect of $(Id - \varepsilon\Delta)^{-1}$ with the price of increasing the norm of the nonlinear operator appearing in (BBM_ε), which roughly speaking is of order $O(\varepsilon)$.
- In order to avoid the above inconvenience we proceed as it follows: we first treat the symmetrical part of the system which gives us "good" estimates. Then, we aim at getting rid of the remaining part by making use of another estimate which, as we will further see is not of a direct use. Finally we give a solution on how to "fix" the later situation.

- The type of bound that we want to obtain is:
$$\partial_t U_j^2 \lesssim \varepsilon P\left(\|\Delta_j(\eta,V)\|_{L^2}, \|\nabla\Delta_j(\eta,V)\|_{L^2}, \|(\eta,V)\|_{B^s_{2,r}}, \|(\eta,V)\|_{B^{s+1}_{2,r}}\right),$$
where $P(x,y,z,w)$ is a polynomial of order less than four.

- In order to successfully apply a Gronwall-type argument we precise the following rule of thumb: the coefficients of the general term of P, $x^\alpha y^\beta z^\gamma w^\delta$ must be at least of order $O\left(\varepsilon^{\frac{1}{2}(\beta+\delta)}\right)$; this is because we want to "absorb" them with the terms U_j or U_{s+1} without "altering" the $O(\varepsilon)$ order of the estimate.

As previously announced, we begin by localizing the equation thus obtaining:

$$\begin{cases} (Id-\varepsilon\Delta)\,\partial_t\eta_j + \nabla\cdot V_j + \varepsilon\nabla\cdot\Delta_j(\eta V) = 0, \\ (Id-\varepsilon\Delta)\,\partial_t V_j + \nabla\eta_j + \varepsilon\nabla\Delta_j\left(\frac{1}{2}|V|^2\right) = 0. \end{cases} \tag{3.1}$$

Owing to the fact that $\nabla\times V = 0$, we have $\nabla V = \nabla^t V$ and we can write the above system as:

$$\begin{cases} (Id-\varepsilon\Delta)\,\partial_t\eta_j + \nabla\cdot V_j + \varepsilon\nabla\eta_j V + \varepsilon\eta\nabla\cdot V_j = R_j^1, \\ (Id-\varepsilon\Delta)\,\partial_t V_j + \nabla\eta_j + \varepsilon\nabla^t V_j V = R_j^2, \end{cases} \tag{3.2}$$

with:

$$\begin{aligned} R_j^1 &= [V,\Delta_j]\,\nabla\eta + [\eta,\Delta_j]\,\nabla\cdot V, \\ R_j^2 &= [V,\Delta_j]\,\nabla^t V. \end{aligned}$$

According to Proposition 4.11, of the Appendix, the above operators satisfy the following estimates:

$$\begin{aligned} \left\|R_j^1\right\|_{L^2} &\lesssim c_j^1(t)\left(\|\nabla\eta\|_{B^{s-1}_{2,r}} + \|\nabla V\|_{B^{s-1}_{2,r}}\right)\left(\|\eta\|_{B^s_{2,r}} + \|V\|_{B^s_{2,r}}\right). \\ \left\|R_j^2\right\|_{L^2} &\lesssim c_j^2(t)\,\|\nabla V\|_{B^{s-1}_{2,r}}\|V\|_{B^s_{2,r}}. \end{aligned}$$

where for all t we have $\left\|\left(2^{js}c_j^k(t)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} \leq 1$, $k=1,2$. Next, we multiply the first equation in (3.2) by η_j, the second one by V_j and adding the results we obtain:

$$\begin{aligned} &(Id-\varepsilon\Delta)\,\partial_t\eta_j\eta_j + (Id-\varepsilon\Delta)\,\partial_t V_j V_j + \nabla\cdot(\eta_j V_j) + \frac{\varepsilon}{2}\nabla|\eta_j|^2\cdot V + \varepsilon\nabla^t V_j V\cdot V_j \\ &= -\varepsilon\eta\eta_j\nabla\cdot V_j + \varepsilon\eta_j R_j^1 + \varepsilon V_j\cdot R_j^2. \end{aligned} \tag{3.3}$$

Integrating equation (3.3) we get that:

$$\begin{aligned} &\frac{1}{2}\partial_t\left(\|(\eta_j,V_j)\|_{L^2}^2 + \varepsilon\|\nabla(\eta_j,V_j)\|_{L^2}^2\right) + \int\varepsilon\eta\eta_j\nabla\cdot V_j \\ &\lesssim \varepsilon\|(\eta_j,V_j)\|_{L^2}^2\|\nabla V\|_{L^\infty} + \varepsilon c_j^3(t)\,\|(\eta_j,V_j)\|_{L^2}\|(\eta,V)\|_{B^s_{2,r}}^2 \\ &\lesssim \varepsilon\|(\eta_j,V_j)\|_{L^2}^2\|(\eta,V)\|_{B^s_{2,r}} + \varepsilon c_j^3(t)\,\|(\eta_j,V_j)\|_{L^2}\|(\eta,V)\|_{B^s_{2,r}}^2 \\ &\lesssim \varepsilon U_j^2 U_{s+1} + \varepsilon c_j^3(t)\,U_j U_{s+1}^2 \end{aligned} \tag{3.4}$$

where for all t we have $\left\|\left(2^{js}c_j^3(t)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} \leq 1$.

In all what follows, we aim at getting rid of the "bad" term, $\int \varepsilon\eta\Delta_j\eta\nabla\cdot V_j$ which prevents us from directly obtaining the desired bound. In order to do so, we multiply the second equation of (3.1) with $\varepsilon\eta V_j$ and integrating, we get that:

$$\varepsilon \left\langle (I-\varepsilon\Delta)\partial_t V_j, \eta V_j \right\rangle_{L^2} + \varepsilon \int \eta\nabla\eta_j V_j + \varepsilon^2 \int \eta V_j \nabla\left(\Delta_j \frac{|V|^2}{2}\right) = 0. \tag{3.5}$$

We write:

$$\varepsilon \left\langle (I-\varepsilon\Delta)\,\partial_t V_j, \eta V_j \right\rangle_{L^2} = \varepsilon \left\langle \partial_t V_j, \eta V_j \right\rangle_{L^2} - \varepsilon^2 \left\langle \Delta\partial_t V_j, \eta V_j \right\rangle_{L^2}. \tag{3.6}$$

Next, write the first term from the right-hand side of (3.6) as:

$$2\varepsilon \left\langle \partial_t V_j, \eta V_j \right\rangle_{L^2} = \varepsilon\partial_t \left\langle V_j, \eta V_j \right\rangle_{L^2} - \varepsilon \left\langle V_j, \partial_t\eta V_j \right\rangle_{L^2}\ . \tag{3.7}$$

The second term from the right-hand side of (3.6) can be written as[7]:

$$\begin{aligned}
-2\varepsilon^2 \left\langle \Delta\partial_t V_j, \eta V_j \right\rangle_{L^2} &= \varepsilon^2 \left\langle \partial_t\nabla V_j, \eta\nabla V_j \right\rangle_{L^2} + \varepsilon^2 \left\langle \partial_t\nabla V_j, \nabla\eta\otimes V_j \right\rangle_{L^2}\\
&= \varepsilon^2\partial_t \left\langle \nabla V_j, \eta\nabla V_j \right\rangle_{L^2} - \varepsilon^2 \left\langle \nabla V_j, \partial_t\eta\nabla V_j \right\rangle +\\
&\quad + \varepsilon^2 \left\langle \partial_t\nabla V_j, \nabla\eta\otimes V_j \right\rangle_{L^2}.
\end{aligned} \tag{3.8}$$

Let us write $\varepsilon^2 \left\langle \partial_t\nabla V_j, \nabla\eta\otimes V_j \right\rangle_{L^2}$ as:

$$\begin{aligned}
\varepsilon^2 \left\langle \partial_t\nabla V_j, \nabla\eta\otimes V_j \right\rangle_{L^2} &= -\varepsilon^2 \left\langle \partial_t V_j, \nabla\cdot(\nabla\eta\otimes V_j) \right\rangle_{L^2}\\
&= -\varepsilon^2 \left\langle \partial_t V_j, \Delta\eta V_j \right\rangle_{L^2} - \varepsilon^2 \sum_{l,m=\overline{1,d}} \int \partial_t V_j^l \partial_m\eta\partial_m V_j^m.
\end{aligned} \tag{3.9}$$

Next, we see that:

$$\begin{aligned}
-\varepsilon^2 \left\langle \partial_t V_j, \Delta\eta V_j \right\rangle_{L^2} &\lesssim \varepsilon^2 \left\|\partial_t V_j\right\|_{L^2} \left\|V_j\right\|_{L^2} \left\|\Delta\eta\right\|_{L^\infty}\\
&\lesssim \varepsilon^2 \left\|(I-\varepsilon\Delta)^{-1}\left(\nabla\eta_j + \varepsilon\nabla\left(\Delta_j |V|^2\right)\right)\right\|_{L^2} \left\|V_j\right\|_{L^2} \left\|\eta\right\|_{B^{s+1}_{2,r}}\\
&\lesssim \varepsilon^2 \left\|\left(\nabla\eta_j + \varepsilon\nabla\left(\Delta_j |V|^2\right)\right)\right\|_{L^2} \left\|V_j\right\|_{L^2} \left\|\eta\right\|_{B^{s+1}_{2,r}}\\
&\lesssim \varepsilon^2 (\left\|\nabla\eta_j\right\|_{L^2} + \varepsilon \left\|\nabla\left(\Delta_j |V|^2\right)\right\|_{L^2}) \left\|V_j\right\|_{L^2} \left\|\eta\right\|_{B^{s+1}_{2,r}}\\
&\lesssim \varepsilon^2 \left\|\nabla\eta_j\right\|_{L^2} \left\|V_j\right\|_{L^2} \left\|\eta\right\|_{B^{s+1}_{2,r}} + \varepsilon^3 c_j^4(t) \left\|V_j\right\|_{L^2} \left\|\eta\right\|_{B^{s+1}_{2,r}} \left\|V\right\|_{B^{s+1}_{2,r}} \left\|V\right\|_{B^{s}_{2,r}}
\end{aligned} \tag{3.10}$$

where for all t we have $\left\|\left(2^{js}c_j^4(t)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} \leq 1$. We search now to estimate the second term in the right-hand side of (3.9):

$$\begin{aligned}
-\varepsilon^2 \sum_{l,m=\overline{1,d}} \int \partial_t V_j^l \partial_m\eta\partial_m V_j^m &\lesssim \varepsilon^2 \left\|\nabla\eta\right\|_{L^\infty} \left\|\partial_t V_j\right\|_{L^2} \left\|\nabla V_j\right\|_{L^2}\\
&\lesssim \varepsilon^2 \left(\left\|\nabla\eta_j\right\|_{L^2} + \varepsilon \left\|\nabla\left(\Delta_j |V|^2\right)\right\|_{L^2}\right) \left\|\nabla V_j\right\|_{L^2} \left\|\eta\right\|_{B^{s}_{2,r}}\\
&\lesssim \varepsilon^2 \left\|\nabla\eta_j\right\|_{L^2} \left\|\nabla V_j\right\|_{L^2} \left\|\eta\right\|_{B^{s}_{2,r}} +\\
&\quad + \varepsilon^3 c_j^4(t) \left\|\nabla V_j\right\|_{L^2} \left\|\eta\right\|_{B^{s}_{2,r}} \left\|V\right\|_{B^{s+1}_{2,r}} \left\|V\right\|_{B^{s}_{2,r}}.
\end{aligned} \tag{3.11}$$

[7] The matrix $\nabla\eta\otimes V_j$ is the $d\times d$ matrix with elements $(\nabla\eta\otimes V_j)_{m,l} = \partial_m\eta V_j^l$

Finally, using:

$$\begin{aligned}\|\partial_t\eta\|_{L^\infty} &= \left\|(I-\varepsilon\Delta)^{-1}\left(\nabla\cdot V+\varepsilon\nabla\cdot(\eta V)\right)\right\|_{L^\infty}\\ &\lesssim \left\|(I-\varepsilon\Delta)^{-1}\left(\nabla\cdot V+\varepsilon\nabla\cdot(\eta V)\right)\right\|_{B^{s-1}_{2,r}}\\ &\lesssim \|V\|_{B^s_{2,r}}+\varepsilon\|\eta\|_{B^s_{2,r}}\|V\|_{B^s_{2,r}} \qquad (3.12)\end{aligned}$$

and putting together relations (3.5)-(3.12), we get that:

$$\begin{aligned}&\frac{1}{2}\partial_t\int\left(\varepsilon\eta|V_j|^2+\varepsilon^2\eta\nabla V_j:\nabla V_j\right)+\varepsilon\int\eta\nabla\eta_jV_j+\varepsilon^2\int\eta V_j\nabla\left(\Delta_j\frac{|V|^2}{2}\right)\\ &\lesssim\left(\|V\|_{B^s_{2,r}}+\varepsilon\|\eta\|_{B^s_{2,r}}\|V\|_{B^s_{2,r}}\right)\left(\varepsilon\|V_j\|^2_{L^2}+\varepsilon^2\|\nabla V_j\|^2_{L^2}\right)\\ &+\varepsilon^2\|\nabla\eta_j\|_{L^2}\|V_j\|_{L^2}\|\eta\|_{B^{s+1}_{2,r}}+\varepsilon^3c_j^4(t)\|V_j\|_{L^2}\|\eta\|_{B^{s+1}_{2,r}}\|V\|_{B^{s+1}_{2,r}}\|V\|_{B^s_{2,r}}+\\ &+\varepsilon^2\|\nabla\eta_j\|_{L^2}\|\nabla V_j\|_{L^2}\|\eta\|_{B^s_{2,r}}+\varepsilon^3c_j^4(t)\|\nabla V_j\|_{L^2}\|\eta\|_{B^s_{2,r}}\|V\|_{B^{s+1}_{2,r}}\|V\|_{B^s_{2,r}} \qquad (3.13)\end{aligned}$$

From now on, all the estimates that we derive follow easily from the arithmetic mean inequality and by simply bounding terms that are linear combination of $\|\eta_j\|_{L^2}$, $\sqrt{\varepsilon}\|\nabla\eta_j\|_{L^2}$, $\|V_j\|_{L^2}$, $\sqrt{\varepsilon}\|\nabla V_j\|_{L^2}$ by U_j and respectively linear combination of Besov norms of order s and $\sqrt{\varepsilon}$-multiplied by Besov norms of order $s+1$ by U_{s+1}. For the sake of completeness we detail this below. By means of Holder's inequality we get that the first term of the right-hand side of (3.13):

$$\begin{aligned}-\varepsilon^2\int\eta V_j\nabla\left(\Delta_j\frac{|V|^2}{2}\right) &\lesssim\varepsilon^2\|\eta\|_{L^\infty}\|V_j\|_{L^2}\left\|\nabla\Delta_j|V|^2\right\|_{L^2}\\ &\lesssim\varepsilon^2c_j^4(t)\|\eta\|_{B^s_{2,r}}\|V_j\|_{L^2}\|V\|_{B^s_{2,r}}\|V\|_{B^{s+1}_{2,r}}\\ &\lesssim\varepsilon^{3/2}c_j^4(t)\|V_j\|_{L^2}\|\eta\|_{B^s_{2,r}}U^2_{s+1}\\ &\lesssim\varepsilon^{3/2}c_j^4(t)\|V_j\|_{L^2}U^3_{s+1}\\ &\lesssim\varepsilon^{3/2}c_j^4(t)U_jU^3_{s+1}\end{aligned}$$

Next:

$$\begin{aligned}\left(\|V\|_{B^s_{2,r}}+\varepsilon\|\eta\|_{B^s_{2,r}}\|V\|_{B^s_{2,r}}\right)\left(\varepsilon\|V_j\|^2_{L^2}+\varepsilon^2\|\nabla V_j\|^2_{L^2}\right) &\lesssim\varepsilon\left(\|V\|_{B^s_{2,r}}+\varepsilon\|\eta\|_{B^s_{2,r}}\|V\|_{B^s_{2,r}}\right)U_j^2\\ &\lesssim\varepsilon U_j^2\left(U_{s+1}+\varepsilon U^2_{s+1}\right).\end{aligned}$$

The third term is bounded by:

$$\begin{aligned}\varepsilon^2\|\nabla\eta_j\|_{L^2}\|V_j\|_{L^2}\|\eta\|_{B^{s+1}_{2,r}} &=\varepsilon(\sqrt{\varepsilon}\|\nabla\eta_j\|_{L^2})\|V_j\|_{L^2}\sqrt{\varepsilon}\|\eta\|_{B^{s+1}_{2,r}}\\ &\lesssim\varepsilon\left(\|V_j\|^2_{L^2}+\varepsilon\|\nabla\eta_j\|^2_{L^2}\right)U_{s+1}\\ &\lesssim\varepsilon U_j^2U_{s+1}.\end{aligned}$$

Next, the forth term

$$\begin{aligned}\varepsilon^3c_j^4(t)\|V_j\|_{L^2}\|\eta\|_{B^{s+1}_{2,r}}\|V\|_{B^{s+1}_{2,r}}\|V\|_{B^s_{2,r}} &\lesssim\varepsilon^2c_j^4(t)\|V_j\|_{L^2}\|V\|_{B^s_{2,r}}\left(\varepsilon\|\eta\|^2_{B^{s+1}_{2,r}}+\varepsilon\|V\|^2_{B^{s+1}_{2,r}}\right)\\ &\lesssim\varepsilon^2c_j^4(t)U_jU^3_{s+1}.\end{aligned}$$

For the fifth term we infer that:

$$\begin{aligned}\varepsilon^2 \|\nabla \eta_j\|_{L^2} \|\nabla V_j\|_{L^2} \|\eta\|_{B^s_{2,r}} &\lesssim \varepsilon^2 \left(\|\nabla \eta_j\|^2_{L^2} + \|\nabla V_j\|^2\right) \|\eta\|_{B^s_{2,r}} \\ &\lesssim \varepsilon U_j^2 U_{s+1}.\end{aligned}$$

Finally, we treat the last term like:

$$\varepsilon^3 c_j^4(t) \|\nabla V_j\|_{L^2} \|\eta\|_{B^s_{2,r}} \|V\|_{B^{s+1}_{2,r}} \|V\|_{B^s_{2,r}} \lesssim \varepsilon^2 c_j^4(t) U_j U_{s+1}^3.$$

Gathering all the above estimates, yields:

$$\begin{aligned}&\frac{1}{2}\partial_t \int \left(\varepsilon\eta |V_j|^2 + \varepsilon^2 \eta \nabla V_j : \nabla V_j\right) + \varepsilon \int \eta \nabla \eta_j V_j \\ &\lesssim \varepsilon^{3/2} c_j^4(t) U_j U_{s+1}^3 + \varepsilon U_j^2 \left(U_{s+1} + \varepsilon U_{s+1}^2\right) + \varepsilon U_j^2 U_{s+1} \\ &+ \varepsilon^2 c_j^4(t) U_j U_{s+1}^3 + \varepsilon U_j^2 U_{s+1} + \varepsilon^2 c_j^4(t) U_j U_{s+1}^3 \end{aligned} \tag{3.14}$$

Combining estimates (3.4) and (3.14) we get that:

$$\begin{aligned}&\frac{1}{2}\partial_t \int \left(\eta_j^2 + \varepsilon |\nabla \eta_j|^2 + (1+\varepsilon\eta)|V_j|^2 + \varepsilon(1+\varepsilon\eta)(\nabla V_j : \nabla V_j)\right) + \varepsilon \int \eta \left(\nabla \eta_j V_j + \eta_j \nabla \cdot V_j\right) \\ &\lesssim \varepsilon U_j \left(c_j^3(t) U_{s+1}^2 + \sqrt{\varepsilon} c_j^4(t) U_{s+1}^3 + 2\varepsilon c_j^4(t) U_{s+1}^3\right) \\ &+ \varepsilon U_j^2 \left(U_{s+1} + U_{s+1} + \varepsilon U_{s+1}^2 + 2U_{s+1}\right).\end{aligned}$$

and owing to the fact that:

$$\begin{aligned}-\varepsilon \int \eta \left(\nabla \eta_j V_j + \eta_j \nabla \cdot V_j\right) = \varepsilon \int \nabla \eta \eta_j V_j &\le \varepsilon \|\nabla \eta\|_{L^\infty} \|\eta_j\|_{L^2} \|V_j\|_{L^2} \\ &\le \varepsilon \|\eta\|_{B^s_{2,r}} \|\eta_j\|_{L^2} \|V_j\|_{L^2} \\ &\le \varepsilon U_j^2 U_{s+1},\end{aligned}$$

we conclude that:

$$\begin{aligned}&\frac{1}{2}\partial_t \int \left(\eta_j^2 + \varepsilon |\nabla \eta_j|^2 + (1+\varepsilon\eta)|V_j|^2 + \varepsilon(1+\varepsilon\eta)(\nabla V_j : \nabla V_j)\right) \\ &\lesssim \varepsilon U_j \left(c_j^3(t) U_{s+1}^2 + c_j^4(t) U_{s+1}^3\right) + \varepsilon U_j^2 \left(U_{s+1} + U_{s+1}^2\right)\end{aligned} \tag{3.15}$$

Remark 3.1. *The estimate* (3.15) *is the analogue of the ones obtained in [4], pages* 617 *and* 625, *in the context of Sobolev spaces with integer coefficients. The fact that such an estimate holds with η_j and V_j instead of η and V is of course the key to obtain the desired result on the lower bound of the maximal time of existence of* (BBM_ε) *but with a lower regularity assumption on the initial data than that needed in [4].*

Remark 3.2. *Unfortunately estimate* (3.15) *is not of direct use because of the "lack of sign" of the quantity $1 + \varepsilon\eta$. We recall that if we suppose that there exists $T < T(\eta_0, V_0)$ such that $1 + \varepsilon\eta > \alpha > 0$ for any $t \in [T, T(\eta_0, V_0))$, the solution emanating from the initial data is global in time, the regularity level required for the initial data (η_0, V_0) being less than $d/2$. See Theorem 2.2 and [1], Theorem* 2.1. *page* 125.

Motivated by the last observation, we will "repair" the announced inconvenience. In order to do so, let us analyze the following quantity:

$$\partial_t \left(\varepsilon \int \|\eta\|_{L^\infty} \left(|\eta_j|^2 + \varepsilon |\nabla \eta_j|^2 + |V_j|^2 + \varepsilon \nabla V_j : \nabla V_j \right) \right) = I_1 + I_2,$$

with

$$I_1 = \varepsilon \|\eta\|_{L^\infty} \partial_t \int \left(|\eta_j|^2 + \varepsilon |\nabla \eta_j|^2 + |V_j|^2 + \varepsilon \nabla V_j : \nabla V_j \right),$$
$$I_2 = \partial_t \|\eta\|_{L^\infty} \varepsilon \int \left(|\eta_j|^2 + \varepsilon |\nabla \eta_j|^2 + |V_j|^2 + \varepsilon \nabla V_j : \nabla V_j \right).$$

Remark 3.3. *The function*

$$t \to \|\eta(t)\|_{L^\infty}$$

is Lipschitz on any compact $[0,T] \subset [0, T(\eta_0, V_0))$ and thus it is absolutely continuous. Therefore, it is almost everywhere differentiable. All the estimates that follow are valid almost everywhere in time.

Owing to (3.4) we get that:

$$\begin{aligned} I_1 &\lesssim \varepsilon^2 \|\eta\|_{L^\infty} \left(U_j^2 U_{s+1} + c_j^3(t) U_j U_{s+1}^2 - \int \eta \eta_j \nabla \cdot V_j \right) \\ &\lesssim \varepsilon^2 U_{s+1} \left(U_j^2 U_{s+1} + c_j^3(t) U_j U_{s+1}^2 + \|\eta\|_{L^\infty} \|\eta_j\|_{L^2} \|\nabla \cdot V_j\|_{L^2} \right) \\ &\lesssim \varepsilon^2 U_{s+1} \left(U_j^2 U_{s+1} + c_j^3(t) U_j U_{s+1}^2 + U_{s+1} \|\eta_j\|_{L^2} \|\nabla V_j\|_{L^2} \right) \\ &\lesssim \varepsilon^2 U_j^2 U_{s+1}^2 + \varepsilon^2 c_j^3(t) U_j U_{s+1}^3 + \varepsilon^{3/2} U_j^2 U_{s+1}^2. \end{aligned}$$

Next, observe that:

$$\begin{aligned} \partial_t \|\eta(t)\|_{L^\infty} &= \lim_{s \to t} \frac{\|\eta(t)\|_{L^\infty} - \|\eta(s)\|_{L^\infty}}{t-s} \leq \lim_{s \to t} \left| \frac{\|\eta(t)\|_{L^\infty} - \|\eta(s)\|_{L^\infty}}{t-s} \right| \\ &\leq \lim_{s \to t} \left\| \frac{\eta(t) - \eta(s)}{t-s} \right\|_{L^\infty} \leq \lim_{s \to t} \left\| \frac{\eta(t) - \eta(s)}{t-s} \right\|_{B^{s-1}_{2,r}} = \|\partial_t \eta(t)\|_{B^{s-1}_{2,r}} \end{aligned}$$

and according to (3.12) we get:

$$\partial_t \|\eta(t)\|_{L^\infty} \lesssim \|\nabla \cdot V\|_{B^{s-1}_{2,r}} + \varepsilon \|\nabla \cdot (\eta V)\|_{B^{s-1}_{2,r}}.$$

Thus, we get:

$$\begin{aligned} I_2 &\lesssim \varepsilon U_j^2 \left(\|(\eta, V)\|_{B^s_{2,r}} + \varepsilon \|(\eta, V)\|^2_{B^s_{2,r}} \right) \\ &\lesssim .\varepsilon U_j^2 \left(U_{s+1} + \varepsilon U_{s+1}^2 \right) \end{aligned}$$

Finally, we infer that:

$$I_1 + I_2 \lesssim \varepsilon^2 c_j^3(t) U_j U_{s+1}^3 + \varepsilon U_j^2 \left(U_{s+1} + (\varepsilon + \sqrt{\varepsilon}) U_{s+1}^2 \right) \tag{3.16}$$

Estimates and (3.15) and (3.16) give us:

$$\begin{aligned} &\partial_t \int \left((1 + \varepsilon \|\eta\|_{L^\infty}) \left(\eta_j^2 + \varepsilon |\nabla \eta_j|^2 \right) + (1 + \varepsilon \eta + \varepsilon \|\eta\|_{L^\infty}) |V_j|^2 + \varepsilon (1 + \varepsilon \eta + \varepsilon \|\eta\|_{L^\infty}) (\nabla V_j : \nabla V_j) \right) \\ &\lesssim \varepsilon U_j \left(c_j^3(t) U_{s+1}^2 + \varepsilon c_j^3(t) U_{s+1}^3 + c_j^4(t) U_{s+1}^3 \right) + \varepsilon U_j^2 \left(2U_{s+1} + (1 + \sqrt{\varepsilon} + \varepsilon) U_{s+1}^2 \right) \end{aligned} \tag{3.17}$$

Now, observe that denoting:

$$N_j^2 = \int \left(\left(1+\varepsilon \left\|\eta\right\|_{L^\infty}\right)\left(\eta_j^2+\varepsilon\left|\nabla\eta_j\right|^2\right) + \left(1+\varepsilon(\eta+\left\|\eta\right\|_{L^\infty})\right)\left|V_j\right|^2 + \varepsilon\left(1+\varepsilon(\eta+\left\|\eta\right\|_{L^\infty})\right)\left(\nabla V_j : \nabla V_j\right)\right),$$

then, according to the definition of U_j we have that:

$$U_j \leq N_j,$$

and thus:

$$\partial_t N_j^2 \lesssim \varepsilon N_j \left(c_j^3\left(t\right)U_{s+1}^2 + c_j^5\left(t\right)U_{s+1}^3 + U_j\left(U_{s+1}+U_{s+1}^2\right)\right),$$

where for all t we have $\left\|\left(2^{js}c_j^5\left(t\right)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} \leq 1$. Invoking a standard real analysis argument one gets that:

$$\partial_t N_j \lesssim \varepsilon \left(c_j^3\left(t\right)U_{s+1}^2 + c_j^5\left(t\right)U_{s+1}^3 + U_j\left(U_{s+1}+U_{s+1}^2\right)\right).$$

Integrating in time and using again that $U_j \leq N_j$, we end up with:

$$U_j\left(t\right) \lesssim N_j\left(0\right) + \varepsilon\int_0^t \left(c_j^3\left(\tau\right)U_{s+1}^2\left(\tau\right) + c_j^5\left(\tau\right)U_{s+1}^3\left(\tau\right) + U_j\left(\tau\right)\left(U_{s+1}\left(\tau\right)+U_{s+1}^2\left(\tau\right)\right)\right)d\tau,$$

for all $t < T\left(\eta_0, V_0\right)$. We multiply the last inequality by 2^{js} and taking the $\ell^r(\mathbb{Z})$ norm we get that:

$$U_{s+1}(t) \lesssim \left\|\left(2^{js}N_j\left(0\right)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} + \varepsilon\left\|\int_0^t \left(2^{js}c_j^3\left(\tau\right)U_{s+1}^2\left(\tau\right) + 2^{js}c_j^5\left(\tau\right)U_{s+1}^3\left(\tau\right) + 2^{js}U_j\left(\tau\right)\left(U_{s+1}\left(\tau\right)+U_{s+1}^2\left(\tau\right)\right)\right)d\tau\right\|_{\ell^r(\mathbb{Z})}.$$

As $r \geq 1$, by means of Minkowski's inequality and the triangle inequality we get that:

$$U_{s+1}(t) \lesssim \left\|\left(2^{js}N_j\left(0\right)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} + \varepsilon\int_0^t \left(U_{s+1}^2(\tau)+U_{s+1}^3(\tau)\right)d\tau. \tag{3.18}$$

Of course, the first term in the right-hand side of (3.18) is bounded by

$$\left(1+2\left\|\eta_0\right\|_{L^\infty}\right)\left\|\left(2^{js}U_j\left(0\right)\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} \lesssim \left(1+2\left\|\eta_0\right\|_{L^\infty}\right)\left\|(\eta_0, V_0)\right\|_{B^{s+1}_{2,r}}.$$

We are now in the position of applying Gronwall's lemma which yields the following bound on the time of existence:

$$\frac{1}{\varepsilon}F\left(\left(1+2\left\|\eta_0\right\|_{L^\infty}\right)\left\|(\eta_0, V_0)\right\|_{B^{s+1}_{2,r}}\right) \lesssim T\left(\eta_0, V_0\right),$$

where $F:(0,\infty)\to(0,\infty)$,

$$F\left(x\right) = \frac{1}{x} - \ln\left(1+\frac{1}{x}\right).$$

4 Appendix: Littlewood-Paley theory

4.1 Functions with compactly supported Fourier transform

We present here a few results on Fourier analysis used through the text. The full proofs along with other complementary results can be found in [5].

Proposition 4.1 (Minkowski's inequality)**.** *Let (X_1, μ_1) and (X_2, μ_2) be two measure spaces and f a nonnegative measurable function over $X_1 \times X_2$. For all $1 \leq p \leq q \leq \infty$, we have*

$$\Big\| \left\| f(\cdot, x_2) \right\|_{L^p(X_1,\mu_1)} \Big\|_{L^q(X_2,\mu_2)} \leq \Big\| \left\| f(x_1, \cdot) \right\|_{L^q(X_2,\mu_2)} \Big\|_{L^p(X_1,\mu_1)}.$$

Proposition 4.2 (Gronwall's inequality)**.** *Let $u : [t_0, t_1] \to \mathbb{R}^+$ be absolutely continuous and non-negative and suppose that u obeys the differential inequality*

$$\partial_t u(t) \leq B(t)\, u(t)$$

for almost every $t \in [t_0, t_1]$, where $B : [t_0, t_1] \to \mathbb{R}^+$ is continuous and non-negative. Then we have

$$u(t) \leq u(t_0) \exp\left(\int_{t_0}^{t} B(s)\, ds \right)$$

for all $t \in [t_0, t_1]$.

In all that follows we shall call an annulus any set of the type:

$$\left\{ \xi \in \mathbb{R}^d : 0 < r_1 \leq |\xi| \leq r_2 \right\}.$$

Lemma 4.1. *Let $\mathcal{C}$ be an annulus $m \in \mathbb{R}$ and $k = 2[1 + d/2]$. Let σ be a k-times differentiable function on $\mathbb{R}^d \backslash \{0\}$ such that for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$, there exists a constant C_α such that for all $\xi \in \mathbb{R}^d$ we have:*

$$|\partial^\alpha \sigma(\xi)| \leq C_\alpha \, |\xi|^{m - |\alpha|}.$$

Then there exists a constant C depending only on the constants C_α such that for any $p \in [1, \infty]$, any $\lambda > 0$ and for any function $u \in L^p$ with Fourier transform supported in $\mathcal{C}$, we have[8]*:*

$$\|\sigma(D) u\|_{L^p} \leq C \lambda^m \, \|u\|_{L^p}.$$

Let us now state the so-called Bernstein lemmas:

Lemma 4.2. *Let $\mathcal{C}$ be an annulus and $\mathcal{B}$ a ball. Then a constant $C > 0$ exists so that for any positive integer k, any couple $(p, q) \in [1, \infty]^2$ with $q \geq p \geq 1$ and any function of L^p, we have:*

$$Supp(\hat{u}) \subset \lambda \mathcal{B} \Rightarrow \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k + d\left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^p};$$

$$Supp(\hat{u}) \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$

[8] By $\sigma(D)$ we mean the operator with symbol σ i.e.

$$\sigma(D) f = \mathcal{F}^{-1}\left(\sigma \hat{f} \right).$$

4.2 Dyadic partition of unity

Proposition 4.3. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3\}$. There exist two radial functions $\chi \in \mathcal{D}(B(0,4/3))$ and $\varphi \in \mathcal{D}(\mathcal{C})$ valued in the interval $[0,1]$ and such that:*

$$\forall \xi \in \mathbb{R}^d, \ \ \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \tag{4.1}$$

$$\forall \xi \in \mathbb{R}^d \backslash \{0\}, \ \ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{4.2}$$

$$2 \leq |j-j'| \Rightarrow Supp(\varphi(2^{-j}\cdot)) \cap Supp(\varphi(2^{-j'}\cdot)) = \emptyset \tag{4.3}$$

$$j \geq 1 \Rightarrow Supp(\chi) \cap Supp(\varphi(2^{-j}\cdot)) = \varnothing \tag{4.4}$$

the set $\tilde{\mathcal{C}} = \mathcal{B}(0,2/3) + \mathcal{C}$ is an annulus and we have

$$|j-j'| \geq 5 \Rightarrow 2^j\mathcal{C} \cap 2^{j'}\tilde{\mathcal{C}} = \varnothing. \tag{4.5}$$

Also the following inequalities hold true:

$$\forall \xi \in \mathbb{R}^d, \ \ \frac{1}{2} \leq \chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1, \tag{4.6}$$

$$\forall \xi \in \mathbb{R}^d \backslash \{0\}, \ \ \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{4.7}$$

From now on we fix two functions χ and φ satisfying the assertions of the above proposition. Let us denote by $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. For all $u \in \mathcal{S}'$, the nonhomogeneous dyadic blocks are defined as follows:

$$\begin{aligned} \Delta_j u &= 0 \ \text{ if } \ j \leq -2, \\ \Delta_{-1} u &= \chi\left(D\right) u = \tilde{h} \star u, \\ \Delta_j u &= \varphi\left(2^{-j} D\right) u = 2^{jd} \int_{\mathbb{R}^d} h\left(2^q y\right) u\left(x-y\right) dy \ \text{ if } \ j \geq 0. \end{aligned}$$

Also, we will introduce the following low frequency cut-off operator:

$$S_j u = \sum_{j' \leq j-1} \Delta_{j'} u.$$

Let us state some basic facts about the dyadic operators:

Lemma 4.3. *For any $u \in \mathcal{S}'$ we have:*

$$u = \sum_{j\in\mathbb{Z}} \Delta_j u \ \ in \ \ \mathcal{S}'\left(\mathbb{R}^d\right).$$

Lemma 4.4. *For any $u \in \mathcal{S}'$ and $v \in \mathcal{S}'$ we have the following properties:*

$$\begin{aligned} \Delta_j \Delta_l u &= 0 \ \ if \ \ |j-l| \geq 2, \\ \Delta_j\left(S_{l-1} u \Delta_l v\right) &= 0 \ \ if \ |j-l| \geq 5. \end{aligned}$$

4.3 Besov spaces

Let us define now the nonhomogeneous Besov spaces.

Definition 4.1. *Let $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$. The Besov space $B^s_{p,r}$ is the set of tempered distributions $u \in \mathcal{S}'$ such that:*

$$\|u\|_{B^s_{p,r}} := \left\| \left(2^{js} \|\Delta_j u\|_{L^p}\right)_{j\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

Lemma 4.5. *Let $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$ and $(u_j)_{j\geq -1}$ a sequence of functions such that:*

$$\sum_{j\geq -1} 2^{jrs} \|u_j\|^r_{L^p} < \infty.$$

- *If Supp $\hat{u}_{-1} \subset B(0,R_2)$ and Supp $\hat{u}_j \subset \mathcal{C}\left(0, 2^j R_1, 2^j R_2\right) := 2^j \mathcal{C}$ for some $0 < R_1 < R_2$, then $u = \sum_{j\geq -1} u_j \in B^s_{p,r}$ and there exists a universal constant C such that:*

$$\|u\|_{B^s_{p,r}} \leq C^{1+|s|} \left(\sum_{j\geq -1} 2^{jrs} \|u_j\|^r_{L^p} \right)^{\frac{1}{r}}.$$

- *If $s > 0$ and Supp $\hat{u}_j \subset 2^j \mathcal{B}$ for some ball $\mathcal{B}$ then $u = \sum_{j\geq -1} u_j \in B^s_{p,r}$ and there exists a universal constant C such that:*

$$\|u\|_{B^s_{p,r}} \leq \frac{C^{1+|s|}}{s} \left(\sum_{j\geq -1} 2^{jrs} \|u_j\|^r_{L^p} \right)^{\frac{1}{r}}.$$

As an immediate consequence we get:

Corollary 4.1. *The definition of the Besov space $B^s_{p,r}$ is independent of the choice of the couple (χ, φ) defining the Littlewood-Paley decomposition.*

The following proposition is a list of important properties of Besov spaces.

Proposition 4.4. *Let $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$.*

- *$B^s_{p,r}$ is a Banach space which is continuously embedded in $\mathcal{S}'$.*
- *The space $\mathcal{D}\left(\mathbb{R}^d\right)$ is dense in $B^s_{p,r}$ if p and r are finite.*
- *The space $B^{-s}_{p',r'}$ is the dual space of $B^s_{p,r}$.*
- *The inclusion $B^s_{p,r} \subset B^{\tilde{s}}_{p,\tilde{r}}$ is continuous whenever $\tilde{s} < s$ or $s = \tilde{s}$ and $\tilde{r} > r$.*
- *The inclusion $B^s_{p,r} \subset B^{s-d\left(\frac{1}{p}-\frac{1}{q}\right)}_{q,r}$ is continuous whenever $q \geq p$.*
- *If $p < \infty$ we have the following continuous inclusion $B^{\frac{d}{p}}_{p,1} \subset \mathcal{C}_0$[9] $(\subset L^\infty)$.*

[9] $\mathcal{C}_0$ is the space of continuous bounded functions which decay at infinity.

- *If* $s < \tilde{s}$ *and* $u \in B^s_{p,r} \cap B^{\tilde{s}}_{p,r}$ *and* $\theta \in [0,1]$ *then* $u \in B^{\theta s+(1-\theta)\tilde{s}}_{p,r}$ *and*

$$\|u\|_{B^{\theta s+(1-\theta)\tilde{s}}_{p,r}} \leq \|u\|^{\theta}_{B^s_{p,r}} \|u\|^{1-\theta}_{B^{\tilde{s}}_{p,r}}.$$

Remark 4.1. *It can be showed that the Sobolev space* H^s *coincides with* $B^s_{2,2}$ *and that if* s *is not an integer, then* $B^s_{\infty,\infty}$ *coincides with the Hölder space* $\mathcal{C}^r$.

From the above proposition we know that $B^s_{p,1} \subset B^s_{p,\infty}$ with the inclusion being continuous. The next proposition gives us a measure of "how far $B^s_{p,1}$ is from $B^s_{p,\infty}$".

Proposition 4.5. *There exists a constant* C *such that for all* $s \in \mathbb{R}$, $\varepsilon > 0$ *and* $1 \leq p \leq \infty$:

$$\|u\|_{B^s_{p,1}} \leq C\frac{1+\varepsilon}{\varepsilon} \|u\|_{B^s_{p,\infty}} \left(1 + \log \frac{\|u\|_{B^{s+\varepsilon}_{p,\infty}}}{\|u\|_{B^s_{p,\infty}}}\right).$$

We end this section with the following proposition which shows how an important class of Fourier multipliers act on Besov spaces.

Proposition 4.6. *Let us consider* $m \in \mathbb{R}$ *and a smooth function* $f : \mathbb{R}^d \to \mathbb{R}$ *such that for all multi-index* α, *there exists a constant* C_α *such that:*

$$\forall \xi \in \mathbb{R}^d \quad |\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$

Then the operator $f(D)$ *is continuous from* $B^s_{p,r}$ *to* $B^{s-m}_{p,r}$.

4.4 Paradifferential calculus

Paradifferential calculus provides an approach in the study of various properties of the product of two temperate distributions when the later makes sense. For two such distributions u and v we have the following formal decomposition:

$$uv = \sum_{j,l} \Delta_j u \Delta_l v$$

The fundamental idea of paradifferential calculus is to split uv into three parts: the first part denoted by $T_u v$ and called paraproduct of v by u corresponds to terms $\Delta_j u \Delta_l v$ where j is small in comparison with l (i.e.multiply low frequencies of u with high frequencies of v), the second part $T_v u$ is the symmetric counterpart of $T_u v$ and the third part, the remainder term, corresponds to the multiplication of dyadic blocks of comparable frequencies.

Definition 4.2. *Let* u *and* v *be two temperate distributions. We denote:*

$$T_u v = \sum_{j \leq l-2} \Delta_j u \Delta_l v = \sum_l S_{l-1} u \Delta_l v$$

and by

$$R(u,v) = \sum_{|j-l|\leq 1} \Delta_j u \Delta_l v.$$

At least formally, we have the following Bony decomposition:

$$uv = T_u v + T_v u + R(u,v).$$

The following propositions give some continuity properties of the above bilinear operators.

Proposition 4.7. *Let $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$.*

- *The paraproduct T is a bilinear continuous operator from $L^\infty \times B^s_{p,r}$ to $B^s_{p,r}$ and there exists a constant C such that:*
$$\|T\|_{\mathcal{L}\left(L^\infty \times B^s_{p,r}; B^s_{p,r}\right)} \leq C^{|s|+1}.$$
- *If $\sigma > 0$ and $1 \leq r, r_1, r_2, \leq \infty$ are such that $1/r = 1/r_1 + 1/r_2$ then T is bilinear continuous from $B^{-\sigma}_{\infty,r_1} \times B^s_{p,r_2}$ to $B^{s-\sigma}_{p,r}$ and there exists a constant C such that:*
$$\|T\|_{\mathcal{L}\left(B^{-\sigma}_{\infty,r_1} \times B^s_{p,r_2}; B^{s-\sigma}_{p,r}\right)} \leq \frac{C^{|s-\sigma|+1}}{\sigma}.$$

Proposition 4.8. *Let $(s_1, s_2) \in \mathbb{R}^2$ and $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$ such that:*
$$\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \quad \frac{1}{r} \leq \frac{1}{r_1} + \frac{1}{r_2} \leq 1, \quad s_1 + s_2 > 0.$$
Then, there exists a constant C such that:
$$\|R(u,v)\|_{B^{s_1+s_2+d\left(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p,r}} \leq \frac{C^{s_1+s_2+1}}{s_1+s_2} \|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}}.$$

With the aid of the Bony decomposition we can establish the following continuity result for the product:

Proposition 4.9. *Let $s > 0$ and $1 \leq p, r \leq \infty$. Then $B^s_{p,r} \cap L^\infty$ is an algebra. Moreover, we have:*
$$\|uv\|_{B^s_{p,r}} \lesssim \|u\|_{L^\infty} \|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}} \|v\|_{L^\infty}.$$
In particular, if $s > \frac{d}{2}$ or $s = \frac{d}{2}$ and $r = 1$, $B^s_{p,r}$ is an algebra.

Proposition 4.10. *Let $p_i, r \in [1,\infty]$ and $s_i \in \mathbb{R}$ for $i = \overline{1,4}$ such that:*
$$s_1 + s_2 - \frac{d}{p_1} = s_3 + s_4 - \frac{d}{p_4}, \quad \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \quad s_1 + s_2 > 0, \quad s_1, s_3 < \frac{d}{p_1}$$
Then, the product is continuous from $\left(B^{s_1}_{p_1,\infty} \cap B^{s_2}_{p_2,r}\right)^2$ to $B^{s_1+s_2-\frac{d}{p_1}}_{p_2,r}$ and we have:
$$\|uv\|_{B^{s_1+s_2-\frac{d}{p_1}}_{p_2,r}} \lesssim \|u\|_{B^{s_1}_{p_1,\infty}} \|v\|_{B^{s_2}_{p_2,r}} + \|v\|_{B^{s_3}_{p_3,\infty}} \|u\|_{B^{s_4}_{p_4,r}}.$$

We end this section with the following result concerning a commutator-type estimate. For a more general form of this result and its proof see [5] page 116, Lemma 2.100.

Proposition 4.11. *Let us consider $s \in \mathbb{R}$, $r \in [0,\infty]$ and $1 \leq p \leq p_1 \leq \infty$ such that $s > 1 + \frac{d}{p_1}$ or $s = 1 + \frac{d}{p_1}$ and $r = 1$. Let $(u,v) \in B^s_{p,r} \times B^s_{p,r}$. We denote by*
$$R_j = [\Delta_j, u]\partial^\alpha v = \Delta_j\left(u\partial^\alpha v\right) - u\Delta_j \partial^\alpha v,$$
where $[\Delta_j, u]$ is the commutator between Δ_j and the multiplication with u and α is any multi-index with $|\alpha| = 1$. Then, the following estimate holds true:
$$\left\|\left(2^{js}\|R_j\|_{L^p}\right)\right\|_{\ell^r(\mathbb{Z})} \lesssim \|\nabla u\|_{B^{s-1}_{p_1,r}} \|v\|_{B^s_{p,r}}.$$

Bibliography

[1] A. A. Alazman, J. P. Albert, J. L. Bona, M. Chen and J.Wu, *Comparisons between the BBM-equation and a Boussinesq system*, Adv. Differential Equations, **11** (2006), 121-166.

[2] J. L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory*, J. Nonlinear Sci., **12** (2002), 283-318.

[3] J. L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory*, Nonlinearity, **17** (2004), 925-952.

[4] J. L. Bona, T. Colin, C. Guillopé, *Propagation of long-cested water waves*, Discrete and continuous dynamical systems, **2** (2013), 599-628.

[5] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, Berlin 2010.