A characterization of Riemann integrability

Cosmin Burtea
Faculty of Mathematics, "Al. I. Cuza" University of Iaşi, Romania

Abstract

We prove a characterization of Riemann integrability by using some Darboux-like sums where $M_k = \sup\{f(x);\ x \in [t_k, t_{k+1}]\}$ and $m_k = \inf\{f(x);\ x \in [t_k, t_{k+1}]\}$ are replaced by $F_k = \max\{f(t_k), f(t_{k+1})\}$ and $f_k = \min\{f(t_k), f(t_{k+1})\}$, respectively.

1 The main result

Throughout, we denote by $\mathcal{D}([a,b])$ the set of all partitions Δ, of the form $\Delta : a = x_0 < x_1 < ... x_{n-1} < x_n = b$, of the interval $[a,b]$. As usual, if $\Delta \in \mathcal{D}([a,b])$, we denote by $\|\Delta\|$ its norm, i.e.,

$$\|\Delta\| = \max\{t_{k+1} - t_k;\ k = 0, 1, \dots, n-1\}.$$

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a real function. Then f is Riemann integrable over $[a,b]$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| (x_{k+1} - x_k) < \varepsilon, \tag{1.1}$$

for each $\Delta \in \mathcal{D}([a,b])$, $\Delta : a = x_0 < x_1 < ... x_{n-1} < x_n = b$, satisfying $\|\Delta\| < \delta$.

Remark 1. One can easily see that

$$|f(x_{k+1}) - f(x_k)| = F_k - f_k,$$

where

$$\begin{cases} F_k = \max\{f(x_k), f(x_{k+1})\}, \\ f_k = \min\{f(x_k), f(x_{k+1})\}. \end{cases} \tag{1.2}$$

Therefore

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| (x_{k+1} - x_k) = \sum_{k=0}^{n-1} (F_k - f_k)(x_{k+1} - x_k),$$

and Theorem 1 can be reformulated as:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a real function. Then f is Riemann integrable over $[a,b]$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{k=0}^{n-1} (F_k - f_k)(x_{k+1} - x_k) < \varepsilon, \tag{1.3}$$

for each $\Delta \in \mathcal{D}([a,b])$, $\Delta : a = x_0 < x_1 < ... x_{n-1} < x_n = b$, satisfying $\|\Delta\| < \delta$, where F_k and f_k, $k = 0, 1, \dots n-1$, are given by (1.2).

We can see that Theorem 2, and thus Theorem 1, is a integrability criterion which is clearly weaker than Darboux's Criterion (see Frunză [1]). In fact, the main idea in the proof of the necessity of the condition (1.1) consists in this simple observation.

Proof. We begin with the necessity. So, let us assume that f is Riemann integrable over $[a,b]$. According to [1], we have we know that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=0}^{n-1} (M_k - m_k)(x_{k+1} - x_k) < \varepsilon \tag{1.4}$$

for each $\Delta \in \mathcal{D}([a,b])$ with $\|\Delta\| < \delta$, where

$$\begin{cases} M_k = \sup\{f(x);\ x \in [x_k, x_{k+1}] \\ m_k = \sup\{f(x);\ x \in [x_k, x_{k+1}]. \end{cases}$$

But

$$F_k - f_k = |f(x_{k+1}) - f(x_k)| \leq M_k - m_k$$

for each $k = 0, 1, \ldots n-1$, and so, in view of (1.4), we deduce:

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|(x_{k+1} - x_k) \leq \sum_{k=0}^{n-1} (M_k - m_k)(x_{k+1} - x_k) < \varepsilon,$$

which concludes the proof of the necessity.

We can now pass to the proof of the sufficiency. To this aim, let us assume that f satisfies (1.1). The idea of the proof consists in showing that f satisfies the following Cauchy type criterion (also proved in Frunză [1]): for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{k=0}^{n-1} f(\xi_k') - f(\xi_k'')(x_{k+1} - x_k) \right| < \varepsilon,$$

for each $\Delta \in \mathcal{D}([a,b])$ satisfying $\|\Delta\| < \delta$, and for each two systems of intermediate points, $\{\xi_i'\}_{i=\overline{1,n-1}}$ and $\{\xi_i''\}_{i=\overline{1,n-1}}$, of the partition Δ.

From (1.1), we know that for each $\varepsilon > 0$ there exists $\delta_0 > 0$ such that

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|(x_{k+1} - x_k) < \frac{\varepsilon}{4}, \tag{1.5}$$

for each $\Delta \in \mathcal{D}([a,b])$ satisfying $\|\Delta\| < \delta_0$

Let Δ be a partition of $[a,b]$ with $\|\Delta\| < \delta_0$ and let $\{\xi_i\}_{i=1,n-1}$ be an arbitrary system of intermediate points of Δ.

Obviously, we have

$$\sum_{k=0}^{n-1} |f(\xi_k) - f(x_k)|(x_{k+1} - x_k)$$

$$= \sum_{k=0}^{n-1} |f(\xi_k) - f(x_k)|(\xi_k - x_k) + \sum_{k=0}^{n-1} |f(\xi_k) - f(x_k)|(x_{k+1} - \xi_k)$$

$$\leq \sum_{k=0}^{n-1} |f(\xi_k) - f(x_k)|(\xi_k - x_k)$$

$$+ \sum_{k=0}^{n-1} |f(\xi_k) - f(x_{k+1})|(x_{k+1} - \xi_k) + \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|(x_{k+1} - \xi_k).$$

Let us consider $\Delta' : a = x_0 < \xi_0 < x_1 < \xi_1 < \ldots < x_{n-1} < \xi_{n-1} < x_n = b$, which obviously is a partition of $[a,b]$. We have $\|\Delta'\| \leq \|\Delta\| < \delta_0$ and so, by virtue of (1.5), we deduce both:

$$\sum_{k=0}^{n-1} |f(\xi_k) - f(x_k)|(\xi_k - x_k) + \sum_{k=0}^{n-1} |f(\xi_k) - f(x_{k+1})|(x_{k+1} - \xi_k) < \frac{\varepsilon}{4}$$

and

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| (x_{k+1} - \xi_k) \leq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| (x_{k+1} - x_k) < \frac{\varepsilon}{4}.$$

By summing side by side the last two inequalities, we obtain

$$\sum_{k=0}^{n-1} |f(\xi_k) - f(x_k)| (x_{k+1} - x_k) < \frac{\varepsilon}{2} \tag{1.6}$$

for each $\Delta \in \mathcal{D}([a,b])$ satisfying $\|\Delta\| < \delta_0$, and for each system of intermediate points $\{\xi_i\}_{i=1,n-1}$.

Let $\{\xi_i\}_{i=1,n-1}$ and $\{\eta_i\}_{i=1,n-1}$ be two different systems of intermediate points of a partition $\Delta \in \mathcal{D}([a,b])$ with $\|\Delta\| < \delta_0$.

Taking into account of (1.6), we get

$$\begin{aligned} &\sum_{k=0}^{n-1} |f(\xi_k) - f(\eta_k)| (x_{k+1} - x_k) \\ &\leq \sum_{k=0}^{n-1} |f(\xi_k) - f(x_k)| (x_{k+1} - x_k) + \sum_{k=0}^{n-1} |f(\eta_k) - f(x_k)| (x_{k+1} - x_k) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Obviously

$$\left|\sum_{k=0}^{n-1} f(\xi_k) - f(\eta_k) (x_{k+1} - x_k)\right| \leq \sum_{k=0}^{n-1} |f(\xi_k) - f(\eta_k)| (x_{k+1} - x_k) < \varepsilon$$

and so

$$\left|\sum_{k=0}^{n-1} f(\xi_k) - f(\eta_k) (x_{k+1} - x_k)\right| < \varepsilon.$$

Because the last inequality holds for every two systems of intermediate points of Δ, the proof is complete. □

References

[1] Ştefan Frunză, *Lecţii de analiză matematică*, Editura Universităţii "Al. I. Cuza", Iaşi, 1992.