Artin-Mazur-Milne duality Theorem for fppf cohomology

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April 11, 2018

Abstract. We provide a complete proof of a duality theorem for the fppf cohomology of either a curve over a finite field or a ring of integers of a number field, which extends the classical Artin-Verdier Theorem in étale cohomology. We also prove some finiteness and vanishing statements.

Keywords : fppf cohomology, arithmetic duality, Artin approximation Theorem.

1 Introduction

Let K be a number field or the function field of a smooth, projective, geometrically integral curve X over a finite field. In the number field case, set $X = \operatorname{Spec} \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K. Let U be a non empty Zariski open subset of X and denote by N a commutative, finite and flat group scheme over U with Cartier dual N^D. Assume that the order of N is invertible on U (in particular N is étale). The classical "étale" *Artin-Verdier Theorem* (cf. [Mi2], Corollary II.3.3.) is a duality statement between étale cohomology $H^{\bullet}_{\text{ét}}(U, N)$ and étale cohomology with compact support $H^{\bullet}_{\text{ét},c}(U, N^D)$. It has been known for a long time that this theorem is especially useful in view of concrete arithmetic applications : for example it yields a very nice method to prove deep results like Cassels–Tate duality for abelian varieties and schemes ([Mi2], section II.5) and their generalizations to 1-motives ([HS], section 4); Artin-Verdier's Theorem also provides a "canonical" path to prove Poitou–Tate's Theorem and its extension to complex of tori ([Dem1]), which in turn turns out to be very fruitful to deal with local-global questions for (non necessarily commutative) linear algebraic groups ([Dem2]).

It is of course natural to try to remove the condition that the order of N is invertible on U. A good framework to do this is provided by fppf cohomology of finite and flat group schemes over U, as introduced by J.S. Milne in the third part of his book [Mi2]. This includes the case of group schemes of order divisible by $p := \operatorname{Char} K$ in the function field case.

Such a fppf duality theorem has been first announced by B. Mazur[1] ([Maz], Prop. 7.2), relying on work by Artin and himself. Special cases have also been proved by Artin and Milne ([AM]). The precise statement of the theorem is as follows (see [Mi2], Corollary III.3.2. for the number field case and Theorem III.8.2 for the function field case):

[1]Thanks to A. Schmidt for having pointed this out to us.

Theorem 1.1 *Let $j : U \hookrightarrow X$ be a non-empty open subscheme of X. Let N be a finite flat commutative group scheme over U with Cartier dual N^D. For all integers r with $0 \leq r \leq 3$, the canonical pairing*

$$H^r(U, N^D) \times H_c^{3-r}(U, N) \to H_c^3(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}$$

(where $H^r(U, N^D)$ is a fppf cohomology group and $H_c^{3-r}(U, N)$ a fppf cohomology group with compact support) induces a perfect duality between the profinite group $H_c^{3-r}(U, N)$ and the discrete group $H^r(U, N^D)$. Besides all groups are finite in the number field case.

For example this extension of the étale Artin-Verdier Theorem is needed to prove Poitou–Tate exact sequence over global fields of characteristic p ([Gon], Th. 4.8. and 4.11) as well as Poitou–Tate Theorem over a global field without restriction on the order ([Čes2], Th. 5.1, which in turn is used in [Ros], §6.4 and 6.5). Results of [Mi2], section III.9. (which rely on the fppf duality Theorem) are also a key ingredient in the proof of the Birch and Swinnerton-Dyer conjecture for abelian varieties over a global field of positive characteristic, in [Bau], §4 and [KT], §2 for instance. Our initial interest in Theorem 1.1 was to try to extend it to complexes of tori in the function field case, following the same method as in the number field case [Dem1]. Such a generalization should then provide results (known in the number field case) about weak and strong approximation for linear algebraic groups defined over a global field of positive characteristic.

However, as K. Česnavičius pointed out to us[2], it seems necessary to add details to the proof in [Mi2], sections III.3. and III.8, for two reasons :

- the functoriality of flat cohomology with compact support and the commutativity of several diagrams is not explained in [Mi2]. Even in the case of an imaginary number field, a definition of $H_c^r(U, \mathcal{F})$ as $H^r(X, j_!\mathcal{F})$ for a fppf sheaf $\mathcal{F}$ (which works for the étale Artin-Verdier Theorem) would not be the right one, because it does not provide the key exact sequence [Mi2] Prop. III.0.4.a) in the fppf setting (the analogue of [Mi2], Prop II.1.1, does not stand anymore). It is therefore necessary to work with an adhoc definition of compact support cohomology as in loc. cit., §III.0. Since this definition involves mapping cones, commutativities of some diagrams have to be checked in the category of complexes and not in the derived category (where there is no good functoriality for the mapping cones). Typically, the isomorphisms that compute $C^\bullet(b)$, $C^\bullet(b \circ a)$ and $C^\bullet(c \circ b \circ a)$ in loc. cit., Prop. III.0.4.c) are not canonical a priori. Hence the required compatibilities in loc. cit., proof of Theorem III.3.1. and Lemma III.8.4. have to be checked carefully.

- in the positive characteristic case, it is necessary (as explained in [Mi2], §III.8.) to work with a definition of cohomology with compact support involving *completions* of the local rings of points in $X \setminus U$ instead of their henselizations. The reason is that a local duality statement (loc. cit., Th. III.6.10), which only works in the context of complete valuation fields, is needed. It turns out that some properties

[2] In particular, he observed that the analogue of [Mi2], Prop. III.0.4.c) is by no means obvious when henselizations are replaced by completions. This analogue is actually false without additional assumptions, as shown by T. Suzuki in [Suz], Rem 2.7.9

of compact support cohomology (in particular loc. cit., Prop. III.0.4.c)) are more difficult to establish in this context : for example the comparison between cohomology of the completion $\widehat{\mathcal{O}_v}$ and of the henselization $\mathcal{O}_v$ is not as straightforward as in the étale case.

The goal of this article is to present a detailed proof of Theorem 1.1 with special regards to the two issues listed above. Section 2 is devoted to general properties of fppf cohomology with compact support (Prop. 2.1), which involves some homological algebra (Lemma 2.3) as well as comparison statements between cohomology of $\mathcal{O}_v$ and $\widehat{\mathcal{O}_v}$ (Lemma 2.6).

We also define a natural topology on the fppf compact support cohomology groups (see section 3). In section 4, we follow the method of [Mi2], §III.8. to prove Theorem 1.1 in the function field case. As a corollary, we get a finiteness statement (Cor. 4.3), which apparently has not been observed before this paper. The case of a number field is simpler once the functorial properties of section 2 have been proved; it is treated in section 5.

One week after the first draft of this article was released, Takeshi Suzuki kindly informed us that in his preprint [Suz], he obtained (essentially at the same time as us) ffpf duality results similar to Theorem 1.1 in a slightly more general context. His methods are somehow more involved than ours, they use the *rational étale site*, which he developed in earlier papers.

Notation. Let X be either a smooth projective curve over a finite field k of characteristic p, or the spectrum of the ring of integers $\mathcal{O}_K$ of a number field K. Let $K := k(X)$ be the function field of X. Throughout the paper X is endowed with the (big) fppf site, and cohomology is fppf cohomology unless stated otherwise.

For any closed point $v \in X$, let $\mathcal{O}_v$ (resp. $\widehat{\mathcal{O}_v}$) be the henselization (resp. the completion) of the local ring $\mathcal{O}_{X,v}$ of X at v. Let K_v (resp. $\widehat{K_v}$) be the fraction field of $\mathcal{O}_v$ (resp. $\widehat{\mathcal{O}_v}$). Let U be a non empty Zariski open subset of X and denote by $j : U \to X$ the corresponding open immersion. By [Mat], §34, the local ring $\mathcal{O}_{X,v}$ of X at v is excellent (indeed $\mathcal{O}_{X,v}$ is either of characteristic zero or the localization of a ring of finite type over a field); hence so are $\mathcal{O}_v$ (by [EGA4], Cor. 18.7.6) as the henselization of an excellent ring, and $\widehat{\mathcal{O}_v}$ as a complete d.v.r. ([Mat], §34).

The piece of notation "$v \notin U$" means that we consider all places v corresponding to closed points of $X \setminus U$ *plus the real places in the number field case.* If v is a real place, we set $K_v = \widehat{K_v} = \mathcal{O}_v = \widehat{\mathcal{O}_v}$ for the completion of K at v, and we denote by $H^*(K_v, M)$ the *Tate (=modified)* cohomology groups of a $\mathrm{Gal}(\overline{K_v}/K_v)$-module M.

If $\mathcal{F}$ is a fppf sheaf on U, define the Cartier dual $\mathcal{F}^D$ to be the fppf sheaf $\mathcal{F}^D := \underline{\mathrm{Hom}}(\mathcal{F}, \mathbf{G}_m)$. Notation as $\Gamma(U, \mathcal{F})$ stands for the group of sections of $\mathcal{F}$ over U, and $\Gamma_Z(U, \mathcal{F})$ for the group of sections with support in Z. If E is a field (e.g. $E = K_v$ or $E = \widehat{K_v}$) and $i : \mathrm{Spec}\, E \to U$ is an E-point of U, we will frequently write $H^r(E, \mathcal{F})$ for $H^r(\mathrm{Spec}\, E, i^*\mathcal{F})$. Similarly for an open subset $V \subset U$, the piece of notation $H^r(V, \mathcal{F})$ (resp. $H^r_c(V, \mathcal{F})$) stands for $H^r(V, \mathcal{F}_{|V})$ (resp. $H^r_c(V, \mathcal{F}_{|V})$).

A finite group scheme N over a field E of characteristic $p > 0$ is *local* if it is connected (in particular this implies $H^0(E', N) = 0$ for every field extension E' of E). Examples of such group schemes are μ_p (defined by the affine equation $y^p = 1$) and α_p (defined by the equation $y^p = 0$).

For any topological abelian group A, let $A^* := \mathrm{Hom}_{\mathrm{cont.}}(A, \mathbf{Q}/\mathbf{Z})$ be the group of continuous homomorphisms from A to $\mathbf{Q}/\mathbf{Z}$ (the latter equipped with the discrete topology). A continuous morphism $f : A \to B$ of topological groups is *strict* if the restriction $f : A \to f(A)$ is an open map (where the topology on $f(A)$ is induced by B). This is equivalent to saying that f induces an isomorphism of the topological quotient $A/\ker f$ with the topological subspace $f(A) \subset B$.

2 Fppf cohomology with compact support

Define $Z := X \setminus U$ and $Z' := \coprod_{v \in Z} \mathrm{Spec}\,(\widehat{K_v})$ (disjoint union).

Then we have a natural morphism $i : Z' \to X$.

Let $\mathcal{F}$ be a sheaf on U (for the fppf topology). Let $I^\bullet(\mathcal{F})$ be an injective resolution of $\mathcal{F}$ over U.

Denote by $\mathcal{F}_v$ and $I^\bullet(\mathcal{F})_v$ their respective pullbacks to $\mathrm{Spec}\, K_v$.

As noticed by A. Schmidt, the definition of the modified fppf cohomology groups in the number field case in [Mi2], III.0.6 (a), has to be written more precisely, because of the non-canonicity of the mapping cone in the derived category. We are grateful to him for the following alternative definition. Let $\Omega_{\mathbf{R}}$ denote the set of real places of K. For $v \in \Omega_{\mathbf{R}}$, let $\varepsilon^v : \mathrm{Spec}\,(K_v)_{\mathrm{fppf}} \to \mathrm{Spec}\,(K_v)_{\text{ét}}$ be the natural morphism of sites, then $\varepsilon^v_* I^\bullet(\mathcal{F})_v$ is a flasque resolution of $\varepsilon^v_*\mathcal{F}_v$. Following [GS] §2, there is a natural acyclic resolution $D^\bullet(\varepsilon^v_*\mathcal{F}_v) \to \varepsilon^v_*\mathcal{F}_v$ of the $\mathrm{Gal}(\overline{K_v}/K_v) = \mathbf{Z}/2\mathbf{Z}$-module $\varepsilon^v_*\mathcal{F}_v$ (identified with $\mathcal{F}_v(\mathrm{Spec}\,(\overline{K_v}))$). Splicing the resolutions $D^\bullet(\varepsilon^v_*\mathcal{F}_v)$ and $\varepsilon^v_* I^\bullet(\mathcal{F})_v$ together, one gets a complete acyclic resolution $\widehat{I}^\bullet(\mathcal{F}_v)$ of the $\mathrm{Gal}(\overline{K_v}/K_v)$-module $\varepsilon^v_*\mathcal{F}_v$, which computes the Tate cohomology of $\varepsilon^v_*\mathcal{F}_v$. And by construction, there is a natural morphism $\widehat{i}_v : \varepsilon^v_* I^\bullet(\mathcal{F})_v \to \widehat{I}^\bullet(\mathcal{F}_v)$.

As suggested by [Mi2], section III.0, define $\Gamma_c(U, I^\bullet(\mathcal{F}))$ to be the following object in the category of complexes of abelian groups:

$$\Gamma_c(U, I^\bullet(\mathcal{F})) := \mathrm{Cone}\left(\Gamma(U, I^\bullet(\mathcal{F})) \to \Gamma(Z', i^* I^\bullet(\mathcal{F})) \oplus \bigoplus_{v \in \Omega_{\mathbf{R}}} \Gamma(K_v, \widehat{I}^\bullet(\mathcal{F}_v))\right)[-1],$$

and $H^r_c(U, \mathcal{F}) := H^r(\Gamma_c(U, I^\bullet(\mathcal{F})))$. From now on, we will abbreviate Cone(...) in $C(...)$.

Proposition 2.1 *1. There is a natural exact sequence, for all $r \geq 0$,*

$$\cdots \to H^r_c(U, \mathcal{F}) \to H^r(U, \mathcal{F}) \to \bigoplus_{v \notin U} H^r(\widehat{K}_v, \mathcal{F}) \to H^{r+1}_c(U, \mathcal{F}) \to \dots.$$

2. For any short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves on U, there is a long exact sequence

$$\cdots \to H^r_c(U, \mathcal{F}') \to H^r_c(U, \mathcal{F}) \to H^r_c(U, \mathcal{F}'') \to H^{r+1}_c(U, \mathcal{F}') \to \dots.$$

3. *For any flat affine commutative group scheme $\mathcal{F}$ locally of finite type on U, and any non empty open subscheme $V \subset U$, there is a canonical exact sequence*

$$\cdots \to H^r_c(V,\mathcal{F}) \to H^r_c(U,\mathcal{F}) \to \bigoplus_{v \in U \setminus V} H^r(\widehat{\mathcal{O}}_v,\mathcal{F}) \to H^{r+1}_c(V,\mathcal{F}) \to \dots,$$

and the following natural diagram commutes:

$$\begin{tikzcd}
 & \bigoplus_{v\notin V} H^{r-1}(\widehat{K_v},\mathcal{F}) \arrow[d] & \bigoplus_{v\notin U} H^{r-1}(\widehat{K_v},\mathcal{F}) \arrow[l, "i_2"'] \arrow[d] & \\
\bigoplus_{v\in U\setminus V} H^{r-1}(\widehat{\mathcal{O}_v},\mathcal{F}) \arrow[r] \arrow[ur, "i_1"'] & H^r_c(V,\mathcal{F}) \arrow[r] \arrow[d] & H^r_c(U,\mathcal{F}) \arrow[r] \arrow[d] & \bigoplus_{v\in U\setminus V} H^r(\widehat{\mathcal{O}_v},\mathcal{F}) \\
 & H^r(V,\mathcal{F}) \arrow[d] & H^r(U,\mathcal{F}) \arrow[l, "\mathrm{Res}"'] \arrow[d] \arrow[ur] & \\
 & \bigoplus_{v\notin V} H^r(\widehat{K_v},\mathcal{F}) \arrow[r, "p"'] & \bigoplus_{v\notin U} H^r(\widehat{K_v},\mathcal{F}) & .
\end{tikzcd}$$

where i_1 (resp. i_2) is obtained by puting 0 at the places $v \notin U$ (resp. $v \in U \setminus V$) and p is the natural projection.

4. *If $\mathcal{F}$ is represented by a smooth group scheme locally of finite type, then for $r \neq 1$, $H^r_c(U,\mathcal{F}) \cong H^r_{\text{ét},c}(U,\mathcal{F})$, where $H^*_{\text{ét},c}$ stands for étale cohomology with compact support as defined in [Mi2], §II.2. In particular those groups are just $H^r_{\text{ét}}(X, j_!\mathcal{F})$ in the function field case. If in addition the generic fiber $\mathcal{F}_K$ is a finite K-group scheme, then $H^1_c(U,\mathcal{F}) \cong H^1_{\text{ét},c}(U,\mathcal{F})$ (=$H^1_{\text{ét}}(X, j_!\mathcal{F})$ in the function field case).*

Remark 2.2 Unlike what happens in étale cohomology, the groups $H^1(\mathcal{O}_v,\mathcal{F})$ and $H^1(\widehat{\mathcal{O}_v},\mathcal{F})$ cannot in general be identified with the group $H^1(k(v),F(v))$, where $k(v)$ is the residue field at v and $F(v)$ the fiber of $\mathcal{F}$ over $k(v)$. For example this already fails for $\mathcal{F} = \mu_p$ and $\widehat{\mathcal{O}_v} = \mathbf{F}_p[[t]]$, because by Kummer exact sequence

$$0 \to \mu_p \to \mathbf{G}_m \stackrel{\cdot p}{\to} \mathbf{G}_m \to 0$$

in fppf cohomology, the group $H^1(\widehat{\mathcal{O}_v},\mathcal{F}) = \widehat{\mathcal{O}_v}^* / \widehat{\mathcal{O}_v}^{*p}$ is an infinite dimensional $\mathbf{F}_p$-vector space, while $H^1(k(v),F(v)) = k(v)^*/k(v)^{*p} = 0$. The situation is better for the groups H^r, $r \geq 2$ ([Toe], Cor. 3.4).

Before proving Proposition 2.1, we need the following lemmas.
We start by a lemma in homological algebra:

Lemma 2.3 *Let $\mathcal{A}$ be an abelian category with enough injectives and let $\mathbf{C}(\mathcal{A})$ (resp. $\mathbf{D}(\mathcal{A})$) denote the category (resp. the derived category) of bounded below cochain complexes in $\mathcal{A}$.*

Consider a commutative diagram in $\mathbf{C}(\mathcal{A})$*:*

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \oplus E \arrow[d, "(\mathrm{id},g)"] \\
A' \arrow[r, "\alpha'"] & B \oplus E' ,
\end{tikzcd}$$

and denote by p_E *(resp.* $p_{E'}$*) the projection* $B \oplus E \to B$ *(resp.* $B \oplus E' \to B$*).*

Assume that the natural morphism $C(f) \to C(g)$ *in* $\mathbf{C}(\mathcal{A})$ *is a quasi-isomorphism. Then there exists a canonical commutative diagram in* $\mathbf{D}(\mathcal{A})$*:*

$$\begin{tikzcd}
(B \oplus E')[-1] \arrow[d] & B[-1] \arrow[l, "i_{E'}"'] \arrow[d] & B \oplus E \arrow[r, "(\mathrm{id},g)"] & B \oplus E' \arrow[d] \\
C(\alpha')[-1] \arrow[r] \arrow[d] & C(p_E \circ \alpha)[-1] \arrow[r] \arrow[d] & E \arrow[r] \arrow[u, "i_B"] & C(\alpha') \\
A' \arrow[d, "\alpha'"] & A \arrow[l, "f"] \arrow[r, "\alpha"] \arrow[d, "p_E \circ \alpha"] & B \oplus E \arrow[u, "p_B"] & \\
B \oplus E' \arrow[r, "p_{E'}"'] & B & &
\end{tikzcd}$$

where the second line and the first two columns are exact triangles.

Proof: The assumption that $C(f) \to C(g)$ is a quasi-isomorphism implies that $C(\alpha) \to C(\alpha')$ is a quasi-isomorphism (see for instance Proposition 1.1.11 in [BBD] or Corollary A.14 in [PS]).

Functoriality of the mapping cone in the category $\mathbf{C}(\mathcal{A})$ gives a diagram in $\mathbf{C}(\mathcal{A})$, where the second line (by [Mi2], Prop. II.0.10, or [KS], proof of Theorem 11.2.6) and the columns are exact triangles in the derived category:

$$\begin{tikzcd}
 & (B \oplus E)[-1] \arrow[dl, "(\mathrm{id},g)"'] \arrow[d] & B[-1] \arrow[l, "i_E"'] \arrow[r, "="] \arrow[d] & B[-1] \arrow[d] & & \\
(B \oplus E')[-1] \arrow[d] & C(\alpha)[-1] \arrow[r] \arrow[dl] \arrow[d] & C(p_E \circ \alpha)[-1] \arrow[r] \arrow[d] & C(p_E)[-1] \arrow[r] \arrow[d] \arrow[dr] & C(\alpha) \arrow[d, phantom, "*"] & \\
C(\alpha')[-1] \arrow[d] & A \arrow[r, "="] \arrow[dl, "f"'] \arrow[d, "\alpha"] & A \arrow[r, "\alpha"] \arrow[d, "p_E \circ \alpha"] & B \oplus E \arrow[r, "p_B"] \arrow[d, "p_E"] & E \arrow[r, "i_B"] & B \oplus E \arrow[ul] \\
A' \arrow[d, "\alpha'"] & B \oplus E \arrow[r, "p_E"] \arrow[dl, "(\mathrm{id},g)"] & B \arrow[r, "="] & B & & \\
B \oplus E' & & & & &
\end{tikzcd}$$

.

As usual, notation as p_B, p_E denotes projections and i_B, i_E are given by puting 0 at the missing piece.

This diagram is commutative in $\mathbf{C}(\mathcal{A})$, except the square $*$ which is commutative up to homotopy. Indeed, this square defines two maps $f, g : C(p_E)[-1] \to C(\alpha)$, which are given in degree n by two maps $f^n, g^n : B^{n-1} \oplus (B^n \oplus E^n) \to (B^n \oplus E^n) \oplus A^{n+1}$, where $f^n(b', b, e) := (b, e, 0)$ and $g^n(b', b, e) := (0, e, 0)$. Consider now the maps $s^n : B^{n-1} \oplus (B^n \oplus E^n) \to (B^{n-1} \oplus E^{n-1}) \oplus A^n$ defined by $s^n(b', e, b) := (-1)^n(b', 0, 0)$. Then the collection (s^n) is a homotopy between f and g. Hence the square $*$ is commutative up to the homotopy (s^n).

Since the map $C(\alpha) \to C(\alpha')$ is a quasi-isomorphism, and since the natural map $C(p_E)[-1] \to E$ is a homotopy equivalence, the lemma follows from the commutativity and the exactness of the previous diagram. □

We now need the following result, for which we didn't find a suitable reference:

Lemma 2.4 *Let A be a henselian valuation ring with fraction field K. Assume that the completion $\widehat{K}$ is separable over K.*

1. *Let G be a K-group scheme locally of finite type.*

   Then the map $H^1(K, G) \to H^1(\widehat{K}, G)$ has dense image.

2. *Assume $\widehat{A}$ is henselian. Let G be a flat A-group scheme locally of finite presentation.*

   Then the map $H^1(A, G) \to H^1(\widehat{A}, G)$ has dense image.

Here the topology on the pointed sets $H^1(\widehat{A}, G)$ and $H^1(\widehat{K}, G)$ are provided by [Čes1], §3.

Remark 2.5

- The assumption that $\widehat{K}$ is separable over K is satisfied if the ring A is excellent.
- In the second statement, the assumption that $\widehat{A}$ is henselian is satisfied if the valuation on A has height 1. This assumption is used in the proof below to apply [Čes1], Theorem B.5.

Proof: We prove both statements at the same time. Let R be either A or K.

Let $\mathbf{B}G$ denote the classifying R-stack of G-torsors. We need to prove that $\mathbf{B}G(R)$ is dense in $\mathbf{B}G(\widehat{R})$.

It is a classical fact that $\mathbf{B}G$ is an algebraic stack (see [LMB], Proposition 10.13.1).

Let $x \in \mathbf{B}G(\widehat{R})$ and $U \subset \mathbf{B}G(\widehat{R})$ be an open subcategory (in the sense of [Čes1], 2.4) contaning x. We need to find an object $x' \in \mathbf{B}G(R)$ that maps to $U \subset \mathbf{B}G(\widehat{R})$.

Using [Čes1], Theorem B.5 and Remark B.6, there exists an affine scheme Y, a smooth R-morphism $\pi : Y \to \mathbf{B}G$ and $y \in Y(\widehat{R})$ such that $\pi_{\widehat{R}}(y) = x$, where $\pi_{\widehat{R}} : Y(\widehat{R}) \to BG(\widehat{R})$ is the map induced by π.

In particular, $Y \to \operatorname{Spec} R$ is smooth because so are π and $\mathbf{B}G \to \operatorname{Spec} R$ (the latter by [Čes1], Prop. A.3). Hence Y is locally of finite presentation over $\operatorname{Spec} R$.

By assumption, $\pi_{\widehat{R}}^{-1}(U) \subset Y(\widehat{R})$ is an open subset containing y. Hence [MB], Corollary 1.2.1 (in the d.v.r. case, it is Greenberg's approximation Theorem) implies that $Y(R) \cap \pi_{\widehat{R}}^{-1}(U) \neq \emptyset$. Applying π_R, we get that $\mathbf{B}G(R) \cap U \neq \emptyset$, which proves the required result. □

The previous lemma is useful to prove the following crucial (in the function field case) statement. For a local integral domain A with maximal ideal $\mathfrak{m}$, fraction field

K and residue field κ, and $\mathcal{F}$ a fppf sheaf on $\operatorname{Spec} A$ with injective resolution $I^\bullet(\mathcal{F})$, define

$$\Gamma_{\mathfrak{m}}(A, I^\bullet(\mathcal{F})) := \operatorname{Cone}\left(\Gamma(\operatorname{Spec} A, I^\bullet(\mathcal{F})) \to \Gamma(\operatorname{Spec} K, I^\bullet(\mathcal{F}))\right)[-1]$$

and $H^r_{\mathfrak{m}}(A, \mathcal{F}) := H^r(\Gamma_{\mathfrak{m}}(A, \mathcal{F}))$ (the cohomology with compact support in $\operatorname{Spec} \kappa$). We have a localization long exact sequence ([Mi2], Prop. III.0.3)

$$\ldots \to H^r_{\mathfrak{m}}(A, \mathcal{F}) \to H^r(A, \mathcal{F}) \to H^r(K, \mathcal{F}) \to H^{r+1}_{\mathfrak{m}}(A, \mathcal{F}) \to \ldots$$

Lemma 2.6 *Let A be an excellent henselian discrete valuation ring, with maximal ideal $\mathfrak{m}$. Let $\mathcal{F}$ be a flat affine commutative group scheme locally of finite type on* $\operatorname{Spec}(A)$ *and $I^\bullet(\mathcal{F})$ be an injective resolution of $\mathcal{F}$.*

Then the natural morphism $\Gamma_{\mathfrak{m}}(A, I^\bullet(\mathcal{F})) \to \Gamma_{\mathfrak{m}}(\widehat{A}, I^\bullet(\mathcal{F}))$ is an isomorphism in the derived category.

Remark 2.7 In the previous statement, the injective resolution $I^\bullet(\mathcal{F})$ can be replaced by any complex of acyclic fppf sheaves that is quasi-isomorphic to $\mathcal{F}$. Also note that Lemma 2.6 is slightly more general than [Suz], Prop. 2.6.2., and answers a question raised page 26 of loc. cit.

Proof of the lemma: By definition, it is sufficient to prove that for all $r \geq 0$, the morphism $H^r_{\mathfrak{m}}(A, \mathcal{F}) \to H^r_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is an isomorphism.

- $r = 0$:

  Since $\mathcal{F}$ is separated (as an affine scheme), the morphisms $H^0(A, \mathcal{F}) \to H^0(K, \mathcal{F})$ and $H^0(\widehat{A}, \mathcal{F}) \to H^0(\widehat{K}, \mathcal{F})$ are injective, which implies that

  $$H^0_{\mathfrak{m}}(A, \mathcal{F}) = H^0_{\mathfrak{m}}(\widehat{A}, \mathcal{F}) = 0\,.$$

- $r = 1$:

  Consider the following commutative diagram with exact rows:

  $$\begin{array}{ccccccccc} H^0(A,\mathcal{F}) & \longrightarrow & H^0(K,\mathcal{F}) & \longrightarrow & H^1_{\mathfrak{m}}(A,\mathcal{F}) & \longrightarrow & H^1(A,\mathcal{F}) & \longrightarrow & H^1(K,\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(\widehat{A},\mathcal{F}) & \longrightarrow & H^0(\widehat{K},\mathcal{F}) & \longrightarrow & H^1_{\mathfrak{m}}(\widehat{A},\mathcal{F}) & \longrightarrow & H^1(\widehat{A},\mathcal{F}) & \longrightarrow & H^1(\widehat{K},\mathcal{F})\,. \end{array} \tag{1}$$

  By Artin approximation (see [Art], Theorem 1.12), the morphism $H^1(A, \mathcal{F}) \to H^1(\widehat{A}, \mathcal{F})$ is injective: indeed, given a $\operatorname{Spec} A$-torsor $\mathcal{P}$ under $\mathcal{F}$, $\mathcal{P}$ is locally of finite presentation, and Artin approximation ensures that $\mathcal{P}(\widehat{A}) \neq \emptyset$ implies that $\mathcal{P}(A) \neq \emptyset$.

  Since $(\operatorname{Spec} \widehat{A}, \operatorname{Spec} K)$ is a fpqc covering[3] of $\operatorname{Spec} A$ with $\widehat{A} \otimes_A K = \widehat{K}$ (A is a dvr, hence $K = A[1/\pi]$, where π is a uniformizing parameter for both A and $\widehat{A}$) and $\mathcal{F}$ is a fpqc sheaf (since it is representable), the square on the left hand side in (1) is cartesian.

[3]Working with fpqc topology is needed here because the map $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} A$ is not of finite presentation.

Hence an easy diagram chase implies that $H^1_{\mathfrak{m}}(A, \mathcal{F}) \to H^1_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is injective.

By Proposition A.6 in [GP], the right hand side square in (1) is cartesian. In addition, $H^0(\widehat{A}, \mathcal{F}) \subset H^0(\widehat{K}, \mathcal{F})$ is open ([GGM], Prop. 3.3.4), and $H^0(K, \mathcal{F}) \subset H^0(\widehat{K}, \mathcal{F})$ is dense by [GGM], proposition 3.5.2 (weak approximation for $\mathcal{F}$).

Therefore, an easy diagram chase implies that the map $H^1_{\mathfrak{m}}(A, \mathcal{F}) \to H^1_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is surjective.

- $r = 2$:

  Consider the commutative diagram with exact rows:

$$\begin{tikzcd} H^1(A,\mathcal{F}) \arrow[r] \arrow[d] & H^1(K,\mathcal{F}) \arrow[r] \arrow[d] & H^2_{\mathfrak{m}}(A,\mathcal{F}) \arrow[r] \arrow[d] & H^2(A,\mathcal{F}) \arrow[r] \arrow[d] & H^2(K,\mathcal{F}) \arrow[d] \\ H^1(\widehat{A},\mathcal{F}) \arrow[r] & H^1(\widehat{K},\mathcal{F}) \arrow[r] & H^2_{\mathfrak{m}}(\widehat{A},\mathcal{F}) \arrow[r] & H^2(\widehat{A},\mathcal{F}) \arrow[r] & H^2(\widehat{K},\mathcal{F}) . \end{tikzcd} \tag{2}$$

  By [Toe], Corollary 3.4, the map $H^2(A, \mathcal{F}) \to H^2(\widehat{A}, \mathcal{F})$ is an isomorphim. And we already explained (in the case $r = 1$) that the left hand side square in (2) is cartesian. Hence a diagram chase proves that the map $H^2_{\mathfrak{m}}(A, \mathcal{F}) \to H^2_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is injective.

  Using [GGM], Proposition 3.5.3.(3), the map $H^2(K, \mathcal{F}) \to H^2(\widehat{K}, \mathcal{F})$ is also an isomorphism. By [Čes1], Proposition 3.10, the map $H^1(\widehat{A}, \mathcal{F}) \to H^1(\widehat{K}, \mathcal{F})$ is open. Lemma 2.4 implies that the map $H^1(K, \mathcal{F}) \to H^1(\widehat{K}, \mathcal{F})$ has dense image. By diagram chase, we get that the map $H^2_{\mathfrak{m}}(A, \mathcal{F}) \to H^2_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is surjective.

- $r \geq 3$:

  Corollary 3.4 in [Toe] implies that the morphisms $H^{r-1}(A, \mathcal{F}) \to H^{r-1}(\widehat{A}, \mathcal{F})$ and $H^r(A, \mathcal{F}) \to H^r(\widehat{A}, \mathcal{F})$ are isomorphisms. Proposition 3.5.3.(3) in [GGM] implies that the maps $H^{r-1}(K, \mathcal{F}) \to H^{r-1}(\widehat{K}, \mathcal{F})$ and $H^r(K, \mathcal{F}) \to H^r(\widehat{K}, \mathcal{F})$ are isomorphisms. Therefore, the five-lemma proves that $H^r_{\mathfrak{m}}(A, \mathcal{F}) \to H^r_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is an isomorphism.

□

Remark 2.8 We will apply the previous lemma to a finite and flat group scheme N. As was pointed out to us by K. Česnavičius, it is then possible to argue without using Corollary 3.4 in [Toe] (whose proof is quite involved) : indeed there exists (cf. [Mi2], Th. III.A.5) an exact sequence

$$0 \to N \to G_1 \to G_2 \to 0$$

of affine A-group schemes such that G_1 and G_2 are smooth. Now for $i > 0$ we have $H^i(A, G_j) \cong H^i(\widehat{A}, G_j)$ $(j = 1, 2)$ by [Mi1], Rem. III.3.11 because A and $\widehat{A}$ are henselian, and fppf cohomology coincides with étale cohomology for smooth group schemes. It remains to apply the five-lemma.

Proof of Proposition 2.1:

1. cf. [Mi2], III, Proposition 0.4.a) and Remark 0.6. b).

2. cf. [Mi2], III, Proposition 0.4.b) and Remark 0.6. b).

3. As in the proof of [Mi2], III, Proposition 0.4.c), let $I^\bullet(\mathcal{F})$ be an injective resolution of $\mathcal{F}$.

   Consider the following natural commutative diagram of complexes in the category of bounded below complexes of abelian groups:

$$\begin{tikzcd}
\Gamma(U, I^\bullet(\mathcal{F})) \arrow[r, "\alpha"] \arrow[d, "f"] & \bigoplus_{v \notin U} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F})) \oplus \bigoplus_{v \in U \setminus V} \Gamma(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F})) \arrow[r, "p_{\mathcal{O}}"] \arrow[d, "(\mathrm{id}, g)"] & \bigoplus_{v \notin U} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F})) \\
\Gamma(V, I^\bullet(\mathcal{F})) \arrow[r, "\alpha'"] & \bigoplus_{v \notin U} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F})) \oplus \bigoplus_{v \in U \setminus V} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F})) \,, \arrow[ur, "p_K"'] &
\end{tikzcd}$$

   where the maps are the natural ones.

   Functoriality of the mapping cone in the category of complexes gives natural morphisms

$$\Gamma_{U \setminus V}(U, I^\bullet(\mathcal{F})) \to \bigoplus_{v \in U \setminus V} \Gamma_v(\mathcal{O}_v, I^\bullet(\mathcal{F})) \to \bigoplus_{v \in U \setminus V} \Gamma_v(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F}))\,,$$

   where $\Gamma_{U \setminus V}(U, I^\bullet(\mathcal{F})) := C(f)[-1]$, $\Gamma_v(\mathcal{O}_v, I^\bullet(\mathcal{F})) := \Gamma_{\mathfrak{m}_v}(\mathcal{O}_v, I^\bullet(\mathcal{F}))$ and $\Gamma_v(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F})) := \Gamma_{\mathfrak{m}_v}(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F}))$.

   Excision property for fppf cohomology and for étale morphisms (see [Mi1], Proposition III.1.27, where étale cohomology can be replaced by fppf cohomology) implies that the first morphism $\Gamma_{U \setminus V}(U, I^\bullet(\mathcal{F})) \to \bigoplus_{v \in U \setminus V} \Gamma_v(\mathcal{O}_v, I^\bullet(\mathcal{F}))$ is a quasi-isomorphism.

   Since for all $v \in X$, the ring $\mathcal{O}_v$ is an excellent henselian discrete valuation ring, Lemma 2.6 ensures that the second map

$$\bigoplus_{v \in U \setminus V} \Gamma_v(\mathcal{O}_v, I^\bullet(\mathcal{F})) \to \bigoplus_{v \in U \setminus V} \Gamma_v(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F}))$$

   is a quasi-isomorphism. Therefore, the natural morphism $C(f) \to C(g)$ is a quasi-isomorphism.

   Apply now Lemma 2.3: one gets a natural commutative diagram in the derived category of abelian groups:

$$\begin{tikzcd}
\left(\bigoplus_{v\notin V}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F}))\right)[-1] \arrow[d] & \left(\bigoplus_{v\notin U}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F}))\right)[-1] \arrow[l, "i_K"'] \arrow[d] & \bigoplus_{v\notin U}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F}))\oplus\bigoplus_{v\in U\setminus V}\Gamma(\widehat{\mathcal{O}_v},I^\bullet(\mathcal{F})) \arrow[r, "(\mathrm{id},g)"] & \bigoplus_{v\notin V}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F})) \arrow[d] \\
\Gamma_c(V,I^\bullet(\mathcal{F})) \arrow[r] \arrow[d] & \Gamma_c(U,I^\bullet(\mathcal{F})) \arrow[r] \arrow[d] & \bigoplus_{v\in U\setminus V}\Gamma(\widehat{\mathcal{O}_v},I^\bullet(\mathcal{F})) \arrow[r] \arrow[u, "i'_{\mathcal{O}}"] & \Gamma_c(V,I^\bullet(\mathcal{F}))[1] \\
\Gamma(V,I^\bullet(\mathcal{F})) \arrow[d, "\alpha'"] & \Gamma(U,I^\bullet(\mathcal{F})) \arrow[l, "f"'] \arrow[r, "\alpha"] \arrow[d, "p_{\mathcal{O}}\circ\alpha"] & \bigoplus_{v\notin U}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F}))\oplus\bigoplus_{v\in U\setminus V}\Gamma(\widehat{\mathcal{O}_v},I^\bullet(\mathcal{F})) \arrow[u, "p'_{\mathcal{O}}"] & \\
\bigoplus_{v\notin V}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F})) \arrow[r, "p_K"'] & \bigoplus_{v\notin U}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F})) & &
\end{tikzcd}, \tag{3}$$

where the first two columns and the second lines are exact triangles.

Now the cohomology of this diagram gives the following canonical commutative diagram, with an exact second line (and the two first columns exact):

$$\begin{tikzcd}
& \bigoplus_{v\notin V}H^{r-1}(\widehat{K_v},\mathcal{F}) \arrow[d] & \bigoplus_{v\notin U}H^{r-1}(\widehat{K_v},\mathcal{F}) \arrow[l] \arrow[d] & \bigoplus_{v\notin U}H^{r}(\widehat{K_v},\mathcal{F})\oplus\bigoplus_{v\in U\setminus V}H^r(\widehat{\mathcal{O}_v},\mathcal{F}) \arrow[r] & \bigoplus_{v\notin V}H^{r}(\widehat{K_v},\mathcal{F}) \arrow[d] & \\
\dots \arrow[r] & H^r_c(V,\mathcal{F}) \arrow[r] \arrow[d] & H^r_c(U,\mathcal{F}) \arrow[r] \arrow[d] & \bigoplus_{v\in U\setminus V}H^r(\widehat{\mathcal{O}_v},\mathcal{F}) \arrow[r] \arrow[u] & H^{r+1}_c(V,\mathcal{F}) \arrow[r] & \dots \\
& H^r(V,\mathcal{F}) \arrow[d] & H^r(U,\mathcal{F}) \arrow[l, "\mathrm{Res}"'] \arrow[r] \arrow[d] & \bigoplus_{v\notin U}H^{r}(\widehat{K_v},\mathcal{F})\oplus\bigoplus_{v\in U\setminus V}H^r(\widehat{\mathcal{O}_v},\mathcal{F}) \arrow[u] & & \\
& \bigoplus_{v\notin V}H^{r}(\widehat{K_v},\mathcal{F}) \arrow[r] & \bigoplus_{v\notin U}H^{r}(\widehat{K_v},\mathcal{F}) & & &
\end{tikzcd},$$

which prove the required exactness and commutativity.

4. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^{r-1}_{\text{ét}}(U,\mathcal{F}) \arrow[r] \arrow[d] & \bigoplus_{v\notin U}H^{r-1}_{\text{ét}}(K_v,\mathcal{F}) \arrow[r] \arrow[d] & H^r_{\text{ét},c}(U,\mathcal{F}) \arrow[r] \arrow[d] & H^r_{\text{ét}}(U,\mathcal{F}) \arrow[r] \arrow[d] & \bigoplus_{v\notin U}H^r_{\text{ét}}(K_v,\mathcal{F}) \arrow[d] \\
H^{r-1}(U,\mathcal{F}) \arrow[r] & \bigoplus_{v\notin U}H^{r-1}(\widehat{K_v},\mathcal{F}) \arrow[r] & H^r_c(U,\mathcal{F}) \arrow[r] & H^r(U,\mathcal{F}) \arrow[r] & \bigoplus_{v\notin U}H^r(\widehat{K_v},\mathcal{F}).
\end{tikzcd}$$

Here $H_{\text{ét}}$ stands for étale cohomology and $H_{\text{ét},c}$ for étale cohomology with compact support (as defined in [Mi2], §III.2; recall that in the number field case, the piece of notation $v \notin U$ means that we consider the places corresponding to closed points of $\operatorname{Spec}(\mathcal{O}_K) \setminus U$ *and* the real places).

By [GGM], Lemma 3.5.3, and [Mi1] III.3, we have

$$H^r_{\text{ét}}(K_v,\mathcal{F}) \cong H^r_{\text{ét}}(\widehat{K}_v,\mathcal{F}) \xrightarrow{\sim} H^r(\widehat{K_v},\mathcal{F})$$

for all $r \geq 1$ (resp. for all $r \geq 0$ if $\mathcal{F}_K$ is finite) and all places v of K, and $H^r_{\text{ét}}(U,\mathcal{F}) \xrightarrow{\sim} H^r(U,\mathcal{F})$ for all $r \geq 0$, therefore the five-lemma gives the result.

□

Remark 2.9 The definition of fppf compact support cohomology and its related properties are specific to schemes of dimension 1. To the best of our knowledge, there is no good analogue in higher dimension, unlike what happens for étale cohomology.

We will need the following complement to Proposition 2.1 :

Proposition 2.10 *Let $\mathcal{F}$ be a flat affine commutative group scheme locally of finite type over U. Let $V \subset U$ be a non empty open subset. Then there is a long exact sequence*

$$. \to \bigoplus_{v \in U \setminus V} H^r_v(\widehat{\mathcal{O}_v}, \mathcal{F}) \to H^r(U, \mathcal{F}) \to H^r(V, \mathcal{F}) \to \bigoplus_{v \in U \setminus V} H^{r+1}_v(\widehat{\mathcal{O}_v}, \mathcal{F}) \to ... \qquad (4)$$

Proof: The map $\bigoplus_{v \in U \setminus V} H^r_v(\widehat{\mathcal{O}_v}, \mathcal{F}) \to H^r(U, \mathcal{F})$ is given by the identification of the first group with $H^r_Z(U, \mathcal{F})$ via excision, where $Z = U \setminus V$ (see proof of Prop. 2.1, 3. and Lemma 2.6). By the localization exact sequence ([Mi2], Prop. III.0.3. c), this identification yields the required long exact sequence.

□

3 Topology on cohomology groups with compact support

With the previous notation, let us define a natural topology on the groups $H^*_c(U, N)$, where N is a finite flat U-group scheme. Th. 1.1 will actually show that $H^2_c(U, N)$ is profinite, but this result will not be used in this paragraph. This "a priori" approach answers a question raised by Milne ([Mi2], Problem III.8.8.).

We restrict ourselves to the function field case, because when K is a number field all groups are finite (cf. [Mi2], Th. III.3.2; see also section 5 of this article).

Recall that as usual, the groups $H^*(U, N)$ are endowed with the discrete topology. Similarly, we endow the groups $H^i_c(U, N)$ with the discrete topology, for $i \neq 2$.

Let us now focus on the case $i = 2$. Given an exact sequence of abelian groups

$$0 \to A \to B \to C \to 0\,,$$

such that the group A is a topological group, there exists a natural topology on B such that B is a topological group, A is an open subgroup of B, and C is discrete when endowed with the quotient topology. Indeed, the topology on B is generated by the subsets $b + U$, where $b \in B$ and U is an open subset of A. In addition, given another abelian group B' with a subgroup $A' \subset B'$ that is a topological group, and a commutative diagram of abelian groups

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"] & B \arrow[d, "g"] \\ A' \arrow[r, hook] & B'\,, \end{tikzcd}$$

then f is continuous if and only if g is continuous, for the aforementioned topologies. And f is open if and only if g is.

Consider now the exact sequence (see Proposition 2.1, 1.)

$$H^1(U,N) \to \bigoplus_{v \notin U} H^1(\widehat{K}_v,N) \to H^2_c(U,N) \to H^2(U,N). \tag{5}$$

and for $i=1,2$, set (cf. loc. cit.)

$$D^i(U,N) = \mathrm{Im}\,[H^i_c(U,N) \to H^i(U,N)] = \mathrm{Ker}\,[H^i(U,N) \to \bigoplus_{v \notin U} H^i(\widehat{K_v},N)].$$

By loc. cit., there is an exact sequence

$$\bigoplus_{v \notin U} H^{i-1}(\widehat{K_v},N) \to H^i_c(U,N) \to D^i(U,N) \to 0. \tag{6}$$

The following result has been proved by Česnavičius ([Čes3], Th. 2.9).[4]

Theorem 3.1 (Česnavičius) *The map $H^1(U,N) \to \bigoplus_{v \notin U} H^1(\widehat{K}_v,N)$ is a strict morphism of topological groups, that is: the image of $H^1(U,N)$ is a discrete subgroup of $\bigoplus_{v \notin U} H^1(\widehat{K}_v,N)$. Besides the group $D^1(U,N)$ is finite.*

Corollary 3.2 *The group $H^1_c(U,N)$ is finite.*

Proof: The group $\bigoplus_{v \notin U} H^0(\widehat{K_v},N)$ is finite (N being a finite U-group scheme). Thus the finiteness of $H^1_c(U,N)$ is equivalent to the finiteness of $D^1(U,N)$ by (6).

□

Put the quotient topology on $(\bigoplus_{v \notin U} H^1(\widehat{K}_v,N))/\mathrm{Im}\,H^1(U,N)$. Using Th. 3.1, the previous facts define a natural topology on $H^2_c(U,N)$, so that morphisms in the exact sequence (5) are continuous (and even strict). This topology makes $H^2_c(U,N)$ a Hausdorff and locally compact group.

Lemma 3.3 *Let $V \subset U$ be a non-empty open subset. Then the natural map $u : H^2_c(V,N) \to H^2_c(U,N)$ is continuous.*

Proof: Since (by definition of the topology) the image I of $A := \bigoplus_{v \notin V} H^1(\widehat{K_v},N)$ is an open subgroup of $H^2_c(V,N)$, it is sufficient to show that the restriction of u to I is continuous. As I is equipped with the quotient topology (induced by the topology of A), this is equivalent to showing that the natural map $s : A \to H^2_c(U,N)$ is continuous. Now we observe that A is the direct sum of $A_1 := \bigoplus_{v \notin U} H^1(\widehat{K_v},N)$ and $A_2 := \bigoplus_{v \in U \setminus V} H^1(\widehat{K_v},N)$. The restriction of s to A_1 is continuous by the commutative diagram of Prop. 2.1, 3. Therefore it only remains to show that the restriction s_2 of s to A_2 is continuous. By loc. cit., the restriction of s_2 to $\bigoplus_{v \in U \setminus V} H^1(\widehat{O_v},N)$ is zero. Since $\bigoplus_{v \in U \setminus V} H^1(\widehat{O_v},N)$ is an open subgroup of $\bigoplus_{v \in U \setminus V} H^1(\widehat{K_v},N)$ ([Čes1], Prop. 3.10), the result follows.

□

Proposition 3.4 *The topology on $H^2_c(U,N)$ is profinite.*

[4] Proposition 2.3 of loc. cit. uses the fppf duality Theorem 1.1, but this proposition is actually not needed to prove Theorem 3.1 because a discrete subgroup of a Hausdorff topological group is automatically closed.

Proof: By Proposition 2.1, 2., [Česl], Propositions 4.2 and 4.3(d), and [Mi2], corollary II.3.3, one can assume that the order of N is a power of p. The generic fiber N_K of N is a finite group scheme over K. By [DG], IV, §3.5, N_K admits a composition series whose quotients are étale (with a dual of height one), local (of height one) with étale dual, or α_p. The schematic closure in N of this composition series provides a composition series defined over U.

Using Proposition 2.1, 2. and [Česl], Propositions 4.2 and 4.3(d), one can therefore reduce to the case where the generic fiber N_K or its dual N_K^D has height one.

Proposition B.4 and Corollary B.5 in [Mi2] now imply that there exists a non-empty open subset $V \subset U$ such that $N_{|V}$ extends to a finite flat group scheme $\widetilde{N}$ over X.

Then Proposition 2.1, 3. gives an exact sequence

$$H^1_c(X, \widetilde{N}) \to \bigoplus_{v \in X \setminus V} H^1(\widehat{\mathcal{O}}_v, \widetilde{N}) \to H^2_c(V, N) \to H^2_c(X, \widetilde{N})\,. \tag{7}$$

By Proposition 2.1, 3., the map $\bigoplus_{v \in X \setminus V} H^1(\widehat{\mathcal{O}}_v, \widetilde{N}) \to H^2_c(V, N)$ factors through $\bigoplus_{v \notin V} H^1(\widehat{K}_v, N)$, hence it is continuous. Therefore all maps in (7) are continuous. In addition, the group $H^2_c(X, \widetilde{N})$ is finite and $\bigoplus_{v \in X \setminus V} H^1(\widehat{\mathcal{O}}_v, \widetilde{N})$ is profinite, hence $H^2_c(V, N)$ is profinite. Since $H^2(\widehat{\mathcal{O}}_v, N) = 0$ for every $v \in U$ ([Mi2], §III.7), Prop. 2.1, 3. gives an exact sequence of topological groups

$$\bigoplus_{v \in U \setminus V} H^1(\widehat{\mathcal{O}}_v, N) \to H^2_c(V, N) \to H^2_c(U, N) \to 0,$$

which implies that $H^2_c(U, N)$ is profinite, the map $H^2_c(V, N) \to H^2_c(U, N)$ being continuous by Lemma 3.3, hence strict because $H^2_c(V, N)$ is compact and $H^2_c(U, N)$ is Hausdorff.

□

Proposition 3.5 *Assume that $\mathcal{F} = N$ is a finite and flat group scheme over U. Then all maps in Proposition 2.1 are strict.*

Proof: For the maps in assertion 1. of Prop. 2.1, this follows from the definition of the topology and Th. 3.1.

Let us consider the maps in assertion 2. The finiteness of the H^1_c groups (Cor. 3.2) implies that it only remains to prove that the maps between H^2_c's and the map $H^2_c(U, \mathcal{F}'') \to H^3_c(U, \mathcal{F}')$ are strict. The former are continuous by [Česl], Prop. 4.2. and the definition of the topology, hence they are strict because the H^2_c groups are compact. The latter is also continuous (thus it is strict by compactness of $H^2_c(U, \mathcal{F}'')$) by the same argument, the maps $H^1(\widehat{K_v}, \mathcal{F}'') \to H^2(\widehat{K_v}, \mathcal{F}')$ being continuous by loc. cit.

Finally, it has already been proven (cf. proof of Prop. 3.4) that the maps in the exact sequence of assertion 3. are continuous. They are strict because $H^1_c(U, \mathcal{F})$ is finite, $H^2_c(U, \mathcal{F})$ (resp. $\bigoplus_{v \in U \setminus V} H^1(\widehat{\mathcal{O}_v}, \mathcal{F})$) is profinite, and the other groups are discrete.

□

4 Proof of Theorem 1.1 in the function field case

In this section K is the function field of a projective, smooth and geometrically integral curve X defined over a finite field k of characteristic p. The proof follows the same lines as the proof of [Mi2], Theorem 8.2, replacing Proposition III.0.4 in [Mi2] by Proposition 2.1 and using the results of section 2.

For every non empty open subset $V \subset U$, the natural map $H^3_c(V, \mathbf{G}_m) \to H^3_c(U, \mathbf{G}_m)$ is an isomorphism, and the trace map identifies $H^3_c(U, \mathbf{G}_m)$ with $\mathbf{Q}/\mathbf{Z}$. Indeed since $\mathbf{G}_m$ is a smooth group scheme we can apply Prop 2.1, 4. and [Mi2], §II.3.

For a fppf sheaf $\mathcal{F}$ on U, let us first define precisely the pairing

$$H^r(U, \mathcal{F}^D) \times H^{3-r}_c(U, \mathcal{F}) \to H^3_c(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}\,.$$

Let A and B be two fppf sheaves.

Let $P \to A$ be a flat resolution, and let $P \to G$ be the truncated Godement resolution of the complex P (see for instance [SGA4], XVII, 4.2.9; Godement resolutions exist on the fppf site because this site has enough points, see Remark 1.6. of [GK]). Let $B \to J$, $\mathrm{Tot}(G \otimes J) \to R$ be injective resolutions. Then the natural morphisms

$$\mathrm{Tot}(\Gamma(U, G) \otimes \Gamma(U, J)) \to \Gamma(U, \mathrm{Tot}(G \otimes J)) \to \Gamma(U, R)$$

define a map of complexes

$$\mathrm{Tot}(\Gamma(U, G) \otimes \Gamma(U, J)) \to \Gamma(U, R)\,,$$

hence a canonical morphism in the derived category of abelian groups

$$\Gamma(U, G) \otimes^{\mathbf{L}} \Gamma(U, J) \to \Gamma(U, R)\,.$$

Since G is acyclic (in small degrees) and quasi-isomorphic to A, $\Gamma(U, G)$ computes the cohomology of A (in small degrees), and since R is an injective resolution of $\mathrm{Tot}(P \otimes B)$ (as a consequence of flatness for P and G), we get the canonical cup-product pairing

$$H^r(U, A) \times H^s(U, B) \to H^{r+s}(U, A \otimes^{\mathbf{L}} B)\,.$$

Considering the local versions of the previous pairings, one gets a commutative diagram of complexes of abelian groups

$$\begin{tikzcd}
\mathrm{Tot}(\Gamma(U, G) \otimes \Gamma(U, J)) \arrow[r] \arrow[d] & \Gamma(U, R) \arrow[d] \\
\prod_{v \notin U} \mathrm{Tot}(\Gamma(U, G) \otimes \Gamma(\widehat{K}_v, J)) \arrow[r] & \prod_{v \notin U} \Gamma(\widehat{K}_v, R)\,,
\end{tikzcd}$$

and functoriality of cones gives a canonical morphism of complexes

$$\mathrm{Tot}(\Gamma(U, G) \otimes \Gamma_c(U, J)) \to \Gamma_c(U, R)\,.$$

Hence one gets a canonical morphism in the derived category

$$\Gamma(U, G) \otimes^{\mathbf{L}} \Gamma_c(U, J) \to \Gamma_c(U, R)\,.$$

Computing cohomology of this morphism gives pairings

$$H^r(U,A) \times H^s_c(U,B) \to H^{r+s}_c(U, A \otimes^{\mathbf{L}} B)\,,$$

whence we deduce the required canonical pairings

$$H^r(U,\mathcal{F}^D) \times H^s_c(U,\mathcal{F}) \to H^{r+s}_c(U,\mathbf{G}_m)\,, \tag{8}$$

using the canonical map $\mathcal{F}^D = \underline{\mathrm{Hom}}(\mathcal{F},\mathbf{G}_m) \to \underline{\mathbf{R}\mathrm{Hom}}(\mathcal{F},\mathbf{G}_m)$.

The pairings above are defined via the cup-product on U and via the local duality pairings $H^r(\widehat{K}_v,\mathcal{F}) \times H^{s-1}(\widehat{K}_v,\mathcal{F}^D) \to H^{r+s-1}(\widehat{K}_v,\mathbf{G}_m)$ which are continuous (see [Čes1], Theorem 5.11 and [Mi2], Lemma 6.5 (e)). Hence if N is a finite and flat U-group scheme of order n, the pairings

$$H^r(U,N^D) \times H^s_c(U,N) \to H^{r+s}_c(U,\mu_n)$$

induced by (8) are continuous for the topologies defined in section 3.

Remark 4.1 In [Mi2] (see for example Th. III.3.1), the pairings are defined via the Ext groups, which is quite convenient for the definition itself but makes the required commutativities of diagrams more difficult to check. Nevertheless, Proposition V.1.20 in [Mi1] provides a comparison between both definitions.

We now want to show that the induced map $H^{3-r}_c(U,N) \to H^r(U,N^D)^*$ is an isomorphism (of topological groups) for every finite flat group scheme N over U and every $r \in \{0,1,2,3\}$ (recall that the groups $H^r(U,N^D)$ are equipped with the discrete topology).

We first note that [Mi2], Lemma III.8.3 is correct, taking into account Prop. 3.5 and that duality $\mathrm{Hom}_{\mathrm{cont.}}(.,\mathbf{Q}/\mathbf{Z})$ is exact for discrete groups. In particular, given a short exact sequence of finite flat group schemes:

$$0 \to N' \to N \to N'' \to 0\,,$$

then Theorem 1.1 for N is a consequence of Theorem 1.1 for both N' and N''.

In order to prove Lemma III.8.4 in [Mi2] (which shows that to prove Theorem 1.1, it is equivalent to prove it for a smaller open subset $V \subset U$), we need to check the compatibility of the pairing in Theorem 1.1 with restriction to an open subset of U and with the local duality pairing. For every fppf sheaf $\mathcal{F}$ on U and every non-empty open subset $V \subset U$, we have (Prop 2.1, 3.) a map (which is continuous if $\mathcal{F}$ is a finite and flat group scheme) $H^{3-r}_c(U,\mathcal{F}) \to \bigoplus_{v\in U\setminus V} H^{3-r}(\widehat{\mathcal{O}_v},\mathcal{F})$ and, assuming $\mathcal{F}$ is a flat affine commutative group scheme locally of finite type, (Prop. 2.10) a map of discrete groups $\bigoplus_{v\in U\setminus V} H^r_v(\widehat{\mathcal{O}_v},\mathcal{F}) \to H^r(U,\mathcal{F})$, which appears in the long exact sequence (4). Besides Prop. 2.1, 1. gives a map of discrete groups $\bigoplus_{v\in U\setminus V} H^2(\widehat{K}_v,\mathbf{G}_m) \to H^3_c(V,\mathbf{G}_m)$. Finally, the canonical map $\bigoplus_{v\in U\setminus V} H^2(\widehat{K_v},\mathbf{G}_m) \to \bigoplus_{v\in U\setminus V} H^3_v(\widehat{\mathcal{O}_v},\mathbf{G}_m)$ is an isomorphism thanks to the localization exact sequence because by smoothness of $\mathbf{G}_m$, the group $H^i(\widehat{\mathcal{O}_v},\mathbf{G}_m) \cong H^i_{\text{ét}}(\widehat{\mathcal{O}_v},\mathbf{G}_m) \simeq H^i(k(v),\mathbf{G}_m)$ is zero for $i \geq 2$ (indeed the finite field $k(v)$ is of cohomological dimension 1).

Lemma 4.2 *Let $\mathcal{F}$ be a flat affine commutative group scheme locally of finite type on U. Let $V \subset U$ be a non-empty open subscheme:*

1. *the natural diagram*

$$\begin{tikzcd} H^r(U,\mathcal{F}^D) \times H^{3-r}_c(U,\mathcal{F}) \arrow[r] \arrow[d] & H^3_c(U,\mathbf{G}_m) \\ H^r(V,\mathcal{F}^D) \times H^{3-r}_c(V,\mathcal{F}) \arrow[r] \arrow[u] & H^3_c(V,\mathbf{G}_m) \arrow[u, "\sim"'] \end{tikzcd}$$

is commutative.

2. *the natural diagram*

$$\begin{tikzcd} H^r(U,\mathcal{F}^D) \times H^{3-r}_c(U,\mathcal{F}) \arrow[r] \arrow[d] & H^3_c(U,\mathbf{G}_m) & H^3_c(V,\mathbf{G}_m) \arrow[l, "\sim"'] \\ \bigoplus_{v\in U\setminus V} H^r_v(\widehat{\mathcal{O}_v},\mathcal{F}^D) \times \bigoplus_{v\in U\setminus V} H^{3-r}(\widehat{\mathcal{O}_v},\mathcal{F}) \arrow[r] \arrow[u] & \bigoplus_{v\in U\setminus V} H^3_v(\widehat{\mathcal{O}_v},\mathbf{G}_m) \arrow[u] & \bigoplus_{v\in U\setminus V} H^2(\widehat{K_v},\mathbf{G}_m) \arrow[l, "\sim"'] \arrow[u] \end{tikzcd}$$

is commutative.

Proof:

1. Let $A := \mathcal{F}^D$ and $B := \mathcal{F}$. Let $P \to A$ be a flat resolution, and $P \to G$ be the truncated Godement resolution of P. Let also $B \to J$ and $\mathrm{Tot}(G \otimes J) \to R$ be injective resolutions. Let $\widetilde{\Gamma}(U,J) := \mathrm{Cone}(\Gamma(U,J) \to \bigoplus_{v\notin U} \Gamma(\widehat{K_v},J) \oplus \bigoplus_{v\in U\setminus V} \Gamma(\widehat{\mathcal{O}_v},J))[-1]$. Then functoriality of the cone gives a commutative diagram (similar to (3), where $I^\bullet(\mathcal{F})$ is replaced by J and by R) of complexes of abelian groups:

$$\begin{tikzcd} \mathrm{Tot}(\Gamma(U,G) \otimes \Gamma_c(U,J)) \arrow[r] & \Gamma_c(U,R) \\ \mathrm{Tot}(\Gamma(U,G) \otimes \widetilde{\Gamma}(U,J)) \arrow[r] \arrow[u, "="] \arrow[d] & \widetilde{\Gamma}(U,R) \arrow[u] \arrow[d, "q"] \\ \mathrm{Tot}(\Gamma(V,G) \otimes \Gamma_c(V,J)) \arrow[r] & \Gamma_c(V,R)\,. \end{tikzcd}$$

Here the maps denoted by q are quasi-isomorphisms (see Remark 2.7 and the proof of the third point in Proposition 2.1, which uses Lemma 2.4). This diagram gives a commutative diagram in the derived category of abelian groups (where all the maps are either the natural ones or the ones constructed above):

$$\begin{tikzcd} \Gamma(U,G) \otimes^{\mathbf{L}} \Gamma_c(U,J) \arrow[r] \arrow[d] & \Gamma_c(U,R) \\ \Gamma(V,G) \otimes^{\mathbf{L}} \Gamma_c(V,J) \arrow[r] \arrow[u] & \Gamma_c(V,R)\,. \arrow[u] \end{tikzcd}$$

Taking cohomology of this diagram gives a commutative diagram of abelian groups:

$$\begin{tikzcd} H^r(U,A) \times H^s_c(U,B) \arrow[r] \arrow[d] & H^{r+s}_c(U, A \otimes^{\mathbf{L}} B) \\ H^r(V,A) \times H^s_c(V,B) \arrow[r] \arrow[u] & H^{r+s}_c(V, A \otimes^{\mathbf{L}} B)\,, \arrow[u] \end{tikzcd}$$

which implies the required commutativity.

2. First, the commutativity of the right hand side square is easy using functoriality of the cone for complexes.

   We now prove the remaining commutativity. Let A and B be two fppf sheaves over U. Let $P \to A$ be a flat resolution, and $P \to G$ be the truncated Godement resolution of P. Let also $B \to J$ and $\mathrm{Tot}(G \otimes J) \to R$ be injective resolutions. Using functoriality of cones, one proves that there is a natural commutative diagram of complexes:

$$\begin{tikzcd}
\mathrm{Tot}(\Gamma(U,G) \otimes \Gamma_c(U,J)) \arrow[r] \arrow[d, shift left=4] & \Gamma_c(U,R) \\
\mathrm{Tot}(\Gamma_c(U,G) \otimes \Gamma(U,J)) \arrow[r] \arrow[u, shift right=4] & \Gamma_c(U,R)\,. \arrow[u, "="']
\end{tikzcd}$$

   Hence the following diagram

$$\begin{tikzcd}
\Gamma(U,G) \otimes^{\mathbf{L}} \Gamma_c(U,J) \arrow[r] \arrow[d, shift left=4] & \Gamma_c(U,R) \\
\Gamma_c(U,G) \otimes^{\mathbf{L}} \Gamma(U,J) \arrow[r] \arrow[u, shift right=4] & \Gamma_c(U,R) \arrow[u, "="']
\end{tikzcd}$$

   commutes in the derived category. Computing cohomology gives a commutative diagram of abelian groups:

$$\begin{tikzcd}
H^r(U,A) \times H^s_c(U,B) \arrow[r] \arrow[d, shift left=4] & H^{r+s}_c(U, A \otimes^{\mathbf{L}} B) \\
H^r_c(U,A) \times H^s(U,B) \arrow[r] \arrow[u, shift right=4] & H^{r+s}_c(U, A \otimes^{\mathbf{L}} B)\,. \arrow[u, "="']
\end{tikzcd}$$

   Let $\Gamma_Z(U,G) := \mathrm{Cone}(\Gamma(U,G) \to \Gamma(V,G))[-1]$. In order to prove the required commutativity, it is enough to prove that the natural diagram

$$\begin{tikzcd}
\Gamma_c(U,G) \otimes^{\mathbf{L}} \Gamma(U,J) \arrow[r] & \Gamma_c(U,R) \\
\Gamma_Z(U,G) \otimes^{\mathbf{L}} \Gamma(U,J) \arrow[r] \arrow[u] \arrow[u, shift left=4, "="'] & \Gamma_Z(U,R) \arrow[u]
\end{tikzcd}$$

   commutes in the derived category. To do this, consider the following diagram in the category of complexes:

$$\begin{tikzcd}
\mathrm{Tot}(\Gamma(U,G) \otimes \Gamma(U,J)) \arrow[rr] \arrow[dr] \arrow[dd] & & \Gamma(U,R) \arrow[dr] \arrow[dd] & \\
& \mathrm{Tot}(\Gamma(V,G) \otimes \Gamma(U,J)) \arrow[rr, crossing over] \arrow[dl] & & \Gamma(V,R) \arrow[dl] \\
\prod_{v \notin U} \mathrm{Tot}(\Gamma(\widehat{K}_v,G) \otimes \Gamma(U,J)) \arrow[rr] & & \prod_{v \notin U} \Gamma(\widehat{K_v},R) & .
\end{tikzcd}$$

This diagram is commutative, hence it induces a natural commutative diagram of complexes at the level of cones:

$$\begin{tikzcd}
 & \mathrm{Tot}(\Gamma_c(U,G) \otimes \Gamma(U,J)) \arrow[rr] \arrow[dd] & & \Gamma_c(U,R) \arrow[dd] \\
\mathrm{Tot}(\Gamma_Z(U,G) \otimes \Gamma(U,J)) \arrow[ur] \arrow[dr] \arrow[rr] & & \Gamma_Z(U,R) \arrow[ur] \arrow[dr] & \\
 & \mathrm{Tot}(\Gamma(U,G) \otimes \Gamma(U,J)) \arrow[rr] & & \Gamma(U,R) .
\end{tikzcd}$$

The commutativity of the upper face of this last diagram concludes the proof.

□

Now Lemma 4.2, Prop. 2.1, 3., [Mi2], Theorem III.7.1. (local duality) and exact sequence (4) immediately imply Lemma III.8.4 in [Mi2], which tells us that Theorem 1.1 holds for N on U if and only if it holds for $N_{|V}$ on V.

The end of the proof of Theorem 1.1 is exactly the same as the end of the proof of Theorem III.8.2 in [Mi2].

□

As observed in [Mi2], §III.8 (remark before Lemma 8.9), the group $H^1(U,N)$ is in general infinite if $U \neq X$ and by duality, the same is true for $H^2_c(U,N)$. However, the situation is better for H^2 and H^1_c :

Corollary 4.3 *Let N be a finite and flat group scheme over U. The groups $H^2(U,N)$ and $H^1_c(U,N)$ are finite.*

Proof: The statement about $H^1_c(U,N)$ is Corollary 3.2. The finiteness of $H^2(U,N)$ follows by the duality Theorem 1.1.

□

The previous corollary can be refined in some cases :

Proposition 4.4 *Let N be a finite and flat group scheme over an affine open subset $U \subset X$, such that the generic fiber N_K is local. Then $H^1_c(U,N) = 0$.*

Proof: It is well known that the restriction map $H^1(U,N) \to H^1(K,N)$ is injective (the proof is as in [Mi2], Lemma III.1.1.). Now the proposition is an immediate consequence of the main theorem of [GT], which means that if we choose $v \notin U$, the restriction map $H^1(K,N) \to H^1(K_v,N)$ is injective when N_K is local. Indeed this implies that $D^1(U,N) = 0$, hence $H^1_c(U,N) = 0$ by exact sequence (6) because $H^0(K_v,N) = 0$ for every completion K_v of K.

□

Remark 4.5 The finiteness of $H^1_c(U,N)$ (Cor. 3.2) relies on the finiteness of $D^1(U,N)$ proven in [Česk3], Th. 2.9. An alternative argument is actually available. By [Mi2], Lemma III.8.9., we can assume that $U \neq X$, namely that U is affine. By loc. cit., Th. II.3.1. and Prop. 2.1, 4., we can also assume that the order of N is a power of p. Let

N_K be the generic fiber of N, it is a finite group scheme over K. By [DG], IV, §3.5, and Prop. 2.1, 2., it is sufficient to prove the required finiteness in the following cases : N_K is étale, N_K is local with étale dual, $N_K = \alpha_p$. The last two cases are taken care of by Prop. 4.4, so we can suppose that N_K is étale. Let $V \subset U$ be a non empty open subset. By Prop. 2.1, we have an exact sequence

$$H^1_c(V, N) \to H^1_c(U, N) \to \bigoplus_{v \in U \setminus V} H^1(\widehat{\mathcal{O}_v}, N).$$

Since the generic fiber of N is étale, the group $H^1(\widehat{\mathcal{O}_v}, N)$ is finite by [Mi2], Rem. III.7.6. (this follows from the fact that $H^1(\widehat{\mathcal{O}_v}, N)$ is a compact subgroup of the discrete group $H^1(K_v, N)$), hence the finiteness of $H^1_c(U, N)$ is equivalent to the finiteness of $H^1_c(V, N)$, which in turn is equivalent to the finiteness of $D^1(V, N)$. The latter holds for V sufficiently small: either apply [Gon], Lemma 4.3. (which relies on an embedding of N_K into an abelian variety) or reduce (as in [Mi2], Lemma III.8.9.) to the case when N^D is of height one. Indeed by loc. cit., Cor. III.B.5., the assumption that N^D is of height one implies that for V sufficiently small, the restriction of N to V extends to a finite and flat group scheme $\widetilde{N}$ over X. Then the finiteness of $H^1_c(X, \widetilde{N})$ implies the finiteness of $H^1_c(V, \widetilde{N}) = H^1_c(V, N)$ by Prop 2.1, 3., because the groups $H^0(\widehat{\mathcal{O}_v}, \widetilde{N})$ are finite.

Remark 4.6 The main result of [GT] also relies on an embedding of N_K into an abelian variety. Another approach is available to prove Prop. 4.4 in the critical cases $N_K = \alpha_p$ and N_K^D étale. For $N_K = \alpha_p$, one checks directly that $H^1(K, N) = K/K^p$ injects into $H^1(\widehat{K_v}, N) = \widehat{K_v}/\widehat{K_v}^p$, using the fact that $\widehat{K_v}$ does not contain any inseparable algebraic element over K because the local ring $\mathcal{O}_{X,v}$ is excellent. If N_K^D is étale, one first reduces by devissage to the case when N_K is of height one. Then we choose a non empty open subset $V \subset U$ such that the restriction N_1^D of N^D to V is étale with dual of height one. We can also assume that $\operatorname{Pic} V = 0$ by finiteness of the ideal class group of K. By [Mi2], Th. III.5.1., there is an exact sequence of fppf sheaves

$$0 \to N_1^D \to \mathcal{F} \to \mathcal{G} \to 0,$$

where $\mathcal{F}$ and $\mathcal{G}$ are coherent locally free sheaves (for Zariski topology) over the affine Dedekind scheme V. Since $\operatorname{Pic} V = 0$, these sheaves are free and of finite type, hence the corresponding fppf sheaves are represented by groups isomorphic to $\mathbf{G}_a^r$ for some r. In particular $H^i(V, \mathcal{F}) = H^i(V, \mathcal{G}) = 0$ for $i > 0$ and $H^i(V, N^D) = H^i(V, N_1^D) = 0$ for $i > 1$. We now have a commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathcal{G}(V) & \longrightarrow & H^1(V, N_1^D) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\bigoplus_{v \in U \setminus V} G(\widehat{K_v}) & \longrightarrow & \bigoplus_{v \in U \setminus V} H^1(\widehat{K_v}, N_1^D) & \longrightarrow & 0.
\end{array}$$

By the strong approximation theorem on the affine Dedekind scheme U, the left vertical map has dense image. Since N_K^D is étale, the group $\bigoplus_{v \in U \setminus V} H^1(\widehat{K_v}, N_1^D)$ is discrete ([Mi2], §III.6). As the bottom horizontal map is continuous ([Čes1], Prop. 4.2), this implies that the right vertical map is surjective. Finally we find that the map $H^1(V, N^D) \to \bigoplus_{v \in U \setminus V} H^1(\widehat{K_v}, N^D)$ is surjective. A fortiori the map $H^1(V, N^D) \to$

$\bigoplus_{v\in U\setminus V} H_v^2(\widehat{\mathcal{O}_v}, N^D)$ is surjective because $H_v^2(\widehat{\mathcal{O}_v}, N^D) = H^1(\widehat{K_v}, N^D)/H^1(\widehat{\mathcal{O}_v}, N^D)$ ([Mi2], §III.7). Dualizing this statement thanks to Theorem III.7.1. of loc. cit. and Theorem 1.1, we obtain that the map

$$\bigoplus_{v\in U\setminus V} H^1(\widehat{\mathcal{O}_v}, N) \to H_c^2(V, N)$$

is injective; by Proposition 2.1, 3., this means that the map $H_c^1(V,N) \to H_c^1(U,N)$ is surjective. Since $H_c^1(V,N) = 0$ (by duality to $H^2(V, N^D)$), we have $H_c^1(U,N) = 0$.

5 The number field case

Assume now that K is a number field and set $X = \operatorname{Spec} \mathcal{O}_K$. Let U be a non empty Zariski open subset of X. Let n be the order of the finite and flat group scheme N. To prove Theorem 1.1 in this case, one follows exactly the same method as in [Mi2], Th. III.3.1. and Cor. III.3.2. once Proposition 2.1 has been proved. Namely Proposition 2.1, 4., shows that on $U[1/n]$, Theorem 1.1 reduces to the étale Artin-Verdier Theorem ([Mi2], II.3.3). Now Proposition 2.1, 3., gives a commutative diagram as in the end of the proof ot [Mi2], Th. III.3.1. (with completions $\widehat{\mathcal{O}}_v$ instead of henselizations $\mathcal{O}_v$). Theorem 1.1 follows by the five-lemma, using the result on $U[1/n]$ and the local duality statement [Mi2], Th. III.1.3.

Remark 5.1 In the number field case, one can as well (as in [Mi2], §III.3) work from the very beginning with henselizations $\mathcal{O}_v$ and not with completions $\widehat{\mathcal{O}}_v$ to define cohomology with compact support. Indeed the local theorem (loc. cit., Th. III.1.3) still holds with henselian (not necessarily complete) d.v.r. with finite residue field when the fraction field is of characteristic zero. Hence the only issue here is commutativity of diagrams. Nevertheless, we felt that it was more convenient to have a uniform statement (Proposition 2.1) in both $\operatorname{Char} 0$ and $\operatorname{Char} p$ situations.

Acknowledgements. We thank K. Česnavičius, C. Pépin, A. Schmidt and T. Szamuely for enlightening comments on the first draft of this paper. We also thank T. Suzuki for sharing his preprint [Suz].

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