2 Measure theoretical tools

Let $d \geq 1$. We describe briefly the Lebesgue mesure on $\mathbb{R}^d$.

Definition 2.0.1 *The Lebesgue outer measure of a set E is defined as*

$$m^*(E) = \inf_{E \subset \bigcup\limits_{n \in \mathbb{N}} D_n} \left\{ \sum_{n=0}^{\infty} vol(D_n) \right\}$$

where D_n is a product of d intervals. We can see $m^* : 2^{\mathbb{R}^d} \to [0, \infty]$ as a function defined on the subsets of $\mathbb{R}^d$.

A subset of $\mathbb{R}^d$ is a null set if, for every $\varepsilon > 0$, it can be covered with countably many products of d intervals whose total volume is at most ε. More precisely, E is a null set if $m^*(E) = 0$.

Definition 2.0.2 *A set $E \subset \mathbb{R}^d$ is called Lebesgue measurable if for every $\varepsilon > 0$ there exists an open set O and a closed set F such that*

$$F \subset E \subset O \text{ and } m^*(O \backslash F) \leq \varepsilon.$$

We will denote by m the restriction of m^* to the class of measurable sets.

Proposition 2.0.1 *1. Borel sets are Lebesgue measurable sets. In particular, open and closed sets are measurable.*

2. Let E_1, E_2 two mesurable sets. Then $E_1 \backslash E_2$ is measurable.

3. Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of measurable sets. Then $\bigcup\limits_{n \in \mathbb{N}} E_n$ is measurable.

Next we gather some important properties of the Lebesgue measure.

Proposition 2.0.2 *The Lebesgue measure has the following properties.*

1. For any finite disjoint collection $\{E_k\}_{k=1}^n$ of measurable sets,

$$m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$$

2. If A and B are measurable sets and $A \subseteq B$, then

$$m(A) \leq m(B)$$

3. If, moreover, $A \subseteq B$ and $\mu(A) < \infty$, then

$$m(B \setminus A) = m(B) - m(A)$$

so that if $m(A) = 0$, then

$$m(B \setminus A) = m(B)$$

4. For any countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets that covers a measurable set E,

$$m(E) \leq \sum_{k=1}^{\infty} m(E_k)$$

5. If $\{E_k\}_{k=1}^{\infty}$ is an increasing sequence of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(E_k).$$

6. If $\{E_k\}_{k=1}^{\infty}$ is a decreasing sequence of measurable sets for which $\mu(E_1) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(E_k).$$

Definition 2.0.3 *A function $f : \Omega \to \overline{\mathbb{R}}$ is called Lebesgue measurable if for every λ the set $\{x \in \Omega : f(x) < \lambda\}$ is Lebesgue measurable.*

The next result is important to verify the measurability of functions

Theorem 2.0.1 *Consider $f_n : \Omega \to \overline{\mathbb{R}}$ a sequence of measurable functions that converge a.e. to a function $f : \Omega \to \mathbb{R}$. Then f is measurable.*

A class of important functions is given in the following definition:

Definition 2.0.4 *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$. Then f is said to be a Carathéodory function if:*

(i) $\xi \mapsto f(x, \xi)$ is continuous for almost every $x \in \Omega$,

(ii) $x \mapsto f(x, \xi)$ is measurable for every $\xi \in \mathbb{R}^N$.

Proposition 2.0.3 *Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function, and $u : \Omega \to \mathbb{R}^N$ be a measurable function. Then the function $g : \Omega \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$g(x) := f(x, u(x))$$

is measurable.

Theorem 2.0.2 (Egorov's Theorem) *Assume that Ω is a bounded open set of $\mathbb{R}^d$. Let (u_n) be a sequence of Lebesgue measurable functions on Ω such that*

$$u_n(x) \to u(x) \quad \text{a.e. on } \Omega \quad \text{with } |u(x)| < \infty \text{ a.e..}$$

Then for every $\epsilon > 0$, there exists a measurable set $\Omega_\varepsilon \subset \Omega$ such that $m(\Omega \setminus \Omega_\varepsilon) < \epsilon$ and $u_n \to u$ uniformly on Ω_ε.

Theorem 2.0.3 (Lusin's theorem) *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set and let $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ be a Lebesgue measurable function which is finite a.e.. Then for every $\epsilon > 0$, there exists a measurable set $\Omega_\varepsilon \subset \Omega$ such that $|\Omega \setminus \Omega_\varepsilon| < \epsilon$ and a continious function $g : \mathbb{R}^d \to \mathbb{R}$ such that*

$$f(x) = g(x) \text{ for all } x \in \Omega_\varepsilon.$$

In order to prove Lusin's theorem we need the following result from general topology:

Theorem 2.0.4 (Tietze) *Let (X, d) be a metric space and $F \subset X$ a closed set of the space X. Consider $u : F \to [a, b]$ where $a < b$ are real numbers. Then, there exists a function $\tilde{u} : X \to [a, b]$ which continiously extends u to X : $u(x) = \tilde{u}(x)$ for all $x \in F$.*

We will need the following theorem which can be seen as a generalization of Liusin's theorem:

Theorem 2.0.5 (Scorza-Dragoni Theorem) *Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable and let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function. Then for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subset \Omega$ such that*

$$meas(\Omega - K_\epsilon) \leq \epsilon$$

and f restricted to $K_\epsilon \times \mathbb{R}^N$ is continuous.

3 Lebesgue spaces

3.1 Basic properties

The following notions and theorems are reminded here for the reader's convenience. These notions will not be adressed in the lectures.

Definition 3.1.1 *Consider $\Omega \subset \mathbb{R}^d$ an open set and $p \in [1, \infty]$. A measurable function $u : \Omega \to \mathbb{R}$ belongs to $L^p(\Omega)$ if $\|u\|_{L^p} < \infty$ where*

$$\|u\|_{L^p} = \begin{cases} \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \inf \{ M : |u(x)| \leq M \ \text{ a.e. on } \Omega \} & \text{if } p = \infty. \end{cases}$$

If $u : \Omega \to \mathbb{R}^m$ belongs to $L^p(\Omega; \mathbb{R}^m)$ if $u = (u_1, u_2, \cdots, u_m)$ and $u_i \in L^p(\Omega)$ for all $i \in 1, m$.

Theorem 3.1.1 (Monotone Convergence Theorem, Beppo Levi) *Let (u_n) be a sequence of functions in $L^1(\Omega)$ that satisfy:*

(a) $u_1 \leq u_2 \leq \cdots \leq u_n \leq u_{n+1} \leq \ldots$ *a.e. on* Ω,

(b) $\sup_n \|u_n\|_{L^1} < \infty$.

Then $u_n(x)$ converges a.e. on Ω to a finite limit, which we denote by $u(x)$. The function u belongs to L^1, and

$$\|u_n - u\|_{L^1} \to 0.$$

Theorem 3.1.2 (Dominated Convergence Theorem, Lebesgue) *Let $(u_n)_n$ be a sequence of functions in L^1 that satisfy:*

(a) $u_n(x) \to u(x)$ *a.e. on* Ω,

(b) there is a function $g \in L^1$ such that for all n, $|u_n(x)| \leq g(x)$ a.e. on Ω.

Then $u \in L^1$, and

$$\|u_n - u\|_{L^1} \to 0.$$

Lemma 3.1.1 (Fatou's Lemma) *Let $(u_n)_n$ be a sequence of functions in L^1 that satisfy:*

(a) For all n, $u_n \geq 0$ a.e.,

(b) $\sup_n \|u_n\|_{L^1} < \infty$.

For almost all $x \in \Omega$, we set $u(x) = \liminf_{n\to\infty} u_n(x) \leq +\infty$. Then $u \in L^1(\Omega)$ and

$$\int_\Omega u = \int_\Omega \liminf_{n\to\infty} u_n(x) \leq \liminf_{n\to\infty} \int_\Omega u_n.$$

Notation. We denote by $C_c(\mathbb{R}^d)$ the space of all continuous functions on $\mathbb{R}^d$ with compact support, i.e.,

$$C_c(\mathbb{R}^d) = \{u \in C(\mathbb{R}^N);\ u(x) = 0\,\forall x \in \mathbb{R}^d \setminus K,\ \text{where } K \text{ is compact}\}.$$

Theorem 3.1.3 *The space $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for any p, $1 \leq p < \infty$.*

Notation. Let $1 \leq p \leq \infty$; we denote by p' the conjugate exponent, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 3.1.4 (Hölder's Inequality) *Assume that $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$. Then $uv \in L^1(\Omega)$ and*

$$\int_\Omega |uv| \leq \|u\|_{L^p} \|v\|_{L^{p'}}. \tag{1}$$

Theorem 3.1.5 *The space $L^p(\Omega)$ is a vector space, and $\|\cdot\|_p$ is a norm for any p, $1 \leq p \leq \infty$.*

Theorem 3.1.6 (Fischer-Riesz) *$L^p(\Omega)$ is a Banach space for any p, $1 \leq p \leq \infty$.*

Theorem 3.1.7 *Let $(u_n)_n$ be a sequence in L^p and let $u \in L^p$ be such that*

$$\|u_n - u\|_p \to 0.$$

Then, there exists a subsequence $(u_{\varphi(n)})$ and a function $h \in L^p(\Omega)$ such that:

(a) $u_{\varphi(n)}(x) \to u(x)$ a.e. on Ω,

(b) $|u_{\varphi(n)}(x)| \leq h(x)$ for all n, a.e. on Ω.

Theorem 3.1.8 *L^p is reflexive for any p, $1 < p < \infty$.*

Theorem 3.1.9 (Riesz Representation Theorem) *Let $1 < p < \infty$ and let $\phi \in (L^p(\Omega))'$. Then there exists a unique function $u \in L^{p'}(\Omega)$ such that*

$$\langle \varphi, v \rangle = \int_\Omega uv \quad \forall v \in L^p(\Omega).$$

Moreover,

$$\|u\|_{L^{p'}} = \|\phi\|_{(L^p)'}.$$

Remark 3.1.1 *The mapping $\phi \mapsto u$, which is a linear surjective isometry, allows us to identify the space $(L^p)'$ with the space $L^{p'}$.*

In what follows, we shall systematically make the identification $(L^p)' = L^{p'}$.

Theorem 3.1.10 *The space $L^p(\Omega)$ is separable for any p, $1 \leq p < \infty$.*

3.2 Convolution and approximation by smooth functions

In this section we gather some results that will be assumed known by the reader. The particular form of theese results is burrowed from H. Brezis "Functional Analysis".

Theorem 3.2.1 (Young's inequality) *Let $u \in L^p(\mathbb{R}^d)$ and let $v \in L^q(\mathbb{R}^d)$ with $1 \le p \le \infty$ and*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Then for almost every $x \in \mathbb{R}^d$, the function $y \mapsto u(x-y)v(y)$ is integrable on $\mathbb{R}^d$ and we define

$$(u * v)(x) = \int_{\mathbb{R}^N} u(x-y)v(y)\, dy.$$

In addition, $u * v \in L^r(\mathbb{R}^d)$ and

$$\|u * v\|_{L^r} \le \|u\|_{L^p} \|v\|_{L^q}.$$

Proposition 3.2.1 *Let $u \in L^1(\mathbb{R}^d)$ and $v \in L^p(\mathbb{R}^d)$ with $1 \le p \le \infty$. Then*

$$\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v.$$

Proposition 3.2.2 *Let $u \in C_c^k(\mathbb{R}^d)$ $(k \ge 1)$ and let $v \in L^1_{loc}(\mathbb{R}^d)$. Then $u * v \in C^k(\mathbb{R}^d)$ and*

$$D^\alpha (u * v) = (D^\alpha u) * v \quad \forall \alpha \text{ with } |\alpha| \le k.$$

In particular, if $u \in C_c^\infty(\mathbb{R}^d)$ and $v \in L^1_{loc}(\mathbb{R}^d)$, then $u * v \in C^\infty(\mathbb{R}^d)$.

Definition 3.2.1 *A sequence of* mollifiers $(\rho_n)_{n \ge 1}$ *is any sequence of functions on $\mathbb{R}^d$ such that*

$$\begin{cases} \rho_n \in C_c^\infty(\mathbb{R}^d), \\ \operatorname{supp} \rho_n \subset B(0, \frac{1}{n}), \\ \int_{\mathbb{R}^d} \rho_n(x)\, dx = 1, \quad \rho_n \ge 0 \text{ on } \mathbb{R}^d. \end{cases}$$

In what follows, we shall systematically use the notation $(\rho_n)_n$ to denote a sequence of mollifiers.

We recall the classical construction which starts from a function $\rho \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} \rho \subset B(0,1)$, $\rho \ge 0$ on $\mathbb{R}^d$, and ρ does not vanish identically for example, the function

$$\rho(x) = \begin{cases} e^{1/(|x|^2-1)} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

We obtain a sequence of mollifiers by letting $\rho_n(x) = Cn^d \rho(nx)$ with $C \times \int_{\mathbb{R}^d} \rho(x)\, dx = 1$.

Proposition 3.2.3 *Assume $u \in C(\mathbb{R}^d)$. Then*

$$\rho_n * u \to u \quad \text{uniformly on compact sets of } \mathbb{R}^d \text{ as } n \to \infty.$$

Theorem 3.2.2 *Assume $u \in L^p(\mathbb{R}^d)$ with $1 \le p < \infty$. Then*

$$\rho_n * u \to u \quad \text{in } L^p(\mathbb{R}^d) \text{ as } n \to \infty.$$

Let $\Omega \subset \mathbb{R}^d$ be an open set. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \le p < \infty$.

Theorem 3.2.3 *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u(x)\, \varphi(x)\, dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega).$$

Then $u = 0$ almost everywhere on Ω.

3.3 Weak-L^p spaces

First, let us recall the following elementary result which links the L^p-norm of a function (this also holds true for the integral on more abstract space) to the integral of a function of a real variable.

Proposition 3.3.1 *Let $p \in [1, \infty[$ and $u \in L^p(\Omega)$. Then*

$$\|u\|_{L^p}^p = p \int_0^\infty \lambda^{p-1} m\{x : |u(x)| > \lambda\} \, d\lambda = \|\lambda m\{x : |u(x)| > \lambda\}\|_{L^p\left((0,\infty), \frac{d\lambda}{\lambda}\right)}.$$

The above identity comes from observing that

$$\begin{aligned} \|u\|_{L^p}^p &= \int_\Omega |u(x)|^p \, dx = \int_\Omega \int_0^{|u(x)|} p\lambda^{p-1} d\lambda dx = \int_\Omega \int_0^\infty p\lambda^{p-1} \mathbf{1}_{\{\lambda \leq |u(x)|\}}(\lambda) \, d\lambda dx \\ &= \int_0^\infty p\lambda^{p-1} \int_\Omega \mathbf{1}_{\{\lambda < |u(x)|\}}(\lambda) \, dx d\lambda = \int_0^\infty p\lambda^{p-1} m\{x : |u(x)| > \lambda\} \, dx d\lambda. \end{aligned} \tag{2}$$

Proposition 3.3.2 (Chebyshev's inequality) *Consider $p \in [1, \infty)$ and $u \in L^p(\Omega)$. For all $\lambda \geq 0$ we have that*

$$\lambda^p m(\{x : |u(x)| > \lambda\}) \leq \int_\Omega |u(x)|^p \, dx.$$

Definition 3.3.1 *For a measurable function $u : \Omega \to \mathbb{R}$ we define its distribution function by*

$$\begin{cases} d_u : (0, \infty) \to \mathbb{R}, \\ d_u(\lambda) = m(\{x \in \Omega : |u(x)| > \lambda\}). \end{cases}$$

Motivated by the previous inequality, we introduce the following class of functions

Definition 3.3.2 *Consider $p \in [1, \infty)$. For any measurable function $u : \Omega \to \mathbb{R}$, we consider*

$$[u]_{p,\infty}^p := \sup_{\lambda > 0} \lambda^p d_u(\lambda).$$

We denote by

$$L^{p,\infty}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ mesurable } : [u]_{p,\infty} < \infty \right\}.$$

Proposition 3.3.3 *Fix, $p \in [1, \infty)$, $u, v \in L^{p,\infty}(\Omega)$. Then*

1. $[u]_{p,\infty} = 0 \Rightarrow u = 0$ *a.e. on* Ω.
2. $\alpha \in \mathbb{R}$ *then* $[\alpha u]_{p,\infty} = |\alpha| [u]_{p,\infty}$.
3. $[u + v]_{p,\infty} \leq 2 \left([u]_{p,\infty} + [v]_{p,\infty} \right)$.

In order to proove the last inequality, we observe that we can use

$$\begin{aligned} &\{x \in \Omega : |u(x) + v(x)| \geq \lambda\} \\ &\subset \left\{ x \in \Omega : |u(x)| \geq \frac{\lambda}{2} \right\} \cup \left\{ x \in \Omega : |v(x)| \geq \frac{\lambda}{2} \right\} \end{aligned}$$

in order to infer that

$$d_{u+v}(\lambda) \leq d_u(\lambda/2) + d_v(\lambda/2).$$

Thus, we obtain that

$$[u+v]_{p,\infty}^{p} \leq 2^{p}\,[u]_{p,\infty}^{p} + 2^{p}\,[v]_{p,\infty}^{p}$$

from which we deduce the desired inequality since for $p \geq 1$ and $x, y \geq 0$ we have

$$x^{p} + y^{p} \leq (x+y)^{p}\,.$$

Proposition 3.3.4 *Consider $p_1, p_2 \in [1,\infty)$ and $u \in L^{p_1,\infty}(\Omega) \cap L^{p_2,\infty}(\Omega)$. For any $\theta \in (0,1)$ let*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

Then $L^{p}(\Omega) \subset L^{p_1,\infty}(\Omega) \cap L^{p_2,\infty}(\Omega)$ and there exists a constant $C = C(\theta, p_1, p_2)$ such that for all $u \in L^{p}(\Omega)$ we have that

$$\|u\|_{L^{p}} \leq C\,[u]_{p_1,\infty}^{\theta}\,[u]_{p_2,\infty}^{1-\theta}\,.$$

Proof : The identity established in (2) says that

$$\frac{1}{p}\,\|u\|_{L^{p}}^{p} = \int_{0}^{\infty} \lambda^{p-1} d_u(\lambda)\, d\lambda.$$

Consider $A > 0$. Let us observe that

$$\begin{aligned}\int_{0}^{A} \lambda^{p-1} d_u(\lambda)\, d\lambda &= \int_{0}^{A} \lambda^{p-p_1-1}\left[\lambda^{p_1} d_u(\lambda)\right] d\lambda \\ &\leq \sup_{\lambda>0}\left[\lambda^{p_1} d_u(\lambda)\right] \int_{0}^{A} \lambda^{p-p_1-1} d\lambda = [u]_{p_1,\infty}^{p_1}\,\frac{A^{p-p_1}}{p-p_1}.\end{aligned}$$

Also, it holds true that

$$\begin{aligned}\int_{A}^{\infty} \lambda^{p-1} d_u(\lambda)\, d\lambda &= \int_{A}^{\infty} \lambda^{p-p_2-1}\left[\lambda^{p_2} d_u(\lambda)\right] d\lambda \\ &\leq \sup_{\lambda>0}\left[\lambda^{p_2} d_u(\lambda)\right] \int_{A}^{\infty} \lambda^{p-p_2-1} d\lambda = [u]_{p_2,\infty}^{p_2}\,\frac{A^{p-p_2}}{p_2-p}.\end{aligned}$$

Thus, for any $A > 0$ it holds true that

$$\frac{1}{p}\,\|u\|_{L^{p}}^{p} \leq [u]_{p_1,\infty}^{p_1}\,\frac{A^{p-p_1}}{p-p_1} + [u]_{p_2,\infty}^{p_2}\,\frac{A^{p-p_2}}{p_2-p}. \tag{3}$$

The optimal A is such that

$$[u]_{p_1,\infty}^{p_1}\,\frac{A^{p-p_1}}{p-p_1} = [u]_{p_2,\infty}^{p_2}\,\frac{A^{p-p_2}}{p_2-p}$$

which leads to

$$A^{p_2-p_1} = \frac{p-p_1}{p_2-p}\,\frac{[u]_{p_2,\infty}^{p_2}}{[u]_{p_1,\infty}^{p_1}}$$

which pluged-in back into (3) leads to

$$\|u\|_{L^{p}}^{p} \leq 2p\,\frac{1}{(p-p_1)^{\frac{p_2-p}{p_2-p_1}}}\,\frac{1}{(p_2-p)^{\frac{p-p_1}{p_2-p_1}}}\,[u]_{p_1,\infty}^{p\theta}\,[u]_{p_2,\infty}^{p(1-\theta)}$$

from which the conclusion follows.

In the following we will try to obtain a generalisation of the above interpolation inequality for operators. For this we will need a special decomposition of a function

Proposition 3.3.5 *Let $u : \Omega \to \mathbb{R}$ be a measurable function and for any $A > 0$ consider*

$$u_{S,A}(x) = u(x)\,\mathbf{1}_{\{|u|\leq A\}}(x) + A\frac{u(x)}{|u(x)|}\mathbf{1}_{\{|u|>A\}}(x)$$

respectively

$$u_{B,A}(x) = \left(u(x) - A\frac{u(x)}{|u(x)|}\right)\mathbf{1}_{\{|u|>A\}}(x).$$

Then

$$\begin{cases} d_{u_{S,A}}(\lambda) = \mathbf{1}_{[0,A]}(\lambda)\, d_u(\lambda), \\ d_{u_{B,A}}(\lambda) = d_u(A+\lambda). \end{cases}$$

Also, we will be needing the Minkowski inequality in the following form

Proposition 3.3.6 *Assume that $(X_i, \mu_i), i \in \{1,2\}$ are two measurable spaces and let $1 \leq p \leq q \leq \infty$. Then*

$$\left\{\int_{X_2}\left(\int_{X_1} f^p(x_1,x_2)\,dx_2\right)^{\frac{q}{p}} dx_1\right\}^{\frac{1}{p}} \leq \left\{\int_{X_2}\left(\int_{X_1} f^q(x_1,x_2)\,dx_1\right)^{\frac{p}{q}} dx_1\right\}^{\frac{1}{q}}$$

with the obvious modification if $q = \infty$.

The next result concerns the interpolation of bounded sublinear operator and it is very usefull in practice.

Theorem 3.3.1 *Consider $\Omega \subset \mathbb{R}^{d_1}, \Psi \subset \mathbb{R}^{d_2}$ two measurable sets. We consider $T : L^{p_i}(\Omega) \to L^{q_i,\infty}(\Psi)$ $i \in \{1,2\}$ with*

$$p_i \leq q_i \tag{4}$$

such that for all $u \in L^{p_i}(\Omega)$

$$[Tu]_{L^{q_i,\infty}} \leq K_i \|u\|_{L^{p_i}}. \tag{5}$$

Moreover, we assume that T is sublinear in the sense that[1]

$$\begin{cases} T(\alpha u) \leq |\alpha|\, T(u), \\ T(u+v) \leq T(u) + T(v) \ \text{ a.e. on } \Psi. \end{cases} \tag{6}$$

For any $\theta \in (0,1)$, consider

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \text{ and } \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Then T extends to an operator mapping $L^p(\Omega)$ into $L^q(\Psi)$ and there exists a constant

$$C = (K_1, K_2, p_1, p_2, q_1, q_2, \theta)$$

such that

$$\|T(u)\|_{L^q} \leq C\|u\|_{L^p}.$$

[1]Slightly more general conditions can be asked : we can take into account operator for which there exists positive constants C_1, C_2 such that

$$\begin{cases} T(\alpha u) \leq C_1 |\alpha|\, T(u), \\ T(u+v) \leq C_2(T(u) + T(v)) \text{ a.e. on } \Psi. \end{cases}$$

Proof. Consider $u \in L^p(\Omega)$ such that

$$\|u\|_{L^p} = 1.$$

We consider $A(\lambda) = \lambda^c$ with c to be chosen later. Using (6) we deduce that

$$d_{T(u)}(2\lambda) \le d_{T\left(u_{S,A(\lambda)}\right)}(\lambda) + d_{T\left(u_{B,A(\lambda)}\right)}(\lambda).$$

We will use (2) to express the q-norm of $T(u)$ with respect to its distribution function and the above inequality to treat separetaly the set where $T(u)$ is large respectively where it is small. Thus, we obtain that

$$\begin{aligned}\frac{1}{2^q q}\|T(u)\|_{L^q}^q &\le \int_0^\infty \lambda^{q-q_2-1}\left[T(u_{S,A})\right]_{q_2,\infty}^{q_2} d\lambda + \int_0^\infty \lambda^{q-q_1-1}\left[T(u_{B,A})\right]_{q_1,\infty}^{q_1} d\lambda \\ &\le K_2^{q_2}\int_0^\infty \lambda^{q-q_2-1}\left\|u_{S,A(\lambda)}\right\|_{L^{p_2}}^{q_2} d\lambda + K_1^{q_1}\int_0^\infty \lambda^{q-q_1-1}\left\|u_{B,A(\lambda)}\right\|_{L^{p_1}}^{q_1} d\lambda. \qquad (7)\end{aligned}$$

Next we estimate the first term. Using again (2) to express the L^{p_2} norm, using Minkowski's inequality for the pair $L^1, L^{\frac{q_2}{p_2}}$ along with Proposition 3.3.5we obtain that

$$\begin{aligned}\int_0^\infty \lambda^{q-q_2-1}\left\|u_{S,A(\lambda)}\right\|_{L^{p_2}}^{q_2} d\lambda &= p_2^{\frac{q_2}{p_2}}\int_0^\infty \lambda^{q-q_2-1}\left(p_2\int_0^\infty \eta^{p_2-1} d_{u_{S,A(\lambda)}}(\eta)\, d\eta\right)^{\frac{q_2}{p_2}} d\lambda \\ &= p_2^{\frac{q_2}{p_2}}\int_0^\infty \left(\int_0^\infty \left[\lambda^{\frac{p_2}{q_2}(q-q_2-1)}\eta^{p_2-1} d_{u_{S,A(\lambda)}}(\eta)\right] d\eta\right)^{\frac{q_2}{p_2}} d\lambda \\ &\overset{\text{Prop. 3.3.5}}{=} p_2^{\frac{q_2}{p_2}}\int_0^\infty \left(\int_0^{A(\lambda)} \left[\lambda^{\frac{p_2}{q_2}(q-q_2-1)}\eta^{p_2-1} d_u(\eta)\right] d\eta\right)^{\frac{q_2}{p_2}} d\lambda \\ &\overset{\text{Minkowski}}{\le} p_2^{\frac{q_2}{p_2}}\left(\int_0^\infty \left(\int_{A^{-1}(\eta)}^{+\infty} \left[\lambda^{\frac{p_2}{q_2}(q-q_2-1)}\eta^{p_2-1} d_u(\eta)\right]^{\frac{q_2}{p_2}} d\lambda\right)^{\frac{p_2}{q_2}} d\eta\right)^{\frac{q_2}{p_2}} \\ &= p_2^{\frac{q_2}{p_2}}\left(\int_0^\infty \eta^{p_2-1} d_u(\eta)\left(\int_{A^{-1}(\eta)}^{+\infty} \lambda^{q-q_2-1} d\lambda\right)^{\frac{p_2}{q_2}} d\eta\right)^{\frac{q_2}{p_2}} \\ &= p_2^{\frac{q_2}{p_2}}\left(\int_0^\infty \eta^{p_2-1} d_u(\eta)\frac{\eta^{\frac{p_2}{q_2}\frac{q-q_2-1}{c}}}{(q_2-q)^{\frac{p_2}{q_2}}} d\eta\right)^{\frac{q_2}{p_2}} \\ &\le \frac{p_2^{\frac{q_2}{p_2}}}{q_2-q}\left(\int_0^\infty \eta^{p_2+\frac{p_2}{q_2}\frac{q-q_2-1}{c}-1} d_u(\eta)\, d\eta\right)^{\frac{q_2}{p_2}}.\end{aligned}$$

A similar computations gives that

$$\begin{aligned}
\int_0^\infty \lambda^{q-q_1-1} \left\|u_{B,A(\lambda)}\right\|_{L^{p_2}}^{q_1} d\lambda &= \int_0^\infty \lambda^{q-q_1-1} \left(p_1 \int_0^\infty \eta^{p_1-1} d_{u_{B,A(\lambda)}}(\eta)\, d\eta\right)^{q_1} d\lambda \\
&= p_1^{\frac{q_1}{p_1}} \int_0^\infty \left(\int_0^\infty \left[\lambda^{\frac{p_1}{q_1}(q-q_1-1)} \eta^{p_1-1} d_{u_{B,A(\lambda)}}(\eta)\right] d\eta\right)^{\frac{q_1}{p_1}} d\lambda \\
&\overset{\text{Prop. 3.3.5}}{=} p_1^{\frac{q_1}{p_1}} \int_0^\infty \left(\int_0^{+\infty} \left[\lambda^{\frac{p_1}{q_1}(q-q_1-1)} \eta^{p_1-1} d_u(\eta + A(\lambda))\right] d\eta\right)^{\frac{q_1}{p_1}} d\lambda \\
&\underset{(\eta-A(\lambda))^{p_1-1} \le \eta^{p_1-1}}{\overset{\text{Change of var.}}{\le}} p_1^{\frac{q_1}{p_1}} \int_0^\infty \left(\int_{A(\lambda)}^{+\infty} \left[\lambda^{\frac{p_1}{q_1}(q-q_1-1)} \eta^{p_1-1} d_u(\eta)\right] d\eta\right)^{\frac{q_1}{p_1}} d\lambda \\
&\overset{\text{Minkowski}}{\le} p_1^{\frac{q_1}{p_1}} \left(\int_0^\infty \left(\int_0^{A^{-1}(\eta)} \left[\lambda^{\frac{p_1}{q_1}(q-q_1-1)} \eta^{p_1-1} d_u(\eta)\right]^{\frac{q_1}{p_1}} d\lambda\right)^{\frac{p_1}{q_1}} d\eta\right)^{\frac{q_1}{p_1}} \\
&= \frac{p_1^{\frac{q_1}{p_1}}}{q-q_1} \left(\int_0^\infty \eta^{p_1 + \frac{p_1}{q_1}\frac{q-q_1}{c} - 1} d_u(\eta)\, d\eta\right)^{\frac{q_1}{p_1}}.
\end{aligned}$$

We still have a degree of freedom which is the parameter $c > 0$. This is chosen, if possible, such that

$$p_1 + \frac{p_1}{q_1}\frac{q-q_1}{c} - 1 = p_2 + \frac{p_2}{q_2}\frac{q_2-q}{c} - 1 = p - 1.$$

This is possible since

$$\begin{aligned}
c &= \frac{p_2}{q_2}\frac{q_2-q}{p_2-p} = \frac{q}{p}\frac{\frac{1}{q}-\frac{1}{q_2}}{\frac{1}{p}-\frac{1}{p_2}} = \frac{q}{p}\frac{\theta\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}{\theta\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \\
&= \frac{q}{p}\frac{(1-\theta)\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}{(1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} = \frac{q}{p}\frac{\frac{1}{q}-\frac{1}{q_1}}{\frac{1}{p}-\frac{1}{p_1}} = \frac{p_1}{q_1}\frac{q_1-q}{p_1-p}.
\end{aligned}$$

Going back to (7) and taking into account the two estimates of the terms, we get that

$$\begin{aligned}
\frac{1}{2^q q}\|T(u)\|_{L^q}^q &\le \frac{K_2^{q_2} p_2^{q_2}}{q_2-q}\frac{1}{p^{\frac{q_2}{p_2}}}\|u\|_{L^p}^{p\frac{q_2}{p_2}} + \frac{K_1^{q_1} p_1^{q_1}}{q-q_1}\frac{1}{p^{\frac{q_1}{p_1}}}\|u\|_{L^p}^{p\frac{q_1}{p_1}} \\
&\le \frac{K_2^{q_2}}{q_2-q}\frac{p_2^{\frac{q_2}{p_2}}}{p^{\frac{q_2}{p_2}}} + \frac{K_1^{q_1}}{q-q_1}\frac{p_1^{\frac{q_1}{p_1}}}{p^{\frac{q_1}{p_1}}}.
\end{aligned} \tag{8}$$

Thus

$$\|T(u)\|_{L^q} \le 2\left(q\frac{K_2^{q_2}}{q_2-q}\frac{p_2^{\frac{q_2}{p_2}}}{p^{\frac{q_2}{p_2}}} + q\frac{K_1^{q_1} p_1^{q_1}}{q-q_1}\frac{p_1^{\frac{q_1}{p^1}}}{p^{\frac{q_1}{p_1}}}\right)^{\frac{1}{q}}.$$

Finnaly, for all $u \in L^p(\Omega)$, $u \neq 0$ we see that

$$|T(u)| = \left|T\left(\|u\|_{L^p}\frac{u}{\|u\|_{L^p}}\right)\right| \le \|u\|_{L^p} T\left(\frac{u}{\|u\|_{L^p}}\right)$$

Thus, $T(u) \in L^q(\Psi)$ and for all $u \in L^p(\Omega)$

$$\|T(u)\|_{L^q} \leq 2\left(q\frac{K_2^{q_2}}{q_2 - q}\frac{p_2^{\frac{q_2}{p_2}}}{p^{\frac{q_2}{p_2}}} + q\frac{K_1^{q_1}p_1^{q_1}}{q - q_1}\frac{p_1^{\frac{q_1}{p_1}}}{p^{\frac{q_1}{p_1}}}\right)^{\frac{1}{q}} \|u\|_{L^p}.$$

This concludes the proof of Theorem 3.3.1. ■

We will us this result in order to derive the Young inequality for convolutions when one of the functions belongs only to a weak-Lebesgue space.

First, let us prove the following

Theorem 3.3.2 *We consider $p, q, r \in (1, \infty)$ such that*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Consider $u \in L^p(\mathbb{R}^d)$ and $v \in L^{q,\infty}(\mathbb{R}^d)$. Then, $u * v \in L^{r,\infty}(\mathbb{R}^d)$ and there exists a constant depending on p, q, r such that

$$[u * v]_{L^{r,\infty}(\mathbb{R}^d)} \leq Cte\,\|u\|_{L^p(\mathbb{R}^d)}\,[v]_{L^{q,\infty}(\mathbb{R}^d)}.$$

Proof. Fix $u \in L^p(\mathbb{R}^d)$ and $v \in L^{q,\infty}(\mathbb{R}^d)$ and consider $A = A(\lambda)$ to be chosen later. We start from the estimate

$$d_{u*v}(2\lambda) \leq d_{u*v_{S,A(\lambda)}}(\lambda) + d_{u*v_{B,A(\lambda)}}(\lambda).$$

We will choose $A(\lambda)$ such that the first term vanishes. For this it is convenient to estimate the $L^\infty-$norm of the first term. Using Young's inequality and Proposition 3.3.5 we get that

$$\begin{aligned}\left|u * v_{S,A(\lambda)}(x)\right| &\leq \|u\|_{L^p}\left\|v_{S,A(\lambda)}\right\|_{L^{p'}} \leq \left(p'\int_0^\infty \eta^{p'-1} d_{v_{S,A(\lambda)}}(\eta)\right)^{\frac{1}{p'}}\|u\|_{L^p} \\ &\leq \left(\frac{p'}{p'-q}\right)^{\frac{1}{p'}} A(\lambda)^{1-\frac{q}{p'}}\,[v]_{q,\infty}^{\frac{q}{p'}}\,\|u\|_{L^p}. \qquad (9)\end{aligned}$$

Observe by direct computation that

$$1 - \frac{q}{p'} = q\left(\frac{1}{q} - \frac{1}{p'}\right) = q\left(\frac{1}{q} + \frac{1}{p} - 1\right) = \frac{q}{r}.$$

We choose λ such that the term appearing in the RHS of (9) is smaller than λ :

$$\left(\frac{p'}{p'-q}\right)^{\frac{1}{p'}} A(\lambda)^{1-\frac{q}{p'}}\,[v]_{q,\infty}^{\frac{q}{p'}}\,\|u\|_{L^p} = \lambda$$

which gives

$$A(\lambda) = \left(\frac{q}{r}\right)^{-\frac{r}{q}}[v]_{q,\infty}^{-\frac{r}{p'}}\,\|u\|_{L^p}^{-\frac{r}{q}}\,\lambda^{\frac{r}{q}}. \qquad (10)$$

It remains to estimate the second term of the RHS of (9). Using Chebychev's and Young's inequalities we obtain that

$$\lambda^p d_{u*v_{B,A(\lambda)}}(\lambda) \leq \left\|u * v_{B,A(\lambda)}\right\|_{L^p}^p \leq \|u\|_{L^p}^p\left\|v_{B,A(\lambda)}\right\|_{L^1}^p. \qquad (11)$$

We observe that

$$\left\|v_{B,A(\lambda)}\right\|_{L^1} = \int_0^\infty d_{v_{B,A}}(\eta)\, d\eta = \int_{A(\lambda)}^{+\infty} d_v(\eta)\, d\eta \leq [v]_{q,\infty}^q \frac{A(\lambda)^{1-q}}{q-1}.$$

Plugging this into (11) we obtain that

$$\lambda^p d_{u*v_{B,A(\lambda)}}(\lambda) \leq \frac{1}{(q-1)^p} [v]_{q,\infty}^{pq} A(\lambda)^{p(1-q)} \|u\|_{L^p}^p .$$

After a lengthy yet straightforward computations that results from replacing $A(\lambda)$ found in (10) into the last inequality we obtain the desired result. This ends the proof of Theorem 3.3.2.

■

Next, by combining Theorem 3.3.2 and Theorem 3.3.1 we get the following stronger version of Young's inequality :

Theorem 3.3.3 *For all $(p,q,r) \in (1,\infty)^3$, $d \geq 1$ such that*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

there exists a constant $C = C(p,q,r)$ such that for any $u \in L^p(\mathbb{R}^d)$, $v \in L^{q,\infty}(\mathbb{R}^d)$ the following inequality holds true

$$\|u * v\|_{L^r(\mathbb{R}^d)} \leq C \|u\|_{L^p(\mathbb{R}^d)} [v]_{L^{q,\infty}(\mathbb{R}^d)} .$$

The result follows by fixing $v \in L^{q,\infty}(\mathbb{R}^d)$ of norm 1, considering $1 < r_1 < r < r_2 < \infty$ and, owing to Theorem 3.3.2, considering the operator

$$\begin{cases} T : L^{p_i}(\Omega) \to L^{r_i,\infty}(\Omega), \\ \qquad\qquad T(u) = u * v \end{cases}$$

where

$$\frac{1}{p_i} + \frac{1}{q} = 1 + \frac{1}{r_i}.$$

We verify easily that $p_i \leq r_i$ such that the conclusion followis from Theorem 3.3.1. In the next chapter we need a particular form of the above variant of Young's inequality in order to recover the Sobolev injection theorem.

Theorem 3.3.4 *Consider $d \geq 2$, $p \in [1,d)$, and $p^* \in (1,\infty)$ given by*

$$\frac{1}{p} = \frac{1}{p^*} + \frac{1}{d}.$$

There exists a constant $C = C(p,d)$ such that for any $u \in L^p(\mathbb{R}^d)$ the following inequality holds true

$$\left\| u * \frac{1}{|\cdot|^{d-1}} \right\|_{L^r(\mathbb{R}^d)} \leq C \|u\|_{L^p(\mathbb{R}^d)} [v]_{L^{q,\infty}(\mathbb{R}^d)} .$$