1 Problem sheet n°1

1.1 A few Euler-Lagrange equations

Exercise 1 Assuming that the functionals are defined on $C^2\left(\overline{\Omega}\right)$ with $\Omega \subset \mathbb{R}^d$, analyse the Euler-Lagrange equation associated to the following functionals and formulate the Dirichlet and Neumann problems in weak and strong forms.

1. Let $p \in (1, \infty)$ and consider

$$J[u] = \int_\Omega \left\{ \frac{|Du(x)|^p}{p} - g(x)\, u(x) \right\} \mathrm{d}x.$$

2. Let $(a_{ij})_{i\in\overline{1,d}, j\in\overline{1,d}}$ be a matrix valued continious function defined on $C\left(\overline{\Omega}\right)$, $g \in C\left(\overline{\Omega}\right)$ and let

$$J[u] = \int_\Omega \left\{ \frac{1}{2} a_{ij}(x)\, \partial_i u(x)\, \partial_j u(x) - g(x)\, u(x) \right\} \mathrm{d}x.$$

3. Consider $\mu, \lambda > 0$ constants, $g \in C\left(\overline{\Omega}\right)$ and

$$J[u] = \int_\Omega (2\mu(\mathbb{D}u : \mathbb{D}u) + \frac{\lambda}{2}(\operatorname{div} u)^2 - g(x)\, u(x))\mathrm{d}x$$

where $u : \Omega \to \mathbb{R}^d$ and

$$\mathbb{D}u = \frac{1}{2}(Du(x) + (Du(x))^T),$$
$$\forall A, B \in \mathbb{R}^{d\times d},\ A : B = A_{ij}B_{ij} \in \mathbb{R}.$$

4. Let a, b, g continious functions and consider

$$J[u] = \int_\Omega \{a(x)\, |Du(x)|^p + b(x)\, |Du(x)|^q - g(x)\, u(x)\} \mathrm{d}x.$$

5. $$J[u] = \frac{1}{2} \int_\Omega |\Delta u|^2 \mathrm{d}x.$$

Exercise 2 In the scalar case, obtain the equation

$$-\Delta u + D\varphi \cdot Du = f$$

as the Euler-Lagrange equation of some functional. Formulate the associated Neumann problem. Hint : multiply the above equation with $\exp(\varphi)$.

Exercise 3 The linear homogeneous wave equation is given by

$$\partial_{tt} u - \Delta u = 0,$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $u = u(t, x)$, $x \in \mathbb{R}^d$. Show that the wave equation can be obtained as the Euler-Lagrange equation associated to a functional to be found.

1.2 Null Lagrangians

There are certains equations witch are obtained as the Euler-Lagrange equations and witch are verified by any smooth function:

Definition 1 *Let $\Omega \subset \mathbb{R}^d$. A C^2-function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ is called Null-Lagrangian if*

$$\frac{\partial f}{\partial u_i}(x, u(x), Du(x)) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial \xi_{ij}}(x, u(x), Du(x))\right) \text{ in } \Omega \tag{1}$$

for all $i \in \overline{1,m}$ and for all $C^2\left(\overline{\Omega}\right)$ function $u : \Omega \to \mathbb{R}^m$.

Exercise 4 Let $m = 1$ and let $f(x, u, \xi) = h(x) + \sum_{j=1}^{d} g_j(u)\,\xi_j$ with C^2 functions h, g. Show that f is a Null-Lagrangian.

Prove the following:

Problem 1 A C^2-function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ is a Null-Lagrangian if and only if the functional

$$J[u] = \int_\Omega f(x, u(x), Du(x))\,\mathrm{d}x,$$

has the following property, for all $u_0, u_1 \in C^2\left(\overline{\Omega}\right)$

$$u_0 = u_1 \text{ on } \partial\Omega \Rightarrow J[u_0] = J[u_1].$$

Hint : Consider $u_t = tu_0 + (1-t)\,u_1$.

Exercise 5

1. Consider the function $h : \mathbb{R}^{d\times d} \to \mathbb{R}$ given by

$$h(\xi) = \det \xi.$$

   Compute the differential of h. Prove that if $I \subset \mathbb{R}$ is a non-empty open interval and $A : I \to \mathbb{R}^{d\times d}$ is C^1 function then

$$\frac{d}{ds}\det A(s) = \operatorname{trace}\left([\operatorname{Cof} A(s)]^T \frac{\partial A}{\partial s}(s)\right).$$

   Above Cof A denote the cofactor matrix : $(\operatorname{Cof} A)_{ij}$ is $(-1)^{i+j}$ multiplied by the $(d-1)\times(d-1)$ determinant obtained from deleting the i^{th} row and the j^{th} column of A.

2. The purpose of the following points is to prove that h defined above is a Null-Lagrangian.
   (a) What are the Euler-Lagrange equations associated the functional $J[u] = \int_\Omega \det Du(x)\,\mathrm{d}x$?
   (b) Show that these equations hold true on Ω for any C^2-function u. Differentiate the identity $(Du)^T \operatorname{Cof} Du = \det(Du) I_d$.

3. Consider $d > 2$ and $p_1 \neq p_2, q_1 \neq q_2 \in \overline{1,d}$ and let $h_2 : \mathbb{R}^{d\times d} \to \mathbb{R}$ be defined by

$$h_2(\xi) = \det\begin{pmatrix} \xi_{p_1q_1} & \xi_{p_1q_2} \\ \xi_{p_2q_1} & \xi_{p_2q_2} \end{pmatrix}.$$

   Show that h_2 is a Null-Lagrangian.

4. Show that $h_2 : \mathbb{R}^{d\times d} \to \mathbb{R}$ given by $h_2(\xi) = \operatorname{trace}\xi^2 - (\operatorname{trace}\xi)^2$ is a Null-Lagrangian.

Problem 2 One nice application of the fact that the function det is a Null-Lagrangians is the proof of Brouwer's fixed point theorem. We now recall the statement of the theorem.

Theorem. Let $B(0,1)$ be the clossed unit ball in $\mathbb{R}^d$. Assume that $u : B(0,1) \to B(0,1)$ is continious. Then u has a fixed point.

Following the indications below one obtains a proof of this theorem under the extra hypothesis that u is C^2.

1. (Application of the fact that det is a Null Lagrangian) Show that there is no function C^2 function $w : B(0,1) \to \partial B(0,1)$ such that $w(x) = x$ for all $|x| = 1$.

2. Show that for all $x, y \in B(0,1)$ one has

$$|x-y|^2 + \langle x, y\rangle^2 \geq |x|^2 |y|^2 ,$$

   where $\langle \cdot, \cdot \rangle$ represents the inner product on $\mathbb{R}^d$. When does equality hold true? Hint : use a suitable transformation to obtain a simplified form of the inequality.

3. For all $\varepsilon > 0$, consider the C^2 function $u_\varepsilon = \frac{1}{1+\varepsilon} u$ which has the extra property that $|u_\varepsilon| \leq \frac{1}{1+\varepsilon}$. Show that u_ε has a fixed point by arguing by contradiction : consider the function obtained as $w_\varepsilon(x) =$ the point where $\overrightarrow{u_\varepsilon(x)\, x}$ intersects $|x| = 1$ and use the previous point.

4. Conclude.