4 Existence of minimizers in the convex case

In all this section, Ω is a bounded open set witch is star-shaped with respect to a ball.

The main objective of this section is to particularize the abstract Direct Method for the case of integral functionals defined on Sobolev spaces :

$$\begin{cases} J : W^{1,p}(\Omega) \to]-\infty, \infty], \\ J[u] = \int_\Omega f(x, u(x), Du(x))\,\mathrm{d}x. \end{cases} \tag{1}$$

4.1 Weak convergence in Sobolev spaces

First of all let us recall the following consequence of Riesz's representation theorem :

Proposition 4.1.1 *Let $p \in [1, \infty)$. A sequence $(u_n)_n \subset L^p(\Omega; \mathbb{R}^m)$ converges weakly to $u \in L^p(\Omega; \mathbb{R}^m)$ if for every $v \in L^{p'}(\Omega; \mathbb{R}^m)$ it holds*

$$\lim_{n\to\infty} \int_\Omega u_n(x)\, v(x)\,\mathrm{d}x = \int_\Omega u(x)\, v(x)\,\mathrm{d}x.$$

A similar result hold in $W^{1,p}(\Omega; \mathbb{R}^m)$. Let us restrict from now on to the case $p \in (1, \infty)$. The following proposition is a consequence of Hahn-Banach's theorem

Proposition 4.1.2 *Consider $p \in (1, \infty)$, $(u_n)_n \subset W^{1,p}(\Omega; \mathbb{R}^m)$ and $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. Then*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^m) \text{ if and only if}$$
$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } L^p(\Omega; \mathbb{R}^m), \\ \frac{\partial u_n}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i} \text{ weakly in } L^p(\Omega; \mathbb{R}^m) \text{ for all } i \in \overline{1, d}. \end{cases}$$

Recall that

Definition 4.1.1 *We denote by $W_0^{1,p}(\Omega; \mathbb{R}^m) = \overline{C_c^\infty(\Omega; \mathbb{R}^m)}^{\|\cdot\|_{W^{1,p}}}$. Two Sobolev functions $u, v \in W^{1,p}(\Omega; \mathbb{R}^m)$ are said to be equal on $\partial\Omega$ if their difference belongs to $W_0^{1,p}$, i.e.*

$$u = v \text{ on } \partial\Omega \iff u - v \in W_0^{1,p}(\Omega; \mathbb{R}^m).$$

An important property is that $W_0^{1,p}(\Omega; \mathbb{R}^m)$ is weakly closed in $W^{1,p}(\Omega; \mathbb{R}^m)$ (convex and closed). The same remains true for affine spaces : let $u_{\partial\Omega} \in W^{1,p}(\Omega; \mathbb{R}^m)$ then $u_{\partial\Omega} + W_0^{1,p}(\Omega; \mathbb{R}^m)$ is weakly closed in $W^{1,p}(\Omega; \mathbb{R}^m)$.

In the context of Sobolev space by weakly lower semicontinuity we mean the following:

Definition 4.1.2 *Let $p \in [1, \infty)$ and Ω, u, f be as above. We say that J is (sequentially) weakly lower semicontinuous in $W^{1,p}(\Omega; \mathbb{R}^m)$ if for every sequence*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^m)$$

we have

$$\liminf_{n\to\infty} J[u_n] \geq J[u].$$

4.2 The case of integrands $f(x,u,\xi)$ convex w.r.t. (u,ξ)

We are now in the position of stating our first general result of existence of minimizers.

Theorem 4.2.1 (Tonelli-Serin) *Consider $u_{\partial\Omega} \in W^{1,p}(\Omega;\mathbb{R}^m)$ and let $f : \Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times d} \to \mathbb{R}$ a Carathéodory function such that a.e. $x \in \Omega$*

1. *a.e. $x \in \Omega$ the function $(u,\xi) \to f(x,u,\xi)$ is convex*
2. *$f(x,u,\xi) \geq \alpha_1 |\xi|^p + \alpha_2 |u|^r + \alpha_3(x)$: $\alpha_1 > 0$, $\alpha_2 \in \mathbb{R}$ and $\alpha_3 \in L^1(\Omega)$, $r \in [1,p)$.*
3. *Assume that $\operatorname{dom} J \cap \left(u_{\partial\Omega} + W_0^{1,p}(\Omega;\mathbb{R}^m)\right) \neq \emptyset$ where J is defined by (1).*

Then, there exists minimizers for J on $u_{\partial\Omega} + W_0^{1,p}(\Omega;\mathbb{R}^m)$ i.e.

$$\operatorname*{Argmin}_{u_{\partial\Omega}+W_0^{1,p}(\Omega;\mathbb{R}^m)} J[u] \neq \emptyset.$$

Moreover, if f is uniformly convex w.r.t. the ξ variable namely :

$$\lambda f(x,u,\xi) + (1-\lambda) f(x,u,\eta) \geq f(x,u,\lambda\xi + (1-\lambda)\eta) + C\lambda(1-\lambda)|\xi-\eta|^p$$

then the minimizer is unique.

Remark 4.2.1 *If f is twice continuously differentiable w.r.t. the ξ-variable then, convexity implies the (strong) Legendre-Hadamard condition*

$$\frac{\partial^2 f}{\partial\xi_{ij}\partial\xi_{\ell p}}(x,u,\xi)\,\xi_{ij}\xi_{\ell p} \geq C\xi_{ij}\xi_{ij}$$

for some constant $C > 0$.

Proof. We want to show that J is weakly l.s.c. in $W^{1,p}(\Omega;\mathbb{R}^m)$ and coercive on $u_{\partial\Omega} + W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Let us also observe that w.l.o.g. we can assume that $f \geq 0$ otherwise we could work with

$$\tilde{f}(x,u,\xi) = f(x,u,\xi) - \alpha_2 |u|^r + \alpha_3(x)$$

which is positive. The assumption $r \in [1,p)$ ensures that

$$u \to \int_\Omega (\alpha_2 |u(x)|^r + \alpha_3(x))\,\mathrm{d}x$$

is continuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ (for $p \in [1,d]$: $r \in [1,p^*)$ is sufficient if only l.s.c. is considered).

J is weak l.s.c. in $W^{1,p}(\Omega;\mathbb{R}^m)$: Condition 1. implies that J is convex. This means that it is sufficient to prove that J is strongly l.s.c.. Let $(u_n)_n \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_n \to u$ strongly in $W^{1,p}(\Omega;\mathbb{R}^m)$. We assume that

$$\lim_{n\to\infty} J[u_n] = \liminf J[u_n]$$

otherwise we work with a subsequence. One can extract furthermore a subsequence (that we do not re-label) such that

$$u_n \to u \text{ and for all } i \in \overline{1,d},\ \frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ a.e. } x \in \Omega.$$

Then, using that f is Carathéodory and Fatou's lemma

$$\begin{aligned}\liminf J[u_n] &= \lim_{n\to\infty} J[u_n] = \lim_{n\to\infty}\int_\Omega f(x,u_n(x),Du_n(x))\,\mathrm{d}x\\ &\geq \int_\Omega \liminf_{n\to\infty} f(x,u_n(x),Du_n(x))\,\mathrm{d}x.\end{aligned}$$

<u>J is coercive on $u_{\partial\Omega} + W^{1,p}(\Omega;\mathbb{R}^m)$</u> : The hypothesis 2. that f obeys implies immediately that

$$J[u] \geq \alpha_1 \|Du\|^p_{L^p(\Omega)} - \|u\|^r_{L^r(\Omega)} - \|\alpha_3\|_{L^1(\Omega)}. \tag{2}$$

Next, one observes that using Poincaré's inequality we obtain that

$$\begin{aligned}\|u\|_{L^r(\Omega)} &\leq \|u-u_{\partial\Omega}\|_{L^r(\Omega)} + \|u_{\partial\Omega}\|_{L^r(\Omega)} \leq C(\Omega)\|Du\|_{L^r(\Omega)} + \|u_{\partial\Omega}\|_{L^r(\Omega)}\\ &\leq C(\Omega,r)\|Du\|_{L^p(\Omega)} + \|u_{\partial\Omega}\|_{L^r(\Omega)}.\end{aligned}$$

Thus, there exists some constant $C = C(\Omega,r)$ such that

$$\begin{aligned}\|u\|^r_{L^r(\Omega)} &\leq C(\Omega,r)\left(\|Du\|^r_{L^p(\Omega)} + \|u_{\partial\Omega}\|^r_{L^r(\Omega)}\right)\\ &\leq C(\Omega,r)\left(\varepsilon\|Du\|^p_{L^p(\Omega)} + \|u_{\partial\Omega}\|^r_{L^r(\Omega)} + C(\varepsilon,p,r)\right).\end{aligned}$$

Thus, for ε sufficiently small there exists some constant $C = C(\Omega,p,r)$

$$\|u\|^r_{L^r(\Omega)} \leq \frac{\alpha_1}{2}\|Du\|^p_{L^p(\Omega)} + \|u_{\partial\Omega}\|^r_{L^r(\Omega)} + C(\Omega,p,r).$$

Combing the last inequality with (2) leads to

$$J[u] \geq \frac{\alpha_1}{2}\|Du\|^p_{L^p(\Omega)} - \|u_{\partial\Omega}\|^r_{L^r(\Omega)} - \|\alpha_3\|_{L^1(\Omega)} - C(\Omega,p,r). \tag{3}$$

Again, Poincaré's inequality ensures that

$$\|u\|_{L^p(\Omega)} \leq \|u-u_{\partial\Omega}\|_{L^p(\Omega)} + \|u_{\partial\Omega}\|_{L^p(\Omega)} \leq C(\Omega)\|Du\|_{L^p(\Omega)} + \|u_{\partial\Omega}\|_{L^p(\Omega)}.$$

We obtain that

$$\|u\|_{W^{1,p}(\Omega)} \leq (1+C(\Omega))\|Du\|_{L^p(\Omega)} + \|u_{\partial\Omega}\|_{L^p(\Omega)}.$$

Combining the last inequality with (3) ends the proof of the fact that J is coercive on $u_{\partial\Omega} + W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Since the domain of J contains some element from $u_{\partial\Omega}+W_0^{1,p}(\Omega;\mathbb{R}^m)$ existence of minimizers follows.

Uniqueness in the case of uniform convexity follows from the fact that if we take $\bar{u}_1, \bar{u}_2$ two minimizers, then

$$\inf J[u] = \frac{1}{2}(J[\bar{u}_1] + J[\bar{u}_2]) \geq J\left[\frac{\bar{u}_1+\bar{u}_2}{2}\right] + \frac{C}{4}\|D\bar{u}_1 - D\bar{u}_2\|^p_{L^p}.$$

This forces $\|D\bar{u}_1 - D\bar{u}_2\|_{L^p} = 0$ and since $\bar{u}_1 - \bar{u}_2 \in W_0^{1,p}$ the conclusion follows. ■

Under appropriate growth conditions more information can be obtained on the minimizers, namely the validity of Euler-Lagrange equations

Theorem 4.2.2 *Assume all the hypothesis and notations of Theorem 4.2.1 and that, moreover* $f \in C^1_{u,\xi}$ *i.e. a.e.* $x \in \Omega$ *and* f *admits partial derivatives with respect to* u *and* ξ *and that theese partial derivatives are Carathéodory functions. Assume moreover the following growth conditions*

$$\begin{cases} |f(x,u,\xi)| \leq C_1(a(x) + |u|^{p^*} + |\xi|^p), \ p\text{-growth}, \\ \left|\frac{\partial f}{\partial u}(x,u,\xi)\right| + \left|\frac{\partial f}{\partial \xi}(x,u,\xi)\right| \leq C_2(b(x) + |u|^{p-1} + |\xi|^{p-1}), \ (p-1)\text{-growth} \end{cases} \tag{4}$$

with $a, b \in L^1(\Omega)$ *positive and* $C_1, C_2 > 0$ *some generic constants. Then, any minimizer of* J *verifies the weak form of the Euler-Lagrange equations : for all* $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ *we have that*

$$\int_\Omega \left(\frac{\partial f}{\partial u_i}(x, \bar{u}, D\bar{u}) \varphi_i(x) + \frac{\partial f}{\partial \xi_{ij}}(x, \bar{u}, D\bar{u}) \frac{\partial \varphi_i}{\partial x_j}(x) \right) \mathrm{d}x = 0. \tag{5}$$

The boundary condition is $u = u_{\partial\Omega}$ *understood in the sense that* $u - u_{\partial\Omega} \in W_0^{1,p}(\Omega; \mathbb{R}^m)$.

Proof. First of all, the growth condition ensures that $J[u] \in \mathbb{R}$ for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. If $\bar{u} \in u_{\partial\Omega} + W_0^{1,p}(\Omega; \mathbb{R}^m)$ is a minimizer then for all $t \in \mathbb{R}$ and for all $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ it holds true that $\bar{u} + t\varphi \in u_{\partial\Omega} + W_0^{1,p}(\Omega; \mathbb{R}^m)$ and the function

$$t \to J[\bar{u} + t\varphi]$$

has a minimum at $t = 0$. By Fermat's theorem the derivative is 0 and the growth conditions ensure that we can differentiate under the integral sign. ■

Remark 4.2.2 *More general growth conditions can be assumed if Euler-Lagrange equations are to be ensured only for* $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$.

The next proposition shows that convexity w.r.t. the variables (u, ξ) and appropriate growth conditions ensure that any weak solution of the Euler-Lagrange equation is a global minimizer.

Proposition 4.2.1 *Consider* f *as in the previous theorem and* $\bar{u}$ *satisfying (5). Then* $\bar{u}$ *is a global minimizer.*

Proof. The proof is immediate once we find the appropriate definition of convexity. In this situation it is the subdifferential form:

$$f(x, u, \xi) \geq f(x, v, \eta) + \frac{\partial f}{\partial u_i}(x, v, \eta)(u_i - v_i) + \frac{\partial f}{\partial \xi_{ij}}(x, v, \eta)(\xi_{ij} - \eta_{ij})$$

which holds a.e. $x \in \Omega$ and for all $(u, \xi) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}$. Observe that if $u \in u_{\partial\Omega} + W_0^{1,p}(\Omega; \mathbb{R}^m)$ then, when taking $(x, u, \xi) = (x, u(x), Du(x))$ respectively $(x, v, \eta) = (x, \bar{u}(x), D\bar{u}(x))$ then we obtain by integration that

$$\int_\Omega f(x, u(x), Du(x))\,\mathrm{d}x \geq \int_\Omega f(x, \bar{u}(x), D\bar{u}(x))\,\mathrm{d}x$$
$$+ \int_\Omega \frac{\partial f}{\partial u_i}(x, \bar{u}(x), D\bar{u}(x))(u_i(x) - \bar{u}_i(x)) + \frac{\partial f}{\partial \xi_{ij}}(x, \bar{u}(x), D\bar{u}(x)) \left(\frac{\partial u_i}{\partial x_j}(x) - \frac{\partial \bar{u}_i}{\partial x_j}(x) \right) \mathrm{d}x.$$

The last term from the above inequality vanishes since $u - \bar{u} \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ and since u was fixed arbitrarily we obtain the conclusion. ■

Theorem 4.2.3 *Consider $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ a Carathéodory integrand with p-growth. Then the integral functional determined by f on a bounded domain Ω is strongly continuous in $W^{1,p}(\Omega;\mathbb{R}^m)$.*

Proof. Using Scorza-Dragoni's theorem there exists an $\varepsilon > 0$ and a compact set $K_\varepsilon^1 \subset \Omega$ such that $m\left(\Omega\backslash K_\varepsilon^1\right) \leq \frac{\varepsilon}{3}$ and $f_{|K_\varepsilon^1\times\mathbb{R}^m\times\mathbb{R}^{m\times d}}$ is continuous.

Assuming that $u_n \to u$ strongly in $W^{1,p}(\Omega;\mathbb{R}^m)$ it is possible to work with a subsequence, that with by slightly abusing the notation we will not relabel, such that

$$u_n \to u \text{ and } \forall i \in \overline{1,d},\ \frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ a.e. on } \Omega.$$

Egorof's theorem ensure the existence of a compact set $K_\varepsilon^2 \subset \Omega$ such that $m\left(\Omega\backslash K_\varepsilon^2\right) \leq \frac{\varepsilon}{3}$

$$u_n \to u \text{ and } \forall i \in \overline{1,d},\ \frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ uniformly on } K_\varepsilon^2.$$

Also, there exists a compact set $K_\varepsilon^3 \subset \Omega$ such that $m\left(\Omega\backslash K_\varepsilon^3\right) \leq \frac{\varepsilon}{3}$ and that u and Du are continuous on K_ε^3. In particular there exists some M_ε such that

$$\sup_{x\in K_\varepsilon^3}|u(x)| + \sup_{x\in K_\varepsilon^3}|Du(x)| \leq M_\varepsilon.$$

Also, for n larger than and $n_\varepsilon \in \mathbb{N}$ owing to the uniform convergence on K_ε^3 we have that

$$\sup_{x\in K_\varepsilon^2\cap K_\varepsilon^3}|u_n(x)| + \sup_{x\in K_\varepsilon^2\cap K_\varepsilon^3}|Du_n(x)| \leq M_\varepsilon + 1.$$

The restriction $f_{|K_\varepsilon^1\times B_{\mathbb{R}^m\times\mathbb{R}^{m\times d}}(0,M_\varepsilon+1)}$ is uniformly continuous and therefore, for any $\varepsilon > 0$ there is some $\delta = \delta(\varepsilon) > 0$ such that

$$|x-y| + |u-v| + |\xi-\eta| \leq \delta \Rightarrow |f(x,u,\xi) - f(y,v,\eta)| \leq \varepsilon.$$

Again, owing to the uniform convergence, for any n large enough

$$\sup_{x\in K_\varepsilon^2\cap K_\varepsilon^3}|u_n(x) - u(x)| + \sup_{x\in K_\varepsilon^2\cap K_\varepsilon^3}|Du_n(x) - Du(x)| \leq \delta$$

Thus, we have that for any n large enough

$$\int_{K_\varepsilon^1\cap K_\varepsilon^2\cap K_\varepsilon^3}|f(x,u_n(x),Du_n(x)) - f(x,u(x),Du(x))|\,\mathrm{d}x \leq \varepsilon m(\Omega).$$

Of course, $K_\varepsilon := K_\varepsilon^1 \cap K_\varepsilon^2 \cap K_\varepsilon^3$

$$m(\Omega\backslash K_\varepsilon) \leq \varepsilon$$

and owing to the $p-$growth

$$\begin{aligned}
&\int_{\Omega\backslash K_\varepsilon}|f(x,u_n(x),Du_n(x))|\,\mathrm{d}x \\
&\leq C\int_{\Omega\backslash K_\varepsilon}(1 + |u_n(x)|^p + |Du_n(x)|^p)\,\mathrm{d}x \\
&\leq Cm(\Omega\backslash K_\varepsilon) + C2^p\int_\Omega |u_n(x) - u(x)|^p + |Du_n(x) - Du(x)|^p\,\mathrm{d}x \\
&+ C\int_{\Omega\backslash K_\varepsilon}(|u(x)|^p + |Du(x)|^p)\mathrm{d}x.
\end{aligned}$$

The previous inequality ensures that the reminder terms can be made as small as we want.

We thus proved that from any $u_n \to u$ strongly in $W^{1,p}(\Omega;\mathbb{R}^m)$ we can extract a subsequence such that $J\left[u_{\varphi(n)}\right]$ converges to $J[u]$. By uniqueness of the limit, the whole sequence converges. ■

Growth condition excludes the so called Lavrentiev phenomen:

Example 4.2.1 *Consider $m = 1$, $d = 1$, $\Omega = (0,1)$ and*

$$J[u] = \int_0^1 \left(u^3(x) - x\right)^2 \left(\frac{du}{dx}\right)^6 \mathrm{d}x$$

for which it can be showed that

$$0 = \min_{u \in x + W^{1,\frac{4}{3}}(0,1)} J[u] < \min_{u \in x + C_0^\infty(0,1)} J[u].$$

The existence result of Theorem 4.2.1 can be seen as an application of the abstract Direct Method for $E = W^{1,p}(\Omega;\mathbb{R}^m)$ with $\mathcal{A} = u_{\partial\Omega} + W_0^{1,p}(\Omega;\mathbb{R}^m)$. In light of this remark, it transpires that we can obtain different results for weakly-closed subsets of $W^{1,p}(\Omega;\mathbb{R}^m)$ on which coercivity holds. Below, we present a general result pertaining to minimization under constraints.

Theorem 4.2.4 *Consider $u_{\partial\Omega} \in W^{1,p}(\Omega;\mathbb{R}^m)$ and let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ a Carathéodory function verifying the assumptions of Theorem 4.2.1. Furthermore, let $h : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function with p-growth. Then, if we let*

$$\mathcal{A} = \left\{ u \in u_{\partial\Omega} + W_0^{1,p}(\Omega;\mathbb{R}^m) : \int_\Omega h(x, u(x))\,\mathrm{d}x = 0 \right\} \tag{6}$$

and we assume that $\operatorname{dom} J \cap \mathcal{A} \neq \emptyset$ *where J is defined by (1) then*

$$\underset{u\in\mathcal{A}}{\operatorname{Argmin}} J[u] \neq \emptyset.$$

The more interesting issue for minimization under constraints of the type (6) is how to obtain Euler-Lagrange equations.

Theorem 4.2.5 *Assume the assumptions and notations of the 4.2.4. Moreover, suppose that the function h admits a.e. $x \in \Omega$ continuous partial derivatives with respect to u. Assume the existence of*

$$\bar{u} \in \underset{u\in\mathcal{A}}{\operatorname{Argmin}} J[u]$$

with the property that

$$\frac{\partial h}{\partial u}(x, \bar{u}(x)) \in L^{\frac{p}{p-1}}(\Omega;\mathbb{R}^m) \neq 0. \tag{7}$$

Then, there exist a $\lambda \in \mathbb{R}$ such that for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ we have that

$$\int_\Omega \left(\frac{\partial f}{\partial u_i}(x, \bar{u}, D\bar{u})\,\varphi_i(x) + \frac{\partial f}{\partial \xi_{ij}}(x, \bar{u}, D\bar{u}) \frac{\partial \varphi_i}{\partial x_j}(x) \right) \mathrm{d}x = \int_\Omega \lambda \frac{\partial h}{\partial u_i}(x, \bar{u}, D\bar{u})\,\varphi_i(x)\,\mathrm{d}x. \tag{8}$$

Remark 4.2.3 *Of course, the strong form of the Euler-Lagrange equations read*

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial \xi_{ij}}(x, \bar{u}, D\bar{u})\right) = \frac{\partial f}{\partial u_i}(x, \bar{u}, D\bar{u}) + \lambda \frac{\partial h}{\partial u_i}(x, \bar{u}, D\bar{u})$$

for all $i \in 1, m$.

Proof. The condition (7) ensures the existence of $\bar{\psi} \in L^p(\Omega; \mathbb{R}^m)$ with the property that

$$\int_\Omega \frac{\partial h}{\partial u_i}(x, \bar{u}(x)) \bar{\psi}_i(x) \, \mathrm{d}x \neq 0.$$

For $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, we consider the function $g_\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g_\varphi(t, s) = \int_\Omega h\left(x, \bar{u}(x) + t\varphi(x) + s\bar{\psi}(x)\right) \mathrm{d}x.$$

Of course, the growth conditions on h and $\frac{\partial h}{\partial u}$ ensure that g_φ is well-defined and C^1. Moreover

$$g_\varphi(0,0) = 0 \text{ and}$$
$$\frac{\partial g_\varphi}{\partial s}(0,0) = \int_\Omega \frac{\partial h}{\partial u_i}(x, \bar{u}(x)) \bar{\psi}_i(x) \, \mathrm{d}x \neq 0.$$

Thus, by the inverse function theorem, we obtain the existence of $\varepsilon_\varphi > 0$ and a C^1-function $s_\varphi : (-\varepsilon_\varphi, \varepsilon_\varphi) \to \mathbb{R}$ such that

$$g_\varphi(t, s_\varphi(t)) = 0,$$

for all $t \in (-\varepsilon_\varphi, \varepsilon_\varphi)$. Moreover

$$s'_\varphi(0) = -\frac{\int_\Omega \frac{\partial h}{\partial u_i}(x, \bar{u}(x)) \varphi_i(x) \, \mathrm{d}x}{\int_\Omega \frac{\partial h}{\partial u_i}(x, \bar{u}(x)) \bar{\psi}_i(x) \, \mathrm{d}x}$$

Then, we have that the function

$$t \in (-\varepsilon_\varphi, \varepsilon_\varphi) \to J\left[\bar{u} + t\varphi + s_\varphi(t)\bar{\psi}\right]$$

has a minimum at $t = 0$ and owing to the growth conditions on f this application is C^1. Writing the fact that the differential at $t = 0$ is 0 we obtain the conclusion with

$$\lambda := \frac{\int_\Omega \left(\frac{\partial f}{\partial u_i}(x, \bar{u}, D\bar{u}) \bar{\psi}_i(x) + \frac{\partial f}{\partial \xi_{ij}}(x, \bar{u}, D\bar{u}) \frac{\partial \bar{\psi}_i}{\partial x_j}(x)\right) \mathrm{d}x}{\int_\Omega \frac{\partial h}{\partial u_i}(x, \bar{u}(x)) \bar{\psi}_i(x) \, \mathrm{d}x}.$$

■

Example 4.2.2 *The first example is the minimization of*

$$J[u] = \int_\Omega \frac{|Du|^2}{2} \mathrm{d}x$$

on $W^{1,2}(\Omega)$ *on the set* $\mathcal{A} = \left\{u \in W_0^{1,2}(\Omega) : \|u\|_{L^2} = 1\right\}$. *This leads to the Euler-Lagrange equations*

$$\begin{cases} -\Delta \bar{u} = \lambda \bar{u}, \\ \bar{u} = 0 \text{ on } \partial\Omega, \\ \int_\Omega \bar{u}^2 \mathrm{d}x = 1. \end{cases}$$

The constant $\lambda > 0$ *is the smallest eigenvalue of the Laplacian with Dirichlet boundary conditions.*

4.3 Weak l.s.c. revisited

The aim of this section is to prove that a sufficient condition ensuring weak lower semicontinuity of integral functions is that the integrand f is convex w.r.t. ξ. The proof is considerably harder than the case when convexity w.r.t. (u, ξ) is assumed. The result we present of L^p-nature more precisely :

Theorem 4.3.1 *Let Ω be a bounded open set of $\mathbb{R}^d$ and $p, q \in (1, \infty)$. Let $f : \Omega \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R} \cup \{+\infty\}$ be a Carathéodory function satisfying*

$$f(x, u, \xi) \geq a(x) \cdot \xi + b(x) + c|u|^p$$

for almost every $x \in \Omega$, for every $(u, \xi) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, for some $a \in L^{q'}(\Omega; \mathbb{R}^{m_2})$, $b \in L^1(\Omega)$, and $c \in \mathbb{R}$. Consider

$$J[u, \xi] := \int_\Omega f(x, u(x), \xi(x))\, dx.$$

Assume that $\xi \mapsto f(x, u, \xi)$ is convex and that

$$u_n \to u \text{ in } L^p(\Omega; \mathbb{R}^{m_1}),$$
$$\xi_n \rightharpoonup \xi \text{ weakly in } L^q(\Omega; \mathbb{R}^{m_2}).$$

Then

$$\liminf_{n\to\infty} J[u_n, \xi_n] \geq J[u, \xi].$$

Proof.

We may suppose without loss of generality that f is positive. We select a sequence that realizes $\liminf_{n\to\infty} J[u_n, \xi_n]$ and we perform a further subsequence extraction such as to guarantee that

$$\lim_{n\to\infty} u_n \to u \text{ in } L^p(\Omega; \mathbb{R}^{m_1}) \text{ and a.e. on } \Omega.$$

by a slight abuse of notations, we do no relabel the subsequence with the above properties.

First of all, let us suppose that since $p, q \in (1, \infty)$ by Chebychev's inequality for any $\varepsilon > 0$ there exists M_ε large enough that can be chosen independently of n such that

$$m\{|u_n(x)| > M_\varepsilon \text{ or } |u(x)| \geq M_\varepsilon\} \leq \frac{\varepsilon}{8} \text{ and}$$
$$m\{|\xi_n(x)| \geq M_\varepsilon \text{ or } |\xi(x)| \geq M_\varepsilon\} \leq \frac{\varepsilon}{8}.$$

From Scorza-Dragoni's theorem we deduce the existence of a compact $K_\varepsilon^1 \subset \Omega$ such that

$$m\left(\Omega \backslash K_\varepsilon^1\right) \leq \frac{\varepsilon}{8}$$

and

$$f_{|K_\varepsilon \times [-M_\varepsilon, M_\varepsilon]^{m_1+m_2}} \text{ is continuous.}$$

Using the uniform continuity on compacts, for any $\varepsilon > 0$ there exists some $\eta = \eta(\varepsilon)$ such that

$$\left|(x, u, \xi) - \left(\tilde{x}, \tilde{u}, \tilde{\xi}\right)\right| \leq \eta(\varepsilon) \Rightarrow \left|f(x, u, \xi) - f\left(\tilde{x}, \tilde{u}, \tilde{\xi}\right)\right| \leq \varepsilon. \quad (9)$$

Using Egorof's theorem we obtain the existence of a compact $K_\varepsilon^2 \subset \Omega$ such that

$$m\left(\Omega \backslash K_\varepsilon^2\right) \leq \frac{\varepsilon}{8}$$

and

$$\lim_{n\to\infty}\sup_{x\in K_\varepsilon^2}|u_n(x)-u(x)|=0.$$

Thus, it is possible to choose n_ε^1 sufficiently large such that

$$\sup_{x\in K_\varepsilon^2}\left|u_{n_\varepsilon^1}(x)-u(x)\right|\le\eta(\varepsilon) \tag{10}$$

and observe that the set

$$\begin{aligned}\Omega_\varepsilon^1=&\left\{\left|u_{n_\varepsilon^1}\right|\le M_\varepsilon\right\}\cap\{|u|\le M_\varepsilon\}\\&\cap\left\{\left|\xi_{n_\varepsilon^1}\right|\le M_\varepsilon\right\}\cap\{|\xi|\le M_\varepsilon\}\cap K_\varepsilon^1\cap K_\varepsilon^2\end{aligned}$$

has the property

$$m\left(\Omega\backslash\Omega_\varepsilon^1\right)\le\frac{\varepsilon}{2}$$

and that owing to (10) and (9) we obtain

$$\int_{\Omega_\varepsilon^1}\left|f\left(x,u_{n_\varepsilon^1}(x),\xi_{n_\varepsilon^1}(x)\right)-f\left(x,u(x),\xi_{n_\varepsilon^1}(x)\right)\right|dx\le\varepsilon m(\Omega).$$

Since ε is arbitrary, for all $j\in\mathbb{N}^*$ we can construct a sequence $\left(n_\varepsilon^j\right)_{j\in\mathbb{N}^*}\subset\mathbb{N}^*$ and a sequence of measurable sets $\left(\Omega_\varepsilon^j\right)_{j\in\mathbb{N}^*}$ such that for all $j\in\mathbb{N}^*$

$$m\left(\Omega\backslash\Omega_\varepsilon^j\right)\le\frac{\varepsilon}{2^j}$$

and

$$\int_{\Omega_\varepsilon^j}\left|f\left(x,u_{n_\varepsilon^j}(x),\xi_{n_\varepsilon^1}(x)\right)-f\left(x,u(x),\xi_{n_\varepsilon^j}(x)\right)\right|dx\le\varepsilon m(\Omega).$$

Denoting

$$\Omega_\varepsilon=\bigcap_{j\ge1}\Omega_\varepsilon^j$$

we see that

$$m(\Omega\backslash\Omega_\varepsilon)\le\varepsilon$$

and that for all $j\in\mathbb{N}^*$

$$\begin{aligned}&\int_\Omega f(x,u_{n_\varepsilon^j}(x),\xi_{n_\varepsilon^j}(x))dx\\&=\int_{\Omega\backslash\Omega_\varepsilon}f(x,u_{n_\varepsilon^j}(x),\xi_{n_\varepsilon^j}(x))+\int_{\Omega_\varepsilon}f(x,u_{n_\varepsilon^j}(x),\xi_{n_\varepsilon^j}(x))dx\\&\ge\int_{\Omega_\varepsilon}f(x,u_{n_\varepsilon^j}(x),\xi_{n_\varepsilon^j}(x))-f(x,u(x),\xi_{n_\varepsilon^j}(x))dx+\int_{\Omega_\varepsilon}f(x,u(x),\xi_{n_\varepsilon^j}(x))dx\\&\ge-\varepsilon m(\Omega)+\int_{\Omega_\varepsilon}f(x,u(x),\xi_{n_\varepsilon^j}(x))dx.\end{aligned}$$

Using the fact that

$$\tilde{f}(x,\xi)=\mathbf{1}_{\Omega_\varepsilon}(x)f(x,u(x),\xi)$$

is a positive Carathéodory function and that $\xi \to \tilde{f}(x,\xi)$ is convex, from Theorem **??** it follows that

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \xi_n(x))\, dx \geq -\varepsilon m(\Omega) + \int_{\Omega_\varepsilon} f(x, u(x), \xi(x)) dx.$$

It is thus possible to construct measurable sets $\Omega_j \subset \Omega$ such that

$$m(\Omega \backslash \Omega_j) \leq \frac{1}{2^j} \tag{11}$$

and

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \xi_n(x))\, dx \geq \liminf_{j\to\infty} \int_{\Omega_j} f(x, u(x), \xi(x)) dx.$$

Since (11) implies that $\mathbf{1}_{\Omega_j}$ converges in $L^1(\Omega)$ to 1_Ω it is possible, by passing to a subsequence if necessary to obtain that $\mathbf{1}_{\Omega_j}(x)$ converges a.e. and owing to the positivity of f and Fatou's lemma we obtain that

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \xi_n(x))\, dx \geq \int_\Omega f(x, u(x), \xi(x))\, dx.$$

This ends the proof of Theorem 4.3.1.

As a corollary we obtain the Tonelli-Serrin theorem :

Theorem 4.3.2 (Tonelli-Serrin) *Let $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary, and*

$$f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \to \mathbb{R} \cup \{+\infty\}, \quad f = f(x,u,\xi),$$

be a Carathéodory function satisfying

$$f(x,u,\xi) \geq \langle a(x), \xi\rangle + b(x) + c|u|^r$$

for almost every $x \in \Omega$, for every $(u,\xi) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$, for some $a \in L^{p'}(\Omega;\mathbb{R}^{m\times d})$, where $1/p + 1/p' = 1$, $b \in L^1(\Omega)$, $c \in \mathbb{R}$, $1 \leq r < \frac{dp}{d-p}$ if $p < d$, and $1 \leq r < \infty$ if $p \geq d$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{d\times m}$. Let

$$J[u] := \int_\Omega f(x, u(x), Du(x))\, dx.$$

Assume that $\xi \mapsto f(x,u,\xi)$ is convex and that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^m).$$

Then

$$\liminf_{n\to\infty} J[u_n] \geq J[u].$$

One can prove that the convexity is almost optimal in the scalar case $m = 1$ or in dimension $d = 1$. Theese will be established in the following chapter.