5 Generalized notions of convexity

In the previous chapter we saw that Carathéodory integrands that are convex with respect to the ξ variable and verify appropiate lower bound (imagine just positive to take out a layer of technicalities) then minimizers for the associated integral functional on Sobolev spaces is weakly l.s.c.. This turns out to be optimal in the scalar case $m = 1$ (or the vectorial case but in $d = 1$) however in the vectorial case the situation is much more interesting. The search for neccesary conditions which ensure weak l.s. continuity gives rise to three generalized notions of convexity that we will present in this chapter.

As it is the case, we restrict in this presentation to the case of bounded domains (=open,connected) set $\Omega \subset \mathbb{R}^d$ witch are star-shaped w.r.t. a ball $B \subset \Omega$.

5.1 Convexity is not necessary for weak l.s.c. in the vectorial case

First of all, let us show that convexity is not a necesary condition for weak l.s.c.

Proposition 5.1.1 *Assume that $\Omega \subset \mathbb{R}^2$ and $(u_n)_n \subset W^{1,2}\left(\Omega; \mathbb{R}^2\right) \ni u$ is such that*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,2}\left(\Omega; \mathbb{R}^2\right),$$
$$\det Du_n \rightharpoonup d \text{ weakly in } L^1(\Omega).$$

Then $d = \det Du$.

Before presenting a sketch of the proof, let us deduce the following corollary.

Corollary 5.1.1 *Assume that $\Omega \subset \mathbb{R}^2$, $p > 2$ and $(u_n)_n \subset W^{1,p}\left(\Omega; \mathbb{R}^2\right) \ni u$ is such that*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,2}\left(\Omega; \mathbb{R}^2\right). \tag{1}$$

Then

$$\det Du_n \rightharpoonup \det Du \text{ weakly in } L^{\frac{p}{2}}(\Omega). \tag{2}$$

Proof. First of all, it is easy to see that if $u \in C^2\left(\overline{\Omega}\right)$ then

$$\det Du = \partial_2 \left(u_2 \partial_1 u_1\right) - \partial_1 \left(u_2 \partial_2 u_1\right).$$

Moreover, for all $\varphi \in C_c^\infty(\Omega)$ it holds true that

$$\int_\Omega \det Du(x)\,\varphi(x)\,\mathrm{d}x = -\int_\Omega \left(u_2(x)\,\partial_1 u_1(x)\,\partial_2\varphi(x) - u_2(x)\,\partial_2 u_1(x)\,\partial_1\varphi(x)\right)\mathrm{d}x. \tag{3}$$

This last identity can be shown to hold true for $u \in W^{1,2}\left(\Omega; \mathbb{R}^2\right)$ by a density argument. Therefore, for a sequence $(u_n)_n$ verifing (1) and (2) the identity (3) and thus, for all $n \in \mathbb{N}$

$$\int_\Omega \det Du_n(x)\,\varphi(x)\,\mathrm{d}x = -\int_\Omega \left(u_{2,n}(x)\,\partial_1 u_{1,n}(x)\,\partial_2\varphi(x) - u_{2,n}(x)\,\partial_2 u_{1,n}(x)\,\partial_1\varphi(x)\right)\mathrm{d}x. \tag{4}$$

Owing to (2) we have that

$$\lim_{n\to\infty} \int_\Omega \det Du_n(x)\,\varphi(x)\,\mathrm{d}x = \int_\Omega d(x)\,\varphi(x)\,\mathrm{d}x. \tag{5}$$

Next, owing to Rellich-Kondrachov's theorem we have that

$$u_n \to u \text{ strongly in } L^q\left(\Omega;\mathbb{R}^2\right) \text{ for all } q \in [1,\infty).$$

Using the weak convergence in L^2 of the differential we infer that

$$\begin{aligned}&\lim_{n\to\infty} -\int_\Omega \left(u_{2,n}(x)\,\partial_1 u_{1,n}(x)\,\partial_2\varphi(x) - u_{2,n}(x)\,\partial_2 u_{1,n}(x)\,\partial_1\varphi(x)\right)\mathrm{d}x\\ &= -\int_\Omega \left(u_2(x)\,\partial_1 u_1(x)\,\partial_2\varphi(x) - u_2(x)\,\partial_2 u_1(x)\,\partial_1\varphi(x)\right)\mathrm{d}x = \int_\Omega \det Du(x)\,\varphi(x)\,\mathrm{d}x. \qquad (6)\end{aligned}$$

Comparing (5) with (6) we end up

$$\int_\Omega d(x)\,\varphi(x)\,\mathrm{d}x = \int_\Omega \det Du(x)\,\varphi(x)\,\mathrm{d}x$$

which holds for all $\varphi \in C_c^\infty(\Omega)$. Thus we have $d = \det Du$.

The proof of the Corollary follows by the fact that for $p > 2$, $(\det Du_n)_n$ are uniformly bounded in $L^{\frac{p}{2}}(\Omega)$ and since $\frac{p}{2} > 1$ we may extract a subsequence that converges weakly to $\det Du$ owing to the previous result. Since the limit is unique, the whole sequence converges. ■

The result stated in Proposition 5.1.1 and Corollary 5.1.1 are surprising since $\det Du$ is a nonlinear function of Du and weak convergence does not "comute" in general with nonlinear functions. For the moment, let us observe that Corollary 5.1.1 implies in particular the weak l.s.c. of

$$\begin{cases} J : W^{1,p}\left(\Omega;\mathbb{R}^2\right) \to \mathbb{R} \\ J[u] = \int_\Omega \det Du(x)\,\mathrm{d}x \end{cases}$$

with $\Omega \subset \mathbb{R}^2$ and $p > 2$. However, it is imeadiate to see that

$$\xi \in \mathbb{R}^{2\times 2} \to \det \xi \in \mathbb{R}$$

is not convex by a classical exemple

$$\xi_1 = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix},\ \xi_2 = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$$

for which $\det \xi_1 = \det \xi_2 = 3$ but $\det\left(\frac{1}{2}(\xi_1 + \xi_2)\right) = 4$.

5.2 Necessary conditions and generalized convexity notions

The point of view that we adopt from now is the following : given a weakly l.s.c. integral functional, we want to construct particular sequences that will allow us to obtain information on the integrand.

In order to simplfy the presentation and to avoid unnecessary (at this stage) complications, we will work only with continious integrands that depend on ξ :

$$f : \mathbb{R}^{m\times d} \to \mathbb{R},\ f = f(\xi).$$

One important tool for constructing and analyizing oscillating sequences is the

Lemma 5.2.1 (Riemann-Lebesgue) *Consider* $p \in (1,\infty)$ *and* $u \in L^p_{loc}\left(\mathbb{R}^d\right)$ *a* 1*-periodic function : a.e.* $x \in \mathbb{R}^d$, $u(x) = u(\{x\})$ *where* $\{x\}$ *is the vector of the fractional parts of the components of* x*. Then,*

$$u_n(x) = u(nx) \in L^p_{loc}\left(\mathbb{R}^d\right)$$

and for any bounded domain $\Omega \subset \mathbb{R}^d$

$$u_n \rightharpoonup \int_{[0,1)^d} u(y)\,\mathrm{d}y \text{ weakly in } L^p(\Omega).$$

Proof. Since we are dealing with $p \in (1,\infty)$ it is enough to show that for any $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$ we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} u_n(x)\,\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x \int_{[0,1)^d} u(y)\,\mathrm{d}y\ .$$

This follows from the following computations

$$\begin{aligned}
\int_{\mathbb{R}^d} u_n(x)\,\varphi(x)\,\mathrm{d}x &= \sum_{k\in\mathbb{Z}^d} \int_{\frac{1}{n}(k+[0,1)^d)} u_n(x)\,\varphi(x)\,\mathrm{d}x = \sum_{k\in\mathbb{Z}^d} \int_{\frac{1}{n}(k+[0,1)^d)} u(\{nx\})\,\varphi(x)\,\mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}^d} \int_{\frac{1}{n}(k+[0,1)^d)} u(\{nx-k\})\,\varphi(x)\,\mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}^d} \frac{1}{n^d} \int_{[0,1)^d} u(\{y\})\,\varphi\left(\frac{k+y}{n}\right)\mathrm{d}y \\
&= \int_{[0,1)^d} u(y) \sum_{k\in\mathbb{Z}^d} \frac{1}{n^d}\varphi\left(\frac{k+y}{n}\right)\mathrm{d}y \to \int_{[0,1)^d} u(y)\,\mathrm{d}y \int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x.
\end{aligned}$$

■

Remark 5.2.1 *Of course, the previous proposition can be adapted to the case* $L^p(\Omega;\mathbb{R}^m)$.

Remark 5.2.2 *The previous proposition admits a generalization to functions* $u = u(x,y) \in L^p_{loc}\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ *that are* $1-$*periodic w.r.t. the second variable as soon as one can quarantee that*

$$u_n(x) = u(x,\{nx\})$$

are measurable functions. In this case, we obviously have that

$$u_n \rightharpoonup \int_{[0,1)^d} u(x,y)\,\mathrm{d}y$$

weakly in $L^p_{loc}(\Omega)$ *for any bounded open* $\Omega \subset \mathbb{R}^d$.

Remark 5.2.3 *Assume that* $T : \bar{Y} \to [0,1]^d$ *is a* C^1*-diffeomorphism and let* $u \in L^p(Y)$*. Then the sequence of (*Y*-periodic) functions*

$$\begin{gathered}
u_n : Y \to \mathbb{R}, \\
u_n(y) = u\left(T^{-1}\circ\{nT(y)\}\right)
\end{gathered}$$

converges weakly in $L^p(\Omega)$ *for any bounded open* $\Omega \subset \mathbb{R}^d$ *to its mean value:*

$$u_n \rightharpoonup \frac{1}{m(Y)}\int_Y u(y)\,\mathrm{d}y.$$

Example 5.2.1 *Let us construct a sequence of L^p-functions that oscillates between two states $\xi, \eta \in \mathbb{R}^m$. For this we construct the building block*

$$\left\{ \begin{array}{l} u_\square : [0,1)^d \to \mathbb{R}^m, \\ u_\square(y) = \left\{ \begin{array}{l} \xi \text{ if } y_1 \in [0,\lambda), \\ \eta \text{ if } y_1 \in (\lambda, 1]. \end{array} \right. \end{array} \right. \tag{7}$$

The mean value is

$$\int_{[0,1)^d} u_\square(y)\,\mathrm{d}y = \lambda\xi + (1-\lambda)\eta.$$

The sequence

$$u_n : \mathbb{R}^d \to \mathbb{R}^m,\ u_n(x) = u_\square(\{nx\})$$

oscillates faster and faster as $n \to \infty$ between the value ξ and η. We obtain that

$$u_n \rightharpoonup u \text{ weakly in } L^p(\Omega;\mathbb{R}^m)$$

for any bounded open domain $\Omega \subset \mathbb{R}^d$ and any $p \in (1,\infty)$.

We remark that the previous exemple implies imediately the following

Proposition 5.2.1 *Assume that $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^d$ is a bounded open domain and $f : \mathbb{R}^m \to \mathbb{R}$ is continious such that the integral functional*

$$\left\{ \begin{array}{l} J : L^p(\Omega;\mathbb{R}^m) \to \mathbb{R}, \\ J[u] = \int_\Omega f(u(x))\,\mathrm{d}x \end{array} \right.$$

is weakly l.s.c. on $L^p(\Omega;\mathbb{R}^m)$. Then f is convex.

Proof. The only thing that we have to do is to plug in the sequence constructed above and to understand why

$$\liminf_{n\to\infty} \int_\Omega f(u_n(x))\,\mathrm{d}x = m(\Omega)\,\lambda f(\xi) + (1-\lambda) f(\eta)$$

(hint : write explicitely what is the sequence $(f(u_n(x)))_n$) ■

5.2.1 Rank-1 convexity

At a first glance, it is not completly obvious why does convexity fail to be necessary for l.s.c. in Sobolev spaces. It is instructive to try to proceed as above to see what changes. To put things in order : we would like to construct this time a sequence of functions such that their gradients oscilate between two fixed matrices $\xi, \eta \in \mathbb{R}^{m\times d}$. For this, we would need a bouilding block like in the exemple (7)

Assume then that $\in W^{1,p}_{loc}(\mathbb{R}^d)$

$$Du_\square(y) = \left\{ \begin{array}{l} \xi \text{ if } y_1 \in [0,\lambda), \\ \eta \text{ if } y_1 \in (\lambda, 1]. \end{array} \right.$$

Then

$$u_\square(y) = \left\{ \begin{array}{l} \xi y + u_1 \text{ if } y_1 \in [0,\lambda), \\ \eta y + u_2 \text{ if } y_1 \in (\lambda, 1]. \end{array} \right.$$

If $u_\square(y)$ is to be a Sobolev function then values at $y_1 = \lambda$ (and on opposities faces) should agree : for all $i \in \overline{1,m}$ we have that

$$\xi_{ij} y_j + u_{1i} = \eta_{ij} y_j + u_{2i}$$

then we see that this imposes some restrictions of ξ and η namely that

$$\left\{ \begin{array}{l} \xi_{ij} = \eta_{ij} \text{ for all } i \in \overline{1,m} \text{ and } j \in \overline{2,d}, \\ \lambda(\xi_{i1} - \eta_{i1}) = u_{2i} - u_{1i} \text{ for all } i \in \overline{1,m}. \end{array} \right. \tag{8}$$

There is not to much room to improve this : the best we can do is to consider an extra-rotation. This leads to the following definition

Definition 5.2.1 *A function $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is called rank-1 convex if for any $\xi, \eta \in \mathbb{R}^{m\times d}$ such that* rank $(\xi - \eta) = 1$ *and for any $\lambda \in [0,1]$ it holds true that*

$$f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda) f(\eta).$$

Equivalently, for any $\xi \in \mathbb{R}^{m\times d}$ and any $a \in \mathbb{R}^m$, $b \in \mathbb{R}^d$ the function of one real variable

$$t \to f(\xi + ta \otimes b)$$

is convex where $a \otimes b \in \mathbb{R}^{m\times d}$ with $(a \otimes b)_{ij} = a_i b_j$.

The hardest part of the proof of the following proposition was allready done in the above considerations.

Proposition 5.2.2 *Assume that $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^d$ is a bounded open domain and $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is continious such that the integral functional*

$$\left\{ \begin{array}{l} J : W^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}, \\ J[u] = \int_\Omega f(Du(x))\,\mathrm{d}x \end{array} \right.$$

is weakly l.s.c. on $W^{1,p}(\Omega;\mathbb{R}^m)$. Then f is rank-1 convex.

Proof. Consider $\xi, \eta \in \mathbb{R}^{m\times d}, a \in \mathbb{R}^m$ and $b \in \mathbb{R}^d$, $a = 0$, $b \neq 0$ are such that

$$\eta = \xi + a \otimes b$$

We consider

$$u : \mathbb{R}^d \to \mathbb{R}^m$$
$$u(x) = \left\{ \begin{array}{l} \eta x - (b\cdot x)a + (1-\lambda)\,na \text{ if } n \leq b\cdot x < n+\lambda \\ \eta x + (1+n)\,\lambda a \text{ if } n+\lambda \leq b\cdot x < n+1, \end{array} \right.$$

which is periodic on a cube with size 1 and having a face perpendicular on b. Then, it is easy to see that for any $p \in (1,\infty)$, $u \in W^{1,p}_{loc}(\mathbb{R}^d)$ (check continuity at $b\cdot x = n, n+\lambda, n+1$ for $n \in \mathbb{Z}$). Then

$$u_n(x) = \frac{1}{n}u(nx) \to 0 \text{ strongly in } L^\infty(\mathbb{R}^d),$$
$$Du_n \rightharpoonup \lambda\xi + (1-\lambda)\eta \text{ weakly in } L^p(\omega) \text{ for any bounded open } \omega \subset \mathbb{R}^d.$$

But then, owing to the periodicity of the gradient and Lemma 5.2.1 we obtain that

$$f(Du_n) \rightharpoonup \lambda f(\xi) + (1-\lambda) f(\eta) \text{ weakly in } L^p(\Omega)$$

for any $p \in (1, \infty)$ and thus

$$\liminf \int_{\Omega} f(Du_n)\,\mathrm{d}x = m(\Omega)(\lambda f(\xi) + (1-\lambda) f(\eta))$$

which is greater or equal than

$$\int_{\Omega} f(\lim Du_n)\,\mathrm{d}x = m(\Omega) f(\lambda \xi + (1-\lambda)\eta).$$

Therefore, we obtain that

$$\lambda f(\xi) + (1-\lambda) f(\eta) \geq f(\lambda \xi + (1-\lambda)\eta)$$

for any ξ, η with rank one difference. ■

5.2.2 Quasiconvexity

The notion of rank-1 convexity appears because we want to construct an explicit patern of oscillations. A more general oscillatory sequence is constructed as follows. If $x_0 \in \Omega$ then for large enough $\ell \in \mathbb{N}$ we can "squiz"-in the unit cube into Ω :

$$x_0 + \frac{1}{\ell}[0,1]^d \subset \Omega.$$

Assume that $\varphi \in C_c^{\infty}((0,1)^d)$, $\xi \in \mathbb{R}^{m\times d}$ and consider

$$u_\ell(x) = \begin{cases} \xi x + \frac{1}{\ell}\varphi(\ell(x-x_0)) \text{ if } x \in x_0 + \frac{1}{\ell}[0,1]^d, \\ \xi x \text{ if } x \in \Omega \backslash \left(x_0 + \frac{1}{\ell}[0,1]^d\right). \end{cases}$$

We observe that $u_\ell \in W^{1,p}(\Omega; \mathbb{R}^m)$ and

$$Du_\ell(x) = \begin{cases} \xi + D\varphi(\ell(x-x_0)) \text{ if } x \in x_0 + \frac{1}{\ell}[0,1]^d, \\ \xi \text{ if } x \in \Omega \backslash \left(x_0 + \frac{1}{\ell}[0,1]^d\right). \end{cases}$$

Then, we create oscillations in $x_0 + \frac{1}{\ell}[0,1]^d$ by putting

$$u_{\ell,n}(x) = \begin{cases} \xi x + \frac{1}{n\ell}\varphi(\{n\ell(x-x_0)\}) \text{ if } x \in x_0 + \frac{1}{\ell}[0,1]^d, \\ \xi x \text{ if } x \in \Omega \backslash x_0 + \frac{1}{\ell}[0,1]^d. \end{cases}$$

and we observe that

$$u_{\ell,n} \to \xi x \text{ in } L^{\infty}(\Omega; \mathbb{R}^m) \text{ and}$$
$$Du_{\ell,n} \rightharpoonup \xi \text{ weakly in } L^p\left(\Omega; \mathbb{R}^{m\times d}\right)$$

for any $p \in (1, \infty)$. Also, we observe that

$$\int_{\Omega} f(Du_{\ell,n}(x))\,\mathrm{d}x = \int_{\Omega \backslash \left(x_0 + \frac{1}{\ell}[0,1]^d\right)} f(\xi)\,\mathrm{d}x + \int_{x_0 + \frac{1}{\ell}[0,1]^d} f(\xi + D\varphi(\{n\ell(x-x_0)\}))\,\mathrm{d}x$$
$$= \left(m(\Omega) - \frac{1}{\ell^d}\right) f(\xi) + \frac{1}{\ell^d}\int_{[0,1]^d} f(\xi + D\varphi(\{ny\}))\,\mathrm{d}y$$

and thus

$$\liminf \int_{\Omega} f(Du_{\ell,n}(x))\,\mathrm{d}x = \left(m(\Omega) - \frac{1}{\ell^d}\right) f(\xi) + \frac{1}{\ell^d}\int_{[0,1]^d} f(\xi + D\varphi(y))\,\mathrm{d}y.$$

This leads to the following

Definition 5.2.2 *A continious function $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is said to be quasiconvex if for every $\xi \in \mathbb{R}^{m\times d}$ and every $\varphi \in C_c^\infty((0,1)^d)$ it holds true that*

$$\int_{[0,1]^d} f\left(\xi + D\varphi\left(y\right)\right) \mathrm{d}y \geq f\left(\xi\right).$$

Of course, an immediate application of Jensen's inequality leads to the conclusion that if f is convex then f is quasi-convex. We observe that without to much effort we obtain

Proposition 5.2.3 *Consider $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ quasiconvex and $E \subset \mathbb{R}^d$ a bounded open set. Then for every $\xi \in \mathbb{R}^{m\times d}$ and every $\psi \in C_c^\infty(E)$ it holds true that*

$$\frac{1}{m\left(E\right)} \int_E f\left(\xi + D\psi\left(z\right)\right) \mathrm{d}z \geq f\left(\xi\right).$$

It can be showed that quasi-convex functions are rank-1 convex. The oposite is not true, an exemple was constructed by Sverak 2003.

The interest of quasi-convexity is that, although looking as a technical artifact, it acctually turns out to be also a sufficient condition for ensuring the weak l.s.c. of $p-$growth functionals. We cite without proof the following theorem due to Morrey established in the $50s$.

Theorem 5.2.1 *Let $\Omega \subset \mathbb{R}^d$ open bounded and $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ continious and quasi-convex and such that*

$$0 \leq f\left(\xi\right) \leq \Lambda(|\xi|^p + 1).$$

Then

$$J : W^{1,p}\left(\Omega; \mathbb{R}^m\right) : J\left[u\right] = \int_\Omega f\left(Du\left(x\right)\right) \mathrm{d}x$$

is weakly l.s.c. on $W^{1,p}\left(\Omega; \mathbb{R}^m\right)$.

5.2.3 Polyconvexity

Although a necessary condition, quasiconvexity is very hard to verify in practice. It turns out that there is a large subclass of quasi-convex functions which are easier to use in practice (we have in mind exemples from nonlinear elasticity). This are polyconvex functions. The ideea is simple. It turns out that any minor of order k of the differential matrix is continious from $W^{1,p}\left(\Omega; \mathbb{R}^m\right)$ with the weak topology to $L^{\frac{p}{k}}\left(\Omega\right)$ with the weak topology. Then convex functions of theese minors are quasi-convex. Although intuitively clear, the general case is quite technical to describe. It is for this reason that we focus on the particular $2d$ and $3d$ case.

Definition 5.2.3

1. *A function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is said to be polyconvex if there exists a convex function $F : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that*

$$f\left(\xi\right) = F\left(\xi, \det \xi\right). \tag{9}$$

2. *A function $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is said to be polyconvex if there exists a convex function $F : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R} \to \mathbb{R}$ such that*

$$f\left(\xi\right) = F\left(\xi, \mathrm{Cof}\, \xi, \det \xi\right). \tag{10}$$

Remark 5.2.4 *Obviously, if f is convex than it is also polyconvex. The exemple of $\xi \to \det \xi$ shows that polyconvex functions are not convex in general.*

Remark 5.2.5 *In general, a polyconvex function f can be written in more than one way under the form (9) or (10) in 3D. For exemple, in $d = 2$*

$$f(\xi) = \frac{|\xi|^2}{2} = \frac{1}{2}(\xi_{11} - \xi_{22})^2 + \frac{1}{2}(\xi_{12} + \xi_{21})^2 + \det \xi.$$

Proposition 5.2.4 *If $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is polyconvex then it is quasi-convex.*

Proof. Consider $\varphi \in C_c^\infty((0,1)^d)$ and let $F : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ be convex s.t.

$$f(\xi) = F(\xi, \det \xi).$$

Then

$$\begin{aligned}\int_{[0,1]^d} f(\xi + D\varphi(y))\,\mathrm{d}y &= \int_{[0,1]^d} F(\xi + D\varphi(y), \det(\xi + D\varphi(y)))\,\mathrm{d}y \\ &\geq F\left(\int_{[0,1]^d} (\xi + D\varphi(y))\,\mathrm{d}y, \int_{[0,1]^d} \det(\xi + D\varphi(y))\,\mathrm{d}y\right) \\ &= F(\xi, \det(\xi))\end{aligned}$$

where

$$\int_{[0,1]^d} \det(\xi + D\varphi(y))\,\mathrm{d}y = \int_{[0,1]^d} \det(D(\xi y + \varphi(y)))\,\mathrm{d}y = \int_{[0,1]^d} \det(\xi y) = \det \xi$$

owing to the fact that the determinant is a null-Lagrangian (see Problem Sheet n°1). ■

We recall for the reader's convenience the following result wich was proved in Corollary 5.1.1

Corollary 5.2.1 *Assume that $\Omega \subset \mathbb{R}^2$, $p > 2$ and $(u_n)_n \subset W^{1,p}(\Omega; \mathbb{R}^2) \ni u$ is such that*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,2}(\Omega; \mathbb{R}^2). \tag{11}$$

Then

$$\det Du_n \rightharpoonup \det Du \text{ weakly in } L^{\frac{p}{2}}(\Omega). \tag{12}$$

This property on its own, combinded with the Tonelli-Serrin result on weak lower semi-continuity for strong-weak convergence implies the following improved Theorem for sufficiency :

Theorem 5.2.2 *Assume that $f : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \times \mathbb{R} \to [0, \infty)$ a Carathéodory function with the the property that a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^2$ the function*

$$\xi \to f(x, u, \xi, \det \xi) \text{ is convex.}$$

Then if $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^2)$ for $p > 2$ we have that

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n(x), Du_n(x), \det Du_n(x))\,\mathrm{d}x \geq \int_\Omega f(x, u(x), Du(x), \det Du(x))\,\mathrm{d}x.$$