COMPARING DESCENT OBSTRUCTION AND BRAUER-MANIN OBSTRUCTION FOR OPEN VARIETIES

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Abstract. We provide a relation between Brauer-Manin obstruction and descent obstruction for torsors over not necessarily proper varieties under a connected linear algebraic group or a group of multiplicative type. Such a relation is also refined for torsors under a torus. The equivalence between descent obstruction and étale Brauer-Manin obstruction for smooth projective varieties is extended to smooth quasi-projective varieties, which provides the perspective to study integral points.

1. Introduction

The descent theory for tori was first established by Colliot-Thélène and Sansuc in [8] and was extended by Skorobogatov to groups of multiplicative type in [34]. In a series of papers [21], [23], [25], Harari and Skorobogatov introduced descent obstruction for a general algebraic group and compared the descent obstruction with the Brauer-Manin obstruction. By various works of Poonen [31], the second named author [14], Stoll [38] and Skorobogatov [35], it was proved that the descent obstruction is equivalent to the étale Brauer-Manin obstruction for smooth projective geometrically integral varieties. In this paper, we study the relation between the descent obstruction and the Brauer-Manin obstruction for open varieties by using new arithmetic tools developed in [2], [6], [9], [17], [22] and [27], and we extend the equivalence between the descent obstruction and the étale Brauer-Manin obstruction to smooth quasi-projective varieties.

Let \( k \) be a number field, \( \Omega_k \) the set of all primes of \( k \) and \( \mathbb{A}_k \) the adelic ring of \( k \). A variety over \( k \) is defined to be a separated scheme \( X \) of finite type over \( k \). Fix an algebraic closure \( \overline{k} \) of \( k \). We denote by \( X_{\overline{k}} \) the fibre product \( X \times_k \overline{k} \). Let

\[
\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m), \quad \text{Br}_1(X) = \ker(\text{Br}(X) \to \text{Br}(X_{\overline{k}})) \quad \text{and} \quad \text{Br}_0(X) = \text{Im}(\text{Br}(k) \xrightarrow{\pi^*} \text{Br}(X))
\]

where \( X \xrightarrow{\pi} \text{Spec}(k) \) is the structure morphism, and \( \text{Br}_n(X) = \text{Br}_1(X)/\text{Br}_0(X) \). For any subgroup \( B \) of \( \text{Br}(X) \), one can define the Brauer-Manin set

\[
X(\mathbb{A}_k)^B = \{(x_v)_{v \in \Omega_k} \in X(\mathbb{A}_k) : \sum_{v \in \Omega_k} \text{inv}_v(\xi(x_v)) = 0 \text{ for all } \xi \in B\}
\]

with respect to \( B \). When \( B = \text{Br}(X) \), we simply write this Brauer-Manin set as \( X(\mathbb{A}_k)^{\text{Br}} \).

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Suppose \( Y \xrightarrow{f} X \) is a left torsor under a linear algebraic group \( G \) over \( k \). The descent obstruction (see [21], [23] and [25]) given by \( f \) is defined by the following set
\[
X(A_k)^f = \{(x_v) \in X(A_k) : ([Y](x_v)) \in \text{Im}(H^1(k, G) \to \prod_{v \in \Omega_k} H^1(k_v, G))\} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k))
\]
where \( Y^\sigma \xrightarrow{f_\sigma} X \) is the twist of \( Y \xrightarrow{f} X \) by a 1-cocycle representing \( \sigma \in H^1(k, G) \). Moreover, one can define
\[
X(A_k)^{\text{desc}} = \bigcap_{Y \xrightarrow{f} X} X(A_k)^f
\]
following [31], where \( Y \xrightarrow{f} X \) runs through all torsors under all linear algebraic groups over \( k \).

The main results in this paper are the following theorems.

**Theorem 1.1.** (Theorem 3.5) Let \( k \) be a number field, \( G \) a connected linear algebraic group or a group of multiplicative type over \( k \), and \( X \) a smooth and geometrically integral variety over \( k \). Suppose \( Y \xrightarrow{f} X \) is a left torsor under \( G \). For any subgroup \( A \subseteq \text{Br}(X) \) which contains the kernel of the natural map \( f^* : \text{Br}(X) \to \text{Br}(Y) \) we have
\[
X(A_k)^A = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k)^{f\sigma(A)})
\]
where \( Y^\sigma \xrightarrow{f_\sigma} X \) is the twist of \( Y \xrightarrow{f} X \) by \( \sigma \) and \( \text{Br}(X) \xrightarrow{f_\sigma} \text{Br}(Y^\sigma) \) is the associated pull-back map, for each \( \sigma \in H^1(k, G) \).

When \( G \) is a torus, this theorem can be refined in order to get Theorem 4.1 in §4. In particular, we prove:

**Theorem 1.2.** (Corollary 4.3) Under the same assumptions as in Theorem 1.1, if \( G \) is assumed to be a torus, then
\[
X(A_k)^{\text{Br}(Y)} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k)^{\text{Br}(Y^\sigma)})
\]
and
\[
X(A_k)^{\text{Br}} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k)^{\text{Br}(Y^\sigma)} + f^*_\sigma(\text{Br}(X))).
\]

This result is inspired by some lectures by Yonatan Harpaz. It should be pointed out that the first part in Theorem 1.2 was first obtained by Dasheng Wei in [39]: his proof uses an argument of Harari and Skorobogatov in [26] together with an exact sequence due to Sansuc (see [2], Theorem 2.8). Theorem 1.2 can be applied to study strong approximation, as in [39]. It should be noted that in general, the image of \( \text{Br}(X) \) in \( \text{Br}(Y^\sigma) \) in Theorem 1.1 and Theorem 1.2 is not easy to describe, even under the assumption \( \bar{k}[X]^\times = \bar{k}^\times \) (see [24, Theorem 1.7(b)]).

**Definition 1.3.** Let \( X \) be a variety over a number field \( k \) and let \( B \) be a subgroup of \( \text{Br}(X) \). For a finite subset \( S \) of \( \Omega_k \), we denote by \( \text{pr}^S : X(A_k) \to X(A_k^S) \) the projection map, where \( A_k^S \) is the set of adeles of \( k \) without \( S \)-components.
We say that $X$ satisfies strong approximation off $S$ if $X(\mathbb{A}_k) \neq \emptyset$ and the diagonal image of $X(k)$ is dense in $pr^S(X(\mathbb{A}_k))$.

We say that $X$ satisfies strong approximation with respect to $B$ off $S$ if $X(\mathbb{A}_k)^B \neq \emptyset$ and the diagonal image of $X(k)$ is dense in $pr^S(X(\mathbb{A}_k)^B)$.

Corollary 3.20 in [17] provides a sufficient condition for strong approximation with Brauer-Manin obstruction to hold for a connected linear algebraic group. As an application of Theorem 1.2, we prove that this sufficient condition is also a necessary condition:

**Theorem 1.4.** (Corollary 5.3) Let $G$ be a connected linear algebraic group over a number field $k$ and let $S$ be a finite subset of $\Omega_k$ containing $\infty_k$. Then $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$ if and only if $\prod_{v \in S} G'(k_v)$ is not compact for any non-trivial simple factor $G'$ of the semi-simple part $G^{ss}$ of $G$.

For any variety $X$ over a number field $k$, one can define, following [31]:

$$X(\mathbb{A}_k)^\text{et,Br} = \bigcap_{Y \xrightarrow{f} X} \bigcup_{\sigma \in H^1(k,F)} f_\sigma(Y^\sigma(\mathbb{A}_k)^{\text{Br}}),$$

where $Y \xrightarrow{f} X$ runs through all torsors under all finite group schemes $F$ over $k$. The last two sections of the paper are devoted to the proof of the following generalization of [14] and [35]:

**Theorem 1.5.** (Corollary 6.7 and Theorem 7.6) If $X$ is a smooth quasi-projective and geometrically integral variety over a number field $k$, then

$$X(\mathbb{A}_k)^{\text{desc}} = X(\mathbb{A}_k)^{\text{et,Br}}.$$
2. Brauer groups of torsors

In this section, we assume that $k$ is an arbitrary field of characteristic 0.

**Lemma 2.1.** Let $H$ be a semi-simple simply connected group or a unipotent group over $k$. Suppose $X$ is a smooth and geometrically integral variety over $k$. If $Z \overset{\rho}{\to} X$ is a torsor under $H$, then the induced map $\text{Br}(X) \xrightarrow{\rho^*} \text{Br}(Z)$ is an isomorphism.

*Proof.* We first show that $\text{Br}(X) \xrightarrow{\rho^*} \text{Br}(X \times_k H)$, where the map is induced by the natural projection $X \times_k H \to X$. Using the spectral sequence

$$H^p(k, H^q(X, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m),$$

one only needs to show that

$$\bar{k}[X_\bar{k}]^\times \bar{k} \xrightarrow{\rho^*} \bar{k}[X_\bar{k} \times \bar{k} H_\bar{k}]^\times \bar{k}, \quad \text{Pic}(X_\bar{k}) \xrightarrow{\rho^*} \text{Pic}(X_\bar{k} \times \bar{k} H_\bar{k}) \quad \text{and} \quad \text{Br}(X_\bar{k}) \xrightarrow{\rho^*} \text{Br}(X_\bar{k} \times \bar{k} H_\bar{k}).$$

Since $\bar{k}[H]^\times = \bar{k}^\times$ and $\text{Pic}(H_\bar{k}) = \text{Br}(H_\bar{k}) = 0$ by [9, Proposition 2.6], the first two parts are true by [32, Proposition 6.10]. To prove the last part, Kummer exact sequence ensures that one only needs to prove that

$$H^2_{\text{ét}}(X_\bar{k}, \mathbb{Z}/n) \xrightarrow{\rho^*} H^2_{\text{ét}}(X_\bar{k} \times \bar{k} H_\bar{k}, \mathbb{Z}/n) \quad (2.2)$$

for all $n \geq 1$. This last isomorphism follows from [37, Proposition 2.2] and [13, Exposé XI, Théorème 4.4] with $H^i_{\text{ét}}(H_\bar{k}, \mathbb{Z}/n) = 0$ for $i = 1, 2$. So we proved the required isomorphism $\text{Br}(X) \xrightarrow{\rho^*} \text{Br}(X \times_k H)$.

Let us now deduce Lemma 2.1: since $\text{Pic}(H) = 0$, [2, Proposition 2.4] gives the following short exact sequence

$$0 \to \text{Br}(X) \to \text{Br}(Z) \xrightarrow{m^* - p^*_Z} \text{Br}(H \times_k Z),$$

where $m^*$ and $p^*_Z$ are induced by the multiplication map $H \times_k Z \xrightarrow{m} Z$ and the projection map $H \times_k Z \xrightarrow{p_Z} Z$ respectively. Since $m \circ (1_H \times \text{id}) = p_Z \circ (1_H \times \text{id}) = \text{id}$, one concludes that $m^* = p^*_Z$ by the above argument. Therefore $\text{Br}(X) \xrightarrow{\rho^*} \text{Br}(Z)$. \hfill \Box

Let $H$ be a closed subgroup of an algebraic group $G$ over $k$, and $Y \overset{f}{\to} X$ be a left torsor under $H$. Let $Z \overset{\rho}{\to} X$ be the left torsor under $G$ defined by the contracted product $Z = G \times^H Y$ (see [36, Example 3 in p.21]): the torsor $Z$ is the push-forward of $Y$ by the homomorphism $H \to G$. The projection map $G \times_k Y \xrightarrow{pr_G} G$ induces the following commutative diagram

$$G \times_k Y \xrightarrow{pr_G} Z = G \times^H Y \quad (2.3)$$

$$\downarrow \quad \theta$$

$$G \xrightarrow{\pi} G/H,$$

where $\theta$ is induced by $pr_G$ via the quotient by $H$.

**Lemma 2.4.** With the above notations, for any $\gamma \in (G/H)(k)$, the composite map $\theta^{-1}(\gamma) \to Z \overset{\rho}{\to} X$ is naturally a left torsor under $H^\gamma$, which is canonically isomorphic to the twist of $Y \overset{f}{\to} X$ by the $k$-torsor $\pi^{-1}(\gamma)$ under $H$. 
Proof. It follows from diagram (2.3) and [36, Example 2 in p.20]. □

Let $G$ be a connected linear algebraic group over $k$, and $Y$ be a smooth variety over $k$. Since $G_\bar{k}$ is rational over $\bar{k}$ by Bruhat decomposition, the projections $G \times_k Y \to G$ and $G \times_k Y \to Y$ induce an isomorphism

$$\text{Br}_a(G) \oplus \text{Br}_a(Y) \xrightarrow{\sim} \text{Br}_a(G \times_k Y)$$

by [32, Lemma 6.6]. If $P$ is a (left) torsor under $G$ over $k$ and $H^3(k, \bar{k}^\times) = 0$, the previous result generalizes to an isomorphism

$$\text{Br}_a(P) \oplus \text{Br}_a(Y) \xrightarrow{\sim} \text{Br}_a(P \times_k Y) \quad (2.5)$$

by [3, Lemma 5.1].

Let $G$ be a connected linear algebraic group over $k$ and let $X$ be a smooth variety over $k$ with $H^3(k, \bar{k}^\times) = 0$. Suppose that $Y \xrightarrow{f} X$ is a left torsor under $G$ and $P$ is a left $k$-torsor under $G$, associated to a cocycle $\sigma \in Z^1(k, G)$. One can consider $P$ as a right torsor under $G$ by defining a right action $x \circ g := g^{-1}x$ (see [36, Example 2 in p.20]). This right torsor is called the inverse right torsor of $P$ under $G$, and is denoted by $P'$. One can now consider the map given by the quotient of $P \times_k Y$ by the diagonal action of $G$ given by

$$\chi_P : P \times_k Y \to Y^\sigma := (p \circ g^{-1}, g \cdot y) = (g \cdot p, g \cdot y):$$

Definition 2.6. With the above notation, assuming that $H^3(k, \bar{k}^\times) = 0$, consider the map

$$\psi_\sigma = \psi_P : \text{Br}_a(Y^\sigma) \xrightarrow{\chi_P^*} \text{Br}_a(P \times_k Y) \xleftarrow{\sim} \text{Br}_a(P) \oplus \text{Br}_a(Y) \to \text{Br}_a(Y).$$

The following lemma, which compares the algebraic Brauer groups of twists of a given torsor, can be regarded as an extension of [39, Lemma 1.3] to torsors under connected linear algebraic groups.

Lemma 2.7. The morphism $\psi_\sigma$ in Definition 2.6 is an isomorphism.

Proof. The natural morphism $(pr_P, \chi_P) : P \times_k Y \to P \times_k Y^\sigma$ is an isomorphism, and we have a commutative diagram:

$$
\begin{array}{ccc}
P \times_k Y & \xrightarrow{(pr_P, \chi_P)} & P \times_k Y^\sigma \\
p \downarrow & & \downarrow^{pr_P} \\
P & \xrightarrow{id} & P.
\end{array}
$$

Therefore $(pr_P, \chi_P)^* : \text{Br}_a(Y^\sigma \times_k P) \to \text{Br}_a(Y \times P)$ induces the identity map on the subgroups $\text{Br}_a(P) \subset \text{Br}_1(Y^\sigma \times_k P)$ and $\text{Br}_a(P) \subset \text{Br}_1(Y \times_k P)$, hence

$$\psi_\sigma : \text{Br}_a(Y^\sigma) \to \text{Br}_a(Y^\sigma \times_k P) \xrightarrow{(pr_P, \chi_P)^*} \text{Br}_a(Y \times P) \to \text{Br}_a(Y)$$

is an isomorphism (using the isomorphism (2.5)). □

Let $f : Y \to X$ be a torsor under a connected linear algebraic group $G$ over $k$ and let

$$a_Y : G \times_k Y \to Y$$
be the action of $G$. There is a canonical map $\lambda : Br_1(Y) \to Br_a(G)$ by [32, Lemma 6.4]. Let $e : Br_a(G) \to Br_1(G)$ be the section of $Br_1(G) \to Br_a(G)$ such that $1_G \circ e = 0$. If $X$ is smooth and geometrically integral, then the following diagram

$$\begin{array}{c}
Br_1(Y) \xrightarrow{\lambda} Br_a(G) \\
\downarrow \quad \quad \quad \downarrow p_G^* \circ e \\
Br(Y) \xrightarrow{a_Y^* - p_Y^*} Br(G \times_k Y)
\end{array}$$

(2.8)

commutes by [2, Theorem 2.8], where $G \times_k Y \xrightarrow{p_G} G$ and $G \times_k Y \xrightarrow{p_Y} Y$ are the projections. One can reformulate the commutative diagram (2.8) in the following proposition:

**Proposition 2.9.** With the above notation, one has

$$b(t \cdot x) = \lambda(b)(t) + b(x)$$

for any $x \in Y(k)$, $t \in G(k)$ and $b \in Br_1(Y)$.

**Proof.** The commutativity of diagram (2.8) implies that

$$a_Y^* - p_Y^* = p_G^* \circ e \circ \lambda : Br_1(Y) \to Br_1(G \times Y),$$

therefore one has

$$b(t \cdot x) = a_Y^*(b)(t, x) = p_Y^*(b)(t, x) + p_G^* \circ e \circ \lambda(b)(t, x) = b(x) + \lambda(b)(t)$$

as required. \qed

3. CONNECTED LINEAR ALGEBRAIC GROUPS OR GROUPS OF MULTIPLICATIVE TYPE

In this section, we study the relation between the descent obstruction and the Brauer-Manin obstruction for a general connected linear group or a group of multiplicative type.

First we need the following fact concerning topological groups:

**Lemma 3.1.** Let $f : M \to N$ be an open homomorphism of topological groups. If $K$ is a closed subgroup of $M$ containing ker($f$), then $f(K)$ is a closed subgroup of $N$.

**Proof.** Since $K$ is a closed subgroup containing ker($f$), one has

$$f(K) = f(M) \setminus f(M \setminus K).$$

Since $f$ is an open homomorphism, $f(M)$ is an open subgroup of $N$. This implies that $f(M)$ is closed in $N$. Since $f(M \setminus K)$ is open in $N$, one concludes that $f(K)$ is closed in $N$. \qed

**Remark 3.2.** The assumption $K \supseteq \ker(f)$ in Lemma 3.1 cannot be removed. For example, the projection map $pr^S : A_k \rightarrow A_k^S$ is open where $A_k^S$ is the set of adeles of $k$ without $S$-component. It is clear that $k$ is a discrete subgroup of $A_k$ by the product formula. However $k$ is dense in $A_k^S$ by strong approximation for $\mathbb{G}_a$, when $S$ is not empty.

For a short exact sequence of connected linear algebraic groups, one has the following result.
Proposition 3.3. Let
\[ 1 \to G_1 \xrightarrow{\psi} G_2 \xrightarrow{\phi} G_3 \to 1 \]
be a short exact sequence of connected linear algebraic groups over a number field \( k \). Then
\begin{enumerate}
\item (1) \( \phi \left( G_2(A_k)^{\Br_1(G_2)} \right) \) is a closed subgroup of \( G_3(A_k) \).
\item (2) If \( G'(k_{\infty}) \) is not compact for each simple factor \( G' \) of the semi-simple part of \( G_3 \), then one has
\[ G_3(A_k)^{\Br_1(G_3)} = G_3(k) \cdot \phi \left( G_2(A_k)^{\Br_1(G_2)} \right) . \]
\end{enumerate}

Proof. Let \( S \) be a sufficiently large finite set of primes of \( \Omega_k \) containing \( \infty_k \) and let \( G_1 \) (resp. \( G_2 \), resp. \( G_3 \)) be a smooth group scheme model of \( G_1 \) (resp. \( G_2 \), resp. \( G_3 \)) over \( O_S \) with connected fibres, such that the short exact sequence of smooth group schemes
\[ 1 \to G_1 \xrightarrow{\psi} G_2 \xrightarrow{\phi} G_3 \to 1 \]
extends the given short exact sequence of their generic fibres. The set \( H^1_{\et}(O_v, G_1) \) is trivial by Hensel’s lemma together with Lang’s theorem, and the following diagram
\[ \begin{array}{cccc}
G_3(O_v) & \xrightarrow{\partial_v} & H^1_{\et}(O_v, G_3) \\
\downarrow & & \downarrow \\
G_3(k_v) & \xrightarrow{\partial_v} & H^1(k_v, G_3) \\
\end{array} \]
commutes, hence we deduce the following commutative diagram of exact sequences in Galois cohomology:
\[ \begin{array}{cccc}
G_1(k) & \xrightarrow{\psi} & G_2(k) & \xrightarrow{\phi} & G_3(k) & \xrightarrow{\partial} & H^1(k, G_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_1(A_k) & \xrightarrow{(\psi_v)} & G_2(A_k) & \xrightarrow{(\phi_v)} & G_3(A_k) & \xrightarrow{(\partial_v)} & \bigoplus_{v \in \Omega_k} H^1(k_v, G_1) .
\end{array} \]

In addition, [17, Theorem 5.1] and [32, Corollary 6.11] gives the following commutative diagram of exact sequences of topological groups and pointed topological spaces:
\[ \begin{array}{cccc}
G_1(A_k) & \xrightarrow{\theta_1} & \Br_a(G_1)^D & \xrightarrow{\III^1(k, G_1)} \\
\downarrow & & \downarrow & & \\
1 & \xrightarrow{\ker(\theta_2)} & G_2(A_k) & \xrightarrow{\theta_2} & \Br_a(G_2)^D \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \bigoplus_{v \in \Omega_k} H^1(k_v, G_1) ,
\end{array} \]
where $\text{Br}_a(G_i)^D$ is the topological dual of the discrete group $\text{Br}_a(G_i)$, for $1 \leq i \leq 3$. Since $\theta_1(G_1(A_k))$ is the kernel of the continuous map $\text{Br}_a(G_1)^D \to \mathbb{III}^1(k, G_1)$, it is a closed subgroup of $\text{Br}_1(G)^D$. Since $(\psi^*)^D$ is a closed map, one obtains that $(\psi^*)^D(\theta_1(G_1(A_k)))$ is a closed subgroup of $\text{Br}_1(G_2)^D$. It implies that

$$\ker(\theta_2) \cdot \psi(G_1(A_k)) = \theta_2^{-1} [(\psi^*)^D(\theta_1(G_1(A_k)))]$$

is a closed subgroup of $G_2(A_k)$ by diagram (3.4). Proposition 6.5 in Chapter 6 of [30] ensures that $\phi : G_2(A_k) \to G_3(A_k)$ is an open homomorphism of topological groups. Then $\phi(\ker(\theta_2)) = (G_2(A_k)^{\text{Br}_1(G_2)})$ is closed by Lemma 3.1, and property (1) follows.

Let us now prove statement (2): Corollary 3.20 in [17] (see also the proof of Proposition 4.5 in [5]) implies that

$$\ker(\theta_3) = G_3(A_k)^{\text{Br}_1(G_3)} = G_3(k) \cdot \phi (G_2(A_k)^{\text{Br}_1(G_2)}),$$

where $G_3(k)$ is the connected component of identity with respect to the topology of $k$. One only needs to show that

$$G_3(A_k)^{\text{Br}_1(G_3)} \subseteq G_3(k) \cdot \phi (G_2(A_k)^{\text{Br}_1(G_2)}).$$

For any $(x_v) \in G_3(k) \cdot G_3(k)$, there is $h \in G_3(k)$ and $h_\infty \in G_3(k)$ such that

$$(\partial_e)(h \cdot h_\infty) = (\partial_e)(x_v),$$

because $(\partial_e)$ is a continuous map with respect to the discrete topology of $\bigoplus_{e \in \Omega_k} H^1(k, G_1)$. Since $\phi_\infty(G_2(k))$ is open and connected, the finiteness of $H^1(k, G_1)$ gives

$$G_3(k) = \phi_\infty(G_2(k)).$$

Therefore

$$(h \cdot h_\infty) \in G_3(k) \cdot \phi (G_2(A_k)^{\text{Br}_1(G_2)})$$

and one can replace $(x_v)$ by $(h \cdot h_\infty)^{-1} \cdot (x_v)$. Without loss of generality, one can therefore assume $(\partial_e)(x_v)$ is the trivial element in $\bigoplus_{e \in \Omega_k} H^1(k, G_1)$.

Since $\mathbb{III}^1(k, G_1)$ is finite, one can fix $\xi_1, \cdots, \xi_n$ in $G_3(k)$ such that each element of $\mathbb{III}^1(k, G_1) \cap \partial(G_3(k))$ is represented by one of the $\xi_i$’s. As $(\partial_e)(h_\infty)$ is trivial for any $h_\infty \in G_3(k) \infty$, one concludes that

$$(x_v) \in \bigcup_{i=1}^n \xi_i \phi(\ker(\theta_2)) = \bigcup_{i=1}^n \xi_i \cdot \phi(\ker(\theta_2)) \subseteq G_3(k) \cdot \phi (G_2(A_k)^{\text{Br}_1(G_2)}).$$

by Corollary 1 in Page 50 of [33] and assertion (1).

The main result of this section is the following theorem:

**Theorem 3.5.** Let $X$ be a smooth and geometrically integral variety and let $G$ be a connected linear algebraic group or a group of multiplicative type over a number field $k$. Suppose that $f : Y \to X$ is a left torsor under $G$. If $A$ is a subgroup of $\text{Br}(X)$ which contains the kernel of the natural map $f^* : \text{Br}(X) \to \text{Br}(Y)$, then

$$X(A_k)^D = \bigcup_{\sigma \in H^1(k, G)} f \sigma (Y^*(A_k)^{f^*_\sigma(A)}) ,$$

where $Y^*(A_k)^{f^*_\sigma(A)}$ is a subgroup of $Y^*(A_k)$.
where $Y^\sigma \xrightarrow{f^\sigma} X$ is the twist of $f$ by $\sigma$ and $\text{Br}(X) \xrightarrow{f^\sigma} \text{Br}(Y^\sigma)$ is the associated pull-back morphism, for each $\sigma \in H^1(k, G)$.

**Proof.** By the functoriality of Brauer-Manin pairing, one only needs to show that

$$X(A_k) \subseteq \bigcup_{\sigma \in H^1(k, G)} f_{\sigma} \left( Y^\sigma(A_k) \right).$$

It is clear that

$$(x_v) \in \bigcup_{\sigma \in H^1(k, G)} f_{\sigma}(Y^\sigma(A_k)) \iff ([Y](x_v)) \in \text{Im} \left[ H^1(k, G) \to \prod_{v \in \Omega_k} H^1(k_v, G) \right]. \quad (3.6)$$

(1) Assume that $G$ is connected.

Recall first that Hensel’s lemma together with Lang’s theorem ensures that $H^1(k, G)$ maps to $\bigoplus_{v \in \Omega_k} H^1(k_v, G)$. Since any element $P \in \text{Pic}(G)$ can be given the structure of a central extension of algebraic groups

$$1 \to G_m \to P \to G \to 1 \quad (3.7)$$

by [6, Corollary 5.7], one obtains a coboundary map

$$\partial_P : H^1(X, G) \to H^2(X, \mathbb{G}_m) = \text{Br}(X)$$

associated to $P$ (see [19, IV.4.4.2]). Then the map defined by

$$\Delta_{Y/X} : \text{Pic}(G) \to \text{Br}(X), \quad P \mapsto \partial_P([Y])$$

appears in the following short exact sequence (see [2, Theorem 2.8])

$$\text{Pic}(G) \xrightarrow{\Delta_{X/Y}} \text{Br}(X) \xrightarrow{f^*} \text{Br}(Y). \quad (3.8)$$

For any $v \in \Omega_k$, the exact sequence (3.7) defines a coboundary map

$$\partial_{P}^{k_v} : H^1(k_v, G) \to H^2(k_v, \mathbb{G}_m) = \text{Br}(k_v).$$

One can therefore define a pairing

$$\delta_v : H^1(k_v, G) \times \text{Pic}(G) \to \text{Br}(k_v) \subseteq \mathbb{Q}/\mathbb{Z}, \quad (\sigma_v, P) \mapsto \partial_{P}^{k_v}(\sigma_v)$$

such that the following diagram commutes (see Proposition 2.9 in [9]). These pairings induce a pairing

$$(\delta_v)_{v \in \Omega_k} : \bigoplus_{v \in \Omega_k} H^1(k_v, G) \times \text{Pic}(G) \to \mathbb{Q}/\mathbb{Z}, \quad ((\sigma_v)_{v \in \Omega_k}, P) \mapsto \sum_{v \in \Omega_k} \delta_v(\sigma_v, P) \in \mathbb{Q}/\mathbb{Z}$$
and a natural exact sequence of pointed sets

$$H^1(k, G) \to \bigoplus_{v \in \Omega_k} H^1(k_v, G) \to \text{Hom}(	ext{Pic}(G), \mathbb{Q}/\mathbb{Z})$$

by [9, Theorem 3.1]. Therefore (3.6) is equivalent to the fact that $$([Y](x_v)) \in \bigoplus_{v \in \Omega_k} H^1(k_v, G)$$ is orthogonal to Pic($G$) for the pairing $\langle \delta_v \rangle_{v \in \Omega_k}$. The commutative diagram (3.9), together with (3.8), gives

$$X(A_k)^{\text{ker}(f^*)} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k)).$$

Since $\ker(f^*) \subseteq A$, one has

$$X(A_k)^A \subseteq X(A_k)^{\text{ker}(f^*)} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k)).$$

Then the functoriality of the Brauer-Manin pairing implies that

$$X(A_k)^A \subseteq \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k)^{G(A)}).$$

(2) When $G$ is a group of multiplicative type, one obtains that (3.6) is equivalent to

$$\sum_{v \in \Omega_k} \text{inv}_v(\chi \cup [Y])(x_v) = 0$$

for all $\chi \in H^1(k, \hat{G})$ by [16, Theorem 6.3]. Let

$$\mathcal{K}_f = \langle \{\chi \cup [Y] : \chi \in H^1(k, \hat{G})\} \rangle$$

be the subgroup of $\text{Br}(X)$ generated by elements $\chi \cup [Y]$, where $\cup$ is the cup product

$$\cup : H^1(k, \hat{G}) \times H^1(X, G) \to H^2(X, \mathbb{G}_m) = \text{Br}(X).$$

Then

$$X(A_k)^{\mathcal{K}_f} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k))$$

by [26, Proposition 3.1]. Functoriality of the cup product proves that the following diagram

$$\begin{array}{ccc}
H^1(k, \hat{G}) \times H^1(X, G) & \overset{\cup}{\longrightarrow} & H^2(X, \mathbb{G}_m) = \text{Br}(X) \\
\text{id} \times f^* & \downarrow & f^* \\
H^1(k, \hat{G}) \times H^1(Y, G) & \overset{\cup}{\longrightarrow} & H^2(Y, \mathbb{G}_m) = \text{Br}(Y)
\end{array}$$

is commutative. Since $Y \overset{f}{\longrightarrow} X$ becomes a trivial torsor over $Y$, the above diagram gives $\mathcal{K}_f \subseteq \ker(f^*)$. Since $\mathcal{K}_f \subseteq \ker(f^*) \subseteq A$, one has

$$X(A_k)^A \subseteq X(A_k)^{\mathcal{K}_f} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k)).$$
Then the functoriality of the Brauer-Manin pairing implies that
\[ X(A_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_{\sigma} \left( Y_{\sigma}(A_k)^{f_{\sigma}(A)} \right). \]

\[ \square \]

4. REFINEMENT IN THE TORIC CASE

In this section, we will refine Theorem 3.5 for torsors under tori.

**Theorem 4.1.** Let \( f : Y \rightarrow X \) be a torsor under a torus \( G \) over a number field \( k \). Assume that \( X \) is smooth and geometrically integral. Let \( \ker(f^*) \subseteq A \subseteq \Br(X) \) be a subgroup, and for all \( \sigma \in H^1(k,G) \), let \( B_\sigma \subseteq \Br_1(Y^\sigma) \) be a subgroup such that
\[
f^* - 1 \left( \sum_{\sigma \in H^1(k,G)} \psi(\overline{B}_\sigma) \right) \subseteq A,
\]
where \( \Br_a(Y^\sigma) \xrightarrow{\psi} \Br_a(Y) \) is the morphism of Definition 2.6 and \( \overline{B}_\sigma \) is the image of \( B_\sigma \) in \( \Br_a(Y^\sigma) \).

Then one has
\[
X(A_k)^A = \bigcup_{\sigma \in H^1(k,G)} f_{\sigma} \left( Y_{\sigma}(A_k)^{B_\sigma + f_{\sigma}(A)} \right)
\]
where \( Y^\sigma \xrightarrow{\sigma} X \) is the twist of \( Y \xrightarrow{\sigma} X \) by \( \sigma \).

**Proof.** Since
\[
\bigcup_{\sigma \in H^1(k,G)} f_{\sigma} \left( Y_{\sigma}(A_k)^{B_\sigma + f_{\sigma}(A)} \right) \subseteq \bigcup_{\sigma \in H^1(k,G)} f_{\sigma} \left( Y_{\sigma}(A_k)^{f_{\sigma}(A)} \right) \subseteq X(A_k)^A
\]
by the functoriality of Brauer-Manin pairing, one only needs to prove the converse inclusion.

Step 1. We first prove the result when \( \tilde{G} \) is a permutation Galois module. In this case, Shapiro Lemma and Hilbert 90 gives \( H^1(K, G) = \{1\} \) for any field extension \( K/k \). This implies that
\[
X(A_k)^A = f \left( Y(A_k)^{f_*(A)} \right)
\]
by the functoriality of Brauer-Manin pairing.

Let \( (x_v) \in X(A_k)^A \). Then there is \( (y_v) \in Y(A_k)^{f_*(A)} \) such that \( (x_v) = f((y_v)) \).

By Proposition 6.10 (6.10.3) in [32], the natural sequence
\[
\Br_1(X) \xrightarrow{f_*} \Br_1(Y) \xrightarrow{\lambda} \Br_a(G)
\]
is exact, and it induces the exact sequence
\[
(f^*)^{-1}(B) \xrightarrow{f_*} B \xrightarrow{\lambda} \Br_a(G)
\]
for any subgroup \( B \subseteq \Br_1(Y) \). Therefore the following sequence
\[
\Br_a(G)^D \xrightarrow{\lambda^D} B^D \xrightarrow{(f^*)^D} ((f^*)^{-1}(B))^D
\]
is exact. Assuming \((f^*)^{-1}(B) \subset A\), one has \((f^*)^D((y_v)) = 0\), where we (abusively) identify \((y_v)\) with its image in \(B^D\) via the Brauer-Manin pairing. By the aforementioned exactness, there is \(\xi \in \Br_a(G)^D\) such that \(\lambda^D(\xi) = (y_v)\). Since \(\mathfrak{M}_1^1(k, G) = \{1\}\), Theorem 2 in [22] implies that every element in \(\Br_a(G)^D\) is given by an element in \(G(A_k)\) via the Brauer-Manin pairing. Namely, there is \((g_v) \in G(A_k)\) such that
\[
b(\lambda(b)(g_v)) = b(\lambda(b)(g_v))\]
for all \(b \in B\). Then \((g_v)^{-1} \cdot (y_v) \in Y(A_k)^{1+f^*(A)}\) by Proposition 2.9, and \((x_v) = f((g_v)^{-1} \cdot (y_v))\).

Step 2. We now prove the case of an arbitrary torus \(G\). By Proposition-Definition 3.1 in [6], there is a short exact sequence of tori
\[
1 \to G \to T_0 \xrightarrow{\theta} T_1 \to 1,
\]
such that \(\hat{T}_0\) is a permutation Galois module and \(\hat{T}_1\) is a coflasque Galois module. Since
\[
H^3(k, \hat{T}_1) \cong \prod_{v \in \mathcal{S}_k} H^3(k_v, \hat{T}_1) \cong \prod_{v \in \mathcal{S}_k} H^1(k_v, \hat{T}_1) = \{1\}
\]
(see for instance Proposition 5.9 in [27]), the map \(\Br_1(T_0) \to \Br_1(G)\) is surjective.

Let \(Z \xrightarrow{\theta} X\) be the torsor under \(T_0\) defined by \(Z := T_0 \times^G Y\). We have a morphism of torsors under \(G\):
\[
Y \xrightarrow{\varepsilon_0 \times \id_Y} T_0 \times_k Y \xrightarrow{\chi} Z = T_0 \times^G Y
\]
where \(\varepsilon_0 \in T_0(k)\) is the unit element, \(p_0\) is the projection map and \(\theta\) is given as in (2.3). For simplicity, denote by \(i := \chi \circ (\varepsilon_0 \times \id_Y) : Y \to Z\) the composite morphism defined in the previous diagram.

Then Proposition 6.10 (6.10.3) in [32] gives the following commutative diagram of exact sequences:
\[
\begin{array}{ccc}
\Br_1(T_1) & \xrightarrow{q^*} & \Br_1(T_0) & \xrightarrow{\rho_0^*} & \Br_a(G) \\
\downarrow \theta^* & & \downarrow \rho_0^* & & \downarrow \id \\
\Br_1(Z) & \xrightarrow{\varepsilon_0^\times \id_Y^*} & \Br_1(T_0 \times_k Y) & \xrightarrow{\chi^*} & \Br_a(G).
\end{array}
\]

Since the following sequence
\[
\begin{array}{ccc}
\Br_1(T_0) & \xrightarrow{p_0^*} & \Br_1(T_0 \times_k Y) \\
& & \xrightarrow{(\varepsilon_0 \times \id_Y)^*} \Br_a(Y) \to 1
\end{array}
\]
is exact by Lemma 6.6 in [32], the surjectivity of the map \(\Br_1(T_0) \to \Br_1(G)\) implies that the morphism
\[
i^* : \Br_1(Z) \to \Br_1(Y)
\]
is surjective, by a simple diagram chase.
Lemma 2.4 implies that for any \( t \in T_1(k) \), the composite morphism \( \theta^{-1}(t) \to Z \xrightarrow{\phi} X \) is canonically isomorphic to the twist \( f_t : Yq^{-1}(t) \to X \) of \( f : Y \to X \) by the Spec(\( k \))-torsor \( q^{-1}(t) \) under \( G \).

Denote by \( i_t : \theta^{-1}(t) \to Z \) the closed immersion. Then \( f_t = \rho \circ i_t \) for any \( t \in T_1(k) \).

Let \( \chi_t \) be the restriction of \( \chi \) to \( q^{-1}(t) \times_k Y \) for any \( t \in T_1(k) \). Then the following diagram

\[
\begin{array}{ccc}
q^{-1}(t) \times_k Y & \xrightarrow{\chi_t} & Yq^{-1}(t) \\
\downarrow j_t \times \text{id}_Y & & \downarrow i_t \\
Y & \xrightarrow{e_0 \times \text{id}_Y} & T_0 \times_k Y \\
\downarrow p_0 & & \downarrow q \\\nG & \xrightarrow{} & T_0 & \xrightarrow{q} & T_1
\end{array}
\]

is commutative, where \( j_t : q^{-1}(t) \to T_0 \) is the closed immersion of the fiber of \( q \) at \( t \). Therefore Definition 2.6 implies that we have a commutative triangle:

\[
\begin{array}{ccc}
\text{Br}_0(Z) & \xrightarrow{i_t^*} & \text{Br}_0(Yq^{-1}(t)) \\
\downarrow i^* & & \downarrow \psi_{q^{-1}(t)} \sim \psi_{q^{-1}(t)} \\
& & \text{Br}_0(Y)
\end{array}
\]

i.e. that \( \psi_{q^{-1}(t)} \circ i_t^* = i^* \).

Let

\[
B = i^* \left( \sum_{t \in T_1(k)} \psi_{q^{-1}(t)} \left( \overline{B_{q^{-1}(t)}} \right) \right) \subseteq \text{Br}_0(Y)
\]

where \( \overline{B_{q^{-1}(t)}} \) is the image of \( B_{q^{-1}(t)} \) in \( \text{Br}_0(Yq^{-1}(t)) \) and \( \psi_{q^{-1}(t)} \) is given by Definition 2.6 for all \( t \in T_1(k) \).

Since \( i^* \circ \rho^* = f^* \), we have

\[
\rho^{-1}(B) = f^{-1} \left( \sum_{t \in T_1(k)} \psi_{q^{-1}(t)} \left( \overline{B_{q^{-1}(t)}} \right) \right) \subseteq A,
\]

hence step 1 applied to the torsor \( Z \xrightarrow{\phi} X \) under \( T_0 \) implies that

\[
X(A_k)^A = \rho \left( (Z(A_k)^{B+\rho^*(A)}) \right). \tag{4.2}
\]

Let \( (x_v) \in X(A_k)^A \). By (4.2), there is \( (z_v) \in Z(A_k)^{B+\rho^*(A)} \) such that \( (x_v) = \rho((z_v)) \). Since

\[
i^* \circ \theta^*(\text{Br}_1(T_1)) = (e_0 \times \text{id}_Y)^* \circ p_0^* \circ q^* (\text{Br}_1(T_1)) = \text{Br}_0(Y)
\]

and \( i^*(\text{Br}_0(Z)) = \text{Br}_0(Y) \), one gets \( \theta^*(\text{Br}_1(T_1)) \subseteq \text{Br}_0(Z) + B \) (by construction, \( B \) contains \( \ker(i^* : \text{Br}_1(Z) \to \text{Br}_1(Y)) \)). Functoriality of the Brauer-Manin pairing now gives

\[
\theta((z_v)) \in T_1(A_k)^{\text{Br}_1(T_1)}.
\]
By Proposition 3.3, there are \( \alpha \in T_1(k) \) and \( (\beta_v) \in T_0(A_k)^{Br_1(T_0)} \) such that \( \theta((z_v)) = \alpha \cdot q(\beta_v) \). Therefore \( (\beta_v)^{-1} \cdot (z_v) \in \theta^{-1}(\alpha) \), hence \( (\beta_v)^{-1} \cdot (z_v) \in Z(A_k)^{G+\rho^*(A)} \).

Since \( i^*: Br_1(Z) \to Br_1(Y) \) is surjective, one has

\[
\psi_{q^{-1}(\alpha)} \circ i^*_\alpha(\tilde{B}) = i^*(\tilde{B}) = \sum_{t \in T_1(k)} \psi_{q^{-1}(t)} \left( \tilde{B}_{q^{-1}} \right) \supseteq \psi_{q^{-1}(\alpha)} \left( \tilde{B}_{q^{-1}(\alpha)} \right),
\]

where \( \tilde{B} \) is the image of \( B \) in \( Br_a(Z) \). It implies that \( i^*_\alpha(B) + Br_0(\theta^{-1}(\alpha)) \supseteq B_{q^{-1}(\alpha)} \) by Lemma 2.7, and

\[
(\beta_v)^{-1} \cdot (z_v) \in \left[ \theta^{-1}(\alpha)(A_k) \right]^{i^*_\alpha(B) + (i^*_\alpha \circ \rho^*)(A)} \subseteq \left[ \theta^{-1}(\alpha)(A_k) \right]^{Br_{q^{-1}(\alpha)} + (i^*_\alpha \circ \rho^*)(A)}
\]
as desired. \( \square \)

The first part of the following result is also proved in Theorem 1.7 of [39].

**Corollary 4.3.** Let \( X \) be a smooth and geometrically integral variety. If \( f: Y \to X \) is a torsor under a torus \( G \) over a number field \( k \), then

\[
X(A_k)^{Br_1(X)} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma \left( Y^\sigma(A_k)^{Br_1(Y^\sigma)} \right)
\]

and

\[
X(A_k)^{Br} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma \left( Y^\sigma(A_k)^{Br_1(Y^\sigma)+f_\sigma(Br_1(X))} \right).
\]

**Proof.** To get the first equality, apply Theorem 4.1 to \( A = Br_1(X) \) and \( B_\sigma = Br_1(Y^\sigma) \) for each \( \sigma \in H^1(k,G) \). Since \( \text{Pic}(G_k) = 0 \), Proposition 6.10 in [32] gives

\[
f^{\sigma^{-1}} \left( \sum_{\sigma \in H^1(k,G)} \psi_\sigma \left( \tilde{B}_\sigma \right) \right) \subseteq f^{\sigma^{-1}}(Br_a(Y)) \subseteq Br_1(X) = A,
\]
as required.

The second equality follows from Theorem 4.1 by taking \( A = Br(X) \) and \( B_\sigma = Br_1(Y^\sigma) \) for each \( \sigma \in H^1(k,G) \). \( \square \)

5. **AN APPLICATION**

In this section, we apply the previous results to study the necessary conditions for a connected linear algebraic group to satisfy strong approximation with Brauer-Manin obstruction.

When \( X \) is affine, the set \( X(k) \) is discrete in \( X(A_k) \) by the product formula. Therefore if such an \( X \) satisfies strong approximation off \( S \), then \( \prod_{v \in S} X(k_v) \) is not compact. However this necessary condition for strong approximation is no longer true for strong approximation with Brauer-Manin obstruction if \( Br(X)/Br(k) \) is not finite. For example, a torus \( X \) always satisfies strong approximation with Brauer-Manin obstruction off \( \infty_k \), \( X \) being anisotropic over \( k_\infty \) or not: see [22, Theorem 2]. When \( X \) is a semi-simple linear algebraic group, the necessary and sufficient condition for \( X \) to satisfy strong approximation with Brauer-Manin obstruction is
given by Proposition 6.1 in [5]. In this section, we extend this result to a general connected linear algebraic group.

The following lemma explains that strong approximation with Brauer-Manin obstruction for a general connected linear algebraic group can be reduced to the reductive case.

**Lemma 5.1.** Let $G$ be a connected linear algebraic group over a number field $k$.

If $\pi : G \to G_{\text{red}}$ is the quotient map, then $G_{\text{red}}(A_k)^{\text{Br}_1(G_{\text{red}})} = \pi(G(A_k)^{\text{Br}_1(G)})$.

In particular, for any finite subset $S$ of $\Omega_k$, $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$ if and only if $G_{\text{red}}$ satisfies strong approximation with respect to $\text{Br}_1(G_{\text{red}})$ off $S$.

**Proof.** By applying Lemma 2.1 for $k$ and $\bar{k}$, one obtains that $\pi^*(\text{Br}_1(G_{\text{red}})) = \text{Br}_1(G)$. The first part follows from Theorem 3.5 and Proposition 6 of §2.1 of Chapter III in [33].

Suppose $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$. For any open subset

$$M = \prod_{v \in S} G_{\text{red}}(k_v) \times \prod_{v \notin S} M_v$$

of $G_{\text{red}}(A_k)$ such that $M \cap \left[ G_{\text{red}}(A_k)^{\text{Br}_1(G_{\text{red}})} \right] \neq \emptyset$, one has that

$$\pi^{-1}(M) = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} \pi^{-1}(M_v)$$

with $\pi^{-1}(M) \cap G(A_k)^{\text{Br}_1(G)} \neq \emptyset$ by the first part. Then by assumption there is $x \in G(k) \cap \pi^{-1}(M)$. It implies that $\pi(x) \in M \cap G_{\text{red}}(k)$, as required.

Conversely, suppose $G_{\text{red}}$ satisfies strong approximation with respect to $\text{Br}_1(G_{\text{red}})$ off $S$. For any open subset

$$N = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} N_v$$

of $G(A_k)$ such that $N \cap G(A_k)^{\text{Br}_1(G)} \neq \emptyset$, we have

$$\pi(N) = \prod_{v \in S} G_{\text{red}}(k_v) \times \prod_{v \notin S} \pi(N_v)$$

and this set is an open subset of $G_{\text{red}}(A_k)$, with $\pi(M) \cap \left[ G_{\text{red}}(A_k)^{\text{Br}_1(G_{\text{red}})} \right] \neq \emptyset$: here we use Proposition 6 of §2.1 of Chapter III in [33], Proposition 6.5 in Chapter 6 of [30] and the functoriality of Brauer-Manin pairing. Then by assumption there is $y \in G_{\text{red}}(k) \cap \pi(N)$. Using Proposition 6 of §2.1 of Chapter III in [33] one more time, one concludes that $\pi^{-1}(y)$ is isomorphic to $R_u(G)$ as an algebraic variety, hence it satisfies strong approximation off $S$. Since

$$\pi^{-1}(y) \cap N = \prod_{v \in S} \pi^{-1}(y)(k_v) \times \prod_{v \notin S} (\pi^{-1}(y)(k_v) \cap N) \neq \emptyset,$$

there is $z \in \pi^{-1}(y)(k) \cap N \subset G(k) \cap N$, as desired. \qed

The main result of this section is the following statement:
**Theorem 5.2.** Let $G$ be a connected linear algebraic group over a number field $k$ and let $G^{\text{qs}} := G/R(G)$, where $R(G)$ is the solvable radical of $G$. If $\pi : G \to G^{\text{qs}}$ is the quotient map, then

$$G^{\text{qs}}(A_k)^{\text{Br}_1(G^{\text{qs}})} = \pi \left( G(A_k)^{\text{Br}_1(G)} \right) \cdot G^{\text{qs}}(k).$$

In particular, if $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off a finite subset $S$ of $\Omega_k$, then $G^{\text{qs}}$ satisfies strong approximation with respect to $\text{Br}_1(G^{\text{qs}})$ off $S$.

**Proof.** For the first part, by functoriality of the Brauer-Manin pairing, one only needs to prove

$$G^{\text{qs}}(A_k)^{\text{Br}_1(G^{\text{qs}})} \subseteq \pi \left( G(A_k)^{\text{Br}_1(G)} \right) \cdot G^{\text{qs}}(k).$$

By Lemma 5.1, we can assume that $G$ is reductive. Then $R(G)$ is a torus contained in the center of $G$ (see Theorem 2.4 in Chapter 2 of [30]) and $\pi : G \to G^{\text{qs}}$ is a torsor under $R(G)$. By Corollary 4.3, for any $(x_v) \in G^{\text{qs}}(A_k)^{\text{Br}_1(G^{\text{qs}})}$, there are $\sigma \in H^1(k, R(G))$ and $(y_v) \in G^{\sigma}(A_k)^{\text{Br}_1(G^{\sigma})}$ such that $(x_v) = \pi_\sigma((y_v))$. Since $G^{\sigma}(k) \neq \emptyset$ by Corollary 8.7 in [32] (see also Theorem 5.2.1 in [36]), there is $\gamma \in G^{\text{qs}}(k)$ such that $\partial(\gamma) = \sigma$, where $\partial$ is the coboundary map in the following exact sequence in Galois cohomology:

$$1 \to R(G)(k) \to G(k) \to G^{\text{qs}}(k) \xrightarrow{\partial} H^1(k, R(G)) \to H^1(k, G).$$

In addition, the choice of an element $\tilde{\gamma} \in G(\bar{k})$ such that $\pi(\tilde{\gamma}) = \gamma$ defines a commutative diagram defined over $k$:

$$\begin{array}{ccc}
G^{\sigma} & \xrightarrow{\tilde{\gamma}} & G \\
\pi_\sigma \downarrow & & \downarrow \pi \\
G^{\text{qs}} & \xrightarrow{\gamma} & G^{\text{qs}}
\end{array}$$

(see for instance Example 2 of p.20 in [36]). This implies that

$$\pi_\sigma \left( G^{\sigma}(A_k)^{\text{Br}_1(G^{\sigma})} \right) = \pi \left( G(A_k)^{\text{Br}_1(G)} \right) \cdot \gamma,$$

as desired.

Suppose now that $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$. For any open subset

$$M = \prod_{v \in S} G^{\text{qs}}(k_v) \times \prod_{v \not\in S} M_v$$

of $G^{\text{qs}}(A_k)$ such that $M \cap G^{\text{qs}}(A_k)^{\text{Br}_1(G^{\text{qs}})} \neq \emptyset$, the first part implies that there is $g \in G^{\text{qs}}(k)$ such that

$$\pi^{-1}(M \cdot g) = \prod_{v \in S} G(k_v) \times \prod_{v \not\in S} \pi^{-1}(M_v \cdot g),$$

with $\pi^{-1}(M \cdot g) \cap G(A_k)^{\text{Br}_1(G)} \neq \emptyset$. Since $G$ satisfies strong approximation with algebraic Brauer-Manin obstruction off $S$, there exists $x \in G(k) \cap \pi^{-1}(M \cdot g)$. This implies that $\pi(x) \cdot g^{-1} \in M \cap G^{\text{qs}}(k)$ as required. \qed
Corollary 5.3. Let $G$ be a connected linear algebraic group over a number field $k$ and let $S$ a finite subset of $\Omega_k$ containing $\infty_k$. Then $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$ if and only if $\prod_{v \in S} G'(k_v)$ is not compact for any non-trivial simple factor $G'$ of the semi-simple part $G^{ss}$ of $G$.

Proof. By Theorem 2.3 and Theorem 2.4 of Chapter 2 in [30], the quotient map

$$G^{\text{red}} \to G/R(G) = G^{qs}$$

induces an isogeny $G^{ss} \to G^{qs}$. One side follows from Corollary 3.20 in [17]. The other side follows from Theorem 5.2 and Proposition 6.1 in [5]. \hfill \square

Remark 5.4. All the results in this section involve the group $\text{Br}_1(G)$, and they remain true with $\text{Br}_1(G)$ replaced by $\text{Br}(G)$. Indeed, there is a sufficiently large subset $S$ of $\Omega_k$ containing $\infty_k$ such that $\prod_{v \in S} G'(k_v)$ is not compact for any non-trivial simple factor $G'$ of $G^{ss}$, therefore Corollary 3.20 in [17], Proposition 2.6 in [9] and the functoriality of Brauer-Manin pairing gives the following inclusions:

$$G(A_k)^{\text{Br}_1(G)} = G(k) \cdot \rho(\prod_{v \in S} G^{\text{scu}}(k_v)) \subseteq G(A_k)^{\text{Br}(G)} \subseteq G(A_k)^{\text{Br}_1(G)},$$

where $G^{\text{scu}} = G^{\text{sc}} \times_{G^{\text{red}}} G$ with the projection map $G^{\text{scu}} \twoheadrightarrow G$ and $G^{\text{sc}}$ is the simply connected covering of $G^{ss}$. In particular, we have $G(A_k)^{\text{Br}(G)} = G(A_k)^{\text{Br}_1(G)}$.

6. Comparison I, $X(A_k)^{\text{desc}} \subseteq X(A_k)^{\text{ét}, \text{Br}}$

Let $Y \xrightarrow{f} X$ be a left torsor under a linear algebraic group $G$ over a number field $k$. The fundamental problem to define the descent obstruction for strong approximation with respect to $Y \xrightarrow{f} X$ is to decide whether the set

$$X(A_k)^{f} = \left\{(x_v) \in X(A_k) : ([Y](x_v)) \in \text{Im} \left(H^1(k, G) \to \prod_v H^1(k_v, G)\right)\right\} = \bigcup_{\sigma \in H^1(k, G)} f_{\sigma}(Y^\sigma(A_k))$$

is closed or not in $X(A_k)$. We already know that this is true when $G$ is either connected or a group of multiplicative type, by Theorem 3.5. For a general linear algebraic group $G$, this result is proved by Skorobogatov in Corollary 2.7 of [35], when $X$ is assumed to be proper over $k$. The proof depends on Proposition 5.3.2 in [36] or Proposition 4.4 in [23], which are not true for open varieties, as explained in the following example.

Example 6.1. The short exact sequence of linear algebraic groups

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{f} \mathbb{G}_m \to 1,$$

where $f(x) = x^2$, can be viewed as torsor over $\mathbb{G}_m$ under $\mu_2$. For any $\sigma \in H^1(k, \mu_2) \cong k^\times/(k^\times)^2$, the twist $\mathbb{G}_m^\sigma$ of $\mathbb{G}_m$ by $\sigma$ is given by the equation $x = a_\sigma y^2$ in $\mathbb{G}_m \times_k \mathbb{G}_m$, where $a_\sigma$ is an element in $k^\times$ representing the class $\sigma$ by the above isomorphism. It is clear that $\mathbb{G}_m^\sigma \cong \mathbb{G}_m$ as varieties over $k$, hence it always contains adelic points.

We use the same definition of an integral model as in [28].
Definition 6.2. Let $X$ be a variety over a number field $k$ and let $S$ be a finite subset of $\Omega_k$ containing $\infty_k$. An integral model of $X$ over $O_S$ is a faithfully flat separated $O_S$-scheme $\mathcal{X}_S$ of finite type such that $\mathcal{X}_S \times_{O_S} k \cong X$.

The replacement for Proposition 5.3.2 in [36] or Proposition 4.4 in [23] is the following proposition:

Proposition 6.3. Let $X$ be a variety over a number field $k$ and let $S$ be a finite subset of $\Omega_k$ containing $\infty_k$. Fix an integral model $\mathcal{X}_S$ of $X$ over $O_S$. If $Y \not\to X$ is a left torsor under a linear algebraic group $G$ over $k$, then the set
\[
\{\sigma \in H^1(k, G) : f_\sigma(Y^\sigma(A_k)) \cap \left[ \prod_{v \in S} X(k_v) \times \prod_{v \not\in S} \mathcal{X}_S(O_v) \right] \neq \emptyset \}
\]
is finite.

Proof. It follows from the same argument as the proof of Proposition 4.4 in [23].

One can now extend Corollary 2.7 in [35] to open varieties by using the above replacement for Proposition 4.4 in [23].

Proposition 6.4. Let $X$ be a (not necessarily proper) variety over a number field $k$. If $Y \not\to X$ is a left torsor under a linear algebraic group $G$ over $k$, then the set $X(A_k)^f$ is closed in $X(A_k)$.

Proof. Take an integral model $\mathcal{X}_{S_0}$ of $X$ over $O_{S_0}$, where $S_0$ is a finite subset of $\Omega_k$ containing $\infty_k$. Then
\[
\left\{ \prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} \mathcal{X}_{S_0}(O_v) \right\}_{S}
\]
is an open covering of $X(A_k)$ (see Theorem 3.6 in [11]), where $S$ runs through all finite subsets of $\Omega_k$ containing $S_0$. By Proposition 6.3 and Corollary 2.5 in [35], the set
\[
X(A_k)^f \cap \left[ \prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} \mathcal{X}_{S_0}(O_v) \right]
\]
is closed in $\prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} \mathcal{X}_{S_0}(O_v)$, therefore the set $X(A_k)^f$ is closed in $X(A_k)$.

Applying Proposition 6.3, one can also extend Lemma 2.2 and Theorem 1.1 in [35] to open varieties. For any variety over a number field $k$, and following [35], we write
\[
X(A_k)^{\text{desc}} = \bigcap_{Y \not\to X} X(A_k)^f,
\]
where $Y \not\to X$ runs through all torsors under all linear algebraic groups over $k$ (see also §1.).
Lemma 6.5. Let $X$ be a (not necessarily proper) variety and let $Y \to X$ be a torsor over a number field $k$. For any $(P_v) \in X(\mathbb{A}_k)^{\text{desc}}$, there is a twist $Y' \to X$ of $Y \to X$ such that the following property holds:

For any surjective $X$-torsor morphism $Z \to Y'$ (see Definition 2.1 in [35]), there is a twist $Z' \to Y'$ of $Z \to Y'$ such that $(P_v)$ lies in the image of $Z'(\mathbb{A}_k)$.

Proof. There are a finite subset $S_0$ of $\Omega_k$ containing $\infty_k$ and an integral model $\mathcal{X}_{S_0}$ over $O_{S_0}$ such that

$$(P_v) \in \prod_{v \in S_0} X(k_v) \times \prod_{v \in \Omega_k \setminus S_0} \mathcal{X}_{S_0}(O_v)$$

(see for instance Theorem 3.6 in [11]), hence Proposition 6.3 implies that there are only finitely many twists of a given torsor over $X$ such that $(P_v)$ lifts as an adelic point of this torsor. As pointed out in the proof of Lemma 2.2 in [35], the finite combinatorics in the first part of the proof of Proposition 5.17 in [38] are still valid. It concludes the proof. □

Proposition 6.6. Let $X$ be a (not necessarily proper) variety over a number field $k$. If $Y \overset{f}{\to} X$ is a left torsor under a finite group scheme $F$ over $k$, then

$$X(\mathbb{A}_k)^{\text{desc}} = \bigcup_{\sigma \in H^1(k,F)} f_\sigma(Y^\sigma(\mathbb{A}_k)^{\text{desc}}).$$

Proof. One only needs to modify the proof of Theorem 1.1 in [35] by replacing Lemma 2.2 in [35] with Lemma 6.5, Corollary 2.7 in [35] with Proposition 6.4. Moreover, since $f$ is finite, the induced map $Y(\mathbb{A}_k) \overset{f}{\to} X(\mathbb{A}_k)$ is topologically proper by Proposition 4.4 in [11]. This implies that $f^{-1}((P_v))$ is compact. □

Recall that, following [31], one can define for any variety $X$ over a number field $k$, the set

$$X(\mathbb{A}_k)^{\text{et,Br}} = \bigcap_{Y \overset{f}{\to} X} \bigcup_{\sigma \in H^1(k,F)} f_\sigma(Y^\sigma(\mathbb{A}_k)^{\text{Br}}),$$

where $Y \overset{f}{\to} X$ runs over all torsors under all finite groups $F$ over $k$ (see §1). Since the induced map $Y(\mathbb{A}_k) \overset{f}{\to} X(\mathbb{A}_k)$ is topologically closed for any finite morphism $Y \overset{f}{\to} X$ by Proposition 4.4 in [11], one concludes that $X(\mathbb{A}_k)^{\text{et,Br}}$ is closed in $X(\mathbb{A}_k)$ by the same argument as in Proposition 6.4.

Corollary 6.7. If $X$ is a smooth quasi-projective variety over a number field $k$, then

$$X(\mathbb{A}_k)^{\text{desc}} \subseteq X(\mathbb{A}_k)^{\text{et,Br}} \subseteq X(\mathbb{A}_k)^{\text{Br}}.$$

Proof. One only needs to show that $X(\mathbb{A}_k)^{\text{desc}} \subseteq X(\mathbb{A}_k)^{\text{et,Br}}$. For any torsor $Y \overset{f}{\to} X$ under a finite group scheme $F$, Proposition 6.6 gives the equality

$$X(\mathbb{A}_k)^{\text{desc}} = \bigcup_{\sigma \in H^1(k,F)} f_\sigma(Y^\sigma(\mathbb{A}_k)^{\text{desc}}).$$
Since \( X \) is quasi-projective, \( Y^\sigma \) is quasi-projective as well. By a theorem of Gabber (see [12]), one has

\[
Y^\sigma(A_k)^{\text{desc}} \subseteq Y^\sigma(A_k)^{\text{Br}}
\]

(see the proof of Lemma 2.8 in [35]) and the result follows. \( \square \)

7. COMPARISON II, \( X(A_k)^{\text{et,Br}} \subseteq X(A_k)^{\text{desc}} \)

In this section, we prove the inclusion \( X(A_k)^{\text{et,Br}} \subseteq X(A_k)^{\text{desc}} \) for open varieties, which implies Theorem 1.5. The strategy of proof is the same as in [14].

The second named author would like to thank Laurent Moret-Bailly warmly for finding a mistake and for suggesting the following alternative proof of Lemma 4 in [14] (which already appeared in [15]). The statement of this lemma is correct, but the proof in [14] uses a result of Stoll (see [38]) that is not. Note that in contrast with [14], all torsors (unless explicitly mentioned) are assumed to be left torsors.

**Lemma 7.1.** Let \( X \) be a smooth geometrically connected \( k \)-variety. Let \( (P_v) \in X(A_k)^{\text{et,Br}} \) and let \( Z \xrightarrow{\varphi} X \) be a torsor under a finite \( k \)-group \( F \).

Then there are a cocycle \( \sigma \in Z^1(k,F) \) and a connected component \( X' \) of \( Z^\sigma \) over \( k \) such that the restriction of \( g_\sigma \) to \( X' \) is a torsor \( X' \to X \) under the stabilizer \( F' \) of \( X' \) for the action of \( F^\sigma \), and the point \( (P_v) \) lifts to a point \( (Q_v') \in X'(A_k)^{\text{Br}} \).

In particular, \( X' \) is geometrically integral.

**Proof.** By assumption, the point \( (P_v) \) lifts to some point \( (Q_v) \in Z^\sigma(A_k)^{\text{Br}} \) for some cocycle \( \sigma \) with values in \( F \). Since \( Z^\sigma \) is smooth, \( Z^\sigma \) is a disjoint union of connected components over \( k \). By Proposition 3.3 in [28], there is a \( k \)-connected component \( X' \) of \( Z^\sigma \) such that
\[
(Q_v)_{v \in \Xi} \in P_\Xi(X'(A_k)^{\text{Br}}),
\]
where \( \Xi \) is the set of all complex places of \( k \), \( A_k^\Xi \) is the ring of adeles without \( \Xi \)-components and \( P_\Xi \) is the projection from \( X'(A_k) \) to \( X'(A_k^\Xi) \). Since for \( v \in \Xi \), \( Z^\sigma \times_k k_v \) is a trivial torsor under the finite constant group scheme \( F^\sigma \times_k k_v \), we have \( g_\sigma(X'(k_v)) = X(k_v) \) for all \( v \in \Xi \). Hence one can assume that \( Q_v \in X'(k_v) \) for \( v \in \Xi \), so that we have \( (Q_v) \in X'(A_k)^{\text{Br}} \).

Since \( X' \) is connected and \( X'(A_k) \neq \emptyset \), the proof of Lemma 5.5 in [38] implies that \( X' \) is geometrically connected. Eventually, \( X' \) being geometrically connected guarantees that the variety \( X' \) is an \( X \)-torsor under the stabilizer \( F' \) of \( X' \) in \( F^\sigma \). \( \Box \)

Let us continue the proof of the aforementioned inclusion. Let \( X \) be a smooth and geometrically integral \( k \)-variety, and \( (P_v) \in X(A_k)^{\text{et,Br}} \). We need to prove that \( (P_v) \in X(A_k)^{\text{desc}} \).

For a linear algebraic group \( G \) over \( k \), one has the following short exact sequence of algebraic groups over \( k \):

\[
1 \to H \to G \to F \to 1,
\]
where \( H \) is the connected component of \( G \) and \( F \) is finite over \( k \). This induces the following diagram of short exact sequences

\[
\begin{array}{ccc}
1 & \longrightarrow & H \\
\downarrow & & \downarrow \\
1 & \longrightarrow & T \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G' \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G \\
\downarrow & & \downarrow \\
1 & \longrightarrow & F \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & \longrightarrow & H \\
\downarrow & & \downarrow \\
1 & \longrightarrow & T \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G' \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G \\
\downarrow & & \downarrow \\
1 & \longrightarrow & F \\
\end{array}
\]
where $T$ denotes the maximal toric quotient of $H$ and $G'$ is the quotient of $G$ by the kernel of $H \to T$.

Let $Y \to X$ be a torsor under $G$ and let $Z \to X$ be the push-forward of $Y \to X$ by the morphism $G \to F$, which is a torsor under $F$. If $\sigma \in Z^1(k, F)$ is a 1-cocycle given by Lemma 7.1 applied to the torsor $Z \to X$ and to the point $(P_v)$, we want to show that the cocycle $\sigma \in Z^1(k, F)$ lifts to a cocycle $\tau \in Z^1(k, G)$, as in Proposition 5 in [14]. The obstruction to lift $\sigma$ to a cocycle in $Z^1(k, G)$ gives a natural cohomology class $\eta_{\sigma} \in H^2(k, \kappa_{\sigma})$ by (5.1) in [18] (see also (7.7) in [1]), where $\kappa_{\sigma}$ is a natural $k$-kernel on $H_k$ associated to $\sigma$. Lemma 6 in [14] implies that there is a canonical map $H^2(k, \kappa_{\sigma}) \to H^2(k, T^\sigma)$ such that the class $\eta_{\sigma}$ is neutral if and only if its image $\eta_{\sigma}' \in H^2(k, T^\sigma)$ is zero.

We now apply the open descent theory and the extended type developed by Harari and Skorobogatov in [26] to establish the analogue of Lemma 7 in [14] for open varieties. As in the proof of [14], the torsor $Y \to Z$ under $H$ induces a torsor $W \xrightarrow{\varpi} Z$ under $T$ by the natural map $H^1(Z, H) \to H^1(Z, T)$. Instead of using the type of the torsor $\varpi$ that was used in [14], we consider the so-called “extended type” of the torsor $\varpi$ that was introduced by Harari and Skorobogatov (see Definition 8.2 in [26]). For a variety $Z$ over $k$, let $KD'(Z)$ denote the complex of Galois modules $[\kappa(Z)^* / \kappa^* \to \text{Div}(Z_k)]$ in the derived category $D^b_k(k)$ of bounded complexes of étale sheaves over $\text{Spec}(k)$. One can associate to the torsor $W \xrightarrow{\varpi} Z$ under $T$ a canonical morphism in this derived category

$$\lambda_W : \widehat{T} \to KD'(Z),$$

called the extended type of $\varpi$. This induces a morphism in the derived category of bounded complexes of abelian groups

$$\lambda_W^\sigma : \widehat{T}^\sigma \to KD'(Z^\sigma)$$

for the above $\sigma \in Z^1(k, F)$.

**Lemma 7.2.** The morphism $\lambda_W^\sigma : \widehat{T}^\sigma \to KD'(Z^\sigma)$ is a morphism in the derived category of bounded complexes of étale sheaves over $\text{Spec}(k)$.

**Proof.** The natural left actions of $F$ on both $T$ and $Z$ induces right actions of $F$ on $\widehat{T}$ and on $KD'(Z)$.

We first prove that the morphism $\lambda_W$ is $F$-equivariant for those actions.

Let $f \in F(\kappa)$. We denote by $f_z : Z_k \to Z_k$ the morphism of $k$-varieties defined by $z \mapsto f \cdot z$. This morphism induces a natural morphism in the derived category $f_z^* : KD'(Z_k) \to KD'(Z_k)$. Similarly, the element $f$ defines a natural morphism of $\kappa$-tori $f_T : T_k \to T_k$ such that $f_T(t) := gtg^{-1}$, where $g \in G^1(\kappa)$ is any point lifting $f \in F(\kappa)$. This morphism $f_T$ induces a morphism of abelian groups $f_T : \widehat{T} \to \widehat{T}$ such that $f_T(\chi) := \chi \circ f_T$. 


One needs to prove that the following diagram
\[
\begin{array}{c}
\hat{T} \xrightarrow{\lambda_{W_k}} KD'(Z_k) \\
\downarrow f_T \downarrow \\
\hat{T} \xrightarrow{\lambda_{W_k}} KD'(Z_k)
\end{array}
\]
is commutative.

Let \(f_{T,s}W_k\) be the push-forward of the torsor \(W_k \to Z_k\) under \(T_k\) by the \(\bar{k}\)-morphism \(T_k \xrightarrow{f_T} T_k\) and let \(f_{Z}^*W_k\) be the pullback of the torsor \(W_k \to Z_k\) under \(T_k\) by the \(k\)-morphism \(f_Z : Z_k \to Z_k\). Then functoriality of the extended type gives:
\[
f_{Z}^* \circ \lambda_{W_k} = \lambda_{f_{Z}^*W_k} \quad \text{and} \quad \lambda_{f_{T,s}W_k} = \lambda_{W_k} \circ f_T.
\]

To prove the required commutativity \(f_{Z}^* \circ \lambda_{W_k} = \lambda_{f_{Z}^*W_k} \circ f_T\), it is enough to show that the torsors \(f_{Z}^*W_k \to Z_k\) and \(f_{T,s}W_k \to Z_k\) under \(T_k\) are isomorphic. Indeed, we have the following commutative diagram
\[
\begin{array}{c}
T_k \times W_k \xrightarrow{g} W_k \\
\downarrow \text{op}_{pW} \downarrow \\
Z_k \xrightarrow{f_z} Z_k,
\end{array}
\]
where \(p_W\) denotes the projection on \(W_k\) and the morphism \(g\) is defined by \((t, w) \mapsto (tg) \cdot w\). This diagram induces a natural \(Z_k\)-morphism \(\phi : T_k \times W_k \to f_{Z}^*W_k\). Consider now the right action of \(T_k\) on \(T_k \times W_k\) defined by \((s, w) \cdot t := (s f_T(t), t^{-1} \cdot w) = (sg t^{-1}, t^{-1} \cdot w)\). Then the morphism \(\phi\) is \(T_k\)-invariant under this action, hence it induces a \(Z_k\)-morphism \(\psi : f_{T,s}W_k \to f_{Z}^*W_k\). One can check by a simple computation that \(\psi\) is \(T_k\)-equivariant, i.e. that \(\psi\) is a morphism of (left) torsors over \(Z_k\) under \(T_k\). It concludes the proof of the required commutativity, hence the morphism \(\lambda_{W_k}\) is \(F\)-equivariant.

By definition of the twists \(T^\sigma\) and \(Z^\sigma\), the fact that \(\lambda_{W_k}\) is \(F\)-equivariant implies that the morphism \(\lambda_{W_k}^\sigma\) is Galois equivariant, i.e. that \(\lambda_{W_k}^\sigma\) is a morphism in the derived category of bounded complexes of étale sheaves over \(\text{Spec}(k)\).

By Proposition 8.1 in [26], there is a natural exact sequence of abelian groups
\[
H^1(k, T^\sigma) \to H^1(X', T^\sigma) \xrightarrow{\lambda} \text{Hom}_k(T^\sigma, KD'(X')) \xrightarrow{\partial} H^2(k, T^\sigma)
\]
where the map \(\lambda\) is the extended type. Let \(\lambda_{\sigma} = \psi^* \circ \lambda_{W_k}^\sigma\), where \(\psi : X' \to W\) is the inclusion of the \(k\)-connected component given by Lemma 7.1, and \(KD'(Z^\sigma) \xrightarrow{\psi^*} KD'(X')\) is the map induced by \(\psi\).

The following lemma, which is an analogue of Lemma 8 in [14], is a crucial step for proving the main result of this section. We give here a more conceptual proof than that in [14], where a similar statement was proven by cocycle computations under the assumption that \(\bar{k}[X]^\times = k^\times\).
Lemma 7.3. With the above notation, one has
\[ \partial(\lambda'_\sigma) = 0 \] if and only if \( \eta'_\sigma = 0 \).

Proof. In the following proof, we work over the small étale site of Spec\( (k) \).

Recall that if we are given a cocycle \( \sigma \in Z^1(k, F) \) as in Lemma 7.1: one can associate to \( \sigma \) a Spec\( (k) \)-torsor \( U \) under \( F \) with a point \( u_0 \in U(\bar{K}) \). This torsor \( U \) is naturally a homogeneous space of the group \( G' \) with geometric stabilizer isomorphic to \( T_k \). Section IV.5.1 in [26] implies that the element \( \eta'_\sigma \in H^2(k, T^\sigma) \) is the class of the Spec\( (k) \)-gerbe \( \mathcal{E}_\sigma \) banded by \( T^\sigma \) such that for all étale schemes \( S \) over Spec\( (k) \), the category \( \mathcal{E}_\sigma(S) \) is defined as follows: the objects of \( \mathcal{E}_\sigma(S) \) are triples \((P, p, \alpha)\) where \( P \to S \) is a torsor under \( G' \), \( p \in P(S_k) \) and \( \alpha : P \to U_S \) is a \( G' \)-equivariant \( S \)-morphism. The morphisms of \( \mathcal{E}_\sigma(S) \) correspond to maps \( \lambda : \mathcal{E}_\sigma(S) \to \mathcal{E}_\sigma(S) \) and \( \sigma \in H^1(S, \mathcal{E}_\sigma) \) is a 1-cocycle associated to \( \alpha \).

Similarly, one can associate to the morphism \( \lambda'_\sigma \) a Spec\( (k) \)-gerbe banded by \( T^\sigma \) that will be the obstruction for the morphism \( \lambda'_\sigma \) to be the extended type of a torsor over \( X' \) under \( T^\sigma \). The morphism \( \lambda'_\sigma \) induces a morphism \( \mathcal{E}'_\sigma : \tilde{T}^\sigma_k \to K \mathcal{D}'(X^\sigma_k) \) in \( D^0_\et(k) \). By construction, \( \mathcal{E}'_\sigma \) is the extended type of the torsor \( Y_0 := W_k \times_{\bar{k}} X'_k \) over \( X'_k \) under \( T^\sigma_k = T_k \).

We now define \( \mathcal{L}_\sigma \) to be the fibered category defined as follows: for all étale schemes \( S \) over Spec\( (k) \), the objects of the category \( \mathcal{L}_\sigma(S) \) are pairs \((V, \varphi)\), where \( V \to X'_k \) is a torsor under \( T^\sigma_S \) of extended type \( \lambda_V \) compatible with \( \lambda'_\sigma \) and \( \varphi : V_k \to Y_0 \times_{\bar{k}} S_k \) is an isomorphism of torsors over \( X' \times_k S_k \) under \( T^\sigma_S \). Given two such objects \((V, \varphi)\) and \((V', \varphi')\), a morphism between \((V, \varphi)\) and \((V', \varphi')\) in the category \( \mathcal{L}_\sigma(S) \) is a pair \((\alpha, t)\), where \( \alpha : V \to V' \) is a morphism of torsors over \( X'_S \) under \( T^\sigma_S \) and \( t \in T^\sigma(S_k) \).

One can check that \( \mathcal{L}_\sigma \) is a stack for the étale topology over Spec\( (k) \), and the fact that this is a gerbe is a consequence of the exact sequence of Proposition 8.1 in [26]
\[ H^1(S, T^\sigma) \to H^1(X'_S, T^\sigma) \xrightarrow{\alpha} \text{Hom}_S(\tilde{T}^\sigma, K \mathcal{D}'(X^\sigma_S)) \xrightarrow{\beta} H^2(S, T^\sigma) \]
(which holds provided that \( S \) is integral, regular and noetherian).

The band of this gerbe is the abelian band represented by \( T^\sigma \).

In addition, it is clear that \( \mathcal{L}_\sigma \) is neutral if and only if \( \mathcal{L}_\sigma(k) \neq \emptyset \) if and only if there exists a torsor over \( X' \) under \( T^\sigma \) of type \( \lambda'_\sigma \) if and only if \( \partial(\lambda'_\sigma) = 0 \).

Let us now construct an equivalence of gerbes between \( \mathcal{E}_\sigma \) and \( \mathcal{L}_\sigma \). For all étale Spec\( (k) \)-schemes \( S \), consider the functor
\[ m_S : \mathcal{E}_\sigma(S) \to \mathcal{L}_\sigma(S) \]
that maps an object \((P, p, \alpha)\) to the object \((V, \varphi)\), where \( V \) is defined to be the contracted product \( V := (P \times^S W_S) \times^S X'_S \) and \( \varphi : V_k \to Y_0 \times_{\bar{k}} S_k \) is induced by the
point \( p \in P(S_k) \). Indeed, by construction, we have a natural map \( P \times^G_S W_S \to U_S \times^F_S Z_S = Z_S \), and a simple computation proves that this map is a torsor under \( T^\sigma \) of extended type compatible with \( \lambda_W^\sigma \).

By definition, the functor \( m_S \) sends a morphism \( \varphi : (P, p, \alpha) \to (P', p', \alpha') \) to the morphism \((\tilde{\varphi}, t_0)\) such that \( \tilde{\varphi} : (P \times^G_S W_S) \times_{Z_S} X'_S \to (P' \times^G_S W_S) \times_{Z_S} X'_S \) is the morphism induced by the morphism of torsors \( \varphi : P \to P' \), and \( t_0 \in T^\sigma(S_F) \) is the element such that \( p' = t_0 \cdot \varphi(p) \) as \( S_F \)-points in \((P' \times^G_S W_S) \times_{Z_S} X'_S \).

Finally, one checks that the collection of functors \( m_S \) defines a morphism of gerbes \( m : \mathcal{E}_\sigma \to \mathcal{L}_\sigma \) banded by the identity of \( T^\sigma \), which implies that \( \eta'_\sigma := [\mathcal{E}_\sigma] = [\mathcal{L}_\sigma] \in H^2(k, T^\sigma) \).

Therefore, \( \eta'_\sigma \) is zero if and only if \( \mathcal{E}_\sigma(k) \neq \emptyset \) if and only if \( \mathcal{L}_\sigma(k) \neq \emptyset \) if and only if \( \partial(\lambda'_\sigma) = 0 \). \hfill \Box

The immediate consequence of Lemma 7.3 is the following result which extends Proposition 5 in [14] to open varieties.

**Proposition 7.4.** Let \( X \) be a smooth geometrically integral \( k \)-variety. Let \((P_v) \in X(A_k)^{\et, Br}\) and let \( Y \to X \) be a torsor under a linear \( k \)-group \( G \). Let

\[
1 \to H \to G \to F \to 1
\]

be an exact sequence of linear \( k \)-groups, where \( H \) is connected and \( F \) finite. Let \( Z \to X \) be the push-forward of \( Y \to X \) by the morphism \( G \to F \), which is a torsor under \( F \). Let \( \sigma \in Z^1(k, F) \) be a 1-cocycle given by Lemma 7.1 applied to the torsor \( Z \to X \) and the point \((P_v)\).

Then the cocycle \( \sigma \in Z^1(k, F) \) lifts to a cocycle \( \tau \in Z^1(k, G) \).

**Proof.** As mentioned above, Construction (5.1) in [18] (see also (7.7) in [1]) gives a class \( \eta_\sigma \) of \( H^2(k, \kappa_\sigma) \) such that \( \sigma \) can be lifted to \( Z^1(k, G) \) if and only if \( \eta_\sigma \) is neutral, where \( \kappa_\sigma \) is a \( k \)-kernel on \( H_k \). By (6.1.2) of [1] and Lemma 6 in [14], there is a canonical map \( H^2(k, \kappa_\sigma) \to H^2(k, T^\sigma) \) such that the class \( \eta_\sigma \) is neutral if and only if its image \( \eta'_\sigma \in H^2(k, T^\sigma) \) is zero. By Lemma 7.3, one only needs to show that \( \partial(\lambda'_\sigma) = 0 \) where \( \lambda'_\sigma = \psi^* \circ \lambda_W^\sigma \), with \( K D'(Z^\sigma) \iso K D'(X') \) given by Lemma 7.1 and \( \lambda_W^\sigma \) defined by Lemma 7.2.

By Lemma 7.1, we know that \( X(A_k)^{Br} \neq \emptyset \). Therefore the map \( \partial \) in the exact sequence (see Proposition 8.1 in [26])

\[
H^1(X', T^\sigma) \xrightarrow{\partial} \operatorname{Hom}_k(T^\sigma, K D'(X')) \xrightarrow{\psi^*} H^2(k, T^\sigma)
\]

is surjective by Corollary 8.17 in [26]. Hence the map \( \partial \) is the zero map and \( \partial(\lambda'_\sigma) = 0 \), which concludes the proof. \hfill \Box

**Remark 7.5.** The proof of Proposition 7.4 also gives the following result:

Let \( X \) be a smooth geometrically integral \( k \)-variety and let \( Y \to X \) be a torsor under a linear algebraic \( k \)-group \( G \). Let

\[
1 \to H \to G \to F \to 1
\]

be an exact sequence of linear \( k \)-groups, where \( H \) is connected and \( F \) finite. Let \( Z \to X \) be the push-forward of \( Y \to X \) by the morphism \( G \to F \).

If \( \sigma \in H^1(k, F) \) satisfies \( Z^\sigma(A_k)^{Br_1(Z^\sigma)} \neq \emptyset \), then \( \sigma \) can be lifted to \( H^1(k, G) \).

One can now prove the main result of this section:
Theorem 7.6. If $X$ is a smooth and geometrically integral variety over a number field $k$, then

$$X(A_k)^{\text{et,Br}} \subseteq X(A_k)^{\text{desc}}.$$

Proof. Since the statement 2 of Theorem 2 in [21] (which we apply to $X'$) holds for any geometrically integral variety (without any assumption on $\overline{k}[X']^\times$), the proof of this theorem using Proposition 7.4 is exactly the same as the proof of Theorem 1 using Proposition 5 in [14] (see in particular [14], p. 244-245). \hfill \Box

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