

## Problem sheet n°3

- Exercise 1**
1. Let  $G \in C^1(\mathbb{R})$  be such that  $G(0) = 0$  and  $|G'(s)| \leq M$  for all  $s \in \mathbb{R}$  and some constant  $M > 0$ . Let  $\Omega$  be a domain in  $\mathbb{R}^d$  and  $u \in W^{1,p}(\Omega)$  with  $p \in [1, \infty)$ . Then  $G \circ u \in W^{1,p}(\Omega)$  and  $\frac{\partial(G \circ u)}{\partial x_i} = G'(u) \frac{\partial u}{\partial x_i}$  for all  $i \in \overline{1, d}$ . (no regularity on  $\Omega$  needed)
  2. Use an approximation procedure to show that if  $u \in W^{1,p}(\Omega)$  then  $\max\{u, 0\}, \min\{u, 0\}, |u| \in W^{1,p}(\Omega)$ . Hint: approximate  $\max\{s, 0\}$  with  $\left(\sqrt{\varepsilon^2 + |s|^2} - \varepsilon\right) \mathbf{1}_{[0, \infty)}(s)$ .
  3. Deduce that  $Du = 0$  a.e. on  $\{x \in \Omega : u(x) = 0\}$ .

**Problem 1** Consider  $\Omega \subset \mathbb{R}^d$  a bounded open set which is star-shaped w.r.t. a ball  $B \subset \Omega$  and  $p \in (1, \infty)$ .

1. Consider  $(u_n)_n \subset W_0^{1,p}(\Omega)$  such that  $u_n \rightharpoonup u$  weakly in  $W^{1,p}(\Omega)$ . Show that  $u \in W_0^{1,p}(\Omega)$ .
2. Consider  $(u_n)_n \subset W^{1,p}(\Omega)$  such that  $u_n \rightharpoonup u$  weakly in  $W^{1,p}(\Omega)$  and  $u_n \rightarrow v$  a.e. on  $\Omega$ . Show that  $v = u$ .
3. Show that if  $u \in C_c(\Omega) \cap W^{1,p}(\Omega)$  then  $u \in W_0^{1,p}(\Omega)$ . Hint : Consider  $G \in C^1(\mathbb{R})$  such that  $|G'(s)| \leq M$  for some constant  $M > 0$  and  $G(s) = 0$  if  $|s| \leq 1$ ,  $G(s) = s$  if  $|s| \geq 2$  and  $u_n = \frac{1}{n}G(nu)$ .

If  $u \in W^{1,p}(\Omega)$  we say that

$$u \leq 0 \text{ on } \partial\Omega \text{ if } \max\{u, 0\} \in W_0^{1,p}(\Omega).$$

In the following, we want to show that if  $u_1, u_2 \in W^{1,p}(\Omega)$  are such that  $u_1 \leq 0$  on  $\partial\Omega$  and  $u_2 \leq 0$  on  $\partial\Omega$  then  $u_1 + u_2 \leq 0$  on  $\partial\Omega$ .

3. For  $i \in \overline{1, 2}$ , consider  $\delta_i^n \in C_c^\infty(\Omega)$  such that  $\delta_i^n \rightarrow \frac{u_i + |u_i|}{2} = \max\{u_i, 0\}$  and  $(u_i^n)_n \in C_c^\infty(\mathbb{R}^d)$  such that  $u_i^n \rightarrow u_i \in W^{1,p}(\Omega)$ .
4. For  $i \in \overline{1, 2}$ , show that up to a subsequence  $|u_i^n| \rightharpoonup |u_i|$  weakly in  $W^{1,p}(\Omega)$  and deduce that up to a subsequence  $2\delta_i^n - |u_i^n| \rightharpoonup u_i$  weakly in  $W^{1,p}(\Omega)$ .
5. Show that  $v_n := |(2\delta_1^n - |u_1^n|) + (2\delta_2^n - |u_2^n|)| - |u_1^n| - |u_2^n| \in W_0^{1,p}(\Omega)$  and that  $(v_n)_n$  is bounded in  $W^{1,p}(\Omega)$ .
6. Conclude that up to a subsequence  $v_n \rightharpoonup v$  weakly in  $W^{1,p}(\Omega)$ . Identify  $v$  using strong convergence and conclude the proof.

If  $u_1, u_2 \in W^{1,p}(\Omega)$  we say that

$$u_1 \leq u_2 \text{ on } \partial\Omega \text{ if } \max\{u_1 - u_2, 0\} \in W_0^{1,p}(\Omega).$$

7. Show that this relation is transitive : if  $u_1 \leq u_2$  on  $\partial\Omega$  and  $u_2 \leq u_3$  on  $\partial\Omega$  then  $u_1 \leq u_3$  on  $\partial\Omega$ .

**Exercise 2** Consider  $p > 2$ ,  $\Omega \subset \mathbb{R}^2$  an open bounded set which is star-shaped w.r.t. a ball  $B \subset \mathbb{R}^3$  and has boundary of class  $C^{2,1}$ .

1. Consider  $u, v \in W^{1,p}(\Omega, \mathbb{R}^2)$ . Show that there exists a  $\gamma > 0$  such that

$$\|\det Du - \det Dv\|_{L^{\frac{p}{2}}} \leq \gamma (\|Du\|_{L^p} + \|Dv\|_{L^p}) \|Du - Dv\|_{L^p}.$$

2. Suppose moreover that  $\partial\Omega$  is smooth. Show that if  $g \in C(\mathbb{R}^2)$  is continuous and  $u - v \in W_0^{1,p}(\Omega)$  then

$$\int_{\Omega} g(u(x)) \det Du(x) dx = \int_{\Omega} g(v(x)) \det Dv(x) dx.$$

Hint : Show that this identity holds true for  $g \in C^1(\mathbb{R}^2)$ . In order to show the above identity for regular functions, show that it is a null Lagrangian by verifying the EL equations for all  $C^2$ -functions.

3. Consider  $(u_n)_n, u \in W^{1,p}(\Omega, \mathbb{R}^2)$  such that  $u_n \rightharpoonup u$  weakly in  $W^{1,p}(\Omega, \mathbb{R}^2)$ . Show that  $\det Du_n \rightharpoonup \det Du$  weakly in  $L^{\frac{p}{2}}(\Omega)$ .

Hint : argue by contradiction by extracting a subsequence  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\det Du_{\varphi} \not\rightharpoonup \det Du$ . Use the boundness in  $L^{p/2}(\Omega)$  to further extract a subsequence that converges weakly.

**Exercise 3** Consider  $\Omega = [0, 1]^d$  and  $p > d$ . We want to prove that

$$\sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq Cte \|Du\|_{L^p}.$$

Fix  $x, y$  and let  $\sigma = \max_{j \in \{1, \dots, d\}} |x_j - y_j|$ . The geometry of  $\Omega$  is used to say that there exists a cube with sides parallel to the axes and side length  $\sigma$  that contains  $x, y$  and that is contained in  $\Omega$ . Denote  $Q_\sigma$  such a cube.

1. Prove that

$$\left| u(x) - \frac{1}{|Q_\sigma|} \int_{Q_\sigma} u(z) dz \right| \leq cte \int_{Q_\sigma} \int_0^1 |Du(x + t(z - x))| dt.$$

2. By changing the order of integration and considering an appropriate change of variables in the RHS of the above inequality show that

$$\left| u(x) - \frac{1}{|Q_\sigma|} \int_{Q_\sigma} u(z) dz \right| \leq cte \times \sigma^{1-\frac{d}{p}} \|Du\|_{L^p}.$$

3. Conclude.

---

<sup>1</sup>In particular, the Gauss-Ostrogradsky holds true for  $C^1(\bar{\Omega})$  functions : for all  $u \in C^1(\bar{\Omega}; \mathbb{R}^d)$  it holds true that  $\int_{\partial\Omega} u \cdot n d\sigma = \int_{\Omega} \operatorname{div} u dx$ .