3 Sobolev spaces

3.1 Definition and basic properties

We first introduce the notion of weak derivative. Recall that if $\Omega \subset \mathbb{R}^d$ is an open set then

$$L^1_{loc}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \forall K \subset \Omega \text{ compact } \int_K |u(x)|\, \mathrm{d}x < \infty. \right\}$$

Definition 3.1.1 *Let $\Omega \subset \mathbb{R}^d$ be an open set and $1 \leq p \leq \infty$ and $u \in L^p(\Omega)$. We say that $v \in L^1_{loc}(\Omega)$ is the weak-derivative of u in the direction $i \in \overline{1,d}$ if for all $\varphi \in C_c^\infty(\Omega)$ we have*

$$\int_\Omega u(x)\, \partial_i \varphi(x)\, \mathrm{d}x = -\int_\Omega v(x)\, \varphi(x)\, \mathrm{d}x.$$

If it exists, this function is unique (why?). We use the notation $v = \partial_i u$ or $v = \frac{\partial u}{\partial x_i}$ to designate the weak derivative. If the function u admits weak-derivatives with respect to all direction $i \in \overline{1,d}$ we denote by $Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_d} \right)$ (or sometimes $Du = (\partial_1 u, \partial_2 u, ..., \partial_d u)$).

Let us take the time to study some examples.

Example 3.1.1 *The weak derivatives of $x \in \mathbb{R}^d \to |x| \in \mathbb{R}$ are the $L^1_{loc}(\mathbb{R}^d)$ functions $x \to \frac{x_i}{|x|}$:*

$$\frac{\partial}{\partial x_i}(|x|) = \frac{x_i}{|x|}.$$

Example 3.1.2 *If $u \in L^1_{loc}(\Omega)$ admits a weak-derivative w.r.t. the i^{th} direction then for all $\alpha \in C^\infty(\mathbb{R}^d)$ the function αu admits a weak-derivative w.r.t. to the i^{th}-direction and*

$$\partial_i(\alpha u) = \alpha \partial_i u + u \partial_i \alpha.$$

Example 3.1.3 *Consider $u_1, u_2 \in C_c^\infty(\mathbb{R}^2)$. The function*

$$u(x,y) = \begin{cases} u_1(x_1, x_2) & \text{if } x_2 > 0, \\ u_2(x_1, x_2) & \text{if } x_2 \leq 0, \end{cases}$$

admits a weak derivative with respect to x_2 if and only if

$$u_1(x_1, 0) = u_2(x_1, 0)$$

for all $x_1 \in \mathbb{R}$. This example shows that gluing together to regular functions is not sufficient in order to obtain a Sobolev function : in some sense, Sobolev functions "see $(d-1)-$manifolds".

Example 3.1.4 *The previous example should also shed some light on the fact that extending to a larger domain by 0 a function u having weak derivatives in some directions does not necessarily lead to a function having weak derivatives. At a more technical level, if $\Omega \subset \Psi \subset \mathbb{R}^d$, we see that a $C^\infty-$function having compact support in Ψ does not have in general a compact support in Ω. However, if we consider a function $\alpha \in C^\infty(\mathbb{R}^d)$ with compact support in Ω then the extension by 0 to a larger domain of the function αu will have weak derivatives in the directions where u has weak derivatives. Indeed, assume that we extend αu to $\mathbb{R}^d$ by 0 :*

$$\widetilde{\alpha u} = \begin{cases} \alpha u & \text{if } x \in \Omega, \\ 0 & \text{otherwise} \end{cases}$$

Any function $\varphi \in C^{\infty}\left(\mathbb{R}^d\right)$, *we have that*

$$\begin{aligned}\int_{\Omega} \widetilde{\alpha u}\,(x)\,\partial_i\varphi\,(x)\,\mathrm{d}x &= \int_{\Omega} \alpha\,(x)\,u\,(x)\,\partial_i\varphi\,(x)\,\mathrm{d}x \\ &= \int_{\Omega} u\,(x)\,\partial_i(\alpha\varphi)\,(x)\,\mathrm{d}x - \int_{\Omega} u\,(x)\,\partial_i\alpha\,(x)\,\varphi\,(x)\,\mathrm{d}x \\ &= -\int_{\Omega} \partial_i u\,(x)\,\alpha\,(x)\,\varphi\,(x)\,\mathrm{d}x - \int_{\Omega} u\,(x)\,\partial_i\alpha\,(x)\,\varphi\,(x)\,\mathrm{d}x \\ &= -\int_{\Omega} \left(\partial_i u\,(x)\,\alpha\,(x) + u\,(x)\,\partial_i\alpha\,(x)\right)\varphi\,(x)\,\mathrm{d}x.\end{aligned}$$

where it was crucial that $\alpha\varphi \in C_c^{\infty}(\Omega)$ *to pass from the second to the third line. Of course, we deduce that*

$$\partial_i\left(\widetilde{\alpha u}\,(x)\right) = \widetilde{\partial_i(\alpha u)}\,(x)\,.$$

Example 3.1.5 *Let* $g \in L^1_{loc}(\Omega)$ *be the weak derivative of* $u \in L^1_{loc}(\Omega)$ *with respect to the* i^{th}*-direction and consider* $\omega \subset \Omega$ *an open set. The restriction of* $u_{|\omega}$ *admits a weak-derivative with respect to the* i^{th}*-direction and we have*

$$\frac{\partial}{\partial x_i}\left(u_{|\omega}\right) := g_{|\omega}.$$

This is a consequence of the fact that if K *is compact included in* ω *then* K *is also included in* Ω.

Definition 3.1.2 *Let* $\Omega \subset \mathbb{R}^d$ *be an open set and* $1 \leq p \leq \infty$. *We let* $W^{1,p}(\Omega)$ *be the set of functions* $u \in L^p(\Omega)$, *whose weak partial derivatives* $\partial_i u \in L^p(\Omega)$ *for every* $i = 1, \ldots, d$. *We endow this space with the following norm:*

$$\|u\|_{W^{1,p}(\Omega)} := \begin{cases} \left(\|u\|_{L^p}^p + \|Du\|_{L^p}^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\left\{\|u\|_{L^\infty}, \|Du\|_{L^\infty}\right\} & \text{if } p = \infty. \end{cases}$$

For $m \geq 2$, *we denote by* $W^{1,p}(\Omega;\mathbb{R}^m)$ *the functions* $u \in L^p(\Omega;\mathbb{R}^m)$, $u = (u_1, \ldots, u_m)$, *such that* $u_i \in W^{1,p}(\Omega)$ *for every* $i = 1, \ldots, m$. *We denote by*

$$\|u\|_{W^{1,p}(\Omega;\mathbb{R}^m)} =:= \begin{cases} \left(\sum_{i=1}^m \|u_i\|_{W^{1,p}(\Omega)}^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max_{i\in\overline{1,m}}\left\{\|u_i\|_{W^{1,\infty}(\Omega)}\right\} & \text{if } p = \infty. \end{cases}$$

Theorem 3.1.1 *Let* $\Omega \subset \mathbb{R}^d$ *be an open set and* $1 \leq p \leq \infty$. *Then* $\|\cdot\|_{W^{1,p}}$ *is a norm and* $(W^{1,p}(\Omega;\mathbb{R}^m), \|\cdot\|_{W^{1,p}(\Omega;\mathbb{R}^m)})$ *is a Banach space.*

A great deal of identities which are valid in the case of regular functions can be proven to hold true for Sobolev functions by using density arguments. We begin with the whole space case

Theorem 3.1.2 *Consider* $u \in W^{1,p}(\mathbb{R}^d)$ *with* $1 \leq p < \infty$, *there exists a sequence* (u_n) *from* $C_c^{\infty}(\mathbb{R}^d)$ *such that*

$$u_n \to u \text{ in } L^p(\mathbb{R}^d)$$

and

$$Du_n \to Du \text{ in } L^p(\mathbb{R}^d)^d.$$

In the case of a bounded domain Ω the regularity of the boundary of Ω plays an important role (unlike the case of Lebesgue spaces).

Definition 3.1.3 *Let $\Omega \subset \mathbb{R}^d$ be an open set. For an open set ω in $\mathbb{R}^d$ we write $\omega \subset\subset \Omega$, if $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact.*

First, let us note the following result for approximating functions far from the boundary:

Theorem 3.1.3 *Let $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence $(u_n)_n$ from $C_c^\infty(\mathbb{R}^d)$, for all n, such that*

1. *$u_n|_\Omega \to u$ in $L^p(\Omega)$,*
2. *$Du_n|_\omega \to Du|_\omega$ in $L^p(\omega)^d$ for all $\omega \subset\subset \Omega$.*

Proof: We consider the following compact sets

$$K_n = \left\{ x \in \Omega \cap [|x| \leq n] : d(x, \partial\Omega) \geq \frac{1}{2^n} \right\}.$$

The interior of K_n is then

$$\mathrm{int}(K_n) = \left\{ x \in \Omega \cap [|x| < n] : d(x, \partial\Omega) > \frac{1}{2^n} \right\}.$$

Of course, for all n we have that

$$K_n \subset \mathrm{int}(K_{n+1}) \subset K_{n+1}.$$

We let $\chi_n \in C^\infty\left(\mathbb{R}^d\right)$, $\chi_n \in [0,1]$ such that $\chi_n = 1$ on K_n and $\mathrm{Supp}\,\chi_n \subset \mathrm{int}(K_{n+1})$. Then, $\chi_n u \in W^{1,p}(\Omega)$ and the extension by 0 outside of Ω is $W^{1,p}(\mathbb{R}^d)$. Observe also that

$$\|u - \chi_n u\|_{L^p(\Omega)}^p = \int_\Omega |1 - \chi_n(x)|^p \, |u(x)|^p \, dx$$

such that

$$\lim_{n \to +\infty} \|u - \chi_n u\|_{L^p(\Omega)} = 0$$

by dominated convergence. Let us observe that for any $\omega \subset\subset \Omega$ there exists some $n_0 = n_0(\omega)$ such that

$$\omega \subset K_{n_0} \subset \mathrm{int}(K_{n_0+1})$$

and thus it belongs to all K_n with $n \geq n_0 + 1$ since the sets are increasing. The restriction of $D(\chi_n u)$ to ω coincides with Du for $n \geq n_0(\omega)$

$$D(\chi_n u)|_\omega = (uD\chi_n + \chi_n Du)|_\omega = Du.$$

Consider $(\rho_m)_m$ an approximation of the identity. Then $\rho_m * (\widetilde{\chi_n u}) \in C_c^\infty(\mathbb{R}^d)$, the compact support resulting from Proposition ??. In fact, we can be even more precise : we know that $\mathrm{Supp}\,\rho_m * (\widetilde{\chi_n u}) \subset \overline{B\left(0, \frac{1}{m}\right) + K_{n+1}}$. Since $x \in K_{n+1} \subset \mathrm{int}(K_{n+2})$ then for any $x \in K_{n+1}$ there is an $r_x > 0$ such that $B(x, r_x) \subset \mathrm{int}(K_{n+2}) \subset\subset \Omega$. The reunion of $B\left(x, \frac{r_x}{2}\right)$ covers K_{n+1} and since K_{n+1} is compact we can extract a finite subcover:

$$K_{n+1} \subset \cup_i B(x_i, \frac{r_i}{2}) \subset \mathrm{int}(K_{n+2}) \subset\subset \Omega.$$

Thus, for all $x \in K_{n+1}$ it is possible to chose m large enough such that

$$x + B\left(0, \frac{1}{m}\right) \subset \mathrm{int}(K_{n+2}) \subset\subset \Omega.$$

Indeed, consider

$$y \in x + B\left(0, \frac{1}{m}\right)$$

Since $x \in K_{n+1}$ $x \in B(x_i, \frac{r_i}{2})$ for some i and

$$\|y - x_i\| \leq \|y - x\| + \|x - x_i\| \leq \frac{1}{m} + \frac{r_i}{2}$$

thus for small m we have that $y \in B(x_i, r_i) \subset \mathrm{int}(K_{n+2}) \subset\subset \Omega$. It follows that for m large enough (with respect to n), $\mathrm{Supp}\, \rho_m * (\widetilde{\chi_n u})$ is compact and included in $\mathrm{int}(K_{n+2})$ and respectively in Ω.

Next, let us show that

$$D\left(\rho_m * (\widetilde{\chi_n u})\right) = \rho_m * D\left(\widetilde{\chi_n u}\right)$$

in Ω (for m large enough). Indeed for all $\varphi \in C_c^\infty(\Omega)$ since $\rho_m * (\widetilde{\chi_n u})$ has compact support in Ω by classical integration by parts and Fubini we deduce that

$$\begin{aligned}
&\int_\Omega D\left(\rho_m * (\widetilde{\chi_n u})\right)(x)\, \varphi(x)\, dx \\
&= -\int_\Omega \rho_m * (\widetilde{\chi_n u})(x)\, D\varphi(x)\, dx \\
&= -\int_\Omega u(x)\, \chi_n(x)\, (\rho_m * D\varphi)(x)\, dx \\
&= -\int_\Omega u(x)\, D\left[\chi_n(x)\, (\rho_m * \varphi)\right](x)\, dx + \int_\Omega u(x)\, (D\chi_n)(x)\, (\rho_m * \varphi)(x)\, dx.
\end{aligned}$$

Since $\chi_n(\rho_m * \varphi)$ has support in K_{n+1} the first integral of the last equation we can "weakly integrate by parts" and obtain that

$$\begin{aligned}
&\int_\Omega D\left(\rho_m * (\widetilde{\chi_n u})\right)(x)\, \varphi(x)\, dx \\
&= \int_\Omega Du(x)\, \chi_n(x)\, (\rho_m * \varphi)(x)\, dx + \int_\Omega u(x)\, (D\chi_n)(x)\, (\rho_m * \varphi)(x)\, dx \\
&= \int_\Omega D(\chi_n u)(x)\, (\rho_m * \varphi)(x)\, dx \\
&= \int_{\mathbb{R}^d} D(\widetilde{\chi_n u})(x)\, (\rho_m * \varphi)(x)\, dx = \int_{\mathbb{R}^d} \rho_m * (D(\widetilde{\chi_n u}))(x)\, \varphi(x)\, dx \\
&= \int_{\mathbb{R}^d} \rho_m * (D(\widetilde{\chi_n u}))(x)\, \varphi(x)\, dx = \int_{\mathbb{R}^d} \rho_m * (\widetilde{D(\chi_n u)})(x)\, \varphi(x)\, dx.
\end{aligned}$$

We infer that for fixed n

$$\lim_{m \to \infty} \left\| D\left(\rho_m * (\widetilde{\chi_n u})\right) - \widetilde{D\left(\chi_n u\right)} \right\|_{L^p(\mathbb{R}^d)} = 0$$

and in particular, for any n we can choose $\rho_{m(n)} * (\widetilde{\chi_n u})$ such that

$$\left\|\rho_{m(n)} * (\widetilde{\chi_n u}) - \widetilde{\chi_n u}\right\|_{L^p(\mathbb{R}^d)} + \left\| D\left(\rho_{m(n)} * (\widetilde{\chi_n u})\right) - \widetilde{D\left(\chi_n u\right)} \right\|_{L^p(\mathbb{R}^d)} \leq \frac{1}{n}.$$

The sequence $\left(\rho_{m(n)} * (\widetilde{\chi_n u})\right)_n$ then has the desired properties.

Theorem 3.1.4 *Consider* Ω_1 *and* Ω_2 *be two open sets in* $\mathbb{R}^d$*, and let* $T : \Omega_1 \to \Omega_2$ *be a bijective map,* $x_2 = T(x_1)$*, such that* $T \in C^1(\Omega_1)$*,* $T^{-1} \in C^1(\Omega_2)$*, such that* $\det DT, \det DT^{-1}$ *are bounded. Let* $u \in W^{1,p}(\Omega_2)$ *with* $1 \leq p \leq \infty$*. Then* $u \circ T \in W^{1,p}(\Omega_1)$ *and*

$$\partial_i u(T(x_1)) = \partial_k u(T(x_1))\partial_i T_k(y), \quad i \in \overline{1,d}.$$

Definition 3.1.4 *(Partition of Unity). Let* K *be a compact subset of* $\mathbb{R}^d$ *and let* $U_1, U_2, \ldots, U_k$ *be an open covering of* K*, i.e.,* $K \subset \bigcup_{i=1}^k U_i$*. Then there exist functions* $\theta_0, \theta_1, \theta_2, \ldots, \theta_k \in C^\infty(\mathbb{R}^d)$ *such that:*

1. $0 \leq \theta_i \leq 1$ *for all* $i = 0, 1, 2, \ldots, k$*,*
2. $\sum_{i=0}^k \theta_i = 1$ *on* $\mathbb{R}^d$*,*
3. $\mathrm{Supp}(\theta_i)$ *is compact and* $\mathrm{Supp}(\theta_i) \subset U_i$ *for all* $i = 1, 2, \ldots, k$*,*
4. $\mathrm{Supp}(\theta_0) \subset \mathbb{R}^d \setminus K$*.*

Moreover, if Ω *is an open bounded set and* $K = \partial\Omega$*, then* $\theta_0|_\Omega \in C_c^\infty(\Omega)$*.*

One can wonder if the result from Theorem 3.1.3 can hold up to the boundary. It turns out that this is the case for reasonable domains. Let us give some definitions first

Definition 3.1.5 *Consider* $\Omega \subset \mathbb{R}^d$*.*

1. *We say that* Ω *is stared-shaped with respect to a point* $x_0 \in \Omega$ *if for all* $x \in \Omega$ *the whole segment* $[x, x_0] = \{x_\lambda = \lambda x + (1-\lambda) x_0 : \lambda \in [0,1]\} \subset \Omega$ *is contained in* Ω*.*
2. *We say that* Ω *is stared-shaped with respect to a ball* $B(x_0, r)$ *if it is star-shaped with respect to all the points of* $B(x_0, r)$*.*
3. *We call a cone of radius* r *and base* $R \subset S(0,1) = \left\{\theta \in \mathbb{R}^d : |\theta| = 1\right\}$ *the set*

   $$C = \{x = s\theta \text{ with } s \in (0,r) \text{ and } \theta \in R\}.$$

   We say that Ω *has the cone property if there exists a cone* C *of radius* r *and base* $R \subset S(0,1)$ *such for every* $x \in \overline{\Omega}$ *there exists a orthogonal transformation* $A_x \in M_{d\times d}\left(\mathbb{R}^d\right)$*,* $A_x A_x^t = I_d$ *such that* $x + A_x C \in \Omega$*.*
4. *We say that* Ω *is a Lipschitz domain (or that* $\partial\Omega$ *is Lipschitz or simply a domain with Lipschitz boundary) if* $\partial\Omega$ *is a Lipschitz manifold. Namely, for each point there exists local coordinates with respect to which* $\partial\Omega$ *can be written as the graph of some Lipschitz function.*

A bounded domain Ω with Lipschtz boundary has the cone property. A domain that is star-shaped with respect to a ball also has the cone property.

Theorem 3.1.5 *Assume that* Ω *is a bounded domain that is star-shaped with respect to a point* $x_0 \in \Omega$ *and let* $u \in W^{1,p}(\Omega)$ *with* $1 \leq p < \infty$*. Then there exists a sequence* (u_n) *from* $C_c^\infty(\mathbb{R}^d)$ *such that* $u_n|_\Omega \to u$ *in* $W^{1,p}(\Omega)$*.*

Proof : We can suppose that Ω is star-shaped with respect to the origin. Then, for all $\eta > 0$ we consider $\Omega_\eta = \{(1+\eta)\, x : x \in \Omega\}$ and $T_\eta : \Omega_\eta \to \Omega$, $T(x) = \frac{x}{1+\eta}$. Of course, this mapping verifies the conditions of Theorem 3.1.4. Then, $u_\eta : \Omega_\eta \to \mathbb{R}$, $u_\eta(x) = u(\frac{x}{1+\eta}) \in W^{1,p}(\Omega_\eta)$. Moreover, using that Ω is star-shaped with respect to 0 we have that $\Omega \subset \overline{\Omega} \subset \Omega_\eta$. Moreover,

$$\lim_{\eta \to 0} u_\eta = u \text{ in } L^p(\Omega),$$

$$\lim Du_\eta = \lim_{\eta \to 0} (1+\eta)\, Du(\frac{x}{1+\eta}) \text{ in } L^p(\Omega)$$

thus

$$\lim_{\eta \to 0} u_\eta = u \text{ in } W^{1,p}(\Omega).$$

Since $\Omega \subset \overline{\Omega} \subset \Omega_\eta$ using Theorem 3.1.3 there exists a $C^\infty\left(\mathbb{R}^d\right)$ function with compact support in Ω_η such that

$$\|u_{\varepsilon,\eta} - u_\eta\|_{W^{1,p}(\Omega)} \leq \varepsilon.$$

Combing the last two observations concludes the proof of Theorem 3.1.5 in the case of domains which are star-shaped w.r.t. a point.

Theorem 3.1.5 holds true for Lipschitz domains or for domains having the cone property see "Sobolev spaces" by Adams.

3.2 Sobolev and Rellich-Kondrachov embedding theorems

We begin with the following result.

Theorem 3.2.1 *(Sobolev embedding theorem) Let $\Omega \subset \mathbb{R}^d$ be a bounded open set which is star-shaped w.r.t. a ball $B \subset \Omega$.*

1. *If $1 \leq p < d$, then*

   $$W^{1,p}(\Omega) \subset L^q(\Omega)$$

   for every $q \in [1, p^*]$, where p^* is the Sobolev exponent defined by

   $$\frac{1}{p^*} + \frac{1}{d} = \frac{1}{p}, \quad \textit{i.e.,} \quad p^* = \frac{dp}{d-p}.$$

   More precisely, for every $q \in [1, p^*]$, there exists a constant $c = c(\Omega, p, q)$ such that

   $$\|u\|_{L^q} \leq c\|u\|_{W^{1,p}}. \tag{1}$$

2. *If $p = d$, then*

   $$W^{1,d}(\Omega) \subset L^q(\Omega) \quad \textit{for every} \quad q \in [1, \infty).$$

   More precisely, for every $q \in [1, \infty)$, there exists a constant $c = c(\Omega, p, q)$ such that

   $$\|u\|_{L^q} \leq c\|u\|_{W^{1,d}}.$$

3. *If $p > d$, then*

   $$W^{1,p}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$$

   for every $\alpha \in [0, 1 - \frac{d}{p}]$. In particular, there exists a constant $c = c(\Omega, p)$ such that

   $$\|u\|_{L^\infty} \leq c\|u\|_{W^{1,p}}.$$

Proof: We will prove only the first two implications. The proof of 3. in the particular case of $\Omega = (0,1)^d$ is proposed as an exercise in the seminar session.

Using the density of smooth functions in $W^{1,p}(\Omega)$ it is sufficient to prove the inequality (1) for such a function. Observe that

$$u(x) = u(y) + \int_0^1 Du(x + t(y-x)) \cdot (y-x)\, dt$$

and we integrate w.r.t. y on C_x the cone associated to x that is included in Ω. Using polar coordinates and Fubini we obtain that

$$\begin{aligned}
r^d |R|\, u(x) &= \int_{C_x} u(y)\, dy + \int_{C_x}\int_0^1 Du(x + t(y-x)) \cdot (y-x)\, dt \\
&= \int_{C_x} u(y)\, dy + \int_{R_x}\int_0^r\int_0^1 Du(x + ts\theta) \cdot \theta s^d ds d\sigma(\theta)\, dt \\
&= \int_{C_x} u(y)\, dy + \int_{R_x}\int_0^1\int_0^{rt} Du(x + s\theta) \cdot \theta \frac{s^d}{t^{d+1}} ds dt d\sigma(\theta) \\
&= \int_{C_x} u(y)\, dy + \int_{R_x}\int_0^r\int_0^1 Du(x + s\theta) \cdot \theta \frac{s^d}{t^{d+1}} \mathbf{1}_{\left[\frac{s}{t},1\right]}(t)\, dt ds d\sigma(\theta) \\
&= \int_{C_x} u(y)\, dy + \int_{R_x}\int_0^r Du(x + s\theta) \cdot \theta s^d \left(\frac{\left(\frac{r}{s}\right)^d - 1}{d}\right) dt ds d\sigma(\theta) \\
&= \int_{C_x} u(y)\, dy + \frac{1}{d}\int_{C_x} Du(y) \cdot \frac{(y-x)}{|y-x|}\left[\frac{r^d}{|y-x|^{d-1}} - |y-x|\right].
\end{aligned}$$

We obtain after this long computation that

$$|u(x)| \leq \frac{1}{|C|}\int_\Omega |u(y)|\, dy + \frac{2}{d}\int_{\mathbb{R}^d} \left|\widetilde{Du}(y)\right| \frac{1}{|y-x|^{d-1}}.$$

Using the weak-Young inequality proved in Theorem **??** ends the proof.

Definition 3.2.1 *Consider* $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ *two Banach spaces such that* $E \subset F$.

We say that E *is continuously embedded in* F *if there exists some constant* $C > 0$ *such that*

$$\forall u \in E : \|u\|_F \leq C\,\|u\|_E\,.$$

We say that E *is compactly embedded in* F *if it is continuously embedded in* F *and if bounded sets in* E *are relatively compact in* F. *We will use the sequentially compactness characterization of this property: if* $(u_n)_n \subset E$ *is such that*

$$\sup_{n\in\mathbb{N}} \|u_n\|_E < \infty,$$

then there exists $u \in F$ *and a subsequence such that* $u_{\varphi(n)} \to u \in F$.

We are now in the position to state the following fundamental result.

Theorem 3.2.2 *(Rellich-Kondrachov theorem) Let* $\Omega \subset \mathbb{R}^d$ *be a bounded open set that is star-shaped with repsect to a ball.*

1. *If $1 \le p < d$, then the embedding of $W^{1,p}(\Omega)$ in $L^q(\Omega)$ is compact for every $q \in [1, p^*)$. This means that any bounded set of $W^{1,p}(\Omega)$ is precompact (i.e., its closure is compact) in L^q for every $1 \le q < p^*$ (the result is false if $q = p^*$).*
2. *If $p = d$, then the embedding of $W^{1,d}(\Omega)$ in $L^q(\Omega)$ is compact for every $q \in [1, \infty)$.*
3. *If $p > d$, then the embedding of $W^{1,p}(\Omega)$ in $C^{0,\alpha}(\overline{\Omega})$ is compact for every $0 \le \alpha < 1 - \frac{d}{p}$.*

In particular, in all cases (i.e., $1 \le p \le \infty$), the embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is compact.

This will be a direct application of the Riesz-Kolmogorov theorem which we state in the following :

Theorem 3.2.3 *Assume that Ω is a bounded open set in $\mathbb{R}^d$ and that $(u_n)_n$ is a sequence in $L^p(\Omega)$ verifying*

1. $\sup_{n\in\mathbb{N}} \|u_n\|_{L^p(\Omega)} < \infty$,
2. $\lim_{h\to 0} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |\tilde{u}_n(x+h) - \tilde{u}_n(x)|^p \,\mathrm{d}x = 0$.

Then, $(u_n)_n$ admits a subsequence that is convergent in $L^p(\Omega)$. We recall that by $\tilde{u}$ we denote the extension by 0 to the whole $\mathbb{R}^d$.

For the Sobolev embedding the following form is particularly useful :

Corollary 3.2.1 *Assume that Ω is a bounded open set in $\mathbb{R}^d$ and that $(u_n)_n$ is a sequence in $L^p(\Omega)$ verifying*

1. $\sup_{n\in\mathbb{N}} \|u_n\|_{L^p(\Omega)} < \infty$,
2. $\lim_{h\to 0} \sup_{n\in\mathbb{N}} \int_{\omega} |u_n(x+h) - u_n(x)|^p \,\mathrm{d}x = 0$, *for all open sets $\omega \subset\subset \Omega$,*
3. $\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|u_n|>R} |u_n|^p \,\mathrm{d}x = 0$.

Then, $(u_n)_n$ admits a subsequence that is convergent in $L^p(\Omega)$.

Proof. First of all, we have to explain a bit the sense of 2. Since $\omega \subset\subset \Omega$ we observe that for $h \in B(0,1)$ with $|h|$ small enough $\omega + h \subset \Omega$. Indeed, by compactness, it is possible to find N_ω balls with centers $(x_i)_{i\in\overline{1,N_\omega}} \subset \omega$ and $(r_i)_{i\in\overline{1,N_\omega}} \subset (0,\infty)$ such that

$$\omega \subset \cup_{i\in\overline{1,N_\omega}} B\left(x_i, \frac{r_i}{2}\right) \subset \cup_{i\in\overline{1,N_\omega}} B(x_i, r_i) \subset \Omega.$$

Then, any $h \in \mathbb{R}^d$ with $|h| \le \frac{\min_{i\in\overline{1,N_\omega}} r_i}{2}$ has the property that $\omega + h \subset \Omega$. Thus, for all $\omega \subset\subset \Omega$ the function

$$h \to \sup_{n\in\mathbb{N}} \int_{\omega} |u_n(x+h) - u_n(x)|^p \,\mathrm{d}x$$

is defined in a small ball centered at the origin and thus the limit is well defined.

Fix $\varepsilon > 0$. For R_ε large enough

$$\int_{\{|u_n|>R_\varepsilon\}} |u_n(x)|^p \,\mathrm{d}x \le \frac{\varepsilon}{3}.$$

For any measurable set $A \subset \Omega$

$$\int_{A\cap\{|u_n|\leq R_\varepsilon\}} |u_n(x)|^p \,\mathrm{d}x \leq R_\varepsilon^p m(A).$$

Thus, we obtain that

$$\sup_{n\in\mathbb{N}} \int_A |u_n(x)|^p \,\mathrm{d}x \leq \frac{\varepsilon}{3} + R_\varepsilon^p m(A). \tag{2}$$

Take $\omega_\varepsilon \subset\subset \Omega$ such that

$$m(\Omega\backslash\omega_\varepsilon) \leq \frac{\varepsilon}{3}\frac{1}{2^{p+1}R_\varepsilon^p}. \tag{3}$$

Then for h small enough we have that

$$\int_{\omega_\varepsilon} |\tilde{u}_n(x+h) - \tilde{u}_n(x)|^p \,\mathrm{d}x = \int_{\omega_\varepsilon} |u_n(x+h) - u_n(x)|^p \,\mathrm{d}x.$$

Next, using (2) with (3) we have that for all $n \in \mathbb{N}$.

$$\int_{\Omega\backslash\omega} |\tilde{u}_n(x+h) - \tilde{u}_n(x)|^p \,\mathrm{d}x \leq 2^p \int_{\tau_{-h}(\Omega\backslash\omega)} |u_n(x)|^p \,\mathrm{d}x + 2^p \int_{\Omega\backslash\omega} |u_n(x)|^p \,\mathrm{d}x \leq \frac{\varepsilon}{3}.$$

Putting togeather the above estimates, we arrive at the conclusion that

$$\lim_{|h|\to 0} \int_\Omega |\tilde{u}_n(x+h) - \tilde{u}_n(x)|^p \,\mathrm{d}x = 0.$$

■

Let us now, resume the proof of Rellich-Kondrachov's theorem in the case $p \in (1,d)$: assume that $(u_n)_n \subset W^{1,p}(\Omega)$ is bounded. We will show that we can extract a convergent subsequence in $L^p(\Omega)$. The convergence in $L^q(\Omega)$ is then automatic if $q \in [1,p)$ and follows by interpolation if $q \in (p,p^*)$. First of all, owing to the Sobolev embedding and since $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$, it follows that $(u_n)_n$ is bounded in $L^{p^*}(\Omega)$. Since $p^* > p$, using Chebyshev's inequality we obtain that (3) from Corollary (3.2.1) is verified.

The integral equicontinuity property is a consequence of the following estimate verified by Sobolev functions $u \in W^{1,p}(\Omega)$: let $\omega \subset\subset \Omega$ and h sufficiently small such that for all $t \in [0,1]$

$$\omega + th \subset \Omega,$$

see the discussion in the beginning of the proof of Corollary 3.2.1. Then

$$\|\tau_h u - u\|_{L^p(\omega)} \leq |h| \int_0^t \int_{\omega+th} |Du(x)|^p \,\mathrm{d}x \leq |h| \int_\Omega |Du(x)|^p \,\mathrm{d}x.$$

for all $u \in W^{1,p}(\Omega)$ (this follows by first working with $C_c^\infty(\mathbb{R}^d)$-functions and argue by density).

Since (1) from Corollary 3.2.1 is trivially verified by the Sobolev bound, the conclusion follows.

The following subspace $W_0^{1,p}(\Omega)$ is very important:

Definition 3.2.2 *If* $1 \leq p < \infty$, *we denote by* $W_0^{1,p}(\Omega)$ *the closure of* $C_0^\infty(\Omega)$ *in* $W^{1,p}(\Omega)$.

We write $u \in u_0 + W_0^{1,p}(\Omega)$, *meaning that* $u, u_0 \in W^{1,p}(\Omega)$ *and* $u - u_0 \in W_0^{1,p}(\Omega)$. *We also say that* $u = u_0$ *on* $\partial\Omega$.

We end this section with the Poincaré inequality.

Theorem 3.2.4 (Poincaré) *Suppose that* $p \in (1, \infty)$ *and that* Ω *is a bounded open set. Then there exists a constant* $C = C(p, \Omega) > 0$ *such that for all* $u \in W_0^{1,p}(\Omega)$ *it holds true that*

$$\|u\|_{L^p} \leq C \|Du\|_{L^p}.$$

The following variant also holds true : there exists a constant $C > 0$ *such that for all* $u \in W^{1,p}(\Omega)$ *it holds true that*

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x \right\|_{L^p} \leq C \|Du\|_{L^p}.$$

Proof. Let us focus on the first inequality : take $u \in C_c^\infty(\Omega)$ and let $\Omega \subset [-a, a]^d$ (which is possible for large enough $a > 0$). The extension, which abusing notation we still denote by u, with 0 of u to $[-a, a]^d$ is also $C_c^\infty([-a, a]^d)$. Then, for all

$$(x', x_d) = (x_1, x_2, ..., x_d) \in [-a, a]^d$$

it follows that

$$\begin{aligned} |u|^p (x', x_d) = |u|^p (x', x_d) - |u|^p (x', -a) &= \int_{-a}^{x_d} p \left(|u|^{p-2} u \frac{\partial u}{\partial x_d} \right) (x', y_d) \, \mathrm{d}y_d \\ &\leq p \int_{-a}^{a} \left(|u|^{p-1} \frac{\partial u}{\partial x_d} \right) (x', y_d) \, \mathrm{d}y_d. \end{aligned}$$

Integration with respect to the first $(d-1)$-variables and Hölder's inequality leads to the conclusion.

In order to prove the second assertion, we will use the Rellich-Kondrachov theorem : assume that for all $n \in \mathbb{N}$ there exists $u_n \in W^{1,p}(\Omega)$ such that

$$\left\| u_n - \frac{1}{|\Omega|} \int_\Omega u_n \, \mathrm{d}x \right\|_{L^p} > n \|Du_n\|_{L^p}.$$

Consider

$$v_n := \frac{u_n - \frac{1}{|\Omega|} \int_\Omega u_n \mathrm{d}x}{\left\| u_n - \frac{1}{|\Omega|} \int_\Omega u_n \mathrm{d}x \right\|_{L^p}} \in W^{1,p}(\Omega)$$

which has the property that

$$\int_\Omega v_n \, \mathrm{d}x = 0, \ \|v_n\|_{L^p} = 1 \text{ and } \|Dv_n\|_{L^p} \leq \frac{1}{n}. \tag{4}$$

It follows that $(\|v_n\|_{W^{1,p}})_n$ is bounded and up to a subsequence we have $v_n \to v$ in L^p. It follows that

$$\|v\|_{L^p} = 1 \text{ and } \int_\Omega v \, \mathrm{d}x = 0. \tag{5}$$

Moreover we have that $Dv \in L^p(\Omega; \mathbb{R}^d)$ (why?) and the last estimate of (4) forces $Dv = 0$. This implies that v is a constant and we obtain a contradiction when analyzing (5). ∎