Degree 4 cohomological invariants of algebraic tori

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Abstract

In this paper, we determine the motive of the classifying torsor of an algebraic torus. As a result, we give an exact sequence describing the degree 4 cohomological invariants of algebraic tori. Using results by Blinstein and Merkurjev, this provides a formula for the degree 4 unramified cohomology group of an algebraic torus, via a flasque resolution.

1 Introduction

Let G be an algebraic group over a field F. An invariant of G [GMS03] is a natural transformation from the functor of G-torsors over fields to a functor H from category of fields to category of abelian groups. We denote this invariant group by $\mathrm{Inv}(G, H)$. We mainly consider the cohomological invariants, i.e. the case that H is a Galois cohomological functor $H^d_{\text{ét}}(-, \mathbb{Q}/\mathbb{Z}(j))$ for $d \geq 0, j \geq 0$, and denote this group by $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(j))$.

In [BM13], Blinstein and Merkurjev provide a way to compute $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(j))$ through the motivic cohomology of a classifying space BG of G. For an algebraic torus T, they provide an exact sequence describing $\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm}$, which is the subgroup of normalized invariants, and they give a similar exact sequence computing the unramified cohomology group $H^3_{nr}(F(T), \mathbb{Q}/\mathbb{Z}(2))$ via a flasque resolution of T. In particular, they provide an isomorphism

$$\gamma_0 : \bar{H}^4(BT, \mathbb{Z}(2))_{bal}/\mathrm{CH}^2(BT) \xrightarrow{\sim} \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm}$$

(see [BM13], diagram before Lemma 4.2) and the exact sequence recalled below in Theorem 2.2. Applying these results, Wei and the second author give explicit formulas in [LW24] that determine completely the third unramified cohomology group for norm 1 tori associated with abelian field extensions. Understanding unramified cohomology of tori is a longstanding problem related to rational points and rationality questions, and in [CT95], Colliot-Thélène asks for a formula computing these groups for an arbitrary torus T, in terms of the module $\widehat{T}$ of characters of T.

Following the approach in [BM13], we propose to compute the groups $\mathrm{Inv}^4(T, \mathbb{Q}/\mathbb{Z}(3))_{norm}$ and $H^4_{nr}(F(T), \mathbb{Q}/\mathbb{Z}(3))$ for all tori T. In the statement below, a classifying space of T is defined as $BT := U/T$, where V is a generically free representation of T and $U \subset V$ in an open subset on which the action of T is free and such that the codimension of $V \setminus U$ in V is big enough. All cohomology groups are étale or étale motivic cohomology groups.

Let I be the kernel of étale cycle map $\mathrm{CH}^3(BT) \to \bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(3))$.

Theorem. *Let T be a torus over a perfect field F. Then there is a natural commutative diagram:*

$$
\begin{tikzcd}
 & \left(\widehat{T} \otimes K_2^M(F_{sep})\right)^{\Gamma} \arrow[d] & & \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm} \otimes F^* \arrow[dll, "\gamma"'] \arrow[dl, "\cup"] & \\
 & H^2(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) \arrow[d] & & 0 \arrow[d] & \\
A^2(BT, K_3^M) \arrow[r, hook] & \bar{H}^5_{\text{ét}}(BT, \mathbb{Z}(3)) \arrow[r] \arrow[d] & \mathrm{Inv}^4(T, \mathbb{Q}/\mathbb{Z}(3))_{norm} \arrow[r] & I \arrow[r] \arrow[d] & 0 \\
 & \left(S^2(\widehat{T}) \otimes F^*_{sep}\right)^{\Gamma} \arrow[dr] & & \mathrm{CH}^3(BT)_{tors} \arrow[d] & \\
 & & H^3(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & \ker\left(\bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(3)) \to S^3(\widehat{T})^{\Gamma}\right) \arrow[r] & H^1(F, S^2(\widehat{T}) \otimes F^*_{sep})
\end{tikzcd}
$$

where the row, the column and the column that turns to a line are exact, where γ is induced by the map γ_0 above and cup-product. This conclusion holds for char $F = 0$*. If* char $F = p > 0$*, then the conclusion still holds after changing $\mathbb{Q}/\mathbb{Z}(2)$ in the Galois cohomology groups of F to $\mathbb{Q}/\mathbb{Z}(2)[1/p]$.*

Note that the group I is also the kernel of étale cycle map $\mathrm{CH}^3(BT) \to \bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(3))$.

In addition, let $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ be a flasque resolution of an arbitrary F-torus S. By results of Blinstein and Merkurjev [BM13], the fourth unramified cohomology group $\bar{H}^4_{nr}(F(S), \mathbb{Q}/\mathbb{Z}(3))$ is isomorphic to $\mathrm{Inv}^4(T, \mathbb{Q}/\mathbb{Z}(3))_{norm}$, therefore the previous theorem also gives an exact sequence computing this unramified cohomology group of S.

The key point of the proof of the main Theorem is to compute the étale motivic cohomology of a classifying T-torsor BT as above, using the so-called slice filtration spectral sequence introduced in [HK06].

2 Preliminaries

In this section, we introduce notations and lemmas which will be helpful in the next sections.

2.1 Algebraic tori

Let F be a field, F_{sep} be a separable closure of F and $\Gamma := \mathrm{Gal}(F_{sep}/F)$. Recall that an algebraic torus of dimension n over F is an algebraic F-group T such that $T_{sep} := T \times_F F_{sep}$ is isomorphic to $\mathbb{G}_m^n$ as a F_{sep}-group. For an algebraic F-torus T, we write $\widehat{T}$ for the Γ-module $\mathrm{Hom}(T, \mathbb{G}_m)$ and call it the character lattice of T. The contravariant functor $T \mapsto \widehat{T}$ defines an anti-equivalence between the category of algebraic tori over F and the category of Γ-lattices.

Let K/F be a finite Galois extension with $G = \mathrm{Gal}(K/F)$. We embed K to F_{sep}, so that the absolute Galois group $\Gamma_K := \mathrm{Gal}(F_{sep}/K)$ is a subgroup of Γ.

A torus T is called flasque (resp. coflasque) if $H^1(L, \widehat{T}^{\circ}_{sep}) = 0$ (resp. $H^1(L, \widehat{T}_{sep}) = 0$) for all finite field extensions L/F. A flasque resolution of a torus T is an exact sequence of tori

$$1 \longrightarrow Q \longrightarrow P \longrightarrow T \longrightarrow 1$$

If T is a norm one torus $R^{(1)}_{L/F}(\mathbb{G}_{m,L})$, the degree 3 normalized cohomological invariants group of T is isomorphic to Br(L/F) (see for instance [BM13], Example 4.14).

2.2 Cohomological invariants

Given a linear algebraic group G over a field F and a functor H from the category of fields over F to the category of abelian groups, an invariant of G with values in H (or an H-invariant) is a morphism of functor:

$$i : H^1(-, G) \to H$$

where $H^1(-, G)$ is the Galois cohomology functor from the category of fields over F to the category of sets. We denote the group of H-invariants of G by $\mathrm{Inv}(G, H)$. An H-invariant is called normalized if it takes the trivial torsor to 0. We denote the subgroup of normalized H-invariants of G by $\mathrm{Inv}(G, H)_{norm}$. There is a natural isomorphism:

$$\mathrm{Inv}(G, H) \cong \mathrm{Inv}(G, H)_{norm} \oplus H(F)$$

where each element h in $H(F)$ defines an H-invariant i_h which takes all G-torsor over K to the natural image of h in $H(K)$ for every field extension K/F. Such an invariant is called a constant invariant. We consider here the cohomological invariants, i.e. the case where H is a Galois cohomological functor $H^d_{\text{ét}}(-, \mathbb{Q}/\mathbb{Z}(j))$ for $d \geq 0, j \geq 0$, and we denote the corresponding group by $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(j))$.

A G-torsor $E \to X$ is said to be classifying if for every field extension K/F with K infinite, every G-torsor over K is isomorphic to the fiber of $E \to X$ at a K-point of X. Classifying torsors exist for all linear F-groups by [BM13, 2b], [GMS03, 5.3]. In [BM13], Blinstein and Merkurjev give an approach to compute cohomological invariants of smooth linear algebraic groups and an exact estimate of $\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm}$, when T is a F-torus. We recall those results here:

Lemma 2.1. *[BM13, Theorem 3.4] Let G be a smooth linear algebraic group over a field F. We assume that G is connected if F is a finite field. Let $E \to X$ be a classifying G-torsor with E a G-rational variety such that $E(F) \neq \emptyset$. Then the following homomorphism*

$$\varphi : \bar{H}^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))_{\mathrm{bal}} \to \mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))_{norm}$$

is an isomorphism.

Here $\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))$ is the Zariski sheaf on X associated to the presheaf $U \mapsto H^n_{\text{ét}}(U, \mathbb{Q}/\mathbb{Z}(j))$ and $H(X)_{\mathrm{bal}}$ is the subgroup of $H(X)$ consists of the elements $h \in H(X)$ such that $p_1^*(h) = p_2^*(h)$ for the projections $p_i : E^2/G \to X$.

Lemma 2.2. *[BM13, Theorem B] Let T be an algebraic torus over a field F. Then there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{CH}^2(BT)_{tors} \longrightarrow H^1(F, T^\circ) \longrightarrow & \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm} \\ \longrightarrow & S^2(\widehat{T})^\Gamma / Dec \longrightarrow H^2(F, T^\circ). \end{aligned}$$

Here Dec denotes the image of $\mathrm{CH}^2(BT)$ in $\mathrm{CH}^2(BT_{sep}) \simeq S^2(\widehat{T})$.

2.3 Motivic cohomology

Let F be a perfect field. We denote by $DM_-^{eff}(F)$ (resp. $DM_{-,\acute{e}t}^{eff}(F)$) the category of complexes of Nisnevich (resp. étale) sheaves with transfers with homotopy invariant cohomology sheaves, and α^* the natural functor $DM_-^{eff}(F) \to DM_{-,\acute{e}t}^{eff}(F)$. Let $DM_{gm}^{eff}(F)$ be the category of effective geometrical motives which was defined by Voevodsky [Voe00]. There is a functor from $DM_{gm}^{eff}(F)$ to $DM_-^{eff}(F)$, induced by $[X] \mapsto \underline{C}_*(X)$, where $\underline{C}_*(X)$ is the Suslin complex of X: this complex will be denoted by $M(X)$.

In [HK06], Huber and Kahn construct spectral sequences as follows, which are called slice spectral sequences :

$$\begin{aligned} E_2^{p,q} &= H^{p-q}(c_q M, \mathbb{Z}(n-q)) \implies H^{p+q}(M, \mathbb{Z}(n)) \\ E_2^{p,q} &= H_{\text{ét}}^{p-q}(\alpha^* c_q M, \mathbb{Z}(n-q)_{\text{ét}}) \implies H_{\text{ét}}^{p+q}(M, \mathbb{Z}(n)_{\text{ét}}) \end{aligned}$$

where

$$\mathbb{Z}(n)_{\text{ét}} = \begin{cases} \alpha^* \mathbb{Z}(n), & n \geq 0 \\ \bigoplus_{l \neq \mathrm{char} F} \mathbb{Q}_l/\mathbb{Z}_l(n)\,[-1], & n < 0. \end{cases}$$

The motives $c_q M$ are called the fundamental invariants of M in [HK06]. There is an exact triangle in $DM_-^{eff}(F)$:

$$\nu_n M \longrightarrow \nu_{<n+1} M \longrightarrow \nu_{<n} M \longrightarrow \nu_n M\,[1]\,.$$

where $\nu_n M = c_n(M)(n)\,[2n]$ and the functors $\nu_{<n}$ are defined in [HK06].

3 Main results

3.1 Motives of classifying space of algebraic tori

We assume that F is a perfect field. In this section, we prove that the motive of the classifying space BT of a F-torus T is a pure Tate motive, in order to write down the slice spectral sequence for BT. As a result, we compute the étale motivic cohomology of BT in low degree.

Let $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ be an exact sequence of algebraic tori such that P is quasi-split. Since P is quasi-split, we can assume that $P = \prod_i R_{K_i/F}(\mathbb{G}_{m,K_i})$, where K_i/F are finite separable field extensions. Then the free diagonal action of $\mathbb{G}_{m,K_i}$ on $\mathbb{A}_{K_i}^{r+1} \setminus \{0\}$ induces, by Weil restriction, a free action of P on $U = \prod_i R_{K_i/F}(\mathbb{A}_{K_i}^{r+1} \setminus \{0\})$. We write X for U/T and Y for U/P.

It is easy to see that Y is geometrically cellular (actually it is cellular since Weil restriction preserves cellular structure). Note that $U \to X$ is a classifying T-torsor and $U \to Y$ is a classifying

P-torsor, by [GMS03, 5.3]. In addition, $X \to Y$ is a Q-torsor. Let BT (resp. BP) be a classifying space of T (resp. P). Then BT (resp. BP) can be approximated by X (resp. Y) when r is large enough (see for instance [BM13], Lemma A.4). In the following, the notation BT will mean the classifying space X constructed here, for r big enough.

In [Kah99] and [HK06], the authors provide a natural filtration of the motive of a cellular variety. In particular, since Y is a geometrically cellular variety, there is a spectral sequence $E_2^{p,q} = H^{p-q}(F, \mathrm{CH}^q(Y_{sep}) \otimes \mathbb{Z}(n-q)) \implies H^{p+q}_{\text{ét}}(Y, \mathbb{Z}(n))$. We show below that this spectral sequence also holds for X, even though we don't know how to find a classifying torsor $U \to X$ such that X is a geometrically cellular variety and X approximates BT.

Proposition 3.1. *The complex $c^q(X_{sep}) \in D^b_f(Ab)$ is isomorphic to the following Koszul-like complex $K(X_{sep}, q)$:*

$$\begin{aligned}\Lambda^q\widehat{Q} \longrightarrow \Lambda^{q-1}\widehat{Q} \otimes \mathrm{CH}^1(Y_{sep}) \longrightarrow &\Lambda^{q-2}\widehat{Q} \otimes \mathrm{CH}^2(Y_{sep}) \longrightarrow \cdots \\ &\longrightarrow \widehat{Q} \otimes \mathrm{CH}^{q-1}(Y_{sep}) \longrightarrow \mathrm{CH}^q(Y_{sep})\end{aligned}$$

where $\mathrm{CH}^q(Y_{sep})$ is in degree 0, and the maps are induced by interior products and the characteristic map $\theta : \widehat{Q} \to \mathrm{CH}^1(Y_{sep})$.

Proof. The proof is the same as Huber and Khan's proof for split reductive groups [HK06, Proposition 9.3]. Since Y_{sep} is cellular, we have an isomorphism $c^q(Y_{sep}) = \mathrm{CH}^q(Y_{sep})\,[0]$ from [HK06, Proposition 4.11]. Note that $X_{sep} \to Y_{sep}$ is a Q_{sep} torsor. Let Ξ be the cocharacter module of Q. Applying [HK06, Proposition 8.10] to $X_{sep} \to Y_{sep}$ and to the functor $H^0(c_m) : TDM^{eff}_{gm} \to Ab$, there is a spectral sequence

$$E_1^{p,q} = H^q(c_{m-q}(Y_{sep}) \otimes \Lambda^p\Xi) \implies H^{p+q}(c_m(X_{sep}))$$

for each m. This spectral sequence is concentrated in the $q = 0$ row and degenerates at E_2. Therefore, we get that $c^q(X_{sep})$ is isomorphic to the dual of the complex in E_1, whence the result. □

Lemma 3.2. *$K(X_{sep}, q)$ is quasi-isomorphic to $S^q(\widehat{T})[0]$.*

Proof. Note that the sequence $1 \to \widehat{Q} \xrightarrow{\alpha} \widehat{P} \xrightarrow{\beta} \widehat{T} \to 1$ is exact and that the natural morphism $\theta : \widehat{P} \to \mathrm{CH}^1(Y_{sep})$ induces isomorphisms of Galois modules $\mathrm{CH}^q(Y_{sep}) \cong S^q(\widehat{P})$ for all p, by [EG97, Lemma 2].

In addition, for all i, j, the diagram

$$\begin{tikzcd}
\Lambda^i\widehat{Q} \otimes S^j(\widehat{P}) \arrow[r, "\varphi_{i,j}"] \arrow[d, "\sim"] & \Lambda^{i-1}\widehat{Q} \otimes S^{j+1}(\widehat{P}) \arrow[d, "\sim"] \\
\Lambda^i\widehat{Q} \otimes \mathrm{CH}^j(Y_{sep}) \arrow[r] & \Lambda^{i-1}\widehat{Q} \otimes \mathrm{CH}^{j+1}(Y_{sep})
\end{tikzcd}$$

is commutative, where the vertical isomorphisms are induced by the aforementionned isomorphisms $S^j(\widehat{P}) \xrightarrow{\sim} \mathrm{CH}^j(Y_{sep})$, the lower horizontal map is the one in the complex $K(X_{sep}, i+j)$, and $\varphi_{i,j}$

is defined by

$$\varphi_{i,j}((\chi_1 \wedge \cdots \wedge \chi_i) \otimes (\xi_1 \dots \xi_j)) := \sum_{k=1}^{i} (-1)^{k+1} (\chi_1 \wedge \cdots \wedge \widehat{\chi_k} \wedge \dots \chi_i) \otimes (\alpha(\chi_k)\xi_1 \dots \xi_j)\,.$$

Therefore, the complex $K(X_{sep}, q)$ is isomorphic to the Koszul complex $K(\alpha, q)$ defined by

$$\Lambda^q \widehat{Q} \xrightarrow{\varphi_{q,0}} \Lambda^{q-1}\widehat{Q} \otimes \widehat{P} \xrightarrow{\varphi_{q-1,1}} \Lambda^{q-2}\widehat{Q} \otimes S^2(\widehat{P}) \xrightarrow{\varphi_{q-2,2}} \dots \xrightarrow{\varphi_{2,q-2}} \widehat{Q} \otimes S^{q-1}(\widehat{P}) \xrightarrow{\varphi_{1,q-1}} S^q(\widehat{P})\,.$$

And by [Ill71], Proposition 4.3.1.6, the natural morphism $S^q(\beta) : S^q(\widehat{P}) \to S^q(\widehat{T})$ induces a quasi-isomorphism

$$K(\alpha, q) \xrightarrow{\sim} S^q(\widehat{T})\,,$$

which proves the lemma. □

Corollary 3.3. *For all $n \geq 0$ there is a spectral sequence*

$$E_2^{p,q} = H^{p-q}(F, S^q(\widehat{T}) \otimes \mathbb{Z}(n-q)) \implies H^{p+q}_{\text{ét}}(X, \mathbb{Z}(n)).$$

Proof. The key point is to show that the motive of X_{sep} is a pure Tate motive in the sense of [HK06, Definition 4.9]. That is, $M(X_{sep}) \cong \bigoplus_{p\geq 0} \mathrm{CH}^p(X_{sep})^* \otimes \mathbb{Z}(p)\,[2p]$ in $DM^{eff}_{gm}(F_{sep})$.

Indeed, there is natural isomorphism (see [Voe00, Corollary 4.2.5]):

$$\mathrm{CH}^p(X_{sep}) \cong \mathrm{Hom}(M(X_{sep}), \mathbb{Z}(p)\,[2p]).$$

Since $\mathrm{CH}^p(X_{sep})$ is natural isomorphic to $S^p(\widehat{T})$ [EG97, Lemma 2], which is a finitely generated free abelian group, we get a natural map:

$$\varphi : M(X_{sep}) \to \bigoplus_{p\geq 0} \mathrm{CH}^p(X_{sep})^* \otimes \mathbb{Z}(p)\,[2p]\,.$$

Let $M = M(X_{sep})$, $M' = \bigoplus_{p\geq 0} \mathrm{CH}^p(X_{sep})^* \otimes \mathbb{Z}(p)\,[2p]$ and we now prove that the above morphism $\varphi : M \to M'$ is an isomorphism. First, it is easy to see that $c_n M' = \mathrm{CH}^p(X_{sep})^*$, and Proposition 3.1 and Lemma 3.2 imply that the morphism φ induces isomorphisms $c_n M \xrightarrow{\sim} c_n M'$, and hence isomorphisms $\nu_n M \xrightarrow{\sim} \nu_n M'$, since by definition $\nu_n M = c_n(M)(n)\,[2n]$. In addition, φ induces a natural map of exact triangles

$$\begin{tikzcd}
\nu_n M \arrow[r] \arrow[d, "\simeq"] & \nu_{<n+1} M \arrow[r] \arrow[d] & \nu_{<n} M \arrow[r] \arrow[d] & \nu_n M\,[1] \arrow[d, "\simeq"] \\
\nu_n M' \arrow[r] & \nu_{<n+1} M' \arrow[r] & \nu_{<n} M' \arrow[r] & \nu_n M'\,[1]\,.
\end{tikzcd}$$

We deduce by induction that for all n, the map $\nu_{<n} M \xrightarrow{\sim} \nu_{<n} M'$ is an isomorphism, hence $\varphi : M \to M'$ itself is an isomorphism since $\nu_{<n} M = M$ and $\nu_{<n} M' = M'$ for n big enough.

This gives us the required spectral sequence by pulling back the filtration to the big étale site of $\mathrm{Spec}(F)$. □

We write K_i^M for the Milnor's K-group, K_i for the general K-group and $K_i(-)_{ind}$ for the quotient K_i/K_i^M.

Corollary 3.4. *Let T be a torus over a perfect field F and X as above. Then we have:*

$$\bar{H}^i_{\text{ét}}(X_{sep}, \mathbb{Z}(3)) = \begin{cases} 0 & i = 0, 1, 2 \\ \widehat{T} \otimes H^1(F_{sep}, \mathbb{Z}(2)) \simeq \widehat{T} \otimes K_3(F_{sep})_{ind} & i = 3 \\ \widehat{T} \otimes K_2^M(F_{sep}) & i = 4 \\ S^2(\widehat{T}) \otimes F^*_{sep} & i = 5 \\ S^3(\widehat{T}) & i = 6 \end{cases}$$

Moreover, there is an exact sequence:

$$\left(\widehat{T} \otimes K_2^M(F_{sep})\right)^\Gamma \longrightarrow H^2(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \bar{H}^5_{\text{ét}}(X, \mathbb{Z}(3)) \longrightarrow \left(S^2(\widehat{T}) \otimes F^*_{sep}\right)^\Gamma \longrightarrow$$
$$H^3(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \ker\left(\bar{H}^6_{\text{ét}}(X, \mathbb{Z}(3)) \longrightarrow S^3(\widehat{T})^\Gamma\right) \longrightarrow H^1(F, S^2(\widehat{T}) \otimes F^*_{sep}).$$

This exact sequence holds for char $F = 0$*. If* char $F = p > 0$*, then the conclusion still holds after changing $\mathbb{Q}/\mathbb{Z}(2)$ in the Galois cohomology groups of F to $\mathbb{Q}/\mathbb{Z}(2)[1/p]$.*

Proof. For $\bar{H}^i_{\text{ét}}(X_{sep}, \mathbb{Z}(3))$, it is a consequence of Corollary 3.3 and [Kah96, Theorem 1.1] over F_{sep}. As for the exact sequence of $\bar{H}^i_{\text{ét}}(X, \mathbb{Z}(3))$, from Lemma 3.5 and the Hochschild–Serre spectral sequence (see [BM13, Appendix B-IV] and [RS18], (3.5)):

$$E_2^{p,q} = H^p(F, \bar{H}^q_{\text{ét}}(X_{sep}, \mathbb{Z}(3))) \implies \bar{H}^{p+q}_{\text{ét}}(X, \mathbb{Z}(3)),$$

we obtain an exact sequence:

$$\left(\widehat{T} \otimes K_2^M(F_{sep})\right)^\Gamma \longrightarrow H^2(F, \widehat{T} \otimes K_3(F_{sep})_{ind}) \longrightarrow \bar{H}^5_{\text{ét}}(X, \mathbb{Z}(3)) \longrightarrow \left(S^2(\widehat{T}) \otimes F^*_{sep}\right)^\Gamma \longrightarrow$$
$$H^3(F, \widehat{T} \otimes K_3(F_{sep})_{ind}) \longrightarrow \ker\left(\bar{H}^6_{\text{ét}}(X, \mathbb{Z}(3)) \longrightarrow S^3(\widehat{T})^\Gamma\right) \longrightarrow H^1(F, S^2(\widehat{T}) \otimes F^*_{sep}).$$

Because of [MS91, Theorem 11.1] and [Wei13, VI.1.6], $K_3(F_{sep})_{ind}$ is divisible and its torsion subgroup is $\mathbb{Q}/\mathbb{Z}(2)$, therefore $H^i(F, \widehat{T} \otimes K_3(F_{sep})_{ind}) \simeq H^i(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2))$ (see [Wei13, VI.1.3.1] for the case of char $F = p > 0$). This proves the results. □

Lemma 3.5. *$K_i(F)$ is uniquely divisible if F is algebraically closed and $i \geq 2$.*

Proof. See [BT73, Proposition 1.2]. □

Corollary 3.6. *Let T be a torus over a perfect field F and X as above. Then we have:*

$$\bar{H}^i_{\text{ét}}(X_{sep}, \mathbb{Z}(4)) = \begin{cases} 0 & i = 0, 1 \\ \widehat{T} \otimes H^{i-2}(F_{sep}, \mathbb{Z}(3)) & i = 2, 3, 4 \\ S^2(\widehat{T}) \otimes K_2^M(F_{sep}) & i = 6 \\ S^3(\widehat{T}) \otimes F^*_{sep} & i = 7 \end{cases}$$

and

$$0 \longrightarrow \widehat{T} \otimes K_3^M(F_{sep}) \longrightarrow \bar{H}^5_{\text{ét}}(X_{sep}, \mathbb{Z}(4)) \longrightarrow S^2(\widehat{T}) \otimes K_3(F_{sep})_{ind} \longrightarrow 0$$

is exact.

Proof. The proof is similar to the proof of the corollary above. □

3.2 Cohomological invariants of tori

We prove here the main result describing cohomological invariants of degree 4 and 5. We use the notation defined above. Let $A^p(-, K_n^M)$ be the pth homology group of complex of cycle modules [Ros96]. For any $0 \leq p \leq n$, from [EG97, Lemma 2] and Künneth formula [EKLV98, Proposition 3.7], we have:

$$\begin{tikzcd}
A^p(BT, K_n^M) \arrow[r, "\alpha_{p,n}"] & A^p(BT_{sep}, K_n^M)^\Gamma \\
\mathrm{CH}^p(BT) \otimes K_{n-p}^M(F) \arrow[u, "\cup"] \arrow[r] & \left(S^p(\widehat{T}) \otimes K_{n-p}^M(F_{sep})\right)^\Gamma . \arrow[u, "\simeq"']
\end{tikzcd}$$

Let I be the kernel of étale cycle map $\mathrm{CH}^3(BT) \to \bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(3))$.

Theorem 3.7. *Let T be a torus over a perfect field F. Then there is a natural commutative diagram:*

$$\begin{tikzcd}
 & \left(\widehat{T} \otimes K_2^M(F_{sep})\right)^\Gamma \arrow[d] & & \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm} \otimes F^* \arrow[ddll, "\gamma"'] \arrow[ddl, "\cup"'] & \\
 & H^2(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) \arrow[d] & & 0 \arrow[d] & \\
A^2(BT, K_3^M) \arrow[r, hook] & \bar{H}^5_{\text{ét}}(BT, \mathbb{Z}(3)) \arrow[r] \arrow[d] & \mathrm{Inv}^4(T, \mathbb{Q}/\mathbb{Z}(3))_{norm} \arrow[r] & I \arrow[r] \arrow[d] & 0 \\
 & \left(S^2(\widehat{T}) \otimes F_{sep}^*\right)^\Gamma \arrow[dr] & & \mathrm{CH}^3(BT)_{tors} \arrow[d] & \\
 & & H^3(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & \ker\left(\bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(3)) \to S^3(\widehat{T})^\Gamma\right) \arrow[r] & H^1(F, S^2(\widehat{T}) \otimes F_{sep}^*)
\end{tikzcd}$$

where the row, the column and the column that turns to a line are exact, and where γ is induced by the map γ_0 in the Introduction. This conclusion holds for $\mathrm{char}\ F = 0$*. If* $\mathrm{char}\ F = p > 0$*, then the conclusion still holds after changing $\mathbb{Q}/\mathbb{Z}(2)$ in the Galois cohomology groups of F to $\mathbb{Q}/\mathbb{Z}(2)[1/p]$.*

Proof. First, the left column of the diagram continuing as the last line in the statement of the theorem is well defined and is exact by Corollary 3.4.

Second, for any smooth variety X over a field F, the coniveau spectral sequence gives an exact sequence:

$$\begin{aligned} 0 \longrightarrow A^2(X, K_3^M) \xrightarrow{f_1} \bar{H}^5_{\text{ét}}(X, \mathbb{Z}(3)) &\longrightarrow \bar{H}^0_{Zar}(X, \mathcal{H}^4(\mathbb{Q}/\mathbb{Z}(3))) \\ &\longrightarrow \mathrm{CH}^3(X) \longrightarrow \bar{H}^6_{\text{ét}}(X, \mathbb{Z}(3)). \end{aligned}$$

Applying this sequence for the classifying T-torsor $U^i \to U^i/T$ for every $i > 0$, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow A^2(U^i/T, K_3^M) \longrightarrow \bar{H}^5_{\text{ét}}(U^i/T, \mathbb{Z}(3)) &\longrightarrow \bar{H}^0_{Zar}(U^i/T, \mathcal{H}^4(\mathbb{Q}/\mathbb{Z}(3))) \\ &\longrightarrow \mathrm{CH}^3(U^i/T) \longrightarrow \bar{H}^6_{\text{ét}}(U^i/T, \mathbb{Z}(3)). \end{aligned}$$

Those sequences for all i give an exact sequence of cosimplicial groups (see [BM13, A-IV]):

$$\begin{aligned}0 \longrightarrow A^2(U^\bullet/T, K_3^M) \longrightarrow \bar{H}^5_{\text{ét}}(U^\bullet/T, \mathbb{Z}(3)) \longrightarrow &\bar{H}^0_{Zar}(U^\bullet/T, \mathcal{H}^4(\mathbb{Q}/\mathbb{Z}(3))) \\ \longrightarrow &\mathrm{CH}^3(U^\bullet/T) \longrightarrow \bar{H}^6_{\text{ét}}(U^\bullet/T, \mathbb{Z}(3)).\end{aligned}$$

Since $A^i(-, K_j^M)$ is homotopy invariant [Ros96, Proposition 8.6], the first and fourth cosimplicial groups in the above sequence are constant [BM13, Lemma A.4]. By Corollary 3.4, each groups $\bar{H}^5_{\text{ét}}(U^i/T, \mathbb{Z}(3))$ only depend on T, hence $\bar{H}^5_{\text{ét}}(U^\bullet/T, \mathbb{Z}(3))$ is also constant cosimplicial group. Therefore [BM13, Lemma A.2] and Lemma 2.1 provide an exact sequence:

$$0 \longrightarrow A^2(BT, K_3^M) \longrightarrow \bar{H}^5_{\text{ét}}(BT, \mathbb{Z}(3)) \longrightarrow \mathrm{Inv}^4(T, \mathbb{Q}/\mathbb{Z}(3))_{norm} \longrightarrow \mathrm{CH}^3(BT) \longrightarrow \bar{H}^6_{\text{ét}}(U/T, \mathbb{Z}(3)). \tag{3.1}$$

The middle line of the diagram in the Theorem comes from the exact sequence (3.1), from the vanishing of the invariant group over F_{sep}, and from the following commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}^3(BT) & \longrightarrow & \bar{H}^6_{\text{ét}}(U/T, \mathbb{Z}(3)) \\ \downarrow & & \downarrow \\ \mathrm{CH}^3(BT_{sep})^\Gamma & \xrightarrow{\cong} & S^3(\widehat{T})^\Gamma . \end{array}$$

Since $\ker(\mathrm{CH}^3(BT) \to \mathrm{CH}^3(BT_{sep})) \cong \mathrm{CH}^3(BT)_{tor}$ by a restriction and corestriction argument, we get the exactness of the whole middle line in the Theorem.

We now prove that the triangle involving the group $\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm}$ is commutative. The following diagram is commutative by definition (the first line is well-defined since $H^4(BT, \mathbb{Z}(2)) \xrightarrow{\sim} \mathrm{CH}^2(BT)$ and $H^5(BT, \mathbb{Z}(2)) = 0$ for the Zariski topology):

$$\begin{array}{ccc} \left(\bar{H}^4_{\text{ét}}(X, \mathbb{Z}(2))_{bal}/\mathrm{CH}^2(BT)\right) \otimes H^1_{\text{ét}}(F, \mathbb{Z}(1)) & \xrightarrow{\cup} & \bar{H}^5_{\text{ét}}(X, \mathbb{Z}(3)) \\ \downarrow\cong & & \downarrow \\ \bar{H}^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))_{bal} \otimes F^* & \xrightarrow{\cup} & \bar{H}^0(X, \mathcal{H}^4(\mathbb{Q}/\mathbb{Z}(3)))_{bal} \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm} \otimes F^* & \xrightarrow{\cup} & \mathrm{Inv}^4(T, \mathbb{Q}/\mathbb{Z}(3))_{norm}. \end{array}$$

Therefore, we get the required commutativity by inverting the isomorphisms. □

Similarly, for degree 5 cohomological invariants, we can also deduce from the coniveau spectral sequence an exact sequence:

$$\begin{aligned}0 \longrightarrow A^2(X, K_4^M) \xrightarrow{f_2} \bar{H}^6_{\text{ét}}(X, \mathbb{Z}(4)) \longrightarrow &\bar{H}^0_{Zar}(X, \mathcal{H}^5(\mathbb{Q}/\mathbb{Z}(4))) \\ \longrightarrow &A^3(X, K_4^M) \longrightarrow \bar{H}^7_{\text{ét}}(X, \mathbb{Z}(4)).\end{aligned}$$

Combining this exact sequence and the argument in the proof above, we obtain an exact sequence involving degree 5 cohomological invariants:

$$0 \longrightarrow A^2(BT, K_4^M) \longrightarrow \bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(4)) \longrightarrow \mathrm{Inv}^5(T, \mathbb{Q}/\mathbb{Z}(4))_{norm} \longrightarrow A^3(BT, K_4^M) \longrightarrow \bar{H}^7_{\text{ét}}(BT, \mathbb{Z}(4))\,.$$

The Hochschild–Serre spectral sequence (see [RS18], (3.5)):

$$E_2^{p,q} = H^p(F, \bar{H}^q_{\text{ét}}(BT_{sep}, \mathbb{Z}(4))) \implies \bar{H}^{p+q}_{\text{ét}}(BT, \mathbb{Z}(4))\,,$$

and Corollary 3.6 give rise to the Theorem below, taking into account that the groups $H^i(F_{sep}, \mathbb{Z}(3))$ are uniquely divisible for $i = 0$ and $i = 2$ (see for instance [Gei17], Theorem 1.1):

Theorem 3.8. *Let T be a torus over a perfect field F. Then there is an exact commutative diagram:*

$$\begin{tikzcd}[column sep=small]
 & H^3(F, \widehat{T} \otimes H^1(F_{sep}, \mathbb{Z}(3))) \arrow[d] & & & H^4(F, \widehat{T} \otimes H^1(F_{sep}, \mathbb{Z}(3))) \arrow[d] & \\
A^2(BT, K_4^M) \arrow[r, hook] & \bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(4)) \arrow[r] \arrow[d] & \mathrm{Inv}^5(T, \mathbb{Q}/\mathbb{Z}(4))_{norm} \arrow[r] & A^3(BT, K_4^M) \arrow[r] & \bar{H}^7_{\text{ét}}(BT, \mathbb{Z}(4)) \arrow[d] & \\
H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2))' \arrow[r, hook] & Q \arrow[r] \arrow[d] & (S^2(\widehat{T}) \otimes K_2^M(F_{sep}))^\Gamma & H^2(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2))' \arrow[r, hook] & R \arrow[r] \arrow[d] & \left(S^3(\widehat{T}) \otimes F^*_{sep}\right)^\Gamma \\
 & 0 & & & 0 &
\end{tikzcd}$$

where $H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2))' := \ker\left(H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2)) \to H^4(F, \widehat{T} \otimes H^1(F_{sep}, \mathbb{Z}(3)))\right)$ *and* $H^2(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2))' := \ker\left(H^2(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2)) \to H^5(F, \widehat{T} \otimes H^1(F_{sep}, \mathbb{Z}(3)))\right)/(S^2(\widehat{T}) \otimes K_2^M(F_{sep}))^\Gamma$.

3.3 Unramified cohomology

Given a field extension L/F, the i-th unramified cohomology group of L/F is defined as the group

$$H^i_{nr}(L/F, \mathbb{Q}/\mathbb{Z}(j))) := \bigcap_{A \in P(L)} \mathrm{im}\ (H^i(A, \mathbb{Q}/\mathbb{Z}(j)) \longrightarrow H^i(L, \mathbb{Q}/\mathbb{Z}(j)))\,,$$

where $P(L)$ is the set of all rank one discrete valuation rings which contain F and have quotient field L. They can also be defined by the intersection of the kernel of the residue maps ∂_A, for $A \in P(L)$ when $\mathrm{char}(F) = 0$ [CT95]. If X is a smooth integral variety over F, the i-th unramified cohomology group of X is defined as $H^i_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(j))$.

Corollary 3.9. *Let S be a torus over a perfect field F, and let*

$$1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$$

be a flasque resolution of S. Then the following commutative diagrams are exact:

$$\begin{tikzcd}
& \left(\widehat{T} \otimes K_2^M(F_{sep})\right)^{\Gamma} \arrow[d] & & \bar{H}^3_{nr}(F(S), \mathbb{Q}/\mathbb{Z}(2)) \otimes F^* \arrow[ddll, "\gamma"'] \arrow[ddl, "\cup"] & \\
& H^2(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) \arrow[d] & & 0 \arrow[d] & \\
A^2(BT, K_3^M) \arrow[r, hook] & \bar{H}^5_{\text{ét}}(BT, \mathbb{Z}(3)) \arrow[r] \arrow[d] & \bar{H}^4_{nr}(F(S), \mathbb{Q}/\mathbb{Z}(3)) \arrow[r] & I \arrow[r] \arrow[d] & 0 \\
& \left(S^2(\widehat{T}) \otimes F^*_{sep}\right)^{\Gamma} \arrow[dr] & & \mathrm{CH}^3(BT)_{tors} \arrow[d] & \\
& & H^3(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) \arrow[r, tail] & \ker\left(\bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(3)) \to S^3(\widehat{T})^{\Gamma}\right) \arrow[r] & H^1(F, S^2(\widehat{T}) \otimes F^*_{sep})
\end{tikzcd}$$

and

$$\begin{tikzcd}
& H^3(F, \widehat{T} \otimes H^1(F_{sep}, \mathbb{Z}(3))) \arrow[d] & & & H^4(F, \widehat{T} \otimes H^1(F_{sep}, \mathbb{Z}(3))) \arrow[d] & \\
A^2(BT, K_4^M) \arrow[r, hook] & \bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(4)) \arrow[r] \arrow[d] & \bar{H}^5_{nr}(F(S), \mathbb{Q}/\mathbb{Z}(4)) \arrow[r] & A^3(BT, K_4^M) \arrow[r] & \bar{H}^7_{\text{ét}}(BT, \mathbb{Z}(4)) \arrow[d] & \\
H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2))' \arrow[r, hook] & Q \arrow[r] \arrow[d] & (S^2(\widehat{T}) \otimes K_2^M(F_{sep}))^{\Gamma} & H^2(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2))' \arrow[r, hook] & R \arrow[r] \arrow[d] & \left(S^3(\widehat{T}) \otimes F^*_{sep}\right)^{\Gamma} \\
& 0 & & & 0 &
\end{tikzcd}$$

where $H^1(F, S^2\widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2))' := \ker\left(H^1(F, S^2\widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) \to H^4(F, \widehat{T} \otimes H^1(F_{sep}, \mathbb{Z}(3)))\right)$ *and* $H^2(F, S^2\widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2))' := \ker\left(H^2(F, S^2\widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2))/(S^2(\widehat{T}) \otimes K_2^M(F_{sep}))^{\Gamma} \to H^5(F, \widehat{T} \otimes H^1(F_{sep}, \mathbb{Z}(3)))\right)$.

This conclusion holds for $\mathrm{char}\ F = 0$*. If* $\mathrm{char}\ F = p > 0$*, then the conclusion still holds after changing* $\mathbb{Q}/\mathbb{Z}(2)$ *in the Galois cohomology groups of* F *to* $\mathbb{Q}/\mathbb{Z}(2)[1/p]$*.*

Proof. It is a consequence of Theorem 5.7 in [BM13] and of Theorems 3.7 and 3.8. □

4 Examples

In this section, we apply the main results to compute explicitely invariant groups for some families of tori. In particular, we provide an example of torus with a non-trivial invariant of degree 4, and we compute the invariant groups in the case of norm tori.

Example. Let us construct a torus T with a non-trivial degree 4 invariant that does not come from a lower degree invariant.

In [Sal22], Sala constructs a torus T over a field F such that $\mathrm{CH}^3(BT)_{tors}$ is nontrivial, as follows: consider the following exact sequence of Q_8-modules

$$0 \longrightarrow \widehat{P} \longrightarrow \widehat{Q} \longrightarrow \widehat{T} := \widehat{Q}/\widehat{P} \longrightarrow 0\,,$$

where $\widehat{Q}$ is $\mathbb{Z}\,[Q_8]$ and $\widehat{P}$ is $\mathbb{Z}\,[Q_8/\,\{1,-1\}]$. Here $Q_8 = \{i,j,k|i^2=j^2=k^2=ijk=-1\}$ is the quaternion group of order 8. Let e,e',x,x',y,y',z,z' denote the elements associated respectively to $1,-1,i,-i,j,-j,k,-k$ inside the character group $\widehat{Q}$. Then $(e+e',x+x',y+y',z+z')$ is a $\mathbb{Z}$-basis of the sublattice $\widehat{P}$. In addition, the classes of e,x,y,z in $\widehat{T}$ are a $\mathbb{Z}$-basis of $\widehat{T}$. Then we have a decomposition as a Q_8-module:

$$S^2(\widehat{T}) = M \oplus P_x \oplus P_y \oplus P_z\,,$$

where the submodules on the right hand side are defined by $M := \mathbb{Z}ee \oplus \mathbb{Z}xx \oplus \mathbb{Z}yy \oplus \mathbb{Z}zz$, $P_x := \mathbb{Z}(ex+yz) \oplus \mathbb{Z}(ex-yz)$, $P_y := \mathbb{Z}(ey+xz) \oplus \mathbb{Z}(ey-xz)$ and $P_z := \mathbb{Z}(ez+xz) \oplus \mathbb{Z}(ez-xz)$. It is easy to see that M is a rank 4 permutation Q_8-module which is isomorphic to $\mathbb{Z}\,[Q_8/\{1,-1\}]$.

Let $N_x := \mathbb{Z}(ex-yz) \subset P_x$ be the rank 1 submodule generated by $ex-yz$, and let $\bar{N}_x := P_x/N_x$. Then we have an exact sequence of Q_8-modules

$$0 \to N_x \to P_x \to \bar{N}_x \to 0$$

and N_x (resp. $\bar{N}_x$) is isomorphic to $\mathbb{Z}$ with the non-trivial action of $Q_8/\langle j\rangle$ (resp. of $Q_8/\langle k\rangle$).

Let now L/F be a Galois extension with Galois group Q_8. Let K be the subfield fixed by $\{1,-1\}$ and K_1, K_2, K_3 be the subfields of K with Galois groups $\{1,\bar{i}\}$, $\{1,\bar{j}\}$, $\{1,\bar{k}\}$ over F respectively.

By Shapiro Lemma and Hilbert 90, we have $H^1(F, M\otimes L^*) \cong H^1(Q_8, M\otimes L^*) \cong H^1(L/K, L^*) = 0$, and exact sequences

$$0 \to H^1(F, N_x \otimes L^*) \to \mathrm{Br}(F) \to \mathrm{Br}(K_2)$$

and

$$0 \to H^1(F, \bar{N}_x \otimes L^*) \to \mathrm{Br}(F) \to \mathrm{Br}(K_3)\,.$$

Hence, the exact sequence

$$0 \to H^1(F, N_x \otimes L^*) \to H^1(F, P_x \otimes L^*) \to H^1(F, \bar{N}_x \otimes L^*)$$

gives rise to an exact sequence

$$0 \to \mathrm{Br}(K_2/F) \to H^1(F, P_x \otimes L^*) \to \mathrm{Br}(K_3/F)\,.$$

Using a similar argument for P_y and P_z, we get an exact sequence:

$$0 \to \mathrm{Br(K_1/F)}\oplus\mathrm{Br(K_2/F)}\oplus\mathrm{Br(K_3/F)} \to H^1(F, S^2(\widehat{T})\otimes F^*_{sep}) \to \mathrm{Br(K_1/F)}\oplus\mathrm{Br(K_2/F)}\oplus\mathrm{Br(K_3/F)}\,.$$

If we restrict to F being the maximal abelian extension of an algebraic extension of $\mathbb{Q}$, then these relative Brauer groups are trivial. Hence $H^1(F, S^2(\widehat{T}) \otimes F^*_{sep}) = 0$, since $\mathbb{Q}^{ab}$ has cohomological dimension 1, therefore the non-trivial element in $\mathrm{CH}^3(BT)_{tor}$ given by Sala [Sal22, Theorem 5.8] defines a non-trivial degree 4 invariant by Theorem 3.7, and this invariant does not come from degree 3 cohomological invariants by cup with F^*. Note that Q_8 is indeed a Galois group of F by [Iwa53].

In particular, we proved the following:

Proposition 4.1. *There exists an explicit field F of characteristic zero and an explicit 4-dimensional F-torus T with a non-trivial degree 4 invariant that does not come from a degree 3 invariant by cup-product with F^*.*

Example. Let us consider now a torus T that fits into an exact sequence:

$$1 \longrightarrow T \longrightarrow P \longrightarrow \mathbb{G}_m^n \longrightarrow 1$$

where P is a quasi-trivial torus. In particular, norm one tori fit into such exact sequences with $n = 1$.

From the construction at the beginning of section 3.1, the natural map $BT \to BP$ is a $\mathbb{G}_m^n$-torsor and BP is approximated by cellular varieties.

Proposition 4.2. *Let T be a torus as above over an infinite field F, then $A^i(BT, K_j^M) \simeq S^i(\widehat{T})^\Gamma \otimes K_{j-i}^M(F)$. In particular, if $i = j$, $\mathrm{CH}^i(BT) \simeq S^i(\widehat{T})^\Gamma$.*

Proof. Let X and Y be as in the beginning of section 3.1, approximating BT and BP respectively. Then the map $X \to Y$ is a $\mathbb{G}_m^n$-torsor. We may assume that Y is a cellular variety. Let d be the dimension of X. From the arguments of [EKLV98, 3.10-3.12] and [Sal22, Proposition 4.4], we obtain a spectral sequence:

$$E^1_{p,q} = \Lambda^p \mathbb{Z}^n \otimes S^{d-p-q}(\widehat{P})^\Gamma \otimes K^M_{j-q}(F) \implies A^{d-p-q}(X, K_j^M).$$

Therefore, the second page is $E^2_{0,q} = S^{d-q}(\widehat{T})^\Gamma \otimes K^M_{j-q}(F)$ and $E^2_{p,q} = 0$ for $p \neq 0$ because of Proposition 3.1 and Lemma 3.2. This proves $A^i(BT, K_j^M) \simeq S^i(\widehat{T})^\Gamma \otimes K^M_{j-i}(F)$. ☐

Theorefore, the group of degree 3 normalized cohomological invariants of T is isomorphic to

$$\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm} \simeq H^1(F, T^\circ)$$

by Theorem 2.2.

If in particular $T = R^{(1)}_{L/F}(\mathbb{G}_m)$, $P = R_{L/F}(\mathbb{G}_m)$ and $n = 1$, we get

$$\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm} \simeq H^1(F, T^\circ) \simeq \mathrm{Br}(L/F)\,.$$

From Proposition 4.2, $A^2(BT, K_3^M) \simeq S^2(\widehat{T})^\Gamma \otimes F^*$ and $A^3(BT, K_3^M) \simeq S^3(\widehat{T})^\Gamma$. Therefore, considering the commutative diagram:

$$\begin{tikzcd} S^2(\widehat{T})^\Gamma \otimes F^* \simeq A^2(BT, K_3^M) \arrow[r, hook] \arrow[dr, hook] & \bar{H}^5_{\text{ét}}(BT, \mathbb{Z}(3)) \arrow[d] \\ & \left(S^2(\widehat{T}) \otimes F^*_{sep}\right)^\Gamma \end{tikzcd},$$

Theorem 3.7 implies that the following exact sequence computes the group of degree 4 invariants:

$$
\begin{aligned}
0 \to H^2(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2)) / \left(\widehat{T} \otimes K_2^M(F_{sep})\right)^{\Gamma} &\to \mathrm{Inv}^4(T, \mathbb{Q}/\mathbb{Z}(3))_{norm} \\
&\to \left(S^2(\widehat{T}) \otimes F_{sep}^*\right)^{\Gamma} / \left(S^2(\widehat{T})^{\Gamma} \otimes F_{sep}^*\right) \to H^3(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}(2))\,.
\end{aligned}
$$

For degree 5 invariants, assume for simplicity that, from now on, F is of cohomological dimension ≤ 1. Then Theorem 3.7 can be simplified as follows:

$$
\begin{tikzcd}
 & H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2)) \arrow[d, hook] & \\
A^2(BT, K_4^M) \arrow[r, hook] & \bar{H}^6_{\text{ét}}(BT, \mathbb{Z}(4)) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Inv}^5(T, \mathbb{Q}/\mathbb{Z}(4))_{norm} \\
 & \left(S^2(\widehat{T}) \otimes K_2^M(F_{sep})\right)^{\Gamma} &
\end{tikzcd}
$$

Therefore we obtain an exact sequence:

$$
\begin{aligned}
0 \to \ker\left(S^2(\widehat{T})^{\Gamma} \otimes K_2^M(F) \to S^2(\widehat{T}) \otimes K_2^M(F_{sep})\right) &\to H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Inv}^5(T, \mathbb{Q}/\mathbb{Z}(4))_{norm} \\
&\to \left(S^2(\widehat{T}) \otimes K_2^M(F_{sep})\right)^{\Gamma} / \left(S^2(\widehat{T})^{\Gamma} \otimes K_2^M(F)\right) \to 0\,.
\end{aligned}
$$

Let K/F be a splitting field of T and G be its Galois group. The assumption that F has dimension 1 provides an exact sequence for $H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2))$ by Hochschild-Serre spectral sequence:

$$
\begin{aligned}
0 \to H^1(G, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2)^{\Gamma_K}) \to H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2)) &\to H^1(K, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2))^G \\
&\to H^2(G, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2)^{\Gamma_K}) \to 0.
\end{aligned}
$$

The proof of Lemma 3.2 provides an exact sequence:

$$
0 \to \Lambda^2 \mathbb{Z}^n \to \mathbb{Z}^n \otimes \widehat{P} \to S^2(\widehat{P}) \to S^2(\widehat{T}) \to 0.
$$

Note that the composition $S^2(\widehat{P}) \to \widehat{P} \otimes \widehat{P} \to S^2(\widehat{P})$ is multiplication by 2, where the first homomorphism maps $x \cdot y$ to $x \otimes y + y \otimes x$ and the second is the natural quotient. If we assume that $\widehat{P}$ is a direct sum of several copies of $\mathbb{Z}[G]$ (for instance, if T is norm one torus) and $|G|$ is odd (or consider the p-part of each groups, $p \neq 2$), then $H^i(G, S^2(\widehat{P}) \otimes \mathbb{Q}/\mathbb{Z}(2)^{\Gamma_K})$ is trivial for $i \geq 1$ because of the Shapiro's lemma and we get $H^i(G, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2)^{\Gamma_K}) \simeq H^{i+2}(G, \mathbb{Q}/\mathbb{Z}(2)^{\Gamma_K})^{\oplus \frac{n(n-1)}{2}}$ for $i \geq 1$. Therefore, in this case, we obtain an exact sequence describing $H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2))$:

$$
\begin{aligned}
0 \to H^3(G, \mathbb{Q}/\mathbb{Z}(2)^{\Gamma_K})^{\oplus \frac{n(n-1)}{2}} \to H^1(F, S^2(\widehat{T}) \otimes \mathbb{Q}/\mathbb{Z}(2)) &\to \left(S^2(\widehat{T}) \otimes H^1(K, \mathbb{Q}/\mathbb{Z}(2))\right)^G \\
&\to H^4(G, \mathbb{Q}/\mathbb{Z}(2)^{\Gamma_K})^{\oplus \frac{n(n-1)}{2}} \to 0.
\end{aligned}
$$

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