Problem sheet n°5

In all the exercises and problems $\Omega \subset \mathbb{R}^d$ will be a bounded domain (open connected) that is star-shaped with respect to a ball.

Exercise 1 Consider $\Omega \subset \mathbb{R}^3$, $p \geq 1$, $g \in L^p(\Omega)$ and $c : \Omega \to \mathbb{R}$ measurable such that there exists positive constants $0 < c_0 < c_1$ for which a.e. on Ω :

$$c_0 \leq c(x) \leq c_1.$$

Let

$$\begin{cases} J : W^{1,2}(\Omega) \to \mathbb{R}, \\ J[u] = \int_\Omega \left(g(x) |u(x)| + c(x) |Du(x)|^2 \right) dx. \end{cases}$$

Show that if $p > \frac{6}{5}$ then J admits minimizers on the set $\left\{ u \in W^{1,2}(\Omega) : \int_\Omega u(x)\, dx = 1 \right\}$.

Exercise 2 Consider $p \in (1, \infty)$ and $g \in W^{1,p}(\Omega)$.

1. Show the existence of an unique minimizer of the functional $J[u] = \int_\Omega |Du(x)|^p\, dx$ on $g + W_0^{1,p}(\Omega)$.
   Hint: For uniqueness use Clarkson's inequalities : for all $\xi \in \mathbb{R}^d, \eta \in \mathbb{R}^d$

$$\left(\left| \frac{\xi + \eta}{2} \right|^2 + \left| \frac{\xi - \eta}{2} \right|^2 \right)^{\frac{p}{2}} \leq \frac{1}{2} \left(|\xi|^p + |\eta|^p \right) \text{ for } p \in [2, \infty)$$

$$\left(\left| \frac{\xi + \eta}{2} \right|^{p'} + \left| \frac{\xi - \eta}{2} \right|^{p'} \right)^{\frac{p}{p'}} \leq \frac{1}{2} \left(|\xi|^p + |\eta|^p \right) \text{ if } p \in (1, 2].$$

2. If $u, v \in W^{1,p}(\Omega)$ we say that

$$u \leq v \text{ on } \partial\Omega$$

   if

$$\max\{u - v, 0\} \in W_0^{1,p}(\Omega).$$

   Consider $\bar{u} \in W^{1,p}(\Omega)$

$$\int_\Omega |D\bar{u}(x)|^p\, dx = \min \left\{ \int_\Omega |Du(x)|^p\, dx : u - g \in W_0^{1,p}(\Omega) \right\}.$$

   Show that if there exists a constant $M \in \mathbb{R}$ such that $g \leq M$ on $\partial\Omega$ then

$$\bar{u}(x) \leq M \text{ a.e. on } \Omega.$$

3. Show that the minimization problem

$$\min \left\{ \int_\Omega \frac{1}{2} |Du|^2 - 2\sqrt{u} : u \geq 0 \text{ and } u - 1 \in W_0^{1,2}(\Omega) \right\}$$

   has an unique solution. Prove that this solution satisfies $u \geq 1$.

Exercise 3 Consider $a \in (0,1)$, $\Omega := (0,a)^2 \subset \mathbb{R}^2$ and the functional

$$J[u] := \int_\Omega \det(Du)\,dx.$$

Show that this is not weakly lower semicontinuous on $W^{1,2}(\Omega;\mathbb{R}^2)$ by considering the sequence

$$u_n(x,y) := \frac{(1-y)^n}{\sqrt{n}} \begin{pmatrix} \sin(nx) \\ \cos(nx) \end{pmatrix}.$$

What can be said if we restrict J to $W^{1,p}(\Omega;\mathbb{R}^2)$ with $p > 2$?

Problem 4 Part 1. For each $n \in \mathbb{N}$ consider the sets

$$I_{s,n} = \left[\frac{s}{2^n}, \frac{s+\lambda}{2^n}\right], J_{s,n} = \left[\frac{s+\lambda}{2^n}, \frac{s+1}{2^n}\right], \; s \in \overline{0, 2^n - 1}.$$

where $\lambda \in (0,1)$. Consider $\alpha, \beta \in \mathbb{R}^m$, $\lambda\alpha + (1-\lambda)\beta = 0$ and

$$u_n(x) = \begin{cases} \alpha\left(x - \frac{s}{2^n}\right) & \text{if } x \in I_{s,n}, \\ \beta\left(x - \frac{s+1}{2^n}\right) & \text{if } x \in J_{s,n}. \end{cases} \tag{1}$$

1. Show that $(u_n) \subset W^{1,\infty}((0,1);\mathbb{R}^m)$ and that $u_n \to 0$ strongly in $L^\infty((0,1);\mathbb{R}^m)$.
2. Show that for any $p \in (1,\infty)$, $(u_n')_n$ is weakly convergent in $L^p((0,1);\mathbb{R}^m)$ and find its weak limit.

   Part 2. Consider $p \in (1,\infty)$, $f : [0,1] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ a continious function and consider $J : W^{1,p}((0,1);\mathbb{R}^m) \to \mathbb{R}$ defined by

   $$J[u] = \int_0^1 f(x, u(x), u'(x))\,dx.$$

   Assume that J is weakly l.s.c. in $W^{1,p}((0,1);\mathbb{R}^m)$. The purpose of this part is to show that for all $x \in [0,1]$ and for all $u \in \mathbb{R}^m$ the application $\xi \to f(x,u,\xi)$ is convex.

   (a) Consider $h > 0$, $x_0 \in (0,1)$, $u, \xi, \alpha, \beta \in \mathbb{R}^m$, $\lambda \in (0,1)$ such that

   $$\lambda\alpha + (1-\lambda)\beta = \xi.$$

   Show that for h sufficiently small, the function

   $$v_{n,h}(x) = u + \xi(x - x_0) + h\mathbf{1}_{[x_0,x_0+h]}(x)\, u_n\left(\frac{x - x_0}{h}\right)$$

   where u_n is given by (1) with α respectively β are replaced with $\alpha - \xi$ respectively with $\beta - \xi$ defines a sequence of functions in $W^{1,p}((0,1);\mathbb{R}^m) \cap L^\infty((0,1);\mathbb{R}^m)$ having the property that

   $$v_{n,h} \underset{n\to+\infty}{\to} u + \xi(x - x_0) \text{ strongly in } L^\infty$$

   and

   $$v_{n,h}' \rightharpoonup_{n\to+\infty} \xi \text{ weakly in } L^p(0,1).$$

(b) Show that

$$\begin{aligned}
&\liminf_{n} \int_0^1 f\left(x, u+\xi\left(x-x_0\right), v_{n,h}'(x)\right) dx \\
&= \int\limits_{[0,1]\setminus[x_0,x_0+h]} f\left(x, u+\xi\left(x-x_0\right), \xi\right) dx \\
&+ \int_{x_0}^{x_0+h} [\lambda f\left(x, u+\xi\left(x-x_0\right), \alpha\right) + (1-\lambda) f\left(x, u+\xi\left(x-x_0\right), \beta\right)] dx.
\end{aligned}$$

(c) Using the continuity of f show that

$$\lim_{n\to\infty} \int_0^1 f\left(x, v_{n,h}(x), v_{n,h}'(x)\right) dx - \int_0^1 f\left(x, u+\xi\left(x-x_0\right), v_{n,h}'(x)\right) dx = 0.$$

(d) Use the weak lower semi-continuity in order to deduce that

$$\begin{aligned}
&\frac{1}{h}\int_{x_0}^{x_0+h} [\lambda f\left(x, u+\xi\left(x-x_0\right), \alpha\right) + (1-\lambda) f\left(x, u+\xi\left(x-x_0\right), \beta\right) dx] \\
&\geq \frac{1}{h}\int_{x_0}^{x_0+h} f\left(x, u+\xi\left(x-x_0\right), \xi\right) dx.
\end{aligned}$$

Conclude by sending $h \to 0$.