Problem sheet n°2

Exercise 1

1. Let $\Omega \subset \mathbb{R}^d$ be a measurable set. Consider $u \in L^1(\Omega)$. Show that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all measurable set $A \subset \Omega$ we have that
$$m(A) \leq \delta \Rightarrow \int_A |u(x)|\, dx \leq \varepsilon.$$
2. Consider $p \in [1, \infty)$ and $u \in L^p(\Omega)$. Show that for all $\eta > 0$
$$\lim_{\eta \to 0} \int_\Omega \left| \tilde{u}\left(\frac{x}{1+\eta}\right) - \tilde{u}(x) \right|^p dx = 0,$$
where $\tilde{u}$ is the extension by 0 to all $\mathbb{R}^d$.

   Hint : If $m(\Omega) < \infty$ fix $\varepsilon > 0$, split the integral into two parts, one which is small owing to the previous point and one that can be made small using Lusin's theorem and the uniform continuity of continious functions defined on compacts.

Problem 1 Consider $d \geq 1$ and $(p, q, r) \in (1, \infty)^3$, such that
$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$
The purpose of the first part of this problem is to prove that there exists a constant $C_1(p, q, r) > 0$ such that for any $u \in L^p(\mathbb{R}^d)$, $v \in L^{q,weak}(\mathbb{R}^d)$ the following inequality holds true
$$[u * v]_{L^{r,weak}(\mathbb{R}^d)} \leq C_1(p, r, q) \|u\|_{L^p(\mathbb{R}^d)} [v]_{L^{q,weak}(\mathbb{R}^d)}.$$

For the reader's convenience, we recall that if $u : \Omega \to \mathbb{R}$ is a measurable function and for any $A > 0$ consider
$$u_{S,A}(x) = u(x) \mathbf{1}_{\{|u| \leq A\}}(x) + A \frac{u(x)}{|u(x)|} \mathbf{1}_{\{|u| > A\}}(x)$$
respectively
$$u_{B,A}(x) = \left(u(x) - A \frac{u(x)}{|u(x)|} \right) \mathbf{1}_{\{|u| > A\}}(x).$$
The distribution functions of theese functions are
$$\begin{cases} d_{u_{S,A}}(\lambda) = \mathbf{1}_{[0,A]}(\lambda)\, d_u(\lambda), \\ d_{u_{B,A}}(\lambda) = d_u(A + \lambda). \end{cases}$$

1. Consider $\lambda \in (0, \infty)$ and find $A(\lambda)$ such that
$$\left\| u * v_{S,A(\lambda)} \right\|_{L^\infty(\mathbb{R}^d)} \leq \frac{\lambda}{2}.$$
2. Deduce that for all λ we have the estimate
$$\lambda^r d_{u*v}(\lambda) \leq \lambda^r d_{u*v_{B,A(\lambda)}}\left(\frac{\lambda}{2}\right).$$

   Use Chebyshev's inequality in order to estimate the RHS of the above term.

The purpose of the second part of this problem is to prove that there exists $C_2(p,r,q)$ such that for any $u \in L^p(\mathbb{R}^d)$, $v \in L^{q,weak}(\mathbb{R}^d)$ the following inequality holds true

$$\|u * v\|_{L^r(\mathbb{R}^d)} \leq C_2(p,r,q)\,\|u\|_{L^p(\mathbb{R}^d)}\,[v]_{L^{q,weak}(\mathbb{R}^d)}.$$

1. Hint : fix $v \in L^{q,weak}(\mathbb{R}^d)$ and use that $r \in (1,\infty)$ to find $1 < r_1 < r < r_2 < \infty$, $1 < p_1 < p < p_2 < \infty$ and apply Marcinkiewicz's theorem for the spaces $\left(L^{p_i}(\mathbb{R}^d), L^{r_i,weak}(\mathbb{R}^d)\right)$
2. Apply this result for the case where $v = |x|^{-\alpha}$ and in particular to the case $\alpha = d-1$ when $d \geq 2$.

Problem 2 The purpose of this problem is to prove the following theorem:

Scorza-Dragoni theorem. Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function. Then for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subset \Omega$ such that

$$m(\Omega - K_\epsilon) \leq \epsilon$$

and f restricted to $K_\epsilon \times \mathbb{R}^N$ is continuous.

First part : fix $S \subset \mathbb{R}^N$ be compact.

1. Consider
$$\omega_n(x) = \sup_{u,v\in S, |u-v| \leq \frac{1}{n}} |f(x,u) - f(x,v)|$$
and show that for every $\varepsilon > 0$ there exists a compact $K^1_\varepsilon \subset \Omega$ such that $m(\Omega - K^1_\epsilon) \leq \frac{\varepsilon}{2}$ and that for every $\eta > 0$ there exists $N(\eta)$ such that
$$n \geq N(\eta) \Rightarrow \sup_{x\in K^1_\varepsilon} \sup_{u,v\in S, |u-v|\leq \frac{1}{n}} |f(x,u) - f(x,v)| \leq \eta.$$
2. Consider $(u_n)_n \subset S$ a dense set and show that there exists a compact $K^2_\varepsilon \subset \Omega$ such that $m(\Omega - K^2_\epsilon) \leq \frac{\varepsilon}{2}$ and for all $n \in \mathbb{N}$, $f(\cdot, u_n)_{|K^2_\varepsilon}$ is continious.
3. Show that the restriction of f to $\left(K^1_\epsilon \cap K^2_\epsilon\right) \times S$ is continious. Hint : for $(x,u), (\bar{x},\bar{u}) \in \left(K^1_\epsilon \cap K^2_\epsilon\right) \times S$ choose a sub-sequence $\varphi : \mathbb{N} \to \mathbb{N}$ of $(u_n)_n$ that approximates u and use that
$$\begin{aligned}|f(x,u) - f(\bar{x},\bar{u})| \leq{}& \left|f(x,u) - f\left(x,u_{\varphi(n)}\right)\right| + \left|f\left(x,u_{\varphi(n)}\right) - f\left(\bar{x},u_{\varphi(n)}\right)\right| \\ &+ \left|f\left(\bar{x},u_{\varphi(n)}\right) - f(\bar{x},u)\right| + |f(\bar{x},u) - f(\bar{x},\bar{u})|.\end{aligned}$$

Second part :

4. Apply the previous conclusion for an increasing sequence of compacts with their reunion covering $\mathbb{R}^N$.