Problem sheet n°4

In all the exercises and problems $\Omega \subset \mathbb{R}^d$ will be a bounded domain (open connected) that is star-shaped with respect to a ball.

Problem 1 Consider $g \in L^2(\Omega)$ and a symmetric matrix field $A(x) = (a_{ij}(x))_{i,j\in\overline{1,d}} \in L^\infty\left(\Omega; \mathbb{R}^{d\times d}\right)$ with the property that there exists some constant $C > 0$ such that for all $\xi \in \mathbb{R}^d$ and[1] a.e.

$$a_{ij}(x)\,\xi_i\xi_j \geq C\,|\xi|^2. \tag{1}$$

Consider the functional J given by

$$J[u] = \int_\Omega \left\{\frac{1}{2}a_{ij}(x)\,\partial_i u(x)\,\partial_j u(x) - g(x)\,u(x)\right\} dx.$$

1. Show that J is bounded on $W^{1,2}(\Omega)$.
2. Show that J is coercive on $W_0^{1,2}(\Omega)$. Is J coercive on $W^{1,2}(\Omega)$?
3. Show that for all $u, v \in W^{1,2}(\Omega)$
$$\begin{aligned} J[tu + (1-t)v] &= tJ[u] + (1-t)J[v] \\ &\quad - \frac{t(1-t)}{2}\int_\Omega a_{ij}\partial_i(u-v)\partial_j(u-v). \end{aligned}$$
4. Show that J is weakly l.s.c. on $W^{1,2}(\Omega)$.
5. Deduce the existence of an unique minimizer $\bar{u}_1$ in $W_0^{1,2}(\Omega)$ for J. What are the Euler-Lagrange equations in this case (weak and strong forms)?
6. Show that J is coercive on $W_{0-avg}^{1,2}(\Omega)$ the subspace of $W^{1,2}(\Omega)$ with $\int_\Omega u\mathrm{d}x = 0$.
7. Deduce the existence of an unique minimizer $\bar{u}_2$ in $W_{0-avg}^{1,2}(\Omega)$.
8. Consider $v \in W^{1,2}(\Omega)$, what is the derivative of the application
$$t \to J\left[\bar{u}_2 + t\left(v - \tfrac{1}{m(\Omega)}\textstyle\int_\Omega v\mathrm{d}x\right)\right]?$$
9. What are the Euler-Lagrange equations in this case (weak and strong forms)?

[1] Recall that we use the summation over repeated indices convention :

$$a_{ij}\xi_i\xi_j = \sum_{i=1,d}\sum_{j=1,d} a_{ij}\xi_i\xi_j.$$

Problem 2 Consider $g \in L^2(\Omega; \mathbb{R}^d)$, $\mu > 0$ and $\lambda \geq 0$. Recall that for matrices $A, B \in \mathbb{R}^{d\times d}$ we denote

$$A{:}B = \sum_{j=1}^{d}\sum_{i=1}^{d} a_{ij}b_{ij} = \operatorname{trace}\left(A^T B\right).$$

Consider the functional : $J : W^{1,2}(\Omega; \mathbb{R}^d) \to \overline{\mathbb{R}}$ given by

$$J[u] = \int_\Omega (2\mu \mathbb{D}u{:}\mathbb{D}u + \frac{\lambda}{2}(\operatorname{div} u)^2 - g\cdot u)\mathrm{d}x$$

where

$$\mathbb{D}u = \frac{1}{2}(Du + (Du)^T).$$

What is the boundary value problem that u solves in a weak sense?

1. Show that J takes finite values.
2. Show that there does not exist a constant $C > 0$ such that

$$\|Du\|_{L^2} \leq C\,\|\mathbb{D}u\|_{L^2}\,, \tag{2}$$

   for all $u \in W^{1,2}(\Omega; \mathbb{R}^d)$. Hint : in $d = 3$ and $\Omega = B(0,1)$ consider

$$u(x) = \omega \times x \text{ where } \omega \in \mathbb{R}^3.$$

3. Show that there exists $C > 0$ such that for all $u \in W_0^{1,2}(\Omega; \mathbb{R}^d)$

$$\|u\|_{W^{1,2}} \leq C(\Omega)\,\|\mathbb{D}u\|_{L^2}\,.$$

   Hint : Start with the case $C_c^\infty(\Omega; \mathbb{R}^d)$ and develop the expression $\int_\Omega |\mathbb{D}u|^2\,\mathrm{d}x$ using integration by parts $\int_\Omega |\mathbb{D}u|^2\,\mathrm{d}x$.

4. Deduce by following the same steps as in the previous problem the existence of an unique minimizer in $W_0^{1,2}(\Omega; \mathbb{R}^d)$ and find the Euler-Lagrange equations.

Problem 3 The purpose of this problem is to prove the existence of non-trivial solution for the boundary value problem

$$\begin{cases} -\Delta u = |u|^{p-2}\, u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded open set of $\mathbb{R}^3$ with Lipschitz boundary and $p \in (2,6)$, and $u : \Omega \to \overline{\mathbb{R}}$.

1. What does the Sobolev embedding theorem say for $W^{1,2}(\Omega)$ in $d = 3$? How does it change in the case $d = 2$?

2. Consider $J : W^{1,2}(\Omega) \to \mathbb{R}$ given by $J[u] = \int_\Omega \frac{|Du|^2}{2} \mathrm{d}x$ and $G[u] = \int_\Omega |u|^p \, \mathrm{d}x$. Show the existence of a minimizer $\bar{u}$ of $J[u]$ on $W_0^{1,2}(\Omega)$ with $G[u] = 1$. What is the problem with the case $p = 6$?

3. What are the Euler-Lagrange equations verified by $\bar{u}$?

Exercise 4 Consider $f : \mathbb{R} \to \mathbb{R}_+$ a l.s.c. function.

1. Show that the sequence of functions $f_k : \mathbb{R} \to \mathbb{R}_+$ defined by

   $$f_k(x) = \inf_{y \in \mathbb{R}} \{ f(y) + k\,|x - y| \}$$

   are Lipschitz and that for all $x \in \mathbb{R}$,

   $$\lim_{k \to +\infty} f_k(x) = f(x).$$

2. Show that the functional

   $$\begin{cases} J : W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}, \\ J[u] = \int_\Omega f(u(x))\, |Du(x)|^p \end{cases}$$

   is well-defined and that it is weakly l.s.c..