Algebraic Groups with Good Reduction

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(joint work with V. Chernousov and A. Rapinchuk)

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Argument heavily relies on *reduction*.

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(In more technical terms, having good reduction means that there exists an abelian scheme $E_{(p)}$ over valuation ring $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ with generic fiber E.)

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$$|E(\mathbb{Q})/2E(\mathbb{Q})| \leqslant 2^{2(1+|S|)}.$$

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Let K be a number field, and let S be a finite set of places of K. Then for every $g \geqslant 1$, there exist only finitely many isomorphism classes of abelian varieties of dimension g that have good reduction at all $\mathfrak{p} \notin S$.

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Consequences include:

- Mordell's conjecture: a smooth projective curve of genus $g \geqslant 2$ over a number field K has finitely many K-rational points;
- Shafarevich's conjecture for curves: for $g \geqslant 2$, there are only finitely many isomorphism classes of smooth projective curves over K of genus g having good reduction at all $\mathfrak{p} \notin S$.

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Our goal is to find analogues of these results for *linear algebraic groups*.

Reduction of algebraic groups modulo p: SL_n

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Thus, reduction of $\mathrm{SL}_n/\mathbb{Q}$ modulo p is $\mathrm{SL}_n/\mathbb{F}_p$.

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There exists algebraic $\mathbb{Q}$-group $G = \mathrm{R}^{(1)}_{L/\mathbb{Q}}(\mathbb{G}_m)$ (norm torus) such that

$$G(\mathbb{Q}) \ = \ \{z \in L^{\times} \mid \mathrm{N}_{L/\mathbb{Q}}(z) = 1\}.$$

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Structurally, G is 1-dimensional ($\mathbb{Q}$-anisotropic) torus.

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So, reduction of (T) modulo p defines *disconnected* $\mathbb{F}_p$-group whose connected component is 1-dimensional *unipotent group*!

On the other hand, reducing (T) modulo any $q > 2$, $q \neq p$, one still gets 1-dimensional *torus*.

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$\mathrm{SL}_{1,D}$ (cont.)

- $G \simeq \mathrm{SL}_2$ over $\overline{\mathbb{Q}}$.
- Group defined by reduction modulo p of equations defining G has *nontrivial* unipotent radical,

$\mathrm{SL}_{1,D}$ (cont.)

Using *regular representation* of D, one can give matrix realization of G (similar to Example 3).

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To summarize:

- Reductive $\mathbb{Q}$-groups in Examples 1 & 2 can be described by polynomial systems with coefficients in $\mathbb{Z}$ (or in $\mathbb{Z}_{(p)} \subset \mathbb{Q}$) such that their reductions modulo p still define reductive groups.
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We say that a reductive $\mathbb{Q}$-group G has *good reduction* at p if it can be defined by a system of equations with coefficients in $\mathbb{Z}_{(p)}$ such that reduced modulo p system defines reductive group; otherwise, it has *bad reduction*.

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- Group in Example 3 has bad reduction at p and good reduction at all odd primes $q \neq p$.

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  (In fact, enough to have $\mathrm{SL}_{1,D} \simeq \mathrm{SL}_2$ over $\mathbb{Q}_p$.)

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Then *special fiber* (reduction)

$$\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} K^{(v)}$$

is *a connected reductive* group over residue field $K^{(v)}$.

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2. $G = \mathrm{Spin}_n(q)$ has good reduction at v if (over K_v)

$$q \ \sim \ \lambda(a_1x_1^2 + \cdots + a_nx_n^2) \ \text{ with } \ \lambda \in K_v^{\times}, \ \ a_i \in \mathcal{O}_v^{\times}$$

(assuming that $\mathrm{char}\, K^{(v)} \neq 2$).

A K-group G' is a *K-form* (or $\overline{K}/K$-form) of G if

$$G' \otimes_K \overline{K} \simeq G \otimes_K \overline{K} \quad \text{(where } \overline{K} \text{ is a sep. closure of } K\text{).}$$

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and $G' = \mathrm{SL}_{1,A}$ is a K-form of $G = \mathrm{SL}_n$.

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If n is *odd*, then these are **all** K-forms.

Otherwise, there may be K-forms coming from hermitian forms over noncommutative division algebras.

General problem

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To make this question meaningful, one needs to specialize

K, V, and G.

- Previous work has dealt mainly with the case where K is fraction field of *Dedekind ring* R, and V consists of valuations associated with *maximal ideals* of R.

- This situation was first studied in detail by G. Harder (Invent. math. **4**(1967), 165-191) and J.L. Colliot-Thélène & J.J. Sansuc (Math. Ann. **244** (1979), no. 2, 105-134).

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Proposition

Let G be an absolutely almost simple simply connected algebraic group over a number field K, and assume that V contains almost all places of K. **Then** *the number of K-forms of G that have good reduction at all $v \in V$ is finite.*

- Case $R = k[x]$, $K = k(x)$, and

$$V = \{ v_{p(x)} \mid p(x) \in k[x] \text{ monic irreducible} \}.$$

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Let k be a field of characteristic zero, and let G_0 be a connected reductive group over k.

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- Chernousov–Gille–Pianzola (2012) considered similar problem for $R = k[x, x^{-1}]$ and $K = k(x)$.

We can generalize the previous examples as follows:

- Let C be smooth geometrically integral affine curve over field k;
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When is number of K-isomorphism classes of K-forms of reductive K-group G having good reduction at all $v \in V$ finite?

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- If G_0' is k-form of a k-group G, then $G' = G_0' \times_k K$ is K-form of $G = G_0 \times_k K$ having good reduction at all $v \in V$.

- **So**, need to ensure there are only *finitely* many non-isomorphic k-forms.

- Serre introduced condition (F) for *perfect* fields to study finiteness properties of Galois cohomology.

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(F) For every $m \geqslant 1$, $\mathrm{Gal}(\bar{k}/k)$ has finitely many open subgroups of index m.

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If $\mathrm{char}\, k = 0$ *and* k *is of type* (F), *then the number of* K*-isomorphism classes of* K*-forms* G' *of* G *having good reduction at all* $v \in V$ *is finite.*

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Algebraically: We find $R \subset K$ such that $K = \mathrm{Frac}(R)$ and

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Then: V corresponds to height one prime ideals of R.

Divisorial valuations: Example

- $\mathfrak{p} = (p(x))$, with $p(x) \in \mathbb{Z}[x]$ irreducible of content 1;
- $\mathfrak{p} = (p)$, $p \in \mathbb{Z}$ a prime.

- "geometric places" V_0;
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Then $V = V_0 \cup V_1$ is divisorial set of discrete valuations associated with the model $\mathfrak{X} = \mathrm{Spec}(R)$ of K.

Main Finiteness Conjecture

- K a finitely generated field;
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Connections and applications of the Main Conjecture

This conjecture has close connections to:

- Hasse principles for algebraic groups.
- Finiteness properties of unramified cohomology.
- Study of simple algebraic groups having same isomorphism classes of maximal tori (genus problem).
- Analysis of weakly commensurable Zariski-dense subgps and applications to classical problems on locally symmetric spaces (G. Prasad-A. Rapinchuk).

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Set-up: Global-to-local map in Galois cohomology

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One says that the Hasse principle holds if global-to-local map

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Kernel of $\theta_{G,V}$ is called *Tate-Shafarevich set*

$$Ш(G, V) := \ker \theta_{G,V}.$$

Hasse principle over number fields

Let k = *number field*, V = set of all places of k.

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Our recent results strongly suggest the following properness conjecture for reductive groups over finitely generated fields.

Properness conjecture

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Conjecture 2.

If G is a (connected) reductive algebraic K-group, then $\theta_{G,V}$ is proper. In particular, the Tate-Shafarevich set $Ш(G,V)$ is finite.

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Assume Main Conjecture holds for an absolutely almost simple simply connected K-group G and all divisorial sets of places of K. **Then** *$\theta_{\overline{G},V}$ is proper for corresponding adjoint group $\overline{G}$ and any divisorial set V.*

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Key points of the proof:

- K-isomorphism classes of d-dimensional K-tori classified by equivalence classes of cont. reps. $\varphi\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_d(\mathbb{Z})$.
- By reduction theory, $\mathrm{GL}_d(\mathbb{Z})$ has finitely many conjugacy classes of finite subgroups, represented by $\Phi_1, \ldots, \Phi_r$.
- Assumption of good reduction implies we actually consider reps. $\varphi\colon \pi_1^{\text{ét}}(\mathfrak{X}) \to \Phi_i$, for model $\mathfrak{X}$ defining V.
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- $\pi_1^{\text{ét}}(\mathfrak{X})$ is of type (F) $\Rightarrow$ finitely many possibilities for φ.

Theorem 5.

Suppose K is a finitely generated field and V is a divisorial set of places.

- Classical proof for tori over number fields relies on Tate-Nakayama duality, which is not available in general.
- Our proof uses adelic methods.

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Suppose $K = k(X)$, *where* k *is of type (F) and* $\operatorname{char} k = 0$, *and* X *is a normal irreducible variety over* k.

- Argument depends on purity results of Nisnevich (for reductive groups over DVRs) and Colliot-Thélène and Sansuc (for tori).

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- Argument depends on purity results of Nisnevich (for reductive groups over DVRs) and Colliot-Thélène and Sansuc (for tori).

Theorem 6.
Suppose $K = k(X)$, where k is of type (F) and $\operatorname{char} k = 0$, and X is a normal irreducible variety over k. Let V be the set of geometric places of K. **Then** *for any K-group D whose connected component is a torus, the global-to-local map*

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We then automatically obtain properness of

$$\theta_{\mathrm{PSL}_{1,A},V} \colon H^1(K, \mathrm{PSL}_{1,A}) \longrightarrow \prod_{v \in V} H^1(K_v, \mathrm{PSL}_{1,A})$$

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Let $\mathfrak{X}$ be a model of a finitely generated field K.

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Proposition 8.

Let $\mathfrak{X}$ be a model of a finitely generated field K. Then for any $n \geqslant 1$ prime to $\mathrm{char}\, K$, the n-torsion subgroup $\mathrm{Br}(\mathfrak{X})[n]$ of $\mathrm{Br}(\mathfrak{X})$ is finite.

Following Kato, we say K is a 2-dimensional global field if

- $K = k(C)$, with C smooth geometrically integral curve over number field k; or
- $K = \mathbb{F}_q(S)$, with S smooth geometrically integral surface over finite field $\mathbb{F}_q$.

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- Similar results for groups of types A_n, C_n that split over a quadratic extension, and G_2.

Proof of Theorem 9 consists of two main parts:

- Using Milnor's conjecture, we reduce the argument to finiteness of unramified cohomology $H^i(K, \mu_2)_V$, for $i \geqslant 1$, where $\mu_2 = \{\pm 1\}$.
- We establish finiteness of $H^i(K, \mu_2)_V$ for all $i \geqslant 1$.

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Properness of $\theta_{G,V}$ for special orthogonal groups

Theorem 10.

Let K be a 2-dimensional global field of char. $\neq 2$, and V divisorial set of places. Fix $n \geqslant 5$.

- This result follows from Theorem 9 only for odd n.
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Further results and a question

For K a 2-dimensional global field and V divisorial set of valuations, we also establish properness of $\theta_{G,V}$ for:

- G of type G_2;
- $G = \mathrm{SU}_n(L/K, h)$, L/K quadratic extension, h nondegenerate hermitian form of $\dim \geqslant 2$;
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We have recent finiteness results for function fields of rational varieties and certain S-B varieties over number fields, but general case is wide open.

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We say: G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

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• Let G be an absolutely almost simple K-group.

$\mathbf{gen}_K(G) =$ set of isomorphism classes of K-forms G' of G having same K-isomorphism classes of maximal K-tori as G.

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Conjecture.

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Connections to groups with good reduction

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In particular, the Main Conjecture yields finiteness results for the genus.

A sampling of results

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Theorem 13.

(1) *Let D be a central division algebra of exponent 2 over $K = k(x_1, \ldots, x_r)$ where k is a number field or a finite field of characteristic $\neq 2$. Then for $G = \mathrm{SL}_{m,D}$ $(m \geqslant 1)$, we have $|\mathbf{gen}_K(G)| = 1$.*

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Results (cont.)

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