HASSE PRINCIPLE AND WEAK APPROXIMATION FOR MULTINORM EQUATIONS

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ABSTRACT. In this note, we are interested in local-global principles for multinorm equations $\prod_{i=1}^n N_{L_i/k}(z_i) = a$ where k is a global field, L_i/k are finite separable field extensions and $a \in k^*$.

In particular, we prove a result relating the Hasse principle and weak approximation for this equation to the Hasse principle and weak approximation for some classical norm equation $N_{F/k}(w) = a$ where $F := \bigcap_{i=1}^n L_i$. It provides a proof of a "weak approximation" analogue of a recent conjecture by Pollio and Rapinchuk about the multinorm principle. We also provide a counterexample to the original conjecture concerning the Hasse principle.

0. INTRODUCTION

Let k be a global field, Ω be the set of places of k and $n \geq 2$. Let $L_1, \dots, L_n$ be finite separable field extensions of k. We fix a separable closure $\overline{k}$ of k that contains all the L_i's. Throughout this text, intersections of fields and composites of fields are taken inside the given separable closure $\overline{k}$.

For any $a \in k^*$, we consider the following equation

$$\prod_{i=1}^n N_{L_i/k}(z_i) = a\,.$$

It defines an affine k-variety X, which is a principal homogeneous space under the k-torus T defined by the following exact sequence of k-tori

$$0 \to T \to \prod_{i=1}^n \mathrm{R}_{L_i/k}\mathbf{G}_m \xrightarrow{\prod_i N_{L_i/k}} \mathbf{G}_m \to 0\,, \tag{1}$$

where the last map is the product of norm maps.

In this text, we say that a family of (smooth and geometrically integral) k-varieties satisfies the Hasse principle (resp. the Hasse principle and weak approximation) when for every variety Z in this family, if $Z(k_v) \neq \emptyset$ for all v, then $Z(k) \neq \emptyset$ (resp. if $Z(k_v) \neq \emptyset$ for all v, then $Z(k) \neq \emptyset$ and $Z(k)$ is dense in $\prod_{v\in\Omega} Z(k_v)$).

It is well-known that for varieties X as above, the obstruction to the Hasse principle is measured by the finite group $Ш^2(k, \widehat{T})$ and the obstruction to weak approximation by the finite group $Ш^2_\omega(k, \widehat{T}) / Ш^2(k, \widehat{T})$, via the Brauer-Manin obstruction (see for instance [S]), where $\widehat{T}$ is the module of characters of T.

Date: June 25, 2013.

2010 *Mathematics Subject Classification.* Primary: 11G35, 14G05.

Key words and phrases. torsor, multinorm torus, Hasse principle, weak approximation, Brauer-Manin obstruction.

Recall that for any Galois module M over k, we have by definition

$$Ш^i(k,M) := \operatorname{Ker}\left(H^i(k,M) \to \prod_{v\in\Omega} H^i(k_v,M)\right)$$

and

$$Ш^i_\omega(k,M) := \left\{\alpha \in H^i(k,M) \text{ s.t. } \alpha_v = 0 \text{ in } H^i(k_v,M) \text{ for almost all places } v \in \Omega\right\}.$$

More precisely, assuming that X has points in every completion of k, there is an isomorphism of finite groups $Ш^1(k,T) \xrightarrow{\simeq} \operatorname{Hom}(Ш^2(k,\widehat{T}),\mathbb{Q}/\mathbb{Z})$ (global duality for tori) such that the class of X in $Ш^1(k,T)$ maps to the Brauer-Manin obstruction to the Hasse principle for X, so that one says that the Brauer-Manin obstruction to the Hasse principle for X is the only one. Concerning weak approximation, assuming $X(k) \neq \emptyset$, i.e. assuming that X is k-isomorphic to T, there is a natural exact sequence, due to Voskresenskiĭ (see [V], 6.38):

$$0 \to \overline{T(k)} \to \prod_{v\in\Omega} T(k_v) \to \operatorname{Hom}(Ш^2_\omega(k,\widehat{T})/Ш^2(k,\widehat{T}),\mathbb{Q}/\mathbb{Z}) \to 0, \tag{2}$$

where $\overline{T(k)}$ denotes the closure of $T(k)$ inside $\prod_{v\in\Omega} T(k_v)$ (for the product topology), and the last map is defined via the Brauer-Manin obstruction (or via Tate-Nakayama local duality for tori): one says that the Brauer-Manin obstruction to weak approximation on T (or on X) is the only one.

The Hasse principle for such a variety X was studied by several authors, including Hürlimann (see [H], especially Proposition 3.3), Colliot-Thélène and Sansuc (unpublished), Platonov and Rapinchuk (see [PlR], sections 6.3 and 9.3, and in particular Proposition 6.11), Prasad and Rapinchuk (see [PrR], section 4, especially Proposition 4.2) and Pollio and Rapinchuk (see [PR], Main Theorem). The local-global principle for X is for instance related to some arithmetic properties of algebraic groups of type A_n (see for instance [PlR] or [PrR]).

The main result of this note (see Theorem 6) compares the defects of Hasse principle and weak approximation for X to the defects of Hasse principle and weak approximation for the k-variety Y defined by $N_{F/k}(w) = a$, where $F := \bigcap_{i=1}^n L_i$, under some technical assumptions. More precisely, if S denotes the norm k-torus $R^1_{F/k}\mathbf{G}_m$, then we prove that under some assumptions, there is a canonical isomorphism

$$Ш^2_\omega(k,\widehat{S}) \xrightarrow{\simeq} Ш^2_\omega(k,\widehat{T}),$$

which essentially means that both Hasse principle and weak approximation hold on X if and only if they both hold on Y. In other words, we can compute the defect of Hasse principle and weak approximation for the multinorm equation related to $(L_1,\dots,L_n)$ via the defect of Hasse principle and weak approximation for the usual norm equation related to the extension F/k.

In particular, it solves an analogue for weak approximation of a conjecture by Pollio and Rapinchuk (see [PR], section 4), which concerns the multinorm Hasse principle and which we recall here:

Conjecture (Pollio-Rapinchuk). *Let L_1 and L_2 be finite Galois extensions of k. If every extension P of k contained in $L_1 \cap L_2$ satisfies the norm principle, then the pair L_1, L_2 satisfies the multinorm principle (it may be enough to require that only the intersection $L_1 \cap L_2$ satisfies the norm principle).*

In addition to the proof of the weak approximation analogue of this conjecture, we also provide in Proposition 12 a counterexample to this conjecture.

1. The linearly disjoint case

Throughout this text, Γ_k denotes the absolute Galois group of the field k. For a k-torus T, we denote by $\widehat{T}$ the Γ_k-module of characters of T.

We start by the following case, where both Hasse principle and weak approximation hold:

Theorem 1. *Let $L_1, \dots, L_n/k$ be finite separable field extensions. Write $\{1, \dots, n\} = I \cup J$, with $I \cap J = \emptyset$ and $I, J \neq \emptyset$. Let L_I (resp. L_J) be the composite of the fields L_i, $i \in I$ (resp. $i \in J$). Define E_I (resp. E_J) to be the Galois closure of the extension L_I/k (resp. L_J/k). Define T to be the k-torus of equation $\prod_{i=1}^n N_{L_i/k}(z_i) = 1$.*

If $L_I \cap E_J = k$, then

$$Ш^2_\omega(k, \widehat{T}) = 0\,.$$

In particular, under these assumptions, for any $a \in k^*$, the k-variety defined by $\prod_{i=1}^n N_{L_i/k}(z_i) = a$ satisfies the Hasse principle. Assuming it has a rational point, it also satisfies weak approximation.

Remark 2. Note that the assumption implies that $\cap_i L_i = k$. Note also that this result generalizes section 5 of [PR] and Corollary 2.3 of [W1], by taking into account more than two field extensions. The proof is inspired by those two results.

Proof. Define M to be the k-torus $M := M_I \times M_J$ where $M_I := \prod_{i \in I} \mathrm{R}_{L_i/k} \mathbf{G}_m$ (same for M_J).

Lemma 3. *(i) As a Γ_{E_J}-module (resp. as a Γ_{L_I}-module), $\widehat{T}$ is a permutation module.*

(ii) As a Γ_{L_I}-module, $\widehat{T} \cong \widehat{T}^{\Gamma_{E_J}} \oplus N$, where N is a permutation Γ_{L_I}-module.

Proof. (i) We have a natural exact sequence of Γ_k-modules (see the dual exact sequence (1)):

$$0 \to \mathbb{Z} \to \widehat{M} \to \widehat{T} \to 0\,. \tag{3}$$

As a Γ_{E_J}-module, $\widehat{M_J} \cong \mathbb{Z}^m$ is a trivial module for some integer m, therefore $\widehat{T}$ is isomorphic to $\widehat{M_I} \oplus \mathbb{Z}^{m-1}$ as a Γ_{E_J}-module, hence it is a permutation Γ_{E_J}-module.

For any $i \in I$, we have an isomorphism of Γ_{L_I}-modules $\mathbb{Z}[L_i/k] \cong \mathbb{Z} \oplus N_i$, where N_i is a permutation module. Hence by (3), $\widehat{T} \cong \mathbb{Z}^{\#I-1} \oplus \widetilde{M}$ as Γ_{L_I}-modules, where $\widetilde{M} := \bigoplus_{i \in I} N_i \oplus M_J$ is a permutation Γ_{L_I}-module. Therefore $\widehat{T}$ itself is a permutation Γ_{L_I}-module.

(ii) Consider the following commutative exact diagram of Γ_k-modules:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widehat{M}^{\Gamma_{E_J}} & \longrightarrow & \widehat{T}^{\Gamma_{E_J}} & \longrightarrow & H^1(\Gamma_{E_J},\mathbb{Z})=0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widehat{M} & \longrightarrow & \widehat{T} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \widehat{M}/\widehat{M}^{\Gamma_{E_J}} & \xrightarrow{\cong} & \widehat{T}/\widehat{T}^{\Gamma_{E_J}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

Since $L_I \cap E_J = k$, we have $\mathbb{Z}[L_i/k]^{\Gamma_{E_J}} = \mathbb{Z}[L_i/k]^{\Gamma_k} = \mathbb{Z}.\varepsilon_i$ for any $i \in I$, where $\varepsilon_i := \sum_{g \in \Gamma_k/\Gamma_{L_i}} g$. We already know that $\mathbb{Z}[L_i/k] \cong \mathbb{Z} \oplus N_i$ as Γ_{L_I}-modules, therefore, since $\widehat{M_I}/\widehat{M_I}^{\Gamma_{E_J}} \cong \bigoplus_{i \in I} \mathbb{Z}[L_i/k]/\mathbb{Z}.\varepsilon_i$, we deduce that the map of Γ_{L_I}-modules $\widehat{M_I} \to \widehat{M_I}/\widehat{M_I}^{\Gamma_{E_J}}$ splits, hence the map of Γ_{L_I}-modules $\widehat{M} \to \widehat{M}/\widehat{M}^{\Gamma_{E_J}} = \widehat{M_I}/\widehat{M_I}^{\Gamma_{E_J}}$ splits. Therefore the map of Γ_{L_I}-modules $\widehat{T} \to \widehat{T}/\widehat{T}^{\Gamma_{E_J}}$ also splits, which concludes the proof of Lemma 3.

□

Lemma 4. *The restriction map* $\rho : H^2(k,\widehat{T}) \to H^2(L_I,\widehat{T}) \oplus H^2(E_J,\widehat{T})$ *is injective.*

Proof. By point (i) of Lemma 3, $H^1(E_J,\widehat{T}) = 0$. Hence the inflation-restriction exact sequence is the following one:

$$0 \to H^2(E_J/k,\widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\mathrm{inf}_{E_J/k}} H^2(k,\widehat{T}) \xrightarrow{\mathrm{res}_{k/E_J}} H^2(E_J,\widehat{T})\,.$$

So it is enough to prove that the composite map

$$\rho' : H^2(E_J/k,\widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\mathrm{inf}_{E_J/k}} H^2(k,\widehat{T}) \xrightarrow{\mathrm{res}_{k/L_I}} H^2(L_I,\widehat{T})$$

is injective. But we have an isomorphism $H^2(E_J/k,\widehat{T}^{\Gamma_{E_J}}) \cong H^2(L_I.E_J/L_I,\widehat{T}^{\Gamma_{E_J}})$ since the natural map $\mathrm{Gal}(E_J.L_I/L_I) \xrightarrow{\cong} \mathrm{Gal}(E_J/k)$ is an isomorphism (because $L_I \cap E_J = k$). So we can identify the map ρ' with the following composite map

$$H^2(L_I.E_J/L_I,\widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\mathrm{inf}} H^2(L_I,\widehat{T}^{\Gamma_{E_J}}) \xrightarrow{i_*} H^2(L_I,\widehat{T})\,,$$

where the first map is the inflation map for the Galois module $\widehat{T}^{\Gamma_{E_J}}$, and the second map is induced by the inclusion $i : \widehat{T}^{\Gamma_{E_J}} \to \widehat{T}$. We have $H^1(L_I.E_J,\widehat{T}^{E_J}) = 0$ since $\widehat{T}^{E_J}$ is a constant torsion-free Γ_{E_J}-module, hence the inflation map $H^2(L_I.E_J/L_I,\widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\mathrm{inf}} H^2(L_I,\widehat{T}^{\Gamma_{E_J}})$ is injective. By point (ii) of Lemma 3, we know that $\widehat{T}^{\Gamma_{E_J}}$ is a direct summand of $\widehat{T}$ as a Γ_{L_I}-module, hence the natural map $H^2(L_I,\widehat{T}^{\Gamma_{E_J}}) \xrightarrow{i_*} H^2(L_I,\widehat{T})$ is also injective, which concludes the proof. □

We now prove Theorem 1.

By Lemma 4, the natural restriction map

$$Ш^2_\omega(k,\widehat{T}) \to Ш^2_\omega(L_I,\widehat{T}) \oplus Ш^2_\omega(E_J,\widehat{T})$$

is injective. By point (i) of Lemma 3, we know that $\widehat{T}$ is a permutation Γ_{L_I}-module (resp. Γ_{E_J}-module). Therefore we have $Ш^2_\omega(L_I,\widehat{T}) = Ш^2_\omega(E_J,\widehat{T}) = 0$, hence $Ш^2_\omega(k,\widehat{T}) = 0$. □

Remark 5.

- Let k be a number field and K/k a Galois extension of group $\mathbf{A_4}$ (the alternating group on four elements). Let L_1 and L_2 be two different degree 4 subfields of K, hence $L_1 \cap L_2 = k$. However $L_1 \cong L_2$ as k-algebras, hence $\mathrm{R}_{L_1/k}\mathbf{G}_m \cong \mathrm{R}_{L_2/k}\mathbf{G}_m$ as k-tori. Therefore exact sequence (1) implies that we have isomorphisms of k-tori
$$T \cong \mathrm{Ker}\left(\mathrm{R}_{L_1/k}\mathbf{G}_m \times \mathrm{R}_{L_1/k}\mathbf{G}_m \to \mathbf{G}_m\right) \cong \mathrm{R}_{L_1/k}\mathbf{G}_m \times \mathrm{R}^1_{L_1/k}\mathbf{G}_m\,,$$
where the last isomorphism is defined by $(z_1,z_2) \mapsto (z_1, z_1.z_2)$. We know that $Ш^2_\omega(k,\widehat{\mathrm{R}^1_{L_1/k}\mathbf{G}_m}) = \mathbb{Z}/2\mathbb{Z}$ by [Ku], and that $Ш^2_\omega(k,\widehat{\mathrm{R}_{L_1/k}\mathbf{G}_m}) = 0$ since $\widehat{\mathrm{R}_{L_1/k}\mathbf{G}_m}$ is a permutation module. Therefore $Ш^2_\omega(k,\widehat{T}) = \mathbb{Z}/2\mathbb{Z}$ while $L_1 \cap L_2 = k$, hence the assumption about the Galois closure is necessary for the conclusion of Theorem 1 to hold.
- Following Theorem 4.1 in [CT2] (see also [S], Remark 1.9.4), for all $a,b \in k^*$ such that $k(\sqrt{a},\sqrt{b})/k$ is a biquadratic extension, if we define $L_1 := k(\sqrt{a})$, $L_2 := k(\sqrt{b})$ and $L_3 := k(\sqrt{ab})$, then we have $Ш^2_\omega(k,\widehat{T}) = \mathbb{Z}/2\mathbb{Z}$ while $L_i \cap L_j = k$ for $1 \leq i \neq j \leq 3$. Therefore the assumption $L_I \cap E_J = L_I \cap L_J = k$ is necessary for the conclusion of Theorem 1 to hold.

2. THE GENERAL CASE

We now state the main result that deals with a more general situation when the field $\bigcap_{i=1}^n L_i$ is bigger than k. In this case, the Hasse principle or weak approximation does not hold in general, but we have the following theorem:

Theorem 6. *Let $L_1,\dots,L_n/k$ be finite separable field extensions. Define $F := \bigcap_{i=1}^n L_i$ and assume that F/k is Galois. Define T to be the k-torus of equation $\prod_{i=1}^n N_{L_i/k}(z_i) = 1$ and S to be the k-torus of equation $N_{F/k}(w) = 1$. Write $\{1,\dots,n\} = I \cup J$, with $I \cap J = \emptyset$ and $I,J \neq \emptyset$. Let F_i be a field extension of L_i such that the natural map $\mathrm{Aut}_k(F_i) \to \mathrm{Aut}_k(F)$ is surjective. Let F_I (resp. F_J) be the composite of the fields F_i, $i \in I$ (resp. $i \in J$). Let E_I (resp. E_J) be the Galois closure of the extension F_I/F (resp. F_J/F).*

If $F_I \cap E_J = F$, then
$$Ш^2_\omega(k,\widehat{S}) \xrightarrow{\simeq} Ш^2_\omega(k,\widehat{T})\,.$$

This theorem implies the following corollary, which proves the "weak approximation analogue" of the conjecture by Pollio and Rapinchuk about Hasse principle for multinorm tori (see the introduction and the conjecture in section 4 of [PR]):

Corollary 7. *Under the same assumptions as in Theorem 6, let $a \in k^*$. Assume that the k-variety of equation $\prod_i N_{L_i/k}(z_i) = a$ has a k-point. Then weak approximation holds for the equation $\prod_i N_{L_i/k}(z_i) = a$ if it holds for the equation $N_{F/k}(w) = a$.*

Proof. Assume that weak approximation holds for the equation $N_{F/k}(w) = a$. It is equivalent to say that $Ш^2_\omega(k,\widehat{S})/\,Ш^2(k,\widehat{S}) = 0$. Theorem 6 implies that the natural map
$$Ш^2_\omega(k,\widehat{S})/\,Ш^2(k,\widehat{S}) \to Ш^2_\omega(k,\widehat{T})/\,Ш^2(k,\widehat{T})$$

is surjective, hence $Ш^2_\omega(k,\widehat{T})/Ш^2(k,\widehat{T})=0$, which implies by Voskresenskiĭ's exact sequence (2) that T, hence the k-variety of equation $\prod_i N_{L_i/k}(z_i)=a$, satisfies weak approximation. □

We also get a particular case of their conjecture concerning the multinorm Hasse principle:

Corollary 8. *Under the same assumptions as in Theorem 6, assume that the Hasse principle and weak approximation hold for equations $N_{F/k}(w)=a$ (for all $a\in k^*$). Then the Hasse principle and weak approximation hold for equations $\prod_i N_{L_i/k}(z_i)=a$ (for all $a\in k^*$).*

In particular, this corollary contains the case of two Galois extensions L_1,L_2 of k such that $L_1\cap L_2$ is a cyclic field extension of k: this case was one motivation for the conjecture in [PR] (see the remark after the conjecture in section 4 of [PR]).

Proof. The assumption means exactly that $Ш^2_\omega(k,\widehat{S})=0$. Hence Theorem 6 implies that $Ш^2_\omega(k,\widehat{T})=0$, which means that both the Hasse principle and weak approximation hold for equations $\prod_i N_{L_i/k}(z_i)=a$ (for all $a\in k^*$). □

We now prove Theorem 6.

Proof. Let R' be the F-torus defined by the equation $\prod_i N_{L_i/F}(z_i)=1$ and $R:=\mathrm{R}_{F/k}(R')$.

We have an exact sequence of k-tori:

$$0\to R\to T\to S\to 0\,,$$

where the morphism $T\to S$ is given by $w=\prod_i N_{L_i/F}(z_i)$.

The dual exact sequence of Galois modules

$$0\to\widehat{S}\to\widehat{T}\to\widehat{R}\to 0$$

induces a long exact sequence

$$H^1(k,\widehat{R})\to H^2(k,\widehat{S})\to H^2(k,\widehat{T})\to H^2(k,\widehat{R})\,.$$

We know that $H^i(k,\widehat{R})=H^i(F,\widehat{R'})$.

We first prove that $Ш^2_\omega(k,\widehat{R})=0$. First, we have a canonical isomorphism $Ш^2_\omega(k,\widehat{R})\cong Ш^2_\omega(F,\widehat{R'})$. Since the Galois closure of L_J/F is contained in the Galois extension E_J/F, the assumption $F_I\cap E_J=F$ implies that the intersection between L_I and the Galois closure of L_J/F is also F, therefore the field extensions L_i/F fulfill the assumptions of Theorem 1 (over the base field F). Therefore Theorem 1 ensures that $Ш^2_\omega(F,\widehat{R'})=0$ (see also [PR], section 5 or [W1], Corollary 2.3 in some particular cases).

Therefore $Ш^2_\omega(k,\widehat{T})$ is contained in the image of the map $H^2(k,\widehat{S})\to H^2(k,\widehat{T})$.

Let us prove now that the map $H^2(k,\widehat{S})\to H^2(k,\widehat{T})$ is injective. We have

$$H^1(k,\widehat{R})=H^1(F,\widehat{R'})=\mathrm{Ker}(H^2(F,\mathbb{Z})\to\bigoplus_i H^2(L_i,\mathbb{Z}))\,.$$

Since $F=\bigcap_i L_i$, we know that the map $H^2(F,\mathbb{Z})\to\bigoplus_i H^2(L_i,\mathbb{Z})$ is injective, therefore $H^1(k,\widehat{R})=0$, hence the map $H^2(k,\widehat{S})\to H^2(k,\widehat{T})$ is injective.

Now we start to prove that the kernel of the map $\psi: H^2(k,\widehat{S})\to H^2(k,\widehat{T})/Ш^2_\omega(k,\widehat{T})$ is equal to $Ш^2_\omega(k,\widehat{S})$. This kernel clearly contains $Ш^2_\omega(k,\widehat{S})$. Let us prove the converse inclusion.

First we show that $Ш^2_\omega(F,\widehat{T_F})=0$.

Since F/k is Galois, $L_i\otimes_k F\cong\prod_{\sigma\in\mathrm{Gal}(F/k)}L_i^\sigma\cong\prod_{\sigma\in\mathrm{Gal}(F/k)}\tilde\sigma(L_i)$ as F-algebras, where L_i^σ denotes the field L_i endowed with the F-algebra structure given by the morphism $F\xrightarrow{\sigma}F\to L_i$, and $\tilde\sigma\in\mathrm{Aut}_k(F_i)$ is a (chosen) lift of σ (by assumption, the natural map $\mathrm{Aut}_k(F_i)\to\mathrm{Gal}(F/k)$ is surjective).

Therefore, the F-torus T_F is defined by the following equation inside $\prod_{i,\sigma}R_{\tilde\sigma(L_i)/F}\mathbf{G}_m$:

$$\prod_{i,\sigma}N_{\tilde\sigma(L_i)/F}(z_{i,\sigma})=1\,.$$

We first notice that $\bigcap_{i,\sigma}\tilde\sigma(L_i)=F$. Let $\tilde L_I$ (resp. $\tilde L_J$) be the composite of the fields $\tilde\sigma(L_i)$ where $\sigma\in\mathrm{Gal}(F/k)$ and $i\in I$ (resp. $i\in J$). Since $\tilde\sigma(L_i)\subset F_i$, it implies that $\tilde L_I\subset F_I$ and $\tilde L_J\subset F_J$. Since $F_I\cap E_J=F$, we have $\tilde L_I\cap E_J=F$, hence the intersection between $\tilde L_I$ and the Galois closure of $\tilde L_J/F$ is also F, therefore $Ш^2_\omega(F,\widehat{T_F})=0$ by Theorem 1 applied to the field extensions $\tilde\sigma(L_i)/F$.

Consider the following commutative diagram

$$\begin{array}{ccc} H^2(k,\widehat S) & \xrightarrow{\ \psi\ } & H^2(k,\widehat T)/\,Ш^2_\omega(k,\widehat T) \\ \downarrow & & \downarrow \\ H^2(F,\widehat{S_F}) & \xrightarrow{\ \psi_F\ } & H^2(F,\widehat{T_F})/\,Ш^2_\omega(F,\widehat{T_F})=H^2(F,\widehat{T_F})\,. \end{array}$$

We now prove that the map ψ_F in the above diagram is injective, by showing that $H^1(F,\widehat R)=0$. Note that by definition R_F is identified with the kernel of the product of norm maps

$$\prod_{i=1}^n \mathrm{R}_{L_i\otimes_k F/F}\mathbb{G}_m\to\mathrm{R}_{F\otimes_k F/F}\mathbb{G}_m\,,$$

hence we get an isomorphism

$$H^1(F,\widehat R)=\bigoplus_{\sigma\in\mathrm{Gal}(F/k)}\mathrm{Ker}\left(H^2(F,\mathbb{Z})\to\bigoplus_i H^2(\tilde\sigma(L_i),\mathbb{Z})\right)\,.$$

Since $F_I\cap E_J=F$, it implies that $F=\bigcap_i\tilde\sigma(L_i)$ for all $\sigma\in\mathrm{Gal}(F/k)$. Therefore the map $H^2(F,\mathbb{Z})\to\bigoplus_i H^2(\tilde\sigma(L_i),\mathbb{Z})$ is injective for all $\sigma\in\mathrm{Gal}(F/k)$, hence $H^1(F,\widehat R)=0$.

Let $\alpha\in\mathrm{Ker}(\psi)$. Then $\alpha_F=0$ in $H^2(F,\widehat{S_F})$. Hence $\alpha\in H^2(F/k,\widehat S)$. Considering the long exact sequence associated to the exact sequence

$$0\to\mathbb{Z}\to\mathbb{Z}[F/k]\to\widehat S\to 0\,,$$

we get an isomorphism $H^2(F/k,\widehat S)\xrightarrow{\simeq}H^3(F/k,\mathbb{Z})$. For any $g\in\mathrm{Gal}(F/k)$, we have $H^3(<g>,\mathbb{Z})=0$, hence we get that $H^2(F/k,\widehat S)\subset Ш^2_\omega(k,\widehat S)$ (in fact, this is an equality). Hence $\alpha\in Ш^2_\omega(k,\widehat S)$, i.e. $\mathrm{Ker}(\psi)\subset Ш^2_\omega(k,\widehat S)$.

It concludes the proof: the map $H^2(k,\widehat S)\to H^2(k,\widehat T)$ induces an isomorphism

$$Ш^2_\omega(k,\widehat S)\cong Ш^2_\omega(k,\widehat T)\,.$$

□

Example 9. (i) The assumptions of Theorem 6 hold if $n=2$ and $L_1,L_2/k$ are Galois.

(ii) They also hold if $n \geq 2$, $L_1, \dots, L_n/k$ are Galois and $(L_1 \dots L_r) \cap (L_{r+1} \dots L_n) = F$ (for some $1 \leq r < n$).
(iii) Let L_I (resp. L_J) be the composite of the fields L_i, $i \in I$ (resp. $i \in J$). Let $\tilde{E}_I$ (resp. $\tilde{E}_J$) be the Galois closure of the extension L_I/k (resp. L_J/k). The assumptions of Theorem 6 also hold if $\tilde{E}_I \cap \tilde{E}_J = F$.
(iv) Let $k = \mathbb{Q}$ and $L_1 := \mathbb{Q}(\sqrt{2}, \sqrt[4]{3})$ and $L_2 := \mathbb{Q}(\sqrt[4]{2})$. Then $F = L_1 \cap L_2 = \mathbb{Q}(\sqrt{2})$. We can choose $F_1 = L_1$ and $F_2 = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})$. Since $E_2 = F_2$, it implies $F_1 \cap E_2 = F$. Therefore we get $Ш^2_\omega(\widehat{T}) \cong Ш^2_\omega(k, \widehat{S}) = 0$ (since $F/\mathbb{Q}$ is cyclic) while $\tilde{E}_1 \cap \tilde{E}_2 = \mathbb{Q}(\sqrt{-1}, \sqrt{2}) \neq F$ since $\tilde{E}_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt[4]{3})$ and $\tilde{E}_2 = E_2$. Therefore Theorem 6 is more general than point (iii).

The following example shows that even in the case of two field extensions, some assumptions about Galois closures have to be made for Theorem 6 to hold.

Example 10. Let k be a number field, $a \in k$ and $b, d \in k^*$. Let $L_1 = k(\sqrt{a - b\sqrt{d}})$ be a field extension of k of degree 4. Suppose $m := a^2 - b^2 d$ is not a square in $k(\sqrt{d})$ (*eg.* $k = \mathbb{Q}$ and $(a, b, d, m) = (1, 1, 2, -1)$), hence L_1/k is non-Galois. Let $L_2 = k(\sqrt{d}, \sqrt{m})$. Then $F = L_1 \cap L_2 = k(\sqrt{d})$ and

$$Ш^2_\omega(k, \widehat{T}) = \mathbb{Z}/2\mathbb{Z} \text{ while } Ш^2_\omega(k, \widehat{S}) = 0.$$

Proof. Note that $F_1 \supset L_1(\sqrt{m})$, hence $F_1 \cap F_2 \supset F_1 \cap L_2 = L_2 \neq F$, hence the assumptions of Theorem 6 are not satisfied.

Let $G := \mathrm{Gal}(L/k) \cong \mathbf{D}_4$ where $L := L_1.L_2$ is the Galois closure of L_1/k, let $\widehat{M}$ be the permutation G-module $\mathbb{Z}[L_2/k]$ and $\widehat{N} := \mathbb{Z}[L_1/k]/\mathbb{Z}$. Then we have an exact sequence of G-modules

$$0 \to \widehat{M} \to \widehat{T} \to \widehat{N} \to 0,$$

hence an exact sequence of cohomology groups:

$$0 = H^1(G, \widehat{M}) \to H^1(G, \widehat{T}) \xrightarrow{\rho} H^1(G, \widehat{N}) \to H^2(G, \widehat{M}) \to H^2(G, \widehat{T}) \to H^2(G, \widehat{N}).$$

But we have isomorphisms of finite groups

$$H^1(G, \widehat{T}) \cong \mathrm{Ker}(H^1(L/k, \mathbb{Q}/\mathbb{Z}) \to H^1(L/L_1, \mathbb{Q}/\mathbb{Z}) \oplus H^1(L/L_2, \mathbb{Q}/\mathbb{Z})) = H^1(F/k, \mathbb{Q}/\mathbb{Z})$$

and

$$H^1(G, \widehat{N}) \cong \mathrm{Ker}(H^1(L/k, \mathbb{Q}/\mathbb{Z}) \to H^1(L/L_1, \mathbb{Q}/\mathbb{Z})) = H^1(F/k, \mathbb{Q}/\mathbb{Z}),$$

therefore ρ is an isomorphism. Hence we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(G, \widehat{M}) & \longrightarrow & H^2(G, \widehat{T}) & \longrightarrow & H^2(G, \widehat{N}) \\ & & \downarrow{\scriptstyle \phi = (\phi_g)} & & \downarrow & & \downarrow \\ & & \prod_{g \in G} H^2(\langle g \rangle, \widehat{M}) & \xrightarrow{\psi = (\psi_g)} & \prod_{g \in G} H^2(\langle g \rangle, \widehat{T}) & \longrightarrow & \prod_{g \in G} H^2(\langle g \rangle, \widehat{N}). \end{array} \tag{4}$$

Since $G \cong \mathbf{D}_4$ and L is the Galois closure of L_1/k, Proposition 1 in [Ku] implies that $Ш^2_\omega(k, \widehat{N}) = 0$. So we deduce from an easy diagram chase in (4) that

$$\mathrm{Ker}(\psi \circ \phi) \xrightarrow{\simeq} Ш^2_\omega(k, \widehat{T}).$$

We now prove that $\psi \circ \phi = 0$. For any $g \in G$, let L^g denote the subfield of L fixed by g. We consider the two following cases:

(i) if $g \notin \mathrm{Gal}(L/F')$, where $F' := k(\sqrt{m})$. Then g has order 2 (since $G \cong \mathbf{D}_4$), and $L^g \neq L_2$. Then there exists a quadratic extension K'/k contained in L_2 such that $L^g.K' = L$. Therefore $L^g \otimes_k L_2 = L \otimes_{K'} L_2 = L \oplus L$. Therefore
$$H^2(\langle g\rangle, \widehat{M}) = H^2(L/L^g, \mathbb{Z}[L_2/k]) = H^2(L/L^g, \mathbb{Z}^2[L/L^g]) = 0\,,$$
hence $\phi_g = 0$ and in particular $\psi_g \circ \phi_g = 0$.

(ii) if $g \in \mathrm{Gal}(L/F') \cong \mathbb{Z}/4\mathbb{Z}$, then by functoriality, it is enough to consider the case when $\langle g\rangle = \mathrm{Gal}(L/F')$ (if $\psi_g \circ \phi_g = 0$ for some g such that $\langle g\rangle = \mathrm{Gal}(L/F')$, then $\psi_{g'} \circ \phi_{g'} = 0$ for all $g' \in \mathrm{Gal}(L/F')$, since $\psi_{g'} \circ \phi_{g'}$ factors through $\psi_g \circ \phi_g$). So we now assume that g has order 4. Let $\widehat{M'} := \mathbb{Z}[L_2/k]/\mathbb{Z}$ and $\widehat{N'} := \mathbb{Z}[L_1/k]$. We have a natural exact sequence:
$$0 \to \widehat{N'} \to \widehat{T} \to \widehat{M'} \to 0\,,$$
hence an exact sequence
$$H^2(L/F', \widehat{N'}) \to H^2(L/F', \widehat{T}) \to H^2(L/F', \widehat{M'})\,.$$
But we have $H^2(L/F', \widehat{N'}) = H^2(L/F', \mathbb{Z}[L_1/k]) = 0$ since $F'.L_1 = L$. Hence $H^2(L/F', \widehat{T}) \to H^2(L/F', \widehat{M'})$ is injective. Therefore it is enough to prove that the composite map
$$H^2(G, \mathbb{Z}[L_2/k]) \xrightarrow{\phi_g} H^2(L/F', \mathbb{Z}[L_2/k]) \xrightarrow{\psi'_g} H^2(L/F', \widehat{M'})$$
is the zero map. But ψ'_g factors through the cokernel of the natural map $i_g : H^2(L/F', \mathbb{Z}) \to H^2(L/F', \mathbb{Z}[L_2/k])$, hence we only need to prove that the composite map
$$H^2(G, \mathbb{Z}[L_2/k]) \xrightarrow{\phi_g} H^2(L/F', \mathbb{Z}[L_2/k]) \xrightarrow{\psi''_g} \mathrm{Coker}(i_g)$$
is zero.

The image of ϕ_g in $H^2(L/F', \mathbb{Z}[L_2/k])$ is $\mathrm{Gal}(F'/k)$-invariant, hence we only need to show that ψ''_g restricted to $H^2(L/F', \mathbb{Z}[L_2/k])^{\mathrm{Gal}(F'/k)}$ is zero. Since $\mathbb{Z}[L_2/k] \cong \mathbb{Z}[F/k] \otimes \mathbb{Z}[F'/k]$ canonically as G-modules, it implies that
$$H^2(L/F', \mathbb{Z}[L_2/k]) \cong H^2(L/F', \mathbb{Z}[F/k]) \otimes \mathbb{Z}[F'/k]$$
as $\mathrm{Gal}(F'/k)$-modules. Let σ is the unique nontrivial element of $\mathrm{Gal}(F'/k)$, then σ induces an isomorphism $H^2(L/F', \mathbb{Z}[F/k]) \to H^2(L/F', \mathbb{Z}[F/k]), \chi \mapsto \chi^\sigma$. Let
$$(\chi_1, \chi_2) \in \prod_{\mathrm{Gal}(F'/k)} H^2(L/F', \mathbb{Z}[F/k]) \cong H^2(L/F', \mathbb{Z}[F/k]) \otimes \mathbb{Z}[F'/k].$$
Then $\sigma(\chi_1, \chi_2) = (\chi_2^\sigma, \chi_1^\sigma)$ by the definition of the action of $\mathrm{Gal}(F'/k)$ on $H^2(L/F', \mathbb{Z}[L_2/k])$. Since $H^2(L/F', \mathbb{Z}[F/k]) \cong H^2(L/L_2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, we have $\chi_i^\sigma = \chi_i$ for $i = 1, 2$. Therefore
$$H^2(L/F', \mathbb{Z}[L_2/k])^{\mathrm{Gal}(F'/k)} = \{(\chi, \chi) \mid \chi \in H^2(L/F', \mathbb{Z}[F/k])\},$$
hence it is just the image of i_g, so its image by ψ''_g is zero.

Then we deduce that $\psi''_g \circ \phi_g = 0$, hence $\psi_g \circ \phi_g = 0$.

So we proved that $\psi \circ \phi = 0$, hence we get
$$\text{Ш}^2_\omega(k, \widehat{T}) = H^2(G, \widehat{M}) \cong H^2(L/L_2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$
which concludes the proof. □

Remark 11. We give here a heuristic reason why corollaries 7 and 8 hold, while the original conjecture by Pollio and Rapinchuk does not (see Proposition 12 below).

From a geometric point of view, if we denote by X the k-variety defined by the equation $\prod_{i=1}^n N_{L_i/k}(z_i) = a$ and by Y the k-variety defined by $N_{F/k}(w) = a$, then we have a natural morphism $\pi : X \to Y$, defined by $w = \pi((z_i)) := \prod_{i=1}^n N_{L_i/F}(z_i)$. Clearly, the map $\pi : X \to Y$ is a torsor under the k-torus R defined at the beginning of the proof of Theorem 6. Theorem 1 ensures that the rational fibers of the morphism π (which are k-torsors under R) satisfy the Hasse principle and weak approximation. In the statements of the conjecture or of the corollaries, one deduces local-global principles for the total space X of the fibration π from local-global principles for the base space Y of this fibration. A classical way to prove such results (since we know that the fibers of π do satisfy local-global principles) is the fibration method (see for instance [CT1], section 3).

And the key point is that in general, for such a fibration $\pi : X \to Y$, one cannot deduce the Hasse principle for X from the Hasse principle for the basis Y and the Hasse principle for rational fibers of π. But by classical results, one can sometimes prove that the Hasse principle holds for X *assuming that Y satisfies both the Hasse principle and weak approximation* (see [CT1], section 3). And indeed, in the multinorm situation, Proposition 12 below shows that the fibration method and the Hasse principle on X fail due to the failure of weak approximation on Y, while Theorem 6 implies that under the stronger assumption that Y does satisfy the Hasse principle and weak approximation, the variety X does satisfy the Hasse principle.

However, we did not manage to write a direct geometric proof of Theorem 6, nor of its corollaries, via the fibration method and Theorem 1, since this method requires either that the basis Y of the fibration is proper or that Y satisfies strong approximation, which is not the case in our situation. Nevertheless, the general framework of fibration methods gives an explanation why our results hold while the original conjecture does not.

3. A COUNTEREXAMPLE TO THE ORIGINAL CONJECTURE

We now construct a counterexample to the original conjecture of Pollio and Rapinchuk (see introduction), relative to the multinorm Hasse principle:

Proposition 12. *Let $k = \mathbb{Q}$. Let $q = 2$ or $q \equiv 5 \mod 8$ be a prime. Suppose m is an integer and*

$$m \equiv \begin{cases} \pm 1 \mod 8 \text{ or } \pm 2 \mod 16, & \text{if } q = 2, \\ \pm 1, \pm 5 \mod 8, & \text{if } q \equiv 5 \mod 8, \end{cases}$$

and none of $\pm m, \pm mq$ is a square in $\mathbb{Q}^*$ (eg. $(q,m) = (2,7)$ or $(5,17)$). Let $L_1 = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{q})$ and $L_2 = \mathbb{Q}(\sqrt{-1}, \sqrt{m\sqrt{q}})$. Then $F = L_1 \cap L_2 = \mathbb{Q}(\sqrt{-1}, \sqrt{q})$, and $Ш^2(\mathbb{Q}, \widehat{T}) = \mathbb{Z}/2\mathbb{Z}$ while $Ш^2(\mathbb{Q}, \widehat{S}) = 0$.

In particular, any subextension of $F/\mathbb{Q}$ satisfies the Hasse norm principle, but $(L_1, L_2; \mathbb{Q})$ does not satisfy the multinorm principle, i.e. there exists $a \in \mathbb{Q}^*$ such that the equation $N_{L_1/\mathbb{Q}}(z_1).N_{L_2/\mathbb{Q}}(z_2) = a$ violates the Hasse principle, while the equation $N_{F/\mathbb{Q}}(w) = a$ has a rational solution.

Proof. Both L_1 and L_2 are Galois over $\mathbb{Q}$. By assumption, $\pm m, \pm mq$ are not squares in $\mathbb{Q}^*$, hence $L_1 \neq L_2$.

First, Sansuc proved that $Ш^2_\omega(\mathbb{Q},\widehat{S}) = \mathbb{Z}/2\mathbb{Z}$ (see [S], (2.16)). Hence by Theorem 6, we get $Ш^2_\omega(\mathbb{Q},\widehat{T}) = \mathbb{Z}/2\mathbb{Z}$. Similarly, we know that in this case $Ш^2(\mathbb{Q},\widehat{S}) = 0$ since $\mathrm{Gal}(F/\mathbb{Q}) = \mathrm{Gal}(F_2/\mathbb{Q}_2)$, where $F_2 = F \otimes_\mathbb{Q} \mathbb{Q}_2$ is a field.

By Theorem 6, we have the following commutative diagram

$$\begin{array}{ccc} H^2(F/\mathbb{Q},\widehat{S}) = Ш^2_\omega(\mathbb{Q},\widehat{S}) & \xrightarrow{\cong} & Ш^2_\omega(\mathbb{Q},\widehat{T}) \\ \downarrow f & & \downarrow \\ \prod_p H^2(F_p/\mathbb{Q}_p,\widehat{S}) & \xrightarrow{g=(g_p)} & \prod_p H^2(\mathbb{Q}_p,\widehat{T}), \end{array}$$

where $F_p := F \otimes_\mathbb{Q} \mathbb{Q}_p$. This implies that $Ш^2(k,\widehat{T}) \cong \mathrm{Ker}(g \circ f)$.

If $p \neq 2$, then F_p is a product of cyclic field extensions of $\mathbb{Q}_p$, therefore $H^3(F_p/\mathbb{Q}_p,\mathbb{Z}) = 0$. We know that $H^2(F_p/\mathbb{Q}_p,\widehat{S}) \cong H^3(F_p/\mathbb{Q}_p,\mathbb{Z})$, therefore for all odd p, $H^2(F_p/\mathbb{Q}_p,\widehat{S}) = 0$.

Hence $Ш^2(k,\widehat{T}) \cong \mathrm{Ker}(g_2 \circ f)$. Therefore, we only need to prove that $g_2 = 0$.

Since m is a square in $\mathbb{Q}_2(\sqrt{-1},\sqrt{q})$, then $L_1 \otimes_\mathbb{Q} \mathbb{Q}_2 = L_2 \otimes_\mathbb{Q} \mathbb{Q}_2$. Denote this degree 8 field extension of $\mathbb{Q}_2$ by L_v. Note that $L_v/\mathbb{Q}_2$ is a degree 8 Galois extension, with Galois group isomorphic to the dihedral group $\mathbf{D}_4$.

We deduce that $T_2 := T \times_\mathbb{Q} \mathbb{Q}_2$ is isomorphic as a $\mathbb{Q}_2$-torus to the torus defined by the equation $N_{L_v/\mathbb{Q}_2}(w_1).N_{L_v/\mathbb{Q}_2}(w_2) = 1$, so $T_2 \cong \mathrm{R}_{L_v/\mathbb{Q}_2}\mathbf{G}_m \times \mathrm{R}^1_{L_v/\mathbb{Q}_2}\mathbf{G}_m$. Let $T' := \mathrm{R}^1_{L_v/\mathbb{Q}_2}\mathbf{G}_m$. Then the natural map $T_2 \to S_2$ factors through T', hence we only need to prove that the map $h : H^2(F_2/\mathbb{Q}_2,\widehat{S_2}) \to H^2(L_v/\mathbb{Q}_2,\widehat{T'})$ is zero.

Define $G := \mathrm{Gal}(L_v/\mathbb{Q}_2) \cong \mathbf{D}_4$, $H := \mathrm{Gal}(L_v/F_2) \cong \mathbb{Z}/2\mathbb{Z}$, so that $G/H \cong \mathrm{Gal}(F_2/\mathbb{Q}_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have an commutative exact diagram of G-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}[G/H] & \longrightarrow & \widehat{S_2} & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & \widehat{T'} & \longrightarrow & 0 \end{array}$$

that induces the following commutative diagram

$$\begin{array}{ccc} H^2(G/H,\widehat{S_2}) & \xrightarrow{\cong} & H^3(G/H,\mathbb{Z}) \\ \downarrow h & & \downarrow \mathrm{inf} \\ H^2(G,\widehat{T'}) & \xrightarrow{\cong} & H^3(G,\mathbb{Z}). \end{array}$$

Therefore we only need to prove that the inflation map $H^3(G/H,\mathbb{Z}) \to H^3(G,\mathbb{Z})$, i.e. the inflation map $H^2(G/H,\mathbb{Q}/\mathbb{Z}) \to H^2(G,\mathbb{Q}/\mathbb{Z})$, is zero. Consider the restriction-inflation exact sequence

$$H^1(G,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^1(H,\mathbb{Q}/\mathbb{Z})^{G/H} \xrightarrow{\delta} H^2(G/H,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{inf}} H^2(G,\mathbb{Q}/\mathbb{Z}). \tag{5}$$

Since H is exactly the derived subgroup of $G \cong \mathbf{D}_4$, the restriction map $H^1(G,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^1(H,\mathbb{Q}/\mathbb{Z})^{G/H}$ is the zero map. So the map δ is injective. Moreover, $H \cong \mathbb{Z}/2\mathbb{Z}$ and H is central in G, therefore $H^1(H,\mathbb{Q}/\mathbb{Z})^{G/H} \cong \mathbb{Z}/2\mathbb{Z}$.

A classical computation of Schur (see for instance [Ka], Corollary 2.2.12) implies that

$$H^2(G/H, \mathbb{Q}/\mathbb{Z}) = H^2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

hence the map δ is surjective.

Eventually, the exact sequence (5) implies that the map inf is zero, which concludes the proof. $\square$

We keep the same notations as in Proposition 12.

We provide here an explicit example for Proposition 12, i.e. an explicit rational number a for which the equation $N_{L_1/\mathbb{Q}}(z_1).N_{L_2/\mathbb{Q}}(z_2) = a$ violates the Hasse principle, while the equation $N_{F/\mathbb{Q}}(w) = a$ has a rational solution.

Example 13. We consider the special case where $q = 2$ and $m = 7$, that is $L_1 := \mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})$ and $L_2 := \mathbb{Q}(\sqrt{-1}, \sqrt{7\sqrt{2}})$ and $F = L_1 \cap L_2 = \mathbb{Q}(\sqrt{-1}, \sqrt{2})$. Then the equation $N_{L_1/\mathbb{Q}}(z_1).N_{L_2/\mathbb{Q}}(z_2) = 97$ is a counterexample to the Hasse principle (while $N_{F/\mathbb{Q}}(3+\sqrt{2}+\sqrt{-2}) = 97$).

Let us prove this fact.

We first prove that the equation $N_{L_1/\mathbb{Q}}(z_1).N_{L_2/\mathbb{Q}}(z_2) = 97$ is locally solvable. First, it is clearly solvable over $\mathbb{Q}_p$, for all $p \neq 2, 97$, and over $\mathbb{R}$. Since $97 \equiv 1$ [32], the equation $x^8 = 97$ is solvable over $\mathbb{Q}_2$, hence the equation $N_{L_1/\mathbb{Q}}(z_1).N_{L_2/\mathbb{Q}}(z_2) = 97$ has a solution over $\mathbb{Q}_2$. We check that $\left(\frac{-1}{97}\right) = \left(\frac{2}{97}\right) = 1$, and that $\left(\frac{2}{97}\right)_4 = -1$ and $\left(\frac{7}{97}\right) = -1$, therefore the extension $L_2/\mathbb{Q}$ is totally split at the prime 97, hence the equation $N_{L_1/\mathbb{Q}}(z_1).N_{L_2/\mathbb{Q}}(z_2) = 97$ is solvable over $\mathbb{Q}_{97}$. Eventually, the multinorm equation under consideration is locally solvable.

We now prove that this equation does not have any global solution. Following [W2] Corollary 1, we consider the following element $A := \mathrm{cor}_{F/\mathbb{Q}}(N_{L_2/F}(z_2), \chi)$ in the Brauer group $\mathrm{Br}(X)$, where χ is the unique non-trivial character of Γ_F that factors through $\mathrm{Gal}(L_1/F)$. We now prove that this element A induces a Brauer-Manin obstruction to the Hasse principle on X.

Let p be a prime number (or $p = \infty$) and let $x_p = (z_{1,p}, z_{2,p}) \in X(\mathbb{Q}_p)$ with $z_{i,p} \in (L_i \otimes \mathbb{Q}_p)^*$.

- if $p = 2$, then $A(x_p) = 0$ since $(N_{L_2/F}(z_{2,2}), \chi) = 0$, because $L_1 \otimes \mathbb{Q}_p = L_2 \otimes \mathbb{Q}_p$ and $(N_{L_1/F}(z), \chi) = 0$.
- if $p = \infty$, then χ_p is trivial since F is totally imaginary, hence $A(x_p) = 0$.

In the following, we will show:

- if $p \neq 2, 97, \infty$, then $A(x_p) = 0$.
- if $p = 97$, then $A(x_p) \neq 0$.

Let $p \neq 2, \infty$ and let us prove that the restriction of A to $\mathrm{Br}(X \times_{\mathbb{Q}} \mathbb{Q}_p)$ is a constant. We fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and let $K = \mathbb{Q}_p \cap F$, more precisely, we will show that the restriction $\mathrm{res}_{\mathbb{Q}/K}(A)$ of A to $\mathrm{Br}(X \times_{\mathbb{Q}} K)$ is the constant $(97, \chi')$, where χ' is a character of Γ_K which lifts χ.

Using the notation in Proposition 1.5.6 ([NSW], chaper 1), let $G = \Gamma_{\mathbb{Q}}, U = \Gamma_K, V = \Gamma_F \subset U$, we have the following decomposition

$$G = \dot{\bigcup_{\sigma}} U\sigma = \dot{\bigcup_{\sigma}} U\sigma V,$$

where σ runs through a finite system of representatives of the double cosets.

Then we have

$$\begin{aligned}\mathrm{res}_{\mathbb{Q}/K}(A) &= \mathrm{res}_{G/U}\mathrm{cor}_{V/G}(N_{L_2/F}(z_2),\chi) = \sum_\sigma \mathrm{cor}_{U/V}(N_{L_2/F}(z_2)^\sigma,\chi^\sigma)\\ &= \sum_\sigma \mathrm{cor}_{U/V}(N_{L_2/F}(z_2)^\sigma,\chi),\end{aligned}$$

where the last equation holds since $\chi = \chi^\sigma$ (note that χ has order 2). The field extension F/K is cyclic since $K = F \cap \mathbb{Q}_p$ and $F/\mathbb{Q}$ is unramified at p. Note that $\mathrm{Gal}(F/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ is non-cyclic, hence $2 \mid [K:\mathbb{Q}]$, hence $\mathrm{Gal}(L_1/K)$ is abelian since $\mathrm{Gal}(L_1/\mathbb{Q}) = \mathbf{D}_4$. Therefore we can lift χ to be a character χ' of Γ_K factoring through $\mathrm{Gal}(L_1/K)$ and we have

$$\begin{aligned}\mathrm{res}_{\mathbb{Q}/K}(A) &= \sum_\sigma \mathrm{cor}_{V/U}(N_{L_2/F}(z_2)^\sigma,\mathrm{res}_{U/V}(\chi')) = \sum_\sigma (N_{F/K}(N_{L_2/F}(z_2)^\sigma),\chi')\\ &= (N_{F/\mathbb{Q}}(N_{L_2/F}(z_2)),\chi') = (N_{L_2/\mathbb{Q}}(z_2),\chi')\\ &= (97,\chi') - (N_{L_1/\mathbb{Q}}(z_1),\chi') = (97,\chi').\end{aligned}$$

Hence we have $A(x_p) = 0$ for $p \neq 2,97,\infty$, and $A(x_p) \neq 0$ for $p = 97$, since $\left(\frac{2}{97}\right)_4 = -1$.

Eventually, we deduce that A is in the unramified Brauer group of X and that $\sum_p A(x_p) = A(x_{97}) \neq 0$ for all $(x_p) \in \prod_p X(\mathbb{Q}_p)$. Therefore, the reciprocity law from class field theory implies that $X(\mathbb{Q}) = \emptyset$, which concludes the proof.

Acknowledgment *The first author acknowledges the support of the French Agence Nationale de la Recherche (ANR) under reference ANR-12-BL01-0005. He also thanks Ulrich Derenthal and the "Center for Advanced Studies" for the invitation to Ludwig-Maximilians-Universität in Munich where this work was initiated. The second author is supported by NSFC grant # 10671104, 973 Program 2013CB834202 and grant DE 1646/2-1 of the Deutsche Forschungsgemeinschaft. Both authors thank Jean-Louis Colliot-Thélène and the referee for useful remarks and suggestions.*

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