COMPARING DESCENT OBSTRUCTION AND BRAUER-MANIN OBSTRUCTION FOR OPEN VARIETIES

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Abstract. We provide a relation between Brauer-Manin obstruction and descent obstruction for torsors over not necessarily proper varieties under a connected linear algebraic group or a group of multiplicative type. Such a relation is also refined for torsors under a torus. The equivalence between descent obstruction and étale Brauer-Manin obstruction for smooth projective varieties is extended to smooth quasi-projective varieties, which provides the perspective to study integral points.

1. Introduction

The descent theory for tori was first established by Colliot-Thélène and Sansuc in [8] and was extended by Skorobogatov to groups of multiplicative type in [34]. In a series of papers [21], [23], [25], Harari and Skorobogatov introduced descent obstruction for a general algebraic group and compared the descent obstruction with the Brauer-Manin obstruction. By various works of Poonen [31], the second named author [14], Stoll [38] and Skorobogatov [35], it was proved that the descent obstruction is equivalent to the étale Brauer-Manin obstruction for smooth projective geometrically integral varieties. In this paper, we study the relation between the descent obstruction and the Brauer-Manin obstruction for open varieties by using new arithmetic tools developed in [2], [6], [9], [17], [22] and [27], and we extend the equivalence between the descent obstruction and the étale Brauer-Manin obstruction to smooth *quasi-projective* varieties.

Let k be a number field, Ω_k the set of all primes of k and $\mathbf{A}_k$ the adelic ring of k. A variety over k is defined to be a separated scheme X of finite type over k. Fix an algebraic closure $\bar{k}$ of k. We denote by $X_{\bar{k}}$ the fibre product $X \times_k \bar{k}$. Let

$$\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m), \quad \mathrm{Br}_1(X) = \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{\bar{k}})) \quad \text{and} \quad \mathrm{Br}_0(X) = \mathrm{Im}(\mathrm{Br(k)} \xrightarrow{\pi^*} \mathrm{Br(X)})$$

where $X \xrightarrow{\pi} Spec(k)$ is the structure morphism, and $\mathrm{Br}_a(X) = \mathrm{Br}_1(X)/\mathrm{Br}_0(X)$. For any subgroup B of $\mathrm{Br}(X)$, one can define the Brauer-Manin set

$$X(\mathbf{A}_k)^B = \{(x_v)_{v\in\Omega_k} \in X(\mathbf{A}_k) : \sum_{v\in\Omega_k} \mathrm{inv}_v(\xi(x_v)) = 0 \text{ for all } \xi \in B\}$$

with respect to B. When $B = \mathrm{Br}(X)$, we simply write this Brauer-Manin set as $X(\mathbf{A}_k)^{\mathrm{Br}}$.

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Suppose $Y \xrightarrow{f} X$ is a left torsor under a linear algebraic group G over k. The descent obstruction (see [21], [23] and [25]) given by f is defined by the following set

$$X(\mathbf{A}_k)^f = \{(x_v) \in X(\mathbf{A}_k) : ([Y](x_v)) \in \mathrm{Im}(H^1(k,G) \to \prod_{v\in\Omega_k} H^1(k_v,G))\} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k))$$

where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of $Y \xrightarrow{f} X$ by a 1-cocycle representing $\sigma \in H^1(k,G)$. Moreover, one can define

$$X(\mathbf{A}_k)^{\mathrm{desc}} = \bigcap_{Y \xrightarrow{f} X} X(\mathbf{A}_k)^f$$

following [31], where $Y \xrightarrow{f} X$ runs through all torsors under all linear algebraic groups over k.

The main results in this paper are the following theorems.

Theorem 1.1. (Theorem 3.5) *Let k be a number field, G a connected linear algebraic group or a group of multiplicative type over k, and X a smooth and geometrically integral variety over k. Suppose $Y \xrightarrow{f} X$ is a left torsor under G. For any subgroup $A \subseteq \mathrm{Br}(X)$ which contains the kernel of the natural map $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ we have*

$$X(\mathbf{A}_k)^A = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)})$$

where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of $Y \xrightarrow{f} X$ by σ and $\mathrm{Br}(X) \xrightarrow{f_\sigma^*} \mathrm{Br}(Y^\sigma)$ is the associated pull-back map, for each $\sigma \in H^1(k,G)$.

When G is a torus, this theorem can be refined in order to get Theorem 4.1 in §4. In particular, we prove:

Theorem 1.2. (Corollary 4.3) *Under the same assumptions as in Theorem 1.1, if G is assumed to be a torus, then*

$$X(\mathbf{A}_k)^{\mathrm{Br}_1(X)} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}_1(Y^\sigma)})$$

and

$$X(\mathbf{A}_k)^{\mathrm{Br}} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}_1(Y^\sigma)+f_\sigma^*(\mathrm{Br}(X))}).$$

This result is inspired by some lectures by Yonatan Harpaz. It should be pointed out that the first part in Theorem 1.2 was first obtained by Dasheng Wei in [39]: his proof uses an argument of Harari and Skorobogatov in [26] together with an exact sequence due to Sansuc (see [2], Theorem 2.8). Theorem 1.2 can be applied to study strong approximation, as in [39]. It should be noted that in general, the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(Y^\sigma)$ in Theorem 1.1 and Theorem 1.2 is not easy to describe, even under the assumption $\bar{k}[X]^\times = \bar{k}^\times$ (see [24, Theorem 1.7(b)]).

Definition 1.3. *Let X be a variety over a number field k and let B be a subgroup of $\mathrm{Br}(X)$. For a finite subset S of Ω_k, we denote by $pr^S : X(\mathbf{A}_k) \to X(\mathbf{A}_k^S)$ the projection map, where $\mathbf{A}_k^S$ is the set of adeles of k without S-components.*

We say that X satisfies strong approximation off S *if $X(\mathbf{A}_k) \neq \emptyset$ and the diagonal image of $X(k)$ is dense in $pr^S(X(\mathbf{A}_k))$.*

We say that X satisfies strong approximation with respect to B off S *if $X(\mathbf{A}_k)^B \neq \emptyset$ and the diagonal image of $X(k)$ is dense in $pr^S(X(\mathbf{A}_k)^B)$.*

Corollary 3.20 in [17] provides a sufficient condition for strong approximation with Brauer-Manin obstruction to hold for a connected linear algebraic group. As an application of Theorem 1.2, we prove that this sufficient condition is also a necessary condition:

Theorem 1.4. (Corollary 5.3) *Let G be a connected linear algebraic group over a number field k and let S be a finite subset of Ω_k containing ∞_k. Then G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ off S if and only if $\prod_{v\in S} G'(k_v)$ is not compact for any non-trivial simple factor G' of the semi-simple part G^{ss} of G.*

For any variety X over a number field k, one can define, following [31]:

$$X(\mathbf{A}_k)^{\text{ét,Br}} = \bigcap_{Y \xrightarrow{f} X} \bigcup_{\sigma\in H^1(k,F)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}}),$$

where $Y \xrightarrow{f} X$ runs through all torsors under all finite group schemes F over k. The last two sections of the paper are devoted to the proof of the following generalization of [14] and [35]:

Theorem 1.5. (Corollary 6.7 and Theorem 7.6) *If X is a smooth quasi-projective and geometrically integral variety over a number field k, then*

$$X(\mathbf{A}_k)^{\mathrm{desc}} = X(\mathbf{A}_k)^{\text{ét,Br}}.$$

Terminology and notations are standard if not explained. For any connected linear algebraic group G over an field k of characteristic zero, the reductive part G^{red} of G is defined by the exact sequence

$$1 \to R_u(G) \to G \to G^{\mathrm{red}} \to 1$$

where $R_u(G)$ is the unipotent radical of G. The semi-simple part G^{ss} of G is defined to be the derived subgroup $[G^{\mathrm{red}}, G^{\mathrm{red}}]$, which is isogenous to the product of its simple factors, and the maximal toric quotient G^{tor} of G is defined to be $G^{\mathrm{red}}/[G^{\mathrm{red}}, G^{\mathrm{red}}]$. We use $\hat{G}$ for the character group of G. For a topological abelian group A, the topological dual of A is defined as $A^D = \mathrm{Hom}_{cont}(A, \mathbb{Q}/\mathbb{Z})$ with the compact-open topology. For any ring R, $R^\times$ stands for the group of invertible elements of R. For a number field k, we denote by ∞_k the set of all archimedean primes of k and by O_S the ring of S-integers, for any finite subset $S \subset \Omega_k$ containing ∞_k. For any $v \in \Omega_k$, k_v is the completion of k with respect to v, and if $v \in \Omega_k \setminus \infty_k$, O_v is the integral ring of k_v.

The paper is organized as follows. In §2, we establish some algebraic results over an arbitrary field of characteristic zero which we need in the next sections. Then we prove Theorem 1.1 in §3, Theorem 1.2 in §4. As an application of those results, we prove Theorem 1.4 in §5. Theorem 1.5 is proved in §6 and §7.

2. Brauer groups of torsors

In this section, we assume that k is an arbitrary field of characteristic 0.

Lemma 2.1. *Let H be a semi-simple simply connected group or a unipotent group over k. Suppose X is a smooth and geometrically integral variety over k. If $Z \xrightarrow{\rho} X$ is a torsor under H, then the induced map $\mathrm{Br}(X) \xrightarrow{\rho^*} \mathrm{Br}(Z)$ is an isomorphism.*

Proof. We first show that $\mathrm{Br}(X) \xrightarrow{\cong} \mathrm{Br}(X \times_k H)$, where the map is induced by the natural projection $X \times_k H \to X$. Using the spectral sequence

$$H^p(k, H^q(X_{\bar{k}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m),$$

one only needs to show that

$$\bar{k}[X_{\bar{k}}]^\times/\bar{k}^\times \xrightarrow{\cong} \bar{k}[X_{\bar{k}} \times_{\bar{k}} H_{\bar{k}}]^\times/\bar{k}^\times, \quad \mathrm{Pic}(X_{\bar{k}}) \xrightarrow{\cong} \mathrm{Pic}(X_{\bar{k}} \times_{\bar{k}} H_{\bar{k}}) \quad \text{and} \quad \mathrm{Br}(X_{\bar{k}}) \xrightarrow{\cong} \mathrm{Br}(X_{\bar{k}} \times_{\bar{k}} H_{\bar{k}}).$$

Since $\bar{k}[H]^\times = \bar{k}^\times$ and $\mathrm{Pic}(H_{\bar{k}}) = \mathrm{Br}(H_{\bar{k}}) = 0$ by [9, Proposition 2.6], the first two parts are true by [32, Proposition 6.10]. To prove the last part, Kummer exact sequence ensures that one only needs to prove that

$$H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}/n) \xrightarrow{\cong} H^2_{\text{ét}}(X_{\bar{k}} \times_{\bar{k}} H_{\bar{k}}, \mathbb{Z}/n) \tag{2.2}$$

for all $n \geq 1$. This last isomorphism follows from [37, Proposition 2.2] and [13, Exposé XI, Théorème 4.4] with $H^i_{\text{ét}}(H_{\bar{k}}, \mathbb{Z}/n) = 0$ for $i = 1, 2$. So we proved the required isomorphism $\mathrm{Br}(X) \xrightarrow{\cong} \mathrm{Br}(X \times_k H)$.

Let us now deduce Lemma 2.1: since $\mathrm{Pic}(H) = 0$, [2, Proposition 2.4] gives the following short exact sequence

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(Z) \xrightarrow{m^* - p_Z^*} \mathrm{Br}(H \times_k Z)\,,$$

where m^* and p_Z^* are induced by the multiplication map $H \times_k Z \xrightarrow{m} Z$ and the projection map $H \times_k Z \xrightarrow{p_Z} Z$ respectively. Since $m \circ (1_H \times \mathrm{id}) = p_Z \circ (1_H \times \mathrm{id}) = \mathrm{id}$, one concludes that $m^* = p_Z^*$ by the above argument. Therefore $\mathrm{Br}(X) \xrightarrow{\cong} \mathrm{Br}(Z)$. □

Let H be a closed subgroup of an algebraic group G over k, and $Y \xrightarrow{f} X$ be a left torsor under H. Let $Z \xrightarrow{\rho} X$ be the left torsor under G defined by the contracted product $Z = G \times^H Y$ (see [36, Example 3 in p.21]): the torsor Z is the push-forward of Y by the homomorphism $H \to G$. The projection map $G \times_k Y \xrightarrow{pr_G} G$ induces the following commutative diagram

$$\begin{array}{ccc} G \times_k Y & \longrightarrow & Z = G \times^H Y \\ {\scriptstyle pr_G}\downarrow & & \downarrow{\scriptstyle \theta} \\ G & \xrightarrow{\pi} & G/H\,, \end{array} \tag{2.3}$$

where θ is induced by pr_G via the quotient by H.

Lemma 2.4. *With the above notations, for any $\gamma \in (G/H)(k)$, the composite map $\theta^{-1}(\gamma) \to Z \xrightarrow{\rho} X$ is naturally a left torsor under H^σ, which is canonically isomorphic to the twist of $Y \xrightarrow{f} X$ by the k-torsor $\pi^{-1}(\gamma)$ under H.*

Proof. It follows from diagram (2.3) and [36, Example 2 in p.20]. □

Let G be a connected linear algebraic group over k, and Y be a smooth variety over k. Since $G_{\bar{k}}$ is rational over $\bar{k}$ by Bruhat decomposition, the projections $G \times_k Y \to G$ and $G \times_k Y \to Y$ induce an isomorphism

$$\mathrm{Br}_a(G) \oplus \mathrm{Br}_a(Y) \xrightarrow{\sim} \mathrm{Br}_a(G \times_k Y)$$

by [32, Lemma 6.6]. If P is a (left) torsor under G over k and $H^3(k, \bar{k}^\times) = 0$, the previous result generalizes to an isomorphism

$$\mathrm{Br}_a(P) \oplus \mathrm{Br}_a(Y) \xrightarrow{\sim} \mathrm{Br}_a(P \times Y) \tag{2.5}$$

by [3, Lemma 5.1].

Let G be a connected linear algebraic group over k and let X be a smooth variety over k with $H^3(k, \bar{k}^\times) = 0$. Suppose that $Y \xrightarrow{f} X$ is a left torsor under G and P is a left k-torsor under G, associated to a cocycle $\sigma \in Z^1(k, G)$. One can consider P as a right torsor under G by defining a right action $x \circ g := g^{-1}x$ (see [36, Example 2 in p.20]). This right torsor is called the inverse right torsor of P under G, and is denoted by P'. One can now consider the map given by the quotient of $P \times_k Y$ by the diagonal action of G given by $g \cdot (p, y) := (p \circ g^{-1}, g \cdot y) = (g \cdot p, g \cdot y)$:

$$\chi_P : P \times_k Y \to Y^\sigma := P' \times^G Y\,.$$

Definition 2.6. *With the above notation, assuming that $H^3(k, \bar{k}^\times) = 0$, consider the map*

$$\psi_\sigma = \psi_P : \mathrm{Br}_a(Y^\sigma) \xrightarrow{\chi_P^*} \mathrm{Br}_a(P \times_k Y) \xleftarrow{\sim} \mathrm{Br}_a(P) \oplus \mathrm{Br}_a(Y) \to \mathrm{Br}_a(Y)\,.$$

The following lemma, which compares the algebraic Brauer groups of twists of a given torsor, can be regarded as an extension of [39, Lemma 1.3] to torsors under connected linear algebraic groups.

Lemma 2.7. *The morphism ψ_σ in Definition 2.6 is an isomorphism.*

Proof. The natural morphism $(pr_P, \chi_P) : P \times_k Y \to P \times_k Y^\sigma$ is an isomorphism, and we have a commutative diagram:

$$\begin{array}{ccc} P \times_k Y & \xrightarrow{(pr_P, \chi_P)} & P \times_k Y^\sigma \\ {\scriptstyle pr_P}\big\downarrow & & \big\downarrow{\scriptstyle pr_P} \\ P & \xrightarrow[\mathrm{id}]{} & P\,. \end{array}$$

Therefore $(pr_P, \chi_P)^* : \mathrm{Br}_a(Y^\sigma \times_k P) \to \mathrm{Br}_a(Y \times P)$ induces the identity map on the subgroups $\mathrm{Br}_a(P) \subset \mathrm{Br}_1(Y^\sigma \times_k P)$ and $\mathrm{Br}_a(P) \subset \mathrm{Br}_1(Y \times_k P)$, hence

$$\psi_\sigma : \mathrm{Br}_a(Y^\sigma) \to \mathrm{Br}_a(Y^\sigma \times_k P) \xrightarrow{(pr_P, \chi_P)^*} \mathrm{Br}_a(Y \times P) \to \mathrm{Br}_a(Y)$$

is an isomorphism (using the isomorphism (2.5)). □

Let $f : Y \to X$ be a torsor under a connected linear algebraic group G over k and let

$$a_Y : \ G \times_k Y \to Y$$

be the action of G. There is a canonical map $\lambda : \mathrm{Br}_1(Y) \to \mathrm{Br}_a(G)$ by [32, Lemma 6.4]. Let $e : \mathrm{Br}_a(G) \to \mathrm{Br}_1(G)$ be the section of $\mathrm{Br}_1(G) \to \mathrm{Br}_a(G)$ such that $1_G^* \circ e = 0$. If X is smooth and geometrically integral, then the following diagram

$$\begin{array}{ccc} \mathrm{Br}_1(Y) & \xrightarrow{\lambda} & \mathrm{Br}_a(G) \\ \downarrow & & \downarrow{\scriptstyle p_G^* \circ e} \\ \mathrm{Br}(Y) & \xrightarrow[a_Y^* - p_Y^*]{} & \mathrm{Br}(G \times_k Y) \end{array} \tag{2.8}$$

commutes by [2, Theorem 2.8], where $G \times_k Y \xrightarrow{p_G} G$ and $G \times_k Y \xrightarrow{p_Y} Y$ are the projections. One can reformulate the commutative diagram (2.8) in the following proposition:

Proposition 2.9. *With the above notation, one has*

$$b(t \cdot x) = \lambda(b)(t) + b(x)$$

for any $x \in Y(k)$, $t \in G(k)$ and $b \in \mathrm{Br}_1(Y)$.

Proof. The commutativity of diagram (2.8) implies that

$$a_Y^* - p_Y^* = p_G^* \circ e \circ \lambda : \ \mathrm{Br}_1(Y) \to \mathrm{Br}_1(G \times Y)\,,$$

therefore one has

$$b(t \cdot x) = a_Y^*(b)(t, x) = p_Y^*(b)(t, x) + p_G^* \circ e \circ \lambda(b)(t, x) = b(x) + \lambda(b)(t)$$

as required. □

3. Connected linear algebraic groups or groups of multiplicative type

In this section, we study the relation between the descent obstruction and the Brauer-Manin obstruction for a general connected linear group or a group of multiplicative type.

First we need the following fact concerning topological groups:

Lemma 3.1. *Let $f : M \to N$ be an open homomorphism of topological groups. If K is a closed subgroup of M containing $\ker(f)$, then $f(K)$ is a closed subgroup of N.*

Proof. Since K is a closed subgroup containing $\ker(f)$, one has

$$f(K) = f(M) \setminus f(M \setminus K).$$

Since f is an open homomorphism, $f(M)$ is an open subgroup of N. This implies that $f(M)$ is closed in N. Since $f(M \setminus K)$ is open in N, one concludes that $f(K)$ is closed in N. □

Remark 3.2. *The assumption $K \supseteq \ker(f)$ in Lemma 3.1 can not be removed. For example, the projection map $pr^S : \mathbf{A}_k \to \mathbf{A}_k^S$ is open where $\mathbf{A}_k^S$ is the set of adeles of k without S-component. It is clear that k is a discrete subgroup of $\mathbf{A}_k$ by the product formula. However k is dense in $\mathbf{A}_k^S$ by strong approximation for $\mathbb{G}_a$, when S is not empty.*

For a short exact sequence of connected linear algebraic groups, one has the following result.

Proposition 3.3. *Let*

$$1 \to G_1 \xrightarrow{\psi} G_2 \xrightarrow{\phi} G_3 \to 1$$

be a short exact sequence of connected linear algebraic groups over a number field k. Then

(1) $\phi\left(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)}\right)$ is a closed subgroup of $G_3(\mathbf{A}_k)$.

(2) If $G'(k_\infty)$ is not compact for each simple factor G' of the semi-simple part of G_3, then one has

$$G_3(\mathbf{A}_k)^{\mathrm{Br}_1(G_3)} = G_3(k) \cdot \phi\left(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)}\right).$$

Proof. Let S be a sufficiently large finite set of primes of Ω_k containing ∞_k and let $\mathbf{G}_1$ (resp. $\mathbf{G}_2$, resp. $\mathbf{G}_3$) be a smooth group scheme model of G_1 (resp. G_2, resp. G_3) over O_S with connected fibres, such that the short exact sequence of smooth group schemes

$$1 \to \mathbf{G}_1 \xrightarrow{\psi} \mathbf{G}_2 \xrightarrow{\phi} \mathbf{G}_3 \to 1$$

extends the given short exact sequence of their generic fibres. The set $H^1_{\mathrm{et}}(O_v, \mathbf{G}_1)$ is trivial by Hensel's lemma together with Lang's theorem, and the following diagram

$$\begin{array}{ccc} \mathbf{G}_3(O_v) & \xrightarrow{\partial_v} & H^1_{\mathrm{et}}(O_v, \mathbf{G}_3) \\ \downarrow & & \downarrow \\ G_3(k_v) & \xrightarrow[\partial_v]{} & H^1(k_v, G_3) \end{array}$$

commutes, hence we deduce the following commutative diagram of exact sequences in Galois cohomology:

$$\begin{array}{ccccccc} G_1(k) & \xrightarrow{\psi} & G_2(k) & \xrightarrow{\phi} & G_3(k) & \xrightarrow{\partial} & H^1(k, G_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G_1(\mathbf{A}_k) & \xrightarrow[(\psi_v)]{} & G_2(\mathbf{A}_k) & \xrightarrow[(\phi_v)]{} & G_3(\mathbf{A}_k) & \xrightarrow[(\partial_v)]{} & \bigoplus_{v\in\Omega_k} H^1(k_v, G_1)\,. \end{array}$$

In addition, [17, Theorem 5.1] and [32, Corollary 6.11] gives the following commutative diagram of exact sequences of topological groups and pointed topological spaces:

$$\begin{array}{ccccccccc} & & & & G_1(\mathbf{A}_k) & \xrightarrow{\theta_1} & \mathrm{Br}_a(G_1)^D & \longrightarrow & Ш^1(k, G_1) \\ & & & & \downarrow{\scriptstyle (\psi_v)} & & \downarrow{\scriptstyle (\psi^*)^D} & & \\ 1 & \longrightarrow & \ker(\theta_2) & \longrightarrow & G_2(\mathbf{A}_k) & \xrightarrow{\theta_2} & \mathrm{Br}_a(G_2)^D & & \\ & & \downarrow & & \downarrow{\scriptstyle (\phi_v)} & & \downarrow{\scriptstyle (\phi^*)^D} & & \\ 1 & \longrightarrow & \ker(\theta_3) & \longrightarrow & G_3(\mathbf{A}_k) & \xrightarrow{\theta_3} & \mathrm{Br}_a(G_3)^D & & \\ & & & & \downarrow{\scriptstyle (\partial_v)} & & & & \\ & & & & \bigoplus_{v\in\Omega_k} H^1(k_v, G_1)\,, & & & & \end{array} \tag{3.4}$$

where $\mathrm{Br}_a(G_i)^D$ is the topological dual of the discrete group $\mathrm{Br}_a(G_i)$, for $1 \leq i \leq 3$. Since $\theta_1(G_1(\mathbf{A}_k))$ is the kernel of the continuous map $\mathrm{Br}_a(G_1)^D \to Ш^1(k, G_1)$, it is a closed subgroup of $\mathrm{Br}_1(G)^D$. Since $(\psi^*)^D$ is a closed map, one obtains that $(\psi^*)^D(\theta_1(G_1(\mathbf{A}_k))$ is a closed subgroup of $\mathrm{Br}_1(G_2)^D$. It implies that

$$\ker(\theta_2) \cdot \psi(G_1(\mathbf{A}_k)) = \theta_2^{-1}\left[(\psi^*)^D(\theta_1(G_1(\mathbf{A}_k))\right]$$

is a closed subgroup of $G_2(\mathbf{A}_k)$ by diagram (3.4). Proposition 6.5 in Chapter 6 of [30] ensures that $\phi : G_2(\mathbf{A}_k) \to G_3(\mathbf{A}_k)$ is an open homomorphism of topological groups. Then $\phi(\ker(\theta_2)) = \phi\left(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)}\right)$ is closed by Lemma 3.1, and property (1) follows.

Let us now prove statement (2): Corollary 3.20 in [17] (see also the proof of Proposition 4.5 in [5]) implies that

$$\ker(\theta_3) = G_3(\mathbf{A}_k)^{\mathrm{Br}_1(G_3)} = \overline{G_3(k) \cdot G_3(k_\infty)^0}\,,$$

where $G_3(k_\infty)^0$ is the connected component of identity with respect to the topology of k_∞. One only needs to show that

$$G_3(\mathbf{A}_k)^{\mathrm{Br}_1(G_3)} \subseteq G_3(k) \cdot \phi\left(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)}\right)\,.$$

For any $(x_v) \in \overline{G_3(k) \cdot G_3(k_\infty)^0}$, there is $h \in G_3(k)$ and $h_\infty \in G_3(k_\infty)$ such that

$$(\partial_v)(h \cdot h_\infty) = (\partial_v)(x_v)\,,$$

because (∂_v) is a continuous map with respect to the discrete topology of $\bigoplus_{v\in\Omega_k} H^1(k_v, G_1)$. Since $\phi_\infty(G_2(k_\infty)^0)$ is open and connected, the finiteness of $H^1(k_\infty, G_1)$ gives

$$G_3(k_\infty)^0 = \phi_\infty(G_2(k_\infty)^0)\,.$$

Therefore

$$(h \cdot h_\infty) \in G_3(k) \cdot \phi\left(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)}\right)$$

and one can replace (x_v) by $(h \cdot h_\infty)^{-1} \cdot (x_v)$. Without loss of generality, one can therefore assume $(\partial_v)(x_v)$ is the trivial element in $\bigoplus_{v\in\Omega_k} H^1(k_v, G_1)$.

Since $Ш^1(k, G_1)$ is finite, one can fix $\xi_1, \cdots, \xi_n$ in $G_3(k)$ such that each element of $Ш^1(k, G_1) \cap \partial(G_3(k))$ is represented by one of the ξ_i's. As $\partial_\infty(h_\infty)$ is trivial for any $h_\infty \in G_3(k_\infty)^0$, one concludes that

$$(x_v) \in \overline{\bigcup_{i=1}^n \xi_i \phi(\ker(\theta_2))} = \bigcup_{i=1}^n \xi_i \cdot \overline{\phi(\ker(\theta_2))} \subseteq G_3(k) \cdot \phi\left(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)}\right)$$

by Corollary 1 in Page 50 of [33] and assertion (1). □

The main result of this section is the following theorem:

Theorem 3.5. *Let X be a smooth and geometrically integral variety and let G be a connected linear algebraic group or a group of multiplicative type over a number field k. Suppose that $f : Y \to X$ is a left torsor under G. If A is a subgroup of* $\mathrm{Br}(X)$ *which contains the kernel of the natural map $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$, then*

$$X(\mathbf{A}_k)^A = \bigcup_{\sigma\in H^1(k,G)} f_\sigma\left(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)}\right)\,,$$

where $Y^\sigma \xrightarrow{f_\sigma} X$ *is the twist of* f *by* σ *and* $\mathrm{Br}(X) \xrightarrow{f_\sigma^*} \mathrm{Br}(Y^\sigma)$ *is the associated pull-back morphism, for each* $\sigma \in H^1(k,G)$.

Proof. By the functoriality of Brauer-Manin pairing, one only needs to show that

$$X(\mathbf{A}_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma\left(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)}\right) .$$

It is clear that

$$(x_v) \in \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)) \quad \Leftrightarrow \quad ([Y](x_v)) \in \mathrm{Im}\left[H^1(k,G) \to \prod_{v \in \Omega_k} H^1(k_v,G)\right] . \tag{3.6}$$

(1) Assume that G is connected.

Recall first that Hensel's lemma together with Lang's theorem ensures that $H^1(k,G)$ maps to $\bigoplus_{v\in\Omega_k} H^1(k_v,G)$. Since any element $P \in \mathrm{Pic}(G)$ can be given the structure of a central extension of algebraic groups

$$1 \to \mathbb{G}_m \to P \to G \to 1 \tag{3.7}$$

by [6, Corollary 5.7], one obtains a coboundary map

$$\partial_P : \quad H^1(X,G) \to H^2(X,\mathbb{G}_m) = \mathrm{Br}(X)$$

associated to P (see [19, IV.4.4.2]). Then the map defined by

$$\Delta_{Y/X} : \mathrm{Pic}(G) \to \mathrm{Br}(X), \quad P \mapsto \partial_P([Y])$$

appears in the following short exact sequence (see [2, Theorem 2.8])

$$\mathrm{Pic}(G) \xrightarrow{\Delta_{X/Y}} \mathrm{Br}(X) \xrightarrow{f^*} \mathrm{Br}(Y) . \tag{3.8}$$

For any $v \in \Omega_k$, the exact sequence (3.7) defines a coboundary map

$$\partial_P^{k_v} : \quad H^1(k_v,G) \to H^2(k_v,\mathbb{G}_m) = \mathrm{Br}(k_v) .$$

One can therefore define a pairing

$$\delta_v : \ H^1(k_v,G) \times \mathrm{Pic}(G) \to \mathrm{Br}(k_v) \subseteq \mathbb{Q}/\mathbb{Z}, \quad (\sigma_v, P) \mapsto \partial_P^{k_v}(\sigma_v)$$

such that the following diagram

$$\begin{array}{ccccc} X(k_v) & \times & \mathrm{Br}(X) & \xrightarrow{ev} & \mathrm{Br}(k_v) \\ \downarrow{\scriptstyle [Y]} & & \uparrow{\scriptstyle \Delta_{X/Y}} & & \downarrow{\scriptstyle \mathrm{id}} \\ H^1(k_v,G) & \times & \mathrm{Pic}(G) & \xrightarrow{\delta_v} & \mathrm{Br}(k_v) \end{array} \tag{3.9}$$

commutes (see Proposition 2.9 in [9]). These pairings induce a pairing

$$(\delta_v)_{v\in\Omega_k} : \quad \bigoplus_{v\in\Omega_k} H^1(k_v,G) \times \mathrm{Pic}(G) \to \mathbb{Q}/\mathbb{Z}, \quad ((\sigma_v)_{v\in\Omega_k}, P) \mapsto \sum_{v\in\Omega_k} \delta_v(\sigma_v,P) \in \mathbb{Q}/\mathbb{Z}$$

and a natural exact sequence of pointed sets

$$H^1(k,G) \to \bigoplus_{v\in\Omega_k} H^1(k_v,G) \to \mathrm{Hom}(\mathrm{Pic}(G),\mathbb{Q}/\mathbb{Z})$$

by [9, Theorem 3.1]. Therefore (3.6) is equivalent to the fact that $([Y](x_v)) \in \bigoplus_{v\in\Omega_k} H^1(k_v,G)$ is orthogonal to $\mathrm{Pic}(G)$ for the pairing $(\delta_v)_{v\in\Omega_k}$. The commutative diagram (3.9), together with (3.8), gives

$$X(\mathbf{A}_k)^{\ker(f^*)} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)).$$

Since $\ker(f^*) \subseteq A$, one has

$$X(\mathbf{A}_k)^A \subseteq X(\mathbf{A}_k)^{\ker(f^*)} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)).$$

Then the functoriality of the Brauer-Manin pairing implies that

$$X(\mathbf{A}_k)^A \subseteq \bigcup_{\sigma\in H^1(k,G)} f_\sigma\left(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)}\right).$$

(2) When G is a group of multiplicative type, one obtains that (3.6) is equivalent to

$$\sum_{v\in\Omega_k} \mathrm{inv}_v(\chi\cup[Y])(x_v) = 0$$

for all $\chi \in H^1(k,\hat{G})$ by [16, Theorem 6.3]. Let

$$\mathcal{K}_f = \langle\{\chi\cup[Y] : \ \chi\in H^1(k,\hat{G})\}\rangle$$

be the subgroup of $\mathrm{Br}(X)$ generated by elements $\chi\cup[Y]$, where $\cup$ is the cup product

$$\cup : H^1(k,\hat{G}) \times H^1(X,G) \to H^2(X,\mathbb{G}_m) = \mathrm{Br}(X).$$

Then

$$X(\mathbf{A}_k)^{\mathcal{K}_f} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k))$$

by [26, Proposition 3.1]. Functoriality of the cup product proves that the following diagram

$$\begin{array}{ccc} H^1(k,\hat{G}) \times H^1(X,G) & \xrightarrow{\cup} & H^2(X,\mathbb{G}_m) = \mathrm{Br}(X) \\ {\scriptstyle \mathrm{id}\times f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^1(k,\hat{G}) \times H^1(Y,G) & \xrightarrow{\cup} & H^2(Y,\mathbb{G}_m) = \mathrm{Br}(Y) \end{array}$$

is commutative. Since $Y \xrightarrow{f} X$ becomes a trivial torsor over Y, the above diagram gives $\mathcal{K}_f \subseteq \ker(f^*)$. Since $\mathcal{K}_f \subseteq \ker(f^*) \subseteq A$, one has

$$X(\mathbf{A}_k)^A \subseteq X(\mathbf{A}_k)^{\mathcal{K}_f} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)).$$

Then the functoriality of the Brauer-Manin pairing implies that

$$X(\mathbf{A}_k)^A \subseteq \bigcup_{\sigma\in H^1(k,G)} f_\sigma\left(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)}\right).$$

□

4. Refinement in the toric case

In this section, we will refine Theorem 3.5 for torsors under tori.

Theorem 4.1. *Let $f : Y \to X$ be a torsor under a torus G over a number field k. Assume that X is smooth and geometrically integral. Let* $\ker(f^*) \subseteq A \subseteq \mathrm{Br}(X)$ *be a subgroup, and for all $\sigma \in H^1(k,G)$, let $B_\sigma \subseteq \mathrm{Br}_1(Y^\sigma)$ be a subgroup such that*

$$f^{*-1}\left(\sum_{\sigma\in H^1(k,G)} \psi_\sigma(\widetilde{B_\sigma})\right) \subseteq A\,,$$

where $\mathrm{Br}_a(Y^\sigma) \xrightarrow{\psi_\sigma} \mathrm{Br}_a(Y)$ *is the morphism of Definition 2.6 and $\widetilde{B_\sigma}$ is the image of B_σ in* $\mathrm{Br}_a(Y^\sigma)$.

Then one has

$$X(\mathbf{A}_k)^A = \bigcup_{\sigma\in H^1(k,G)} f_\sigma\left(Y^\sigma(\mathbf{A}_k)^{B_\sigma+f_\sigma^*(A)}\right)$$

where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of $Y \xrightarrow{f} X$ by σ.

Proof. Since

$$\bigcup_{\sigma\in H^1(k,G)} f_\sigma\left(Y^\sigma(\mathbf{A}_k)^{B_\sigma+f_\sigma^*(A)}\right) \subseteq \bigcup_{\sigma\in H^1(k,G)} f_\sigma\left(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)}\right) \subseteq X(\mathbf{A}_k)^A$$

by the functoriality of Brauer-Manin pairing, one only needs to prove the converse inclusion.

Step 1. We first prove the result when $\hat{G}$ is a permutation Galois module. In this case, Shapiro Lemma and Hilbert 90 gives $H^1(K,G)=\{1\}$ for any field extension K/k. This implies that

$$X(\mathbf{A}_k)^A = f\left(Y(\mathbf{A}_k)^{f^*(A)}\right)$$

by the functoriality of Brauer-Manin pairing.

Let $(x_v) \in X(\mathbf{A}_k)^A$. Then there is $(y_v) \in Y(\mathbf{A}_k)^{f^*(A)}$ such that $(x_v) = f((y_v))$.

By Proposition 6.10 (6.10.3) in [32], the natural sequence

$$\mathrm{Br}_1(X) \xrightarrow{f^*} \mathrm{Br}_1(Y) \xrightarrow{\lambda} \mathrm{Br}_a(G)$$

is exact, and it induces the exact sequence

$$(f^*)^{-1}(B) \xrightarrow{f^*} B \xrightarrow{\lambda} \mathrm{Br}_a(G)$$

for any subgroup $B \subseteq \mathrm{Br}_1(Y)$. Therefore the following sequence

$$\mathrm{Br}_a(G)^D \xrightarrow{\lambda^D} B^D \xrightarrow{(f^*)^D} ((f^*)^{-1}(B))^D$$

is exact. Assuming $(f^*)^{-1}(B) \subseteq A$, one has $(f^*)^D((y_v)) = 0$, where we (abusively) identify (y_v) with its image in B^D via the Brauer-Manin pairing. By the aforementioned exactness, there is $\xi \in \mathrm{Br}_a(G)^D$ such that $\lambda^D(\xi) = (y_v)$. Since $\text{Ш}^1(k, G) = \{1\}$, Theorem 2 in [22] implies that every element in $\mathrm{Br}_a(G)^D$ is given by an element in $G(\mathbf{A}_k)$ via the Brauer-Manin pairing. Namely, there is $(g_v) \in G(\mathbf{A}_k)$ such that

$$b(y_v) = \lambda(b)(g_v)$$

for all $b \in B$. Then $(g_v)^{-1} \cdot (y_v) \in Y(\mathbf{A}_k)^{B+f^*(A)}$ by Proposition 2.9, and $(x_v) = f((g_v)^{-1} \cdot (y_v))$.

Step 2. We now prove the case of an arbitrary torus G. By Proposition-Definition 3.1 in [6], there is a short exact sequence of tori

$$1 \to G \to T_0 \xrightarrow{q} T_1 \to 1\,,$$

such that $\hat{T_0}$ is a permutation Galois module and $\hat{T_1}$ is a coflasque Galois module. Since

$$H^3(k, \hat{T_1}) \cong \prod_{v \in \infty_k} H^3(k_v, \hat{T_1}) \cong \prod_{v \in \infty_k} H^1(k_v, \hat{T_1}) = \{1\}$$

(see for instance Proposition 5.9 in [27]), the map $\mathrm{Br}_1(T_0) \to \mathrm{Br}_1(G)$ is surjective.

Let $Z \xrightarrow{\rho} X$ be the torsor under T_0 defined by $Z := T_0 \times^G Y$. We have a morphism of torsors under G:

$$\begin{array}{ccccc} Y & \xrightarrow{e_0 \times \mathrm{id}_Y} & T_0 \times_k Y & \xrightarrow{\chi} & Z = T_0 \times^G Y \\ & & {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle \theta} \\ & & T_0 & \xrightarrow[q]{} & T_1 \end{array}$$

where $e_0 \in T_0(k)$ is the unit element, p_0 is the projection map and θ is given as in (2.3). For simplicity, denote by $i := \chi \circ (e_0 \times \mathrm{id}_Y) : Y \to Z$ the composite morphism defined in the previous diagram.

Then Proposition 6.10 (6.10.3) in [32] gives the following commutative diagram of exact sequences:

$$\begin{array}{ccccc} \mathrm{Br}_1(T_1) & \xrightarrow{q^*} & \mathrm{Br}_1(T_0) & \longrightarrow & \mathrm{Br}_a(G) \\ {\scriptstyle \theta^*}\downarrow & & \downarrow{\scriptstyle p_0^*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Br}_1(Z) & \xrightarrow[\chi^*]{} & \mathrm{Br}_1(T_0 \times_k Y) & \longrightarrow & \mathrm{Br}_a(G)\,. \end{array}$$

Since the following sequence

$$\mathrm{Br}_1(T_0) \xrightarrow{p_0^*} \mathrm{Br}_1(T_0 \times_k Y) \xrightarrow{(e_0 \times \mathrm{id}_Y)^*} \mathrm{Br}_a(Y) \to 1$$

is exact by Lemma 6.6 in [32], the surjectivity of the map $\mathrm{Br}_1(T_0) \to \mathrm{Br}_1(G)$ implies that the morphism

$$i^* : \ \mathrm{Br}_1(Z) \to \mathrm{Br}_1(Y)$$

is surjective, by a simple diagram chase.

Lemma 2.4 implies that for any $t \in T_1(k)$, the composite morphism $\theta^{-1}(t) \to Z \xrightarrow{\rho} X$ is canonically isomorphic to the twist $f_t : Y^{q^{-1}(t)} \to X$ of $f : Y \to X$ by the $\mathrm{Spec}(k)$-torsor $q^{-1}(t)$ under G.

Denote by $i_t : \theta^{-1}(t) \to Z$ the closed immersion. Then $f_t = \rho \circ i_t$ for any $t \in T_1(k)$.

Let χ_t be the restriction of χ to $q^{-1}(t) \times_k Y$ for any $t \in T_1(k)$. Then the following diagram

$$\begin{array}{ccccc}
 & & q^{-1}(t) \times_k Y & \xrightarrow{\chi_t} & Y^{q^{-1}(t)} \\
 & & \downarrow{\scriptstyle j_t \times \mathrm{id}_Y} & & \downarrow{\scriptstyle i_t} \\
Y & \xrightarrow{e_0 \times \mathrm{id}_Y} & T_0 \times_k Y & \xrightarrow{\chi} & Z \\
 & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle \theta} \\
G & \longrightarrow & T_0 & \xrightarrow{q} & T_1
\end{array}$$

is commutative, where $j_t : q^{-1}(t) \to T_0$ is the closed immersion of the fiber of q at t. Therefore Definition 2.6 implies that we have a commutative triangle:

$$\begin{array}{ccc}
\mathrm{Br}_a(Z) & \xrightarrow{i_t^*} & \mathrm{Br}_a(Y^{q^{-1}(t)}) \\
 & {\scriptstyle i^*}\searrow & \sim\downarrow{\scriptstyle \psi_{q^{-1}(t)}} \\
 & & \mathrm{Br}_a(Y)\,,
\end{array}$$

i.e. that $\psi_{q^{-1}(t)} \circ i_t^* = i^*$.

Let

$$B = i^{*-1}\left(\sum_{t \in T_1(k)} \psi_{q^{-1}(t)}\left(\widetilde{B_{q^{-1}(t)}}\right)\right) \subset \mathrm{Br}_a(Y)$$

where $\widetilde{B_{q^{-1}(t)}}$ is the image of $B_{q^{-1}(t)}$ in $\mathrm{Br}_a(Y^{q^{-1}(t)})$ and $\psi_{q^{-1}(t)}$ is given by Definition 2.6 for all $t \in T_1(k)$.

Since $i^* \circ \rho^* = f^*$, we have

$$\rho^{*-1}(B) = f^{*-1}\left(\sum_{t \in T_1(k)} \psi_{q^{-1}(t)}\left(\widetilde{B_{q^{-1}(t)}}\right)\right) \subseteq A\,,$$

hence step 1 applied to the torsor $Z \xrightarrow{\rho} X$ under T_0 implies that

$$X(\mathbf{A}_k)^A = \rho\left(Z(\mathbf{A}_k)^{B+\rho^*(A)}\right)\,. \tag{4.2}$$

Let $(x_v) \in X(\mathbf{A}_k)^A$. By (4.2), there is $(z_v) \in Z(\mathbf{A}_k)^{B+\rho^*(A)}$ such that $(x_v) = \rho((z_v))$. Since

$$i^* \circ \theta^*(\mathrm{Br}_1(T_1)) = (e_0 \times \mathrm{id}_Y)^* \circ p_0^* \circ q^*(\mathrm{Br}_1(T_1)) = \mathrm{Br}_0(Y)$$

and $i^*(\mathrm{Br}_0(Z)) = \mathrm{Br}_0(Y)$, one gets $\theta^*(\mathrm{Br}_1(T_1)) \subseteq \mathrm{Br}_0(Z) + B$ (by construction, B contains $\ker(i^* : \mathrm{Br}_1(Z) \to \mathrm{Br}_1(Y))$). Functoriality of the Brauer-Manin pairing now gives

$$\theta((z_v)) \in T_1(\mathbf{A}_k)^{\mathrm{Br}_1(T_1)}\,.$$

By Proposition 3.3, there are $\alpha \in T_1(k)$ and $(\beta_v) \in T_0(\mathbf{A}_k)^{\mathrm{Br}_1(T_0)}$ such that $\theta((z_v)) = \alpha \cdot q(\beta_v)$.

Therefore $(\beta_v)^{-1} \cdot (z_v) \in \theta^{-1}(\alpha)$, hence $(\beta_v)^{-1} \cdot (z_v) \in Z(\mathbf{A}_k)^{B+\rho^*(A)}$.

Since $i^* : \mathrm{Br}_1(Z) \to \mathrm{Br}_1(Y)$ is surjective, one has

$$\psi_{q^{-1}(\alpha)} \circ i_\alpha^*(\widetilde{B}) = i^*(\widetilde{B}) = \sum_{t \in T_1(k)} \psi_{q^{-1}(t)} \left(\widetilde{B_{q^{-1}}}\right) \supseteq \psi_{q^{-1}(\alpha)} \left(\widetilde{B_{q^{-1}(\alpha)}}\right),$$

where $\widetilde{B}$ is the image of B in $\mathrm{Br}_a(Z)$. It implies that $i_\alpha^*(B) + \mathrm{Br}_0(\theta^{-1}(\alpha)) \supseteq B_{q^{-1}(\alpha)}$ by Lemma 2.7, and

$$(\beta_v)^{-1} \cdot (z_v) \in \left[\theta^{-1}(\alpha)(\mathbf{A}_k)\right]^{i_\alpha^*(B)+(i_\alpha^* \circ \rho^*)(A)} \subseteq \left[\theta^{-1}(\alpha)(\mathbf{A}_k)\right]^{B_{q^{-1}(\alpha)}+(i_\alpha^* \circ \rho^*)(A)}$$

as desired. □

The first part of the following result is also proved in Theorem 1.7 of [39].

Corollary 4.3. *Let X be a smooth and geometrically integral variety. If $f : Y \to X$ is a torsor under a torus G over a number field k, then*

$$X(\mathbf{A}_k)^{\mathrm{Br}_1(X)} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma \left(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}_1(Y^\sigma)}\right)$$

and

$$X(\mathbf{A}_k)^{\mathrm{Br}} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma \left(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}_1(Y^\sigma)+f_\sigma^*(\mathrm{Br}(X))}\right).$$

Proof. To get the first equality, apply Theorem 4.1 to $A = \mathrm{Br}_1(X)$ and $B_\sigma = \mathrm{Br}_1(Y^\sigma)$ for each $\sigma \in H^1(k, G)$. Since $\mathrm{Pic}(G_{\bar{k}}) = 0$, Proposition 6.10 in [32] gives

$$f^{*-1}\left(\sum_{\sigma \in H^1(k,G)} \psi_\sigma \left(\widetilde{B_\sigma}\right)\right) \subseteq f^{*-1}(\mathrm{Br}_a(Y)) \subseteq \mathrm{Br}_1(X) = A\,,$$

as required.

The second equality follows from Theorem 4.1 by taking $A = \mathrm{Br}(X)$ and $B_\sigma = \mathrm{Br}_1(Y^\sigma)$ for each $\sigma \in H^1(k, G)$. □

5. An application

In this section, we apply the previous results to study the necessary conditions for a connected linear algebraic group to satisfy strong approximation with Brauer-Manin obstruction.

When X is affine, the set $X(k)$ is discrete in $X(\mathbf{A}_k)$ by the product formula. Therefore if such an X satisfies strong approximation off S, then $\prod_{v \in S} X(k_v)$ is not compact. However this necessary condition for strong approximation is no longer true for strong approximation with Brauer-Manin obstruction if $\mathrm{Br}(X)/\mathrm{Br}(k)$ is not finite. For example, a torus X always satisfies strong approximation with Brauer-Manin obstruction off ∞_k, X being anisotropic over k_∞ or not: see [22, Theorem 2]. When X is a semi-simple linear algebraic group, the necessary and sufficient condition for X to satisfy strong approximation with Brauer-Manin obstruction is given by Proposition 6.1 in [5]. In this section, we extend this result to a general connected linear algebraic group.

The following lemma explains that strong approximation with Brauer-Manin obstruction for a general connected linear algebraic group can be reduced to the reductive case.

Lemma 5.1. *Let G be a connected linear algebraic group over a number field k.*

If $\pi : G \to G^{\rm red}$ is the quotient map, then $G^{\rm red}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\rm red})} = \pi\left(G(\mathbf{A}_k)^{\mathrm{Br}_1(G)}\right)$.

In particular, for any finite subset S of Ω_k, G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ off S if and only if $G^{\rm red}$ satisfies strong approximation with respect to $\mathrm{Br}_1(G^{\rm red})$ off S.

Proof. By applying Lemma 2.1 for k and $\bar{k}$, one obtains that $\pi^*(\mathrm{Br}_1(G^{\rm red})) = \mathrm{Br}_1(G)$. The first part follows from Theorem 3.5 and Proposition 6 of §2.1 of Chapter III in [33].

Suppose G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ off S. For any open subset

$$M = \prod_{v\in S} G^{\rm red}(k_v) \times \prod_{v\notin S} M_v$$

of $G^{\rm red}(\mathbf{A}_k)$ such that $M \cap \left[G^{\rm red}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\rm red})}\right] \neq \emptyset$, one has that

$$\pi^{-1}(M) = \prod_{v\in S} G(k_v) \times \prod_{v\notin S} \pi^{-1}(M_v)$$

with $\pi^{-1}(M) \cap G(\mathbf{A}_k)^{\mathrm{Br}_1(G)} \neq \emptyset$ by the first part. Then by assumption there is $x \in G(k) \cap \pi^{-1}(M)$. It implies that $\pi(x) \in M \cap G^{\rm red}(k)$, as required.

Conversely, suppose $G^{\rm red}$ satisfies strong approximation with respect to $\mathrm{Br}_1(G^{\rm red})$ off S. For any open subset

$$N = \prod_{v\in S} G(k_v) \times \prod_{v\notin S} N_v$$

of $G(\mathbf{A}_k)$ such that $N \cap G(\mathbf{A}_k)^{\mathrm{Br}_1(G)} \neq \emptyset$, we have

$$\pi(N) = \prod_{v\in S} G^{\rm red}(k_v) \times \prod_{v\notin S} \pi(N_v)$$

and this set is an open subset of $G^{\rm red}(\mathbf{A}_k)$, with $\pi(M) \cap \left[G^{\rm red}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\rm red})}\right] \neq \emptyset$: here we use Proposition 6 of §2.1 of Chapter III in [33], Proposition 6.5 in Chapter 6 of [30] and the functoriality of Brauer-Manin pairing. Then by assumption there is $y \in G^{\rm red}(k) \cap \pi(N)$. Using Proposition 6 of §2.1 of Chapter III in [33] one more time, one concludes that $\pi^{-1}(y)$ is isomorphic to $R_u(G)$ as an algebraic variety, hence it satisfies strong approximation off S. Since

$$\pi^{-1}(y) \cap N = \prod_{v\in S} \pi^{-1}(y)(k_v) \times \prod_{v\notin S} (\pi^{-1}(y)(k_v) \cap N) \neq \emptyset,$$

there is $z \in \pi^{-1}(y)(k) \cap N \subset G(k) \cap N$, as desired. □

The main result of this section is the following statement:

Theorem 5.2. *Let G be a connected linear algebraic group over a number field k and let $G^{\mathrm{qs}} := G/R(G)$, where $R(G)$ is the solvable radical of G. If $\pi : G \to G^{\mathrm{qs}}$ is the quotient map, then*

$$G^{\mathrm{qs}}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\mathrm{qs}})} = \pi\left(G(\mathbf{A}_k)^{\mathrm{Br}_1(G)}\right) \cdot G^{\mathrm{qs}}(k)\,.$$

In particular, if G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ *off a finite subset S of Ω_k, then G^{qs} satisfies strong approximation with respect to* $\mathrm{Br}_1(G^{\mathrm{qs}})$ *off S.*

Proof. For the first part, by functoriality of the Brauer-Manin pairing, one only needs to prove that

$$G^{\mathrm{qs}}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\mathrm{qs}})} \subseteq \pi\left(G(\mathbf{A}_k)^{\mathrm{Br}_1(G)}\right) \cdot G^{\mathrm{qs}}(k)\,.$$

By Lemma 5.1, we can assume that G is reductive. Then $R(G)$ is a torus contained in the center of G (see Theorem 2.4 in Chapter 2 of [30]) and $\pi : G \to G^{\mathrm{qs}}$ is a torsor under $R(G)$. By Corollary 4.3, for any $(x_v) \in G^{\mathrm{qs}}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\mathrm{qs}})}$, there are $\sigma \in H^1(k, R(G))$ and $(y_v) \in G^{\sigma}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\sigma})}$ such that $(x_v) = \pi_{\sigma}((y_v))$. Since $G^{\sigma}(k) \neq \emptyset$ by Corollary 8.7 in [32] (see also Theorem 5.2.1 in [36]), there is $\gamma \in G^{\mathrm{qs}}(k)$ such that $\partial(\gamma) = \sigma$, where ∂ is the coboundary map in the following exact sequence in Galois cohomology:

$$1 \to R(G)(k) \to G(k) \to G^{\mathrm{qs}}(k) \xrightarrow{\partial} H^1(k, R(G)) \to H^1(k, G)\,.$$

In addition, the choice of an element $\bar{\gamma} \in G(\bar{k})$ such that $\pi(\bar{\gamma}) = \gamma$ defines a commutative diagram defined over k:

$$\begin{array}{ccc} G^{\sigma} & \xrightarrow[\sim]{\bar{\gamma}\cdot} & G \\ {\scriptstyle \pi_{\sigma}}\downarrow & & \downarrow{\scriptstyle \pi} \\ G^{\mathrm{qs}} & \xrightarrow[\sim]{\gamma\cdot} & G^{\mathrm{qs}} \end{array}$$

(see for instance Example 2 of p.20 in [36]). This implies that

$$\pi_{\sigma}\left(G^{\sigma}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\sigma})}\right) = \pi\left(G(\mathbf{A}_k)^{\mathrm{Br}_1(G)}\right) \cdot \gamma\,,$$

as desired.

Suppose now that G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ off S. For any open subset

$$M = \prod_{v \in S} G^{\mathrm{qs}}(k_v) \times \prod_{v \notin S} M_v$$

of $G^{\mathrm{qs}}(\mathbf{A}_k)$ such that $M \cap G^{\mathrm{qs}}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\mathrm{qs}})} \neq \emptyset$, the first part implies that there is $g \in G^{\mathrm{qs}}(k)$ such that

$$\pi^{-1}(M \cdot g) = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} \pi^{-1}(M_v \cdot g)\,,$$

with $\pi^{-1}(M \cdot g) \cap G(\mathbf{A}_k)^{\mathrm{Br}_1(G)} \neq \emptyset$. Since G satisfies strong approximation with algebraic Brauer-Manin obstruction off S, there exists $x \in G(k) \cap \pi^{-1}(M \cdot g)$. This implies that $\pi(x) \cdot g^{-1} \in M \cap G^{\mathrm{qs}}(k)$ as required. □

Corollary 5.3. *Let G be a connected linear algebraic group over a number field k and let S a finite subset of Ω_k containing ∞_k. Then G satisfies strong approximation with respect to* $\mathrm{Br}_1(G)$ *off S if and only if $\prod_{v\in S} G'(k_v)$ is not compact for any non-trivial simple factor G' of the semi-simple part G^{ss} of G.*

Proof. By Theorem 2.3 and Theorem 2.4 of Chapter 2 in [30], the quotient map

$$G^{\mathrm{red}} \to G/R(G) = G^{\mathrm{qs}}$$

induces an isogeny $G^{ss} \to G^{\mathrm{qs}}$. One side follows from Corollary 3.20 in [17]. The other side follows from Theorem 5.2 and Proposition 6.1 in [5]. □

Remark 5.4. *All the results in this section involve the group* $\mathrm{Br}_1(G)$*, and they remain true with* $\mathrm{Br}_1(G)$ *replaced by* $\mathrm{Br}(G)$*. Indeed, there is a sufficiently large subset S of Ω_k containing ∞_k such that $\prod_{v\in S} G'(k_v)$ is not compact for any non-trivial simple factor G' of G^{ss}, therefore Corollary 3.20 in* [17]*, Proposition 2.6 in* [9] *and the functoriality of Brauer-Manin pairing gives the following inclusions:*

$$G(\mathbf{A}_k)^{\mathrm{Br}_1(G)} = \overline{G(k)\cdot \rho(\prod_{v\in S} G^{scu}(k_v))} \subseteq G(\mathbf{A}_k)^{\mathrm{Br}(G)} \subseteq G(\mathbf{A}_k)^{\mathrm{Br}_1(G)},$$

where $G^{scu} = G^{sc}\times_{G^{\mathrm{red}}} G$ with the projection map $G^{scu} \xrightarrow{\rho} G$ and G^{sc} is the simply connected covering of G^{ss}. In particular, we have $G(\mathbf{A}_k)^{\mathrm{Br}(G)} = G(\mathbf{A}_k)^{\mathrm{Br}_1(G)}$.

6. Comparison I, $X(\mathbf{A}_k)^{\mathrm{desc}} \subseteq X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$

Let $Y \xrightarrow{f} X$ be a left torsor under a linear algebraic group G over a number field k. The fundamental problem to define the descent obstruction for strong approximation with respect to $Y \xrightarrow{f} X$ is to decide whether the set

$$X(\mathbf{A}_k)^f = \left\{(x_v)\in X(\mathbf{A}_k) : ([Y](x_v)) \in \mathrm{Im}\left(H^1(k,G)\to \prod_v H^1(k_v,G)\right)\right\} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k))$$

is closed or not in $X(\mathbf{A}_k)$. We already know that this is true when G is either connected or a group of multiplicative type, by Theorem 3.5. For a general linear algebraic group G, this result is proved by Skorobogatov in Corollary 2.7 of [35], when X is assumed to be proper over k. The proof depends on Proposition 5.3.2 in [36] or Proposition 4.4 in [23], which are not true for open varieties, as explained in the following example.

Example 6.1. *The short exact sequence of linear algebraic groups*

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{f} \mathbb{G}_m \to 1,$$

where $f(x) = x^2$, can be viewed as torsor over $\mathbb{G}_m$ under μ_2. For any $\sigma \in H^1(k,\mu_2) \cong k^\times/(k^\times)^2$, the twist $\mathbb{G}_m^\sigma$ of $\mathbb{G}_m$ by σ is given by the equation $x = a_\sigma y^2$ in $\mathbb{G}_m \times_k \mathbb{G}_m$, where a_σ is an element in $k^\times$ representing the class σ by the above isomorphism. It is clear that $\mathbb{G}_m^\sigma \cong \mathbb{G}_m$ as varieties over k, hence it always contains adelic points.

We use the same definition of an integral model as in [28].

Definition 6.2. *Let X be a variety over a number field k and let S be a finite subset of Ω_k containing ∞_k. An integral model of X over O_S is a faithfully flat separated O_S-scheme $\mathcal{X}_S$ of finite type such that $\mathcal{X}_S \times_{O_S} k \cong X$.*

The replacement for Proposition 5.3.2 in [36] or Proposition 4.4 in [23] is the following proposition:

Proposition 6.3. *Let X be a variety over a number field k and let S be a finite subset of Ω_k containing ∞_k. Fix an integral model $\mathcal{X}_S$ of X over O_S. If $Y \xrightarrow{f} X$ is a left torsor under a linear algebraic group G over k, then the set*

$$\left\{[\sigma] \in H^1(k, G) : \ f_\sigma(Y^\sigma(\mathbf{A}_k)) \cap \left[\prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}_S(O_v)\right] \neq \emptyset\right\}$$

is finite.

Proof. It follows from the same argument as the proof of Proposition 4.4 in [23]. □

One can now extend Corollary 2.7 in [35] to open varieties by using the above replacement for Proposition 4.4 in [23].

Proposition 6.4. *Let X be a (not necessarily proper) variety over a number field k. If $Y \xrightarrow{f} X$ is a left torsor under a linear algebraic group G over k, then the set $X(\mathbf{A}_k)^f$ is closed in $X(\mathbf{A}_k)$.*

Proof. Take an integral model $\mathcal{X}_{S_0}$ of X over O_{S_0}, where S_0 is a finite subset of Ω_k containing ∞_k. Then

$$\left\{\prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} \mathcal{X}_{S_0}(O_v)\right\}_S$$

is an open covering of $X(\mathbf{A}_k)$ (see Theorem 3.6 in [11]), where S runs through all finite subsets of Ω_k containing S_0. By Proposition 6.3 and Corollary 2.5 in [35], the set

$$X(\mathbf{A}_k)^f \cap \left[\prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} \mathcal{X}_{S_0}(O_v)\right]$$

is closed in $\prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} \mathcal{X}_{S_0}(O_v)$, therefore the set $X(\mathbf{A}_k)^f$ is closed in $X(\mathbf{A}_k)$. □

Applying Proposition 6.3, one can also extend Lemma 2.2 and Theorem 1.1 in [35] to open varieties. For any variety over a number field k, and following [35], we write

$$X(\mathbf{A}_k)^{\mathrm{desc}} = \bigcap_{Y \xrightarrow{f} X} X(\mathbf{A}_k)^f \,,$$

where $Y \xrightarrow{f} X$ runs through all torsors under all linear algebraic groups over k (see also §1.).

Lemma 6.5. *Let X be a (not necessarily proper) variety and let $Y \to X$ be a torsor over a number field k. For any $(P_v) \in X(\mathbf{A}_k)^{\mathrm{desc}}$, there is a twist $Y' \to X$ of $Y \to X$ such that the following property holds:*

For any surjective X-torsor morphism $Z \to Y'$ (see Definition 2.1 in [35]*), there is a twist $Z' \to Y'$ of $Z \to Y'$ such that (P_v) lies in the image of $Z'(\mathbf{A}_k)$.*

Proof. There are a finite subset S_0 of Ω_k containing ∞_k and an integral model $\mathcal{X}_{S_0}$ over O_{S_0} such that

$$(P_v) \in \prod_{v \in S_0} X(k_v) \times \prod_{v \in \Omega_k \setminus S_0} \mathcal{X}_{S_0}(O_v)$$

(see for instance Theorem 3.6 in [11]), hence Proposition 6.3 implies that there are only finitely many twists of a given torsor over X such that (P_v) lifts as an adelic point of this torsor. As pointed out in the proof of Lemma 2.2 in [35], the finite combinatorics in the first part of the proof of Proposition 5.17 in [38] are still valid. It concludes the proof. □

Proposition 6.6. *Let X be a (not necessarily proper) variety over a number field k. If $Y \xrightarrow{f} X$ is a left torsor under a finite group scheme F over k, then*

$$X(\mathbf{A}_k)^{\mathrm{desc}} = \bigcup_{\sigma \in H^1(k,F)} f_\sigma\left(Y^\sigma(\mathbf{A}_k)^{\mathrm{desc}}\right).$$

Proof. One only needs to modify the proof of Theorem 1.1 in [35] by replacing Lemma 2.2 in [35] with Lemma 6.5, Corollary 2.7 in [35] with Proposition 6.4. Moreover, since f is finite, the induced map $Y(\mathbf{A}_k) \xrightarrow{f} X(\mathbf{A}_k)$ is topologically proper by Proposition 4.4 in [11]. This implies that $f^{-1}((P_v))$ is compact. □

Recall that, following [31], one can define for any variety X over a number field k, the set

$$X(\mathbf{A}_k)^{\text{ét,Br}} = \bigcap_{Y \xrightarrow{f} X} \bigcup_{\sigma \in H^1(k,F)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}}),$$

where $Y \xrightarrow{f} X$ runs over all torsors under all finite groups F over k (see §1). Since the induced map $Y(\mathbf{A}_k) \xrightarrow{f} X(\mathbf{A}_k)$ is topologically closed for any finite morphism $Y \xrightarrow{f} X$ by Proposition 4.4 in [11], one concludes that $X(\mathbf{A}_k)^{\text{ét,Br}}$ is closed in $X(\mathbf{A}_k)$ by the same argument as in Proposition 6.4.

Corollary 6.7. *If X is a smooth quasi-projective variety over a number field k, then*

$$X(\mathbf{A}_k)^{\mathrm{desc}} \subseteq X(\mathbf{A}_k)^{\text{ét,Br}} \subseteq X(\mathbf{A}_k)^{\mathrm{Br}}.$$

Proof. One only needs to show that $X(\mathbf{A}_k)^{\mathrm{desc}} \subseteq X(\mathbf{A}_k)^{\text{ét,Br}}$. For any torsor $Y \xrightarrow{f} X$ under a finite group scheme F, Proposition 6.6 gives the equality

$$X(\mathbf{A}_k)^{\mathrm{desc}} = \bigcup_{\sigma \in H^1(k,F)} f_\sigma\left(Y^\sigma(\mathbf{A}_k)^{\mathrm{desc}}\right).$$

Since X is quasi-projective, Y^σ is quasi-projective as well. By a theorem of Gabber (see [12]), one has

$$Y^\sigma(\mathbf{A}_k)^{\mathrm{desc}} \subseteq Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}}$$

(see the proof of Lemma 2.8 in [35]) and the result follows. □

7. Comparison II, $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \subseteq X(\mathbf{A}_k)^{\mathrm{desc}}$

In this section, we prove the inclusion $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \subseteq X(\mathbf{A}_k)^{\mathrm{desc}}$ for open varieties, which implies Theorem 1.5. The strategy of proof is the same as in [14].

The second named author would like to thank Laurent Moret-Bailly warmly for finding a mistake and for suggesting the following alternative proof of Lemma 4 in [14] (which already appeared in [15]). The statement of this lemma is correct, but the proof in [14] uses a result of Stoll (see [38]) that is not. Note that in contrast with [14], all torsors (unless explicitely mentioned) are assumed to be left torsors.

Lemma 7.1. *Let X be a smooth geometrically connected k-variety. Let $(P_v) \in X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$ and let $Z \xrightarrow{g} X$ be a torsor under a finite k-group F.*

Then there are a cocycle $\sigma \in Z^1(k, F)$ and a connected component X' of Z^σ over k such that the restriction of g_σ to X' is a torsor $X' \to X$ under the stabilizer F' of X' for the action of F^σ, and the point (P_v) lifts to a point $(Q'_v) \in X'(\mathbf{A}_k)^{\mathrm{Br}}$.

In particular, X' is geometrically integral.

Proof. By assumption, the point (P_v) lifts to some point $(Q_v) \in Z^\sigma(\mathbf{A}_k)^{\mathrm{Br}}$ for some cocycle σ with values in F. Since Z^σ is smooth, Z^σ is a disjoint union of connected components over k. By Proposition 3.3 in [28], there is a k-connected component X' of Z^σ such that $(Q_v)_{v \notin \Xi} \in P_\Xi(X'(\mathbf{A}_k)^{\mathrm{Br}})$, where Ξ is the set of all complex places of k, $\mathbf{A}_k^\Xi$ is the ring of adeles without Ξ-components and P_Ξ is the projection from $X'(\mathbf{A}_k)$ to $X'(\mathbf{A}_k^\Xi)$. Since for $v \in \Xi$, $Z^\sigma \times_k k_v$ is a trivial torsor under the finite constant group scheme $F^\sigma \times_k k_v$, we have $g_\sigma(X'(k_v)) = X(k_v)$ for all $v \in \Xi$. Hence one can assume that $Q_v \in X'(k_v)$ for $v \in \Xi$, so that we have $(Q_v) \in X'(\mathbf{A}_k)^{\mathrm{Br}}$.

Since X' is connected and $X'(\mathbf{A}_k) \neq \emptyset$, the proof of Lemma 5.5 in [38] implies that X' is geometrically connected. Eventually, X' being geometrically connected guarantees that the variety X' is an X-torsor under the stabilizer F' of X' in F^σ. □

Let us continue the proof of the aforementioned inclusion. Let X be a smooth and geometrically integral k-variety, and $(P_v) \in X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$. We need to prove that $(P_v) \in X(\mathbf{A}_k)^{\mathrm{desc}}$.

For a linear algebraic group G over k, one has the following short exact sequence of algebraic groups over k:

$$1 \to H \to G \to F \to 1\,,$$

where H is the connected component of G and F is finite over k. This induces the following diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T & \longrightarrow & G' & \longrightarrow & F & \longrightarrow & 1 \end{array}$$

where T denotes the maximal toric quotient of H and G' is the quotient of G by the kernel of $H \to T$.

Let $Y \to X$ be a torsor under G and let $Z \to X$ be the push-forward of $Y \to X$ by the morphism $G \to F$, which is a torsor under F. If $\sigma \in Z^1(k, F)$ is a 1-cocycle given by Lemma 7.1 applied to the torsor $Z \to X$ and to the point (P_v), we want to show that the cocycle $\sigma \in Z^1(k, F)$ lifts to a cocycle $\tau \in Z^1(k, G)$, as in Proposition 5 in [14]. The obstruction to lift σ to a cocycle in $Z^1(k, G)$ gives a natural cohomology class $\eta_\sigma \in H^2(k, \kappa_\sigma)$ by (5.1) in [18] (see also (7.7) in [1]), where κ_σ is a natural k-kernel on $H_{\bar{k}}$ associated to σ. Lemma 6 in [14] implies that there is a canonical map $H^2(k, \kappa_\sigma) \to H^2(k, T^\sigma)$ such that the class η_σ is neutral if and only if its image $\eta'_\sigma \in H^2(k, T^\sigma)$ is zero.

We now apply the open descent theory and the extended type developed by Harari and Skorobogatov in [26] to establish the analogue of Lemma 7 in [14] for open varieties. As in the proof of [14], the torsor $Y \to Z$ under H induces a torsor $W \xrightarrow{\varpi} Z$ under T by the natural map $H^1(Z, H) \to H^1(Z, T)$. Instead of using the type of the torsor ϖ that was used in [14], we consider the so-called "extended type" of the torsor ϖ that was introduced by Harari and Skorobogatov (see Definition 8.2 in [26]). For a variety Z over k, let $KD'(Z)$ denote the complex of Galois modules $[\overline{k}(Z)^*/\overline{k}^* \to \mathrm{Div}(Z_{\bar{k}})]$ in the derived category $D^b_{\text{ét}}(k)$ of bounded complexes of étale sheaves over $\mathrm{Spec}(k)$. One can associate to the torsor $W \xrightarrow{\varpi} Z$ under T a canonical morphism in this derived category

$$\lambda_W : \widehat{T} \to KD'(Z)\,,$$

called the extended type of ϖ. This induces a morphism in the derived category of bounded complexes of abelian groups

$$\lambda_W^\sigma : \widehat{T^\sigma} \to KD'(Z^\sigma)$$

for the above $\sigma \in Z^1(k, F)$.

Lemma 7.2. *The morphism* $\lambda_W^\sigma : \widehat{T^\sigma} \to KD'(Z^\sigma)$ *is a morphism in the derived category of bounded complexes of étale sheaves over* $\mathrm{Spec}(k)$.

Proof. The natural left actions of F on both T and Z induces right actions of F on $\widehat{T}$ and on $KD'(Z)$.

We first prove that the morphism λ_W is F-equivariant for those actions.

Let $f \in F(\overline{k})$. We denote by $f_Z : Z_{\bar{k}} \to Z_{\bar{k}}$ the morphism of $\bar{k}$-varieties defined by $z \mapsto f \cdot z$. This morphism induces a natural morphism in the derived category $f_Z^* : KD'(Z_{\bar{k}}) \to KD'(Z_{\bar{k}})$. Similarly, the element f defines a natural morphism of $\bar{k}$-tori $f_T : T_{\bar{k}} \to T_{\bar{k}}$ such that $f_T(t) := gtg^{-1}$, where $g \in G'(\overline{k})$ is any point lifting $f \in F(\overline{k})$. This morphism f_T induces a morphism of abelian groups $\widehat{f_T} : \widehat{T} \to \widehat{T}$ such that $\widehat{f_T}(\chi) := \chi \circ f_T$.

One needs to prove that the following diagram

$$\begin{array}{ccc} \widehat{T} & \xrightarrow{\lambda_{W_{\bar{k}}}} & KD'(Z_{\bar{k}}) \\ {\scriptstyle \widehat{f_T}}\downarrow & & \downarrow{\scriptstyle f_Z^*} \\ \widehat{T} & \xrightarrow[\lambda_{W_{\bar{k}}}]{} & KD'(Z_{\bar{k}}) \end{array}$$

is commutative.

Let $f_{T,*}W_{\bar{k}}$ be the push-forward of the torsor $W_{\bar{k}} \to Z_{\bar{k}}$ under $T_{\bar{k}}$ by the $\bar{k}$-morphism $T_{\bar{k}} \xrightarrow{f_T} T_{\bar{k}}$ and let $f_Z^*W_{\bar{k}}$ be the pullback of the torsor $W_{\bar{k}} \to Z_{\bar{k}}$ under $T_{\bar{k}}$ by the $\bar{k}$-morphism $f_Z : Z_{\bar{k}} \to Z_{\bar{k}}$. Then functoriality of the extended type gives:

$$f_Z^* \circ \lambda_{W_{\bar{k}}} = \lambda_{f_Z^* W_{\bar{k}}} \quad \text{and} \quad \lambda_{f_{T,*}W_{\bar{k}}} = \lambda_{W_{\bar{k}}} \circ \widehat{f_T} \, .$$

To prove the required commutativity $f_Z^* \circ \lambda_{W_{\bar{k}}} = \lambda_{W_{\bar{k}}} \circ \widehat{f_T}$, it is enough to show that the torsors $f_Z^*W_{\bar{k}} \to Z_{\bar{k}}$ and $f_{T,*}W_{\bar{k}} \to Z_{\bar{k}}$ under $T_{\bar{k}}$ are isomorphic. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} T_{\bar{k}} \times W_{\bar{k}} & \xrightarrow{g} & W_{\bar{k}} \\ {\scriptstyle \varpi \circ p_W}\downarrow & & \downarrow{\scriptstyle \varpi} \\ Z_{\bar{k}} & \xrightarrow[f_Z]{} & Z_{\bar{k}} \, , \end{array}$$

where p_W denotes the projection on $W_{\bar{k}}$ and the morphism g is defined by $(t, w) \mapsto (tg) \cdot w$. This diagram induces a natural $Z_{\bar{k}}$-morphism $\phi : T_{\bar{k}} \times W_{\bar{k}} \to f_Z^* W_{\bar{k}}$. Consider now the right action of $T_{\bar{k}}$ on $T_{\bar{k}} \times W_{\bar{k}}$ defined by $(s, w) \cdot t := (s f_T(t), t^{-1} \cdot w) = (sgtg^{-1}, t^{-1} \cdot w)$. Then the morphism ϕ is $T_{\bar{k}}$-invariant under this action, hence it induces a $Z_{\bar{k}}$-morphism $\psi : f_{T,*}W_{\bar{k}} \to f_Z^* W_{\bar{k}}$. One can check by a simple computation that ψ is $T_{\bar{k}}$-equivariant, i.e. that ψ is a morphism of (left) torsors over $Z_{\bar{k}}$ under $T_{\bar{k}}$. It concludes the proof of the required commutativity, hence the morphism λ_W is F-equivariant.

By definition of the twists T^σ and Z^σ, the fact that λ_W is F-equivariant implies that the morphism λ_W^σ is Galois equivariant, i.e. that λ_W^σ is a morphism in the derived category of bounded complexes of étale sheaves over $\mathrm{Spec}(k)$. □

By Proposition 8.1 in [26], there is a natural exact sequence of abelian groups

$$H^1(k, T^\sigma) \to H^1(X', T^\sigma) \xrightarrow{\lambda} \mathrm{Hom}_k(\widehat{T^\sigma}, KD'(X')) \xrightarrow{\partial} H^2(k, T^\sigma)$$

where the map λ is the extended type. Let $\lambda'_\sigma = \psi^* \circ \lambda_W^\sigma$, where $\psi : X' \to W$ is the inclusion of the k-connected component given by Lemma 7.1, and $KD'(Z^\sigma) \xrightarrow{\psi^*} KD'(X')$ is the map induced by ψ.

The following lemma, which is an analogue of Lemma 8 in [14], is a crucial step for proving the main result of this section. We give here a more conceptual proof than that in [14], where a similar statement was proven by cocycle computations under the assumption that $\bar{k}[X]^\times = \bar{k}^\times$.

Lemma 7.3. *With the above notation, one has*

$$\partial(\lambda'_\sigma) = 0 \text{ if and only if } \eta'_\sigma = 0\,.$$

Proof. In the following proof, we work over the small étale site of $\mathrm{Spec}(k)$.

Recall that we are given a cocycle $\sigma \in Z^1(k, F)$ as in Lemma 7.1: one can associate to σ a $\mathrm{Spec}(k)$-torsor U under F with a point $u_0 \in U(\overline{k})$. This torsor U is naturally a homogeneous space of the group G' with geometric stabilizer isomorphic to $T_{\bar k}$. Section IV.5.1 in [19] implies that the element $\eta'_\sigma \in H^2(k, T^\sigma)$ is the class of the $\mathrm{Spec}(k)$-gerbe $\mathcal{E}_\sigma$ banded by T^σ such that for all étale schemes S over $\mathrm{Spec}(k)$, the category $\mathcal{E}_\sigma(S)$ is defined as follows: the objects of $\mathcal{E}_\sigma(S)$ are triples (P, p, α) where $P \to S$ is a torsor under G', $p \in P(S_{\bar k})$ and $\alpha : P \to U_S$ is a G'-equivariant S-morphism. The morphisms of $\mathcal{E}_\sigma(S)$ between triples (P, p, α) and (P', p', α') are given by morphisms of torsors $P \to P'$ over S under G' that commute with α and α'.

Similarly, one can associate to the morphism λ'_σ a $\mathrm{Spec}(k)$-gerbe banded by T^σ that will be the obstruction for the morphism λ'_σ to be the extended type of a torsor over X' under T^σ. The morphism λ'_σ induces a morphism $\overline{\lambda'_\sigma} : \widehat{T^\sigma_{\bar k}} \to KD'(X'_{\bar k})$ in $D^b_{\text{ét}}(\bar k)$. By construction, $\overline{\lambda'_\sigma}$ is the extended type of the torsor $Y_0 := W_{\bar k} \times_{Z_{\bar k}} X'_{\bar k}$ over $X'_{\bar k}$ under $T^\sigma_{\bar k} = T_{\bar k}$.

We now define $\mathcal{L}_\sigma$ to be the fibered category defined as follows : for all étale schemes S over $\mathrm{Spec}(k)$, the objects of the category $\mathcal{L}_\sigma(S)$ are pairs (V, φ), where $V \to X'_S$ is a torsor under T^σ_S of extended type λ_V compatible with λ'_σ and $\varphi : V_{\bar k} \to Y_0 \times_{\bar k} S_{\bar k}$ is an isomorphism of torsors over $X' \times_k S_{\bar k}$ under $T^\sigma_{S_{\bar k}}$. Given two such objects (V, φ) and (V', φ'), a morphism between (V, φ) and (V', φ') in the category $\mathcal{L}_\sigma(S)$ is a pair (α, t), where $\alpha : V \to V'$ is a morphism of torsors over X'_S under T^σ_S and $t \in T^\sigma(S_{\bar k})$ such that the diagram

$$\begin{array}{ccc}
V_{\bar k} & \xrightarrow{\ \overline{\alpha}\ } & V'_{\bar k} \\
\Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \varphi'} \\
Y_0 \times_{\bar k} S_{\bar k} & \xrightarrow[\ t\ ]{} & Y_0 \times_{\bar k} S_{\bar k}
\end{array}$$

commutes.

One can check that $\mathcal{L}_\sigma$ is a stack for the étale topology over $\mathrm{Spec}(k)$, and the fact that this is a gerbe is a consequence of the exact sequence of Proposition 8.1 in [26]

$$H^1(S, T^\sigma) \to H^1(X'_S, T^\sigma) \xrightarrow{\lambda} \mathrm{Hom}_S(\widehat{T^\sigma}, KD'(X'_S)) \xrightarrow{\partial} H^2(S, T^\sigma)$$

(which holds provided that S is integral, regular and noetherian).

The band of this gerbe is the abelian band represented by T^σ.

In addition, it is clear that $\mathcal{L}_\sigma$ is neutral if and only if $\mathcal{L}_\sigma(k) \neq \emptyset$ if and only if there exists a torsor over X' under T^σ of type λ'_σ if and only if $\partial(\lambda'_\sigma) = 0$.

Let us now construct an equivalence of gerbes between $\mathcal{E}_\sigma$ and $\mathcal{L}_\sigma$.

For all étale $\mathrm{Spec}(k)$-schemes S, consider the functor

$$m_S : \mathcal{E}_\sigma(S) \to \mathcal{L}_\sigma(S)$$

that maps an object (P, p, α) to the object (V, φ), where V is defined to be the contracted product $V := (P \times_S^{G'} W_S) \times_{Z^\sigma_S} X'_S$ and $\varphi : V_{\bar k} \to Y_0 \times_{\bar k} S_{\bar k} = (W_{\bar k} \times_{Z_{\bar k}} X'_{\bar k}) \times_{\bar k} S_{\bar k}$ is induced by the point $p \in P(S_{\bar{k}})$. Indeed, by construction, we have a natural map $P \times_S^{G'} W_S \to U_S \times_S^F Z_S = Z_S^\sigma$, and a simple computation proves that this map is a torsor under T^σ of extended type compatible with λ_W^σ.

By definition, the functor m_S sends a morphism $\varphi : (P, p, \alpha) \to (P', p', \alpha')$ to the morphism $(\widetilde{\varphi}, t_0)$ such that $\widetilde{\varphi} : (P \times_S^{G'} W_S) \times_{Z_S^\sigma} X'_S \to (P' \times_S^{G'} W_S) \times_{Z_S^\sigma} X'_S$ is the morphism induced by the morphism of torsors $\varphi : P \to P'$, and $t_0 \in T^\sigma(S_{\bar{k}})$ is the element such that $p' = t_0 \cdot \varphi(p)$ as $S_{\bar{k}}$-points in $(P' \times_S^{G'} W_S) \times_{Z_S^\sigma} X'_S$.

Finally, one checks that the collection of functors m_S defines a morphism of gerbes $m : \mathcal{E}_\sigma \to \mathcal{L}_\sigma$ banded by the identity of T^σ, which implies that $\eta'_\sigma := [\mathcal{E}_\sigma] = [\mathcal{L}_\sigma] \in H^2(k, T^\sigma)$.

Therefore, $\eta'_\sigma = 0$ if and only if $\mathcal{E}_\sigma(k) \neq \emptyset$ if and only if $\mathcal{L}_\sigma(k) \neq \emptyset$ if and only if $\partial(\lambda'_\sigma) = 0$. □

The immediate consequence of Lemma 7.3 is the following result which extends Proposition 5 in [14] to open varieties.

Proposition 7.4. *Let X be a smooth geometrically integral k-variety. Let $(P_v) \in X(\mathbf{A}_k)^{\text{ét, Br}}$ and let $Y \to X$ be a torsor under a linear k-group G. Let*

$$1 \to H \to G \to F \to 1$$

be an exact sequence of linear k-groups, where H is connected and F finite. Let $Z \to X$ be the push-forward of $Y \to X$ by the morphism $G \to F$, which is a torsor under F. Let $\sigma \in Z^1(k, F)$ be a 1-cocycle given by Lemma 7.1 applied to the torsor $Z \to X$ and the point (P_v).

Then the cocycle $\sigma \in Z^1(k, F)$ lifts to a cocycle $\tau \in Z^1(k, G)$.

Proof. As mentionned above, Construction (5.1) in [18] (see also (7.7) in [1]) gives a class η_σ of $H^2(k, \kappa_\sigma)$ such that σ can be lifted to $Z^1(k, G)$ if and only if η_σ is neutral, where κ_σ is a k-kernel on $H_{\bar{k}}$. By (6.1.2) of [1] and Lemma 6 in [14], there is a canonical map $H^2(k, \kappa_\sigma) \to H^2(k, T^\sigma)$ such that the class η_σ is neutral if and only if its image $\eta'_\sigma \in H^2(k, T^\sigma)$ is zero. By Lemma 7.3, one only needs to show that $\partial(\lambda'_\sigma) = 0$ where $\lambda'_\sigma = \psi^* \circ \lambda_W^\sigma$, with $KD'(Z^\sigma) \xrightarrow{\psi^*} KD'(X')$ given by Lemma 7.1 and λ_W^σ defined by Lemma 7.2.

By Lemma 7.1, we know that $X'(\mathbf{A}_k)^{\text{Br}} \neq \emptyset$. Therefore the map λ in the exact sequence (see Proposition 8.1 in [26])

$$H^1(X', T^\sigma) \xrightarrow{\lambda} \text{Hom}_k(\widehat{T^\sigma}, KD'(X')) \xrightarrow{\partial} H^2(k, T^\sigma)$$

is surjective by Corollary 8.17 in [26]. Hence the map ∂ is the zero map and $\partial(\lambda'_\sigma) = 0$, which concludes the proof. □

Remark 7.5. *The proof of Proposition 7.4 also gives the following result:*

Let X be a smooth geometrically integral k-variety and let $Y \to X$ be a torsor under a linear algebraic k-group G. Let

$$1 \to H \to G \to F \to 1$$

be an exact sequence of linear k-groups, where H is connected and F finite. Let $Z \to X$ be the push-forward of $Y \to X$ by the morphism $G \to F$.

If $\sigma \in H^1(k, F)$ satisfies $Z^\sigma(\mathbf{A}_k)^{\text{Br}_1(Z^\sigma)} \neq \emptyset$, then σ can be lifted to $H^1(k, G)$.

One can now prove the main result of this section:

Theorem 7.6. *If X is a smooth and geometrically integral variety over a number field k, then*

$$X(\mathbf{A}_k)^{\text{ét,Br}} \subseteq X(\mathbf{A}_k)^{\text{desc}}.$$

Proof. Since the statement 2 of Theorem 2 in [21] (which we apply to X') holds for any geometrically integral variety (without any assumption on $\bar{k}[X']^{\times}$), the proof of this theorem using Proposition 7.4 is exactly the same as the proof of Theorem 1 using Proposition 5 in [14] (see in particular [14], p. 244-245). □

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