COMPARING DESCENT OBSTRUCTION AND BRAUER-MANIN OBSTRUCTION FOR OPEN VARIETIES

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Abstract. We provide a relation between Brauer-Manin obstruction and descent obstruction for torsors over open varieties under a connected linear algebraic group or a group of multiplicative type. Such a relation is further refined for torsors under a torus. As an application, we prove that the semi-simple part of a connected linear algebraic group G will satisfy strong approximation with Brauer-Manin obstruction if G itself satisfies strong approximation with Brauer-Manin obstruction. The equivalence between descent obstruction and étale Brauer-Manin obstruction for smooth projective varieties is extended to smooth quasi-projective varieties.

1. Introduction

The descent theory for tori was first established by Colliot-Thélène and Sansuc in [7] and was extended by Skorobogatov for groups of multiplicative type in [30]. In a series of papers [19], [21], [22], Harari and Skorobogatov introduced descent obstruction for a general algebraic group and compared descent obstruction with Brauer-Manin obstruction. By various efforts of Poonen in [27], the second named author in [12], Stoll in [33] and Skorobogatov in [31], it has been proved that the descent obstruction is equivalent to the étale Brauer-Manin obstruction for geometrically integral projective smooth varieties. In this paper, we study the relation between descent obstruction and Brauer-Manin obstruction for open varieties by using the new arithmetic tools developed in [2], [5], [8], [15], [20] and [24] and extend the equivalence between descent obstruction and étale Brauer-Manin obstruction to smooth quasi-projective varieties.

Let k be a number field, Ω_k be the set of all primes of k and $\mathbf{A}_k$ be the adelic ring of k. A separated scheme X of finite type over k is called a variety over k. Fix an algebraic closure $\bar{k}$ of k, we use $X_{\bar{k}}$ to denote $X \times_k \bar{k}$. Let

$$\mathrm{Br}(X) = H^2_{et}(X, \mathbb{G}_m), \quad \mathrm{Br}_1(X) = \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{\bar{k}})) \quad \text{and} \quad \mathrm{Br}_0(X) = \mathrm{Im}(\mathrm{Br(k)} \xrightarrow{\pi^*} \mathrm{Br(X)})$$

where $X \xrightarrow{\pi} Spec(k)$ is the structure morphism and $\mathrm{Br}_a(X) = \mathrm{Br}_1(X)/\mathrm{Br}_0(X)$. For any subgroup B of $\mathrm{Br}(X)$, one can define the Brauer-Manin set

$$X(\mathbf{A}_k)^B = \{(x_v)_{v\in\Omega_k} \in X(\mathbf{A}_k) : \sum_{v\in\Omega_k} \mathrm{inv}_v(\xi(x_v)) = 0 \ \text{ for all } \xi \in B\}$$

with respect to B. When $B = \mathrm{Br}(X)$, we simply write this Brauer-Manin set as $X(\mathbf{A}_k)^{\mathrm{Br}}$.

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Suppose $Y \xrightarrow{f} X$ is a left torsor under a linear algebraic group G over k. The descent obstruction (see [19], [21] and [22]) given by f is defined by the following set

$$X(\mathbf{A}_k)^f = \{(x_v) \in X(\mathbf{A}_k) : ([Y](x_v)) \in \mathrm{Im}(H^1(k,G) \to \prod_{v\in\Omega_k} H^1(k_v,G))\} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k))$$

where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of $Y \xrightarrow{f} X$ by $\sigma \in H^1(k,G)$. Moreover, one can define

$$X(\mathbf{A}_k)^{\mathrm{desc}} = \bigcap_{Y \xrightarrow{f} X} X(\mathbf{A}_k)^f$$

following [27], where $Y \xrightarrow{f} X$ runs all torsors under all linear algebraic groups over k.

The main results in this paper are the following theorems.

Theorem 1.1. (Theorem 3.4) *Let k be a number field, G be a connected linear algebraic group or a group of multiplicative type over k and X be a smooth and geometrically integral variety over k. Suppose $Y \xrightarrow{f} X$ is a left torsor under G over k. For any subgroup* $\ker(f^*) \subseteq A \subseteq \mathrm{Br}(X)$, *one has*

$$X(\mathbf{A}_k)^A = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)})$$

where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of $Y \xrightarrow{f} X$ by σ and $\mathrm{Br}(X) \xrightarrow{f_\sigma^*} \mathrm{Br}(Y^\sigma)$ for each $\sigma \in H^1(k,G)$.

When G is a torus, this result can be refined as follows.

Theorem 1.2. (Theorem 4.1) *Suppose G is a torus in Theorem 1.1. For any subgroup* $\ker(f^*) \subseteq A \subseteq \mathrm{Br}(X)$ *and any subgroup $B_\sigma \subseteq \mathrm{Br}_1(Y^\sigma)$ such that*

$$f^{*-1}\left(\sum_{\sigma\in H^1(k,G)} \psi_\sigma(\widetilde{B_\sigma})\right) \subseteq A$$

where $\widetilde{B_\sigma}$ is the image of B_σ in $\mathrm{Br}_a(Y^\sigma)$ and $\mathrm{Br}_a(Y^\sigma) \xrightarrow{\psi_\sigma} \mathrm{Br}_a(Y)$ is a canonical isomorphism for each $\sigma \in H^1(k,G)$, one has

$$X(\mathbf{A}_k)^A = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{B_\sigma + f_\sigma^*(A)})$$

where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of $Y \xrightarrow{f} X$ by σ and $\mathrm{Br}(X) \xrightarrow{f_\sigma^*} \mathrm{Br}(Y^\sigma)$ for each $\sigma \in H^1(k,G)$.

This result is inspired by some of Harpaz's lectures. It should be pointed out that Dasheng Wei in [34] put the algebraic Brauer-Manin obstruction in the argument of Harari and Skorobogatov in [23] by using Sansuc exact sequence in [2] and obtained the same result as in Theorem 1.2 for the special case $A = \mathrm{Br}_1(X)$ and $B_\sigma = \mathrm{Br}_1(Y^\sigma)$ (see the first part of Corollary 4.2). Such a result can be applied to study strong approximation, as in [34].

Definition 1.3. *Let X be a variety over a number field k and B be a subgroup of* $\mathrm{Br}(X)$. *For a finite subset S of Ω_k, we denote the projection map $pr^S : X(\mathbf{A}_k) \to X(\mathbf{A}_k^S)$ where $\mathbf{A}_k^S$ is the adeles of k without S-components.*

We say that X satisfies strong approximation off S *if the diagonal image of $X(k)$ is dense in $pr^S(X(\mathbb{A}_k)) \neq \emptyset$.*

We say that X satisfies strong approximation with respect to B off S *if the diagonal image of $X(k)$ is dense in $pr^S(X(\mathbb{A}_k)^B) \neq \emptyset$.*

As an application, we prove the sufficient condition for strong approximation with Brauer-Manin obstruction for a connected linear algebraic group given in Corollary 3.20 of [15] is also necessary.

Theorem 1.4. (Corollary 5.3) *Let G be a connected linear algebraic group over a number field k and S be a finite subset of Ω_k containing ∞_k. Then G satisfies strong approximation with respect to* $\mathrm{Br}_1(G)$ *off S if and only if $\prod_{v\in S} G'(k_v)$ is not compact for any non-trivial simple factor G' of the semi-simple part G^{ss} of G.*

For any variety X over a number field k, following [27], one can define

$$X(\mathbf{A}_k)^{\mathrm{et,Br}} = \bigcap_{Y \xrightarrow{f} X} \bigcup_{\sigma \in H^1(k,F)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}})$$

where $Y \xrightarrow{f} X$ runs over all torsors under all finite group schemes F over k.

Theorem 1.5. (Theorem 7.5) *If X is a smooth quasi-projective and geometrically integral variety over a number field k, then*

$$X(\mathbf{A}_k)^{desc} = X(\mathbf{A}_k)^{\mathrm{et,Br}}.$$

Terminology and notation are standard if not explained. For any connected linear algebraic group G over an field k of characteristic zero, the reductive part G^{red} of G is defined by

$$1 \to R_u(G) \to G \to G^{red} \to 1$$

where $R_u(G)$ is the unipotent radical of G. The semi-simple part G^{ss} of G is defined to be $[G^{red}, G^{red}]$ which is isogenous to the product of the simple factors and the maximal toric quotient G^{tor} of G is defined to be $G^{red}/[G^{red}, G^{red}]$. We use $\hat{G}$ for the character group of G. For a topological abelian group A, the topological dual of A is defined as $A^D = \mathrm{Hom}_{cont}(A, \mathbb{Q}/\mathbb{Z})$ with the compact-open topology. For any ring R, $R^\times$ stands for all invertible elements of R with respect to multiplication. For a number field k, we use ∞_k to denote the set of all archimedean primes of k and O_S for S-integers with any finite subset S containing ∞_k. For any $v \in \Omega_k$, k_v is the completion of k with respect to v and O_v is the integral ring of k_v with $v \in \Omega_k \setminus \infty_k$.

The paper is organized as follows. In §2, we establish some algebraic results over an arbitrary field of characteristic zero which we need in the next sections. Then we prove Theorem 1.1 in §3, Theorem 1.2 in §4. As an application of such results, we show Theorem 1.4 in §5. Theorem 1.5 is proved in §6 and §7.

2. Some lemmas

In this section, we assume that k is an arbitrary field of characteristic 0.

Lemma 2.1. *Let H be a semi-simple simply connected group or a unipotent group over k. Suppose X is a smooth and geometrically integral variety over k. If $Z \xrightarrow{\rho} X$ is a torsor under H, then the induced map $\mathrm{Br}(X) \xrightarrow{\rho^*} \mathrm{Br}(Z)$ is an isomorphism.*

Proof. We first show that $\mathrm{Br}(X) \xrightarrow{\cong} \mathrm{Br}(X \times_k H)$, where the map is induced by the natural projection $X \times_k H \to X$. By using the spectral sequence

$$H^p(k, H^q(X_{\bar{k}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m),$$

one only needs to show that

$$\bar{k}[X_{\bar{k}}]^\times/\bar{k}^\times \xrightarrow{\cong} \bar{k}[X_{\bar{k}} \times_{\bar{k}} H_{\bar{k}}]^\times/\bar{k}^\times, \quad \mathrm{Pic}(X_{\bar{k}}) \xrightarrow{\cong} \mathrm{Pic}(X_{\bar{k}} \times_{\bar{k}} H_{\bar{k}}) \quad \text{and} \quad \mathrm{Br}(X_{\bar{k}}) \xrightarrow{\cong} \mathrm{Br}(X_{\bar{k}} \times_{\bar{k}} H_{\bar{k}}).$$

Since $\bar{k}[H]^\times = \bar{k}^\times$ and $\mathrm{Pic}(H_{\bar{k}}) = \mathrm{Br}(H_{\bar{k}}) = 0$, the first two parts are true by Proposition 6.10 in [28]. To prove the last part, Kummer exact sequence ensures that one only needs to show

$$H^2_{et}(X_{\bar{k}}, \mathbb{Z}/n) \xrightarrow{\cong} H^2_{et}(X_{\bar{k}} \times_{\bar{k}} H_{\bar{k}}, \mathbb{Z}/n) \tag{2.2}$$

for all $n \geq 0$. Since $H^1_{et}(H_{\bar{k}}, \mathbb{Z}/n) = H^2_{et}(H_{\bar{k}}, \mathbb{Z}/n) = 0$, the above (2.2) follows from Künneth formula (see P.240, Corollary 1.11 in SGA4$\frac{1}{2}$).

In general, since $\mathrm{Pic}(H) = 0$, one has the following short exact sequence

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(Z) \xrightarrow{m^* - p_Z^*} \mathrm{Br}(H \times_k Z)$$

by Proposition 2.4 in [2], where m^* and p_Z^* are induced by the multiplication $H \times_k Z \xrightarrow{m} Z$ and the projection $H \times_k Z \xrightarrow{p_Z} Z$ respectively. Since $m \circ (1_H \times id) = p_Z \circ (1_H \times id) = id$, one concludes $m^* = p_Z^*$ by the above argument. Therefore $\mathrm{Br}(X) \xrightarrow{\cong} \mathrm{Br}(Z)$. □

Let H be a closed subgroup of an algebraic group G over k and $Y \xrightarrow{f} X$ be a left torsor under H. Suppose $Z \xrightarrow{\rho} X$ is a left torsor under G which represents the image of $[Y]$ under the natural map $H^1(X, H) \to H^1(X, G)$. Then one can assume that $Z = G \times^H Y$ by Example 3 of P.21 in [32]. The projection map $G \times_k Y \xrightarrow{pr_G} G$ induces the following commutative diagram

$$\begin{array}{ccc} G \times_k Y & \longrightarrow & Z = G \times^H Y \\ {\scriptstyle pr_G}\downarrow & & \downarrow{\scriptstyle \theta} \\ G & \xrightarrow{\pi} & G/H \end{array} \tag{2.3}$$

where θ is given via the quotient by H.

Lemma 2.4. *With the above notations, for any $\gamma \in (G/H)(k)$, the restriction of $Z \xrightarrow{\rho} X$ to $\theta^{-1}(\gamma) \xrightarrow{\rho} X$ is a left torsor under H^σ obtained by twisting $Y \xrightarrow{f} X$ by the k-torsor $\pi^{-1}(\gamma)$ under H.*

Proof. It follows from the diagram (2.3) and Example 2 of P.20 in [32]. □

Lemma 2.5. *Let G be a connected linear algebraic group over k and Y be a smooth variety over k. If P is a (left) torsor under G over k and $H^3(k,\bar{k}^\times)=0$, then*

$$h:\ \mathrm{Br}_a(G\times_k Y)=\mathrm{Br}_a(G)\oplus\mathrm{Br}_a(Y)\cong\mathrm{Br}_a(P)\oplus\mathrm{Br}_a(Y)\xrightarrow{(pr_P^*,pr_Y^*)}\mathrm{Br}_a(P\times_k Y)$$

is an isomorphism, where the first identification is defined via the element $1\in G(k)$ by Lemma 6.6 in [28], the second one is Lemma 6.8 in [28] and

$$P\times_k Y\xrightarrow{pr_P}P\ ,\ \ P\times_k Y\xrightarrow{pr_Y}Y$$

are the projection maps.

Proof. By the spectral sequence $H^p(k,H^q_{et}(X_{\bar{k}},\mathbb{G}_m))\Rightarrow H^{p+q}_{et}(X,\mathbb{G}_m)$, using $H^3(k,\bar{k}^\times)=0$, one has the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
\mathrm{Pic}(G_Y)_{\bar{k}} & \longrightarrow & H^2(k,(G_Y^*)_{\bar{k}}) & \longrightarrow & \mathrm{Br}_a(G_Y) & \longrightarrow & H^1(k,\mathrm{Pic}(G_Y)_{\bar{k}})) & \longrightarrow & H^3(k,(G_Y^*)_{\bar{k}}) \\
\cong\downarrow & & \cong\downarrow & & \downarrow h & & \downarrow\cong & & \downarrow\cong \\
\mathrm{Pic}(P_Y)_{\bar{k}} & \longrightarrow & H^2(k,(P_Y^*)_{\bar{k}}) & \longrightarrow & \mathrm{Br}_a(P_Y) & \longrightarrow & H^1(k,\mathrm{Pic}(P_Y)_{\bar{k}})) & \longrightarrow & H^3(k,(P_Y^*)_{\bar{k}})
\end{array}$$

where the vertical morphisms come from Lemma 6.5, Lemma 6.6 and Lemma 6.7 in [28], with

$$G_Y=G\times_k Y,\quad P_Y=P\times_k Y,\quad (G_Y^*)_{\bar{k}}=\bar{k}[G_Y]^\times/\bar{k}^\times\quad\text{and}\quad (P_Y^*)_{\bar{k}}=\bar{k}[P_Y]^\times/\bar{k}^\times.$$

This implies that h an isomorphism. □

Definition 2.6. *Let G be a connected linear algebraic group over k and X be a smooth variety over k. Assuming $H^3(k,\bar{k}^\times)=0$. Suppose $Y\xrightarrow{f}X$ is a left torsor under G over k and P is a left torsor under G over k representing $\sigma\in H^1(k,G)$. The quotient map*

$$\chi_P:P\times_k Y\to Y^\sigma=P'\times^G Y$$

defines

$$\psi_\sigma:\mathrm{Br}_a(Y^\sigma)\xrightarrow{\chi_P^*}\mathrm{Br}_a(P\times_k Y)\xrightarrow{h^{-1}}\mathrm{Br}_a(G\times_k Y)\xrightarrow{(1_G\times id_Y)^*}\mathrm{Br}_a(Y)$$

by Lemma 2.5, where P' is inverse right torsor of P under G over k and 1_G is the unit element of $G(k)$.

The following lemma can be regarded as an extension Lemma 1.3 in [34] to torsors under connected linear algebraic groups.

Lemma 2.7. *The morphism ψ_σ in Definition 2.6 is an isomorphism.*

Proof. Since P can be viewed a right torsor under G^σ over k, the inverse map of the twisting bijection

$$H^1(X,G)\xrightarrow{t_P}H^1(X,G^\sigma);\quad [Y]\mapsto[P'\times_k^G Y]$$

is given by

$$H^1(X,G^\sigma)\xrightarrow{t_P^{-1}}H^1(X,G);\quad [Y]\mapsto[P\times_k^{G^\sigma}Y].$$

Then one has the quotient map $P \times_k Y^\sigma \xrightarrow{\delta_P} P \times_k^{G^\sigma} Y^\sigma = Y$. Moreover, the following diagram

$$\begin{array}{ccc} P \times_k Y & \xrightarrow{(pr_P, \chi_P)} & P \times_k Y^\sigma \\ {\scriptstyle pr_P}\downarrow & & \downarrow{\scriptstyle pr_P} \\ P & \xrightarrow[id]{} & P \end{array}$$

commutes and (pr_P, χ_P) is an isomorphism with the inverse map (pr_P, δ_P). By Lemma 2.5, one has

$$\mathrm{Br}_a(Y^\sigma) = \mathrm{Br}_a(P \times_k Y^\sigma)/pr_P^*(\mathrm{Br}_a(P)) \cong \mathrm{Br}_a(P \times_k Y)/pr_P^*(\mathrm{Br}_a(P)) = \mathrm{Br}_a(Y)$$

induced by (pr_P, χ_P). Since the following diagram

$$\begin{array}{ccccc} & & \mathrm{Br}_a(Y^\sigma) & & \\ & & \downarrow{\scriptstyle \chi_P^*} & & \\ \mathrm{Br}_a(P) & \xrightarrow{pr_P^*} & \mathrm{Br}_a(P \times_k Y) & & \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle h^{-1}} & & \\ \mathrm{Br}_a(G) & \xrightarrow[pr_G^*]{} & \mathrm{Br}_a(G \times_k Y) & \xrightarrow[(1_G \times id_Y)^*]{} & \mathrm{Br}_a(Y) \end{array}$$

commutes with the exact bottom line, one concludes that ψ_σ is an isomorphism as desired. □

One can interpret Theorem 2.8 in [2] as the following proposition.

Proposition 2.8. *Let $f : Y \to X$ be a torsor under a connected linear algebraic group G over a number field k and $\lambda : \mathrm{Br}_1(Y) \to \mathrm{Br}_a(G)$ be the canonical map of Lemma 6.4 in* [28]*. Assume X is smooth and geometrically integral. For any $x \in Y(k)$ and $t \in G(k)$, one has*

$$b(t \cdot x) = \lambda(b)(t) + b(x)$$

for any $b \in \mathrm{Br}_1(Y)$.

Proof. Let $a_Y : G \times_k Y \to Y$ be the action of G and

$$p_G : G \times_k Y \to G \quad \text{and} \quad p_Y : G \times_k Y \to Y$$

be the two respective projections, and $e : \mathrm{Br}_a(G) \to \mathrm{Br}_1(G)$ be a section of $\mathrm{Br}_1(G) \to \mathrm{Br}_a(G)$ such that $1_G^* \circ e = 0$. Then

$$a_Y^* - p_Y^* = p_G^* \circ e \circ \lambda : \ \mathrm{Br}_1(Y) \to \mathrm{Br}_1(G \times Y)$$

by the commutative diagram of Theorem 2.8 in [2]. Therefore

$$b(t \cdot x) = a_Y^*(b)(t, x) = p_Y^*(b)(t, x) + p_G^* \circ e \circ \lambda(b)(t, x) = b(x) + \lambda(b)(t)$$

for any $x \in Y(k)$, $t \in G(k)$ and $b \in \mathrm{Br}_1(Y)$. □

3. Connected linear algebraic groups or groups of multiplicative type

In this section, we study the relation between descent obstruction and Brauer-Manin obstruction for a general connected algebraic group or a group of multiplicative type. First we need the following fact of topological groups.

Lemma 3.1. *Let $f : M \to N$ be an open homomorphism of topological groups. If K is a closed subgroup of M containing* $\ker(f)$*, then $f(K)$ is a closed subset of N.*

Proof. Since K is a closed subgroup containing $\ker(f)$, one has

$$f(K) = f(M) \setminus f(M \setminus K).$$

Since f is an open homomorphism, $f(M)$ is an open subgroup of N. This implies $f(M)$ is closed in N. Since $f(M \setminus K)$ is open in N, one concludes that $f(K)$ is closed in N. □

Remark 3.2. *The assumption $K \supseteq \ker(f)$ in Lemma 3.1 can not be removed. For example, the projection map $pr^S : \mathbf{A}_k \to \mathbf{A}_k^S$ is open where $\mathbf{A}_k^S$ is the adeles of k without S-component. It is clear that k is a discrete subgroup of $\mathbf{A}_k$ by the product formula. However k will be dense in $\mathbf{A}_k^S$ by strong approximation of $\mathbb{G}_a$ when S is not empty.*

For a short exact sequence of connected linear algebraic groups, one has the following result.

Proposition 3.3. *Let*

$$1 \to G_1 \xrightarrow{\psi} G_2 \xrightarrow{\phi} G_3 \to 1$$

be a short exact sequence of connected linear algebraic groups over a number field k. Then

(1) $\phi(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)})$ is a closed subset of $G_3(\mathbf{A}_k)$.

(2) If $G'(k_\infty)$ is not compact for each simple factor G' of semi-simple part of G_3, one has

$$G_3(\mathbf{A}_k)^{\mathrm{Br}_1(G_3)} = G_3(k) \cdot \phi(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)}).$$

Proof. Galois cohomology gives the following diagram of exact sequences

$$\begin{array}{ccccccc}
G_1(k) & \xrightarrow{\psi} & G_2(k) & \xrightarrow{\phi} & G_3(k) & \xrightarrow{\partial} & H^1(k, G_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_1(\mathbf{A}_k) & \xrightarrow[(\psi_v)]{} & G_2(\mathbf{A}_k) & \xrightarrow[(\phi_v)]{} & G_3(\mathbf{A}_k) & \xrightarrow[(\partial_v)]{} & \bigoplus_{v\in\Omega_k} H^1(k_v, G_1)\,.
\end{array}$$

One further has the following commutative diagram of exact sequences of topological groups

$$\begin{array}{ccccccccc}
 & & & & G_1(\mathbf{A}_k) & \xrightarrow{\theta_1} & \mathrm{Br}_a(G_1)^D & \longrightarrow & Ш^1(k,G_1) \\
 & & & & \downarrow{\scriptstyle(\psi_v)} & & \downarrow{\scriptstyle(\psi^*)^D} & & \\
1 & \longrightarrow & \ker(\theta_2) & \longrightarrow & G_2(\mathbf{A}_k) & \xrightarrow{\theta_2} & \mathrm{Br}_a(G_2)^D & & \\
 & & \downarrow & & \downarrow{\scriptstyle(\phi_v)} & & \downarrow{\scriptstyle(\phi^*)^D} & & \\
1 & \longrightarrow & \ker(\theta_3) & \longrightarrow & G_3(\mathbf{A}_k) & \xrightarrow{\theta_3} & \mathrm{Br}_a(G_3)^D & & \\
 & & & & \downarrow{\scriptstyle(\partial_v)} & & & & \\
 & & & & \bigoplus_{v\in\Omega_k} H^1(k_v,G_1) & & & &
\end{array}$$

by Theorem 5.1 [15] and Corollary 6.11 in [28], where $\mathrm{Br}_a(G_i)^D$ is the topological dual of discrete group $\mathrm{Br}_a(G_i)$ for $1 \le i \le 3$. Then $\theta_1(G_1(\mathbf{A}_k))$ is a closed subgroup of $\mathrm{Br}_1(G)^D$. Since $(\psi^*)^D$ is a closed map, one obtains that $(\psi^*)^D(\theta_1(G_1(\mathbf{A}_k))$ is a closed subgroup of $\mathrm{Br}_1(G_2)^D$. This implies that

$$\ker(\theta_2)\cdot\psi(G_1(\mathbf{A}_k)) = \theta_2^{-1}[(\psi^*)^D(\theta_1(G_1(\mathbf{A}_k))]$$

is a closed subgroup of $G_2(\mathbf{A}_k)$ by the above commutative diagram. By Proposition 6.5 in Chapter 6 of [26], one has that $\phi : G_2(\mathbf{A}_k) \to G_3(\mathbf{A}_k)$ is an open homomorphism of topological groups. Then $\phi(\ker(\theta_2)) = \phi(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)})$ is closed by Lemma 3.1 and (1) follows.

For (2), one has

$$\ker(\theta_3) = G_3(\mathbf{A}_k)^{\mathrm{Br}_1(G_3)} = \overline{G_3(k)\cdot G_3(k_\infty)^0}$$

by Corollary 3.20 in [15] (see also the proof of Proposition 4.5 in [4]), where $G_3(k_\infty)^0$ is the connected component of identity with respect to the topology of k_∞. One only needs to show

$$G_3(\mathbf{A}_k)^{\mathrm{Br}_1(G_3)} \subseteq G_3(k)\cdot\phi(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)}).$$

For any $(x_v) \in \overline{G_3(k)\cdot G_3(k_\infty)^0}$, there is $h \in G_3(k)$ and $h_\infty \in G_3(k_\infty)$ such that

$$(\partial_v)(h\cdot h_\infty) = (\partial_v)(x_v)$$

because (∂_v) is a continuous map with respect to the discrete topology of $\bigoplus_{v\in\Omega_k} H^1(k_v,G_1)$. Since $\phi_\infty(G_2(k_\infty)^0)$ is open and connected, one concludes

$$G_3(k_\infty)^0 = \phi_\infty(G_2(k_\infty)^0)$$

by finiteness of $H^1(k_\infty,G_1)$. Therefore

$$(h\cdot h_\infty) \in G_3(k)\cdot\phi(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)})$$

and one can replace (x_v) by $(h\cdot h_\infty)^{-1}\cdot(x_v)$. Without loss of generality, one can assume $(\partial_v)(x_v)$ is a trivial element in $\bigoplus_{v\in\Omega_k} H^1(k_v,G_1)$.

Since $Ш^1(k,G_1)$ is finite, one can fix $\xi_1,\cdots,\xi_n$ in $G_3(k)$ such that each element of $Ш^1(k,G_1)$ from $\partial(G_3(k))$ is represented by them. Noting $\partial_\infty(h_\infty)$ is trivial for any $h_\infty \in G_3(k_\infty)^0$, one concludes that

$$(x_v) \in \overline{\bigcup_{i=1}^{n} \xi_i \phi(\ker(\theta_2))} = \bigcup_{i=1}^{n} \xi_i \cdot \overline{\phi(\ker(\theta_2))} \subseteq G_3(k) \cdot \phi(G_2(\mathbf{A}_k)^{\mathrm{Br}_1(G_2)})$$

by Corollary 1 in Page 50 of [29] and (1). □

The main result of this section is the following theorem.

Theorem 3.4. *Let X be a smooth and geometrically integral variety and $f : Y \to X$ be a left torsor under a linear algebraic group G over a number field k. Then one has*

$$X(\mathbf{A}_k)^A = \bigcup_{\sigma \in H^1(k,G)} f_\sigma \left(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)} \right)$$

where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of f by σ and $\mathrm{Br}(X) \xrightarrow{f_\sigma^*} \mathrm{Br}(Y^\sigma)$ for each $\sigma \in H^1(k,G)$ if one of the following conditions holds:

(1) G is connected and $\ker(f^*) \subseteq A \subseteq \mathrm{Br}(X)$.

(2) G is a group of multiplicative type and $\mathcal{K} \subseteq A \subseteq \mathrm{Br}(X)$ where $\mathcal{K}$ is a group generated by $\chi \cup [Y]$ for all $\chi \in H^1(k, \hat{G})$.

Proof. By the functoriality of Brauer-Manin pairing, one only needs to show that

$$X(\mathbf{A}_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)}) .$$

It is clear that

$$(x_v) \in \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)) \quad \Leftrightarrow \quad ([Y](x_v)) \in \mathrm{Im}[H^1(k,G) \to \prod_{v \in \Omega_k} H^1(k_v, G)] . \tag{3.5}$$

(1) When G is connected, (3.5) is equivalent to the fact that $([Y](x_v))$ is orthogonal to $\mathrm{Pic}(G)$ by Theorem 3.1 in [8]. Therefore

$$X(\mathbf{A}_k)^{\ker(f^*)} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k))$$

by Proposition 2.9 in [8] and Theorem 2.8 in [2]. Since $\ker(f^*) \subseteq A$, one has

$$X(\mathbf{A}_k)^A \subseteq X(\mathbf{A}_k)^{\ker(f^*)} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)).$$

Then the functoriality of Brauer-Manin pairing implies that

$$X(\mathbf{A}_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)}).$$

(2) When G is a group of multiplicative type, (3.5) is equivalent to

$$\sum_{v \in \Omega_k} \mathrm{inv}_v(\chi \cup [Y])(x_v) = 0$$

for all $\chi \in H^1(k, \hat{G})$ by Theorem 6.3 in [14]. Then

$$X(\mathbf{A}_k)^{\mathcal{K}} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k))$$

by Proposition 3.1 in [23]. Since $\mathcal{K} \subseteq A$, one has

$$X(\mathbf{A}_k)^A \subseteq X(\mathbf{A}_k)^{\mathcal{K}} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)).$$

Then the functoriality of Brauer-Manin pairing implies that

$$X(\mathbf{A}_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)}).$$

□

Remark 3.6. *Since Y becomes trivial over Y, one can check that $\mathcal{K} \subseteq \ker(f^*)$ in case (2) of Theorem 3.4.*

4. Refinement in the toric case

In this section, we will refine Theorem 3.4 for torsors under tori.

Theorem 4.1. *Let $f : Y \to X$ be a torsor under a torus G over a number field k. Assume that X is smooth and geometrically integral. Let $\ker(f^*) \subseteq A \subseteq \mathrm{Br}(X)$ be a subgroup and for all $\sigma \in H^1(k, G)$, let $B_\sigma \subseteq \mathrm{Br}_1(Y^\sigma)$ be a subgroup such that*

$$f^{*-1}\left(\sum_{\sigma \in H^1(k,G)} \psi_\sigma(\widetilde{B_\sigma})\right) \subseteq A$$

where $\mathrm{Br}_a(Y^\sigma) \xrightarrow{\psi_\sigma} \mathrm{Br}_a(Y)$ is the morphism of Definition 7.2 and $\widetilde{B_\sigma}$ is the image of B_σ in $\mathrm{Br}_a(Y^\sigma)$.

Then one has

$$X(\mathbf{A}_k)^A = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{B_\sigma + f_\sigma^*(A)})$$

where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of $Y \xrightarrow{f} X$ by σ and $\mathrm{Br}(X) \xrightarrow{f_\sigma^*} \mathrm{Br}(Y^\sigma)$ for each $\sigma \in H^1(k, G)$.

Proof. Since

$$\bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{B_\sigma + f_\sigma^*(A)}) \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{f_\sigma^*(A)}) \subseteq X(\mathbf{A}_k)^A$$

by the functoriality of Brauer-Manin pairing, one only needs to show the converse inclusion.

Step 1. We first prove the result is true when $\hat{G}$ is a permutation Galois module. In this case, one has $H^1(K, G) = \{1\}$ for any field extension K/k by Shapiro Lemma and Hilbert 90. This implies that

$$X(\mathbf{A}_k)^A = f(Y(\mathbf{A}_k)^{f^*(A)})$$

by the functoriality of Brauer-Manin pairing. For any $(x_v) \in X(\mathbf{A}_k)^A$, there is $(y_v) \in Y(\mathbf{A}_k)^{f^*(A)}$ such that $(x_v) = f((y_v))$. By Proposition 6.10 (6.10.3) in [28], one obtains the exact sequence

$$\mathrm{Br}_1(X) \xrightarrow{f^*} \mathrm{Br}_1(Y) \xrightarrow{\lambda} \mathrm{Br}_a(G)$$

which induces the exact sequence

$$(f^*)^{-1}(B) \xrightarrow{f^*} B \xrightarrow{\lambda} \mathrm{Br}_a(G)$$

for any subgroup $B \subseteq \mathrm{Br}_1(Y)$. Therefore the following sequence

$$\mathrm{Br}_a(G)^D \xrightarrow{\lambda^D} B^D \xrightarrow{(f^*)^D} ((f^*)^{-1}(B))^D$$

is exact. Assuming $(f^*)^{-1}(\widetilde{B}) \subseteq A$, one has $(f^*)^D((y_v)) = 0$ by viewing $(y_v) \in B^D$ through Brauer-Manin pairing. By the aforementioned exactness, there is $\xi \in \mathrm{Br}_a(G)^D$ such that $\lambda^D(\xi) = (y_v)$. Since $Ш^1(k,G) = \{1\}$, Theorem 2 in [20] implies that every element in $\mathrm{Br}_a(G)^D$ is given by an element in $G(\mathbf{A}_k)$ through the Brauer-Manin pairing. Namely, there is $(g_v) \in G(\mathbf{A}_k)$ such that

$$b(y_v) = \lambda(b)(g_v)$$

for all $b \in B$. Then $(g_v)^{-1} \cdot (y_v) \in Y(\mathbf{A}_k)^{B+f^*(A)}$ by Proposition 2.8 and $(x_v) = f((g_v)^{-1} \cdot (y_v))$.

Step 2. For general G, we have the following short exact sequence of tori

$$1 \to G \to T_0 \xrightarrow{q} T_1 \to 1$$

such that $\hat{T}_0$ is a permutation Galois module and $\hat{T}_1$ is a coflasque Galois module by using a flasque resolution of the Galois module $\mathrm{Hom}_{\mathbb{Z}}(\hat{G}, \mathbb{Z})$, see Proposition-Definition 3.1 in [5]. Since

$$H^3(k, \hat{T}_1) \cong \prod_{v \in \infty_k} H^3(k_v, \hat{T}_1) \cong \prod_{v \in \infty_k} H^1(k_v, \hat{T}_1) = \{1\}$$

by Proposition 5.9 in [24], one concludes that the map $\mathrm{Br}_1(T_0) \to \mathrm{Br}_1(G)$ is surjective.

Let $Z \xrightarrow{\rho} X$ be a torsor under T_0 which represents the image of $[Y]$ in $H^1(T_0, X)$. Then Z is isomorphic to $T_0 \times^G Y$, hence we have the following morphism of torsors under G

$$\begin{array}{ccccc} Y & \xrightarrow{e_0 \times id_Y} & T_0 \times_k Y & \xrightarrow{\chi} & Z = T_0 \times^G Y \\ & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle \theta} \\ & & T_0 & \xrightarrow[q]{} & T_1 \end{array}$$

where e_0 is the unit element in $T_0(k)$, p_0 is the projection map and θ is given as in diagram (2.3). Then one has the following commutative diagram of exact sequences

$$\begin{array}{ccccc} \mathrm{Br}_1(T_1) & \xrightarrow{q^*} & \mathrm{Br}_1(T_0) & \longrightarrow & \mathrm{Br}_a(G) \\ \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle p_0^*} & & \downarrow{\scriptstyle id} \\ \mathrm{Br}_1(Z) & \xrightarrow[\chi^*]{} & \mathrm{Br}_1(T_0 \times_k Y) & \longrightarrow & \mathrm{Br}_a(G) \end{array}$$

by Proposition 6.10 (6.10.3) in [28]. Since the following sequence

$$\mathrm{Br}_1(T_0) \xrightarrow{p_0^*} \mathrm{Br}_1(T_0 \times_k Y) \xrightarrow{(e_0 \times id_Y)^*} \mathrm{Br}_a(Y) \to 1$$

is exact by Lemma 6.6 in [28] and the map $\mathrm{Br}_1(T_0) \to \mathrm{Br}_1(G)$ is surjective, one concludes that the map

$$(e_0 \times id_Y)^* \circ \chi^* : \quad \mathrm{Br}_1(Z) \to \mathrm{Br}_1(Y)$$

is surjective by diagram chase.

For any $t \in T_1(k)$, the restriction of $Z \xrightarrow{\rho} X$ to the closed set $\theta^{-1}(t)$ is the twist

$$f^{\partial(t)} : Y^{\partial(t)} \to X \quad \text{of} \quad f : Y \to X$$

by Lemma 2.4, where $T_1(k) \xrightarrow{\partial} H^1(k, G)$ is given by Galois cohomology. Let $i_t : \theta^{-1}(t) \to Z$ be the closed immersion for any $t \in T_1(k)$. Then $f^{\partial(t)} = \rho \circ i_t$ for any $t \in T_1(k)$. Since one has the following commutative diagram

$$\begin{array}{ccc} q^{-1}(t) \times_k Y & \xrightarrow{\chi_{q^{-1}(t)}} & Y^{\partial(t)} \\ {\scriptstyle j_t \times id_Y}\downarrow & & \downarrow{\scriptstyle i_t} \\ T_0 \times_k Y & \xrightarrow[\chi]{} & Z \end{array}$$

where $j_t : q^{-1}(t) \to T_0$ is the closed immersion of fiber of q at t and $\chi_{q^{-1}(t)}$ is the restriction of χ to $q^{-1}(t) \times_k Y$ for any $t \in T_1(k)$, one obtains that

$$\begin{array}{ccc} \mathrm{Br}_1(Z) & \xrightarrow{\chi^*} & \mathrm{Br}_1(T_0 \times_k Y) \\ {\scriptstyle i_t^*}\downarrow & & \downarrow{\scriptstyle (j_t \times id_Y)^*} \\ \mathrm{Br}_1(Y^{\partial(t)}) & \xrightarrow[\chi^*_{q^{-1}(t)}]{} & \mathrm{Br}_1(q^{-1}(t) \times_k Y) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Br}_1(Y) & \xrightarrow{=} & \mathrm{Br}_1(Y) \\ {\scriptstyle pr_Y^*}\downarrow & & \downarrow{\scriptstyle pr_Y^*} \\ \mathrm{Br}_1(T_0 \times_k Y) & \xrightarrow[(j_t \times id_Y)^*]{} & \mathrm{Br}_1(q^{-1}(t) \times_k Y) \end{array}$$

where both pr_Y^* are injective by Lemma 2.5. By Definition 2.6, one has

$$\psi_{\partial(t)} = (e_0 \times id_Y)^* \circ (j_t \times id_Y)^*|_{pr_Y^*(\mathrm{Br}_1(Y))}{}^{-1} \circ \chi^*_{q^{-1}(t)}$$

and $\psi_{\partial(t)} \circ i_t^* = (e_0 \times id_Y)^* \circ \chi^*$.

Let

$$B = [(\chi \circ (e_0 \times id_Y))^{*-1}(\sum_{t \in T_1(k)} \psi_{\partial(t)}(\widetilde{B_{\partial(t)}}))] \cap \mathrm{Br}_1(Z)$$

where $\widetilde{B_{\partial(t)}}$ is the image of $B_{\partial(t)}$ in $\mathrm{Br}_a(Y^{\partial(t)})$ and $\psi_{\partial(t)}$ is given by Definition 2.6 for all $t \in T_1(k)$. Since

$$(\chi \circ (e_0 \times id_Y))^* \circ \rho^* = f^* \quad \text{implies} \quad \rho^{*-1}(B) = f^{*-1}(\sum_{t\in T_1(k)} \psi_{\partial(t)}(\widetilde{B_{\partial(t)}})) \subseteq A\,,$$

by our assumption, one has

$$X(\mathbf{A}_k)^A = \rho(Z(\mathbf{A}_k)^{B+\rho^*(A)})$$

by Step 1 for torsor $Z \xrightarrow{\rho} X$ under T_0. For any $(x_v) \in X(\mathbf{A}_k)^A$, there is $(z_v) \in Z(\mathbf{A}_k)^{B+\rho^*(A)}$ such that $(x_v) = \rho((z_v))$. Since

$$(\chi \circ (e_0 \times id_Y))^* \circ \theta^*(\mathrm{Br}_1(T_1)) = (e_0 \times id_Y)^* \circ p_0^* \circ q^*(\mathrm{Br}_1(T_1)) = \mathrm{Br}_0(Y)$$

and $(\chi \circ (e_0 \times id_Y))^*(\mathrm{Br}_0(Z)) = \mathrm{Br}_0(Y)$, one obtains $\theta^*(\mathrm{Br}_1(T_1)) \subseteq \mathrm{Br}_0(Z) + B$. This implies that

$$\theta((z_v)) \in T_1(\mathbf{A}_k)^{\mathrm{Br}_1(T_1)}$$

by the functoriality of Brauer-Manin pairing. By Proposition 3.3, there are

$$\alpha \in T_1(k) \quad \text{and} \quad (\beta_v) \in T_0(\mathbf{A}_k)^{\mathrm{Br}_1(T_0)}$$

such that $\theta((z_v)) = \alpha \cdot (\beta_v)$. Therefore

$$(\beta_v)^{-1} \cdot (z_v) \in \theta^{-1}(\alpha) \quad \text{and} \quad (\beta_v)^{-1} \cdot (z_v) \in Z(\mathbf{A}_k)^{B+\rho^*(A)}.$$

Since $(\chi \circ (e_0 \times id_Y))^*$ is surjective, one has

$$\psi_{\partial(\alpha)} \circ i_\alpha^*(\widetilde{B}) = (\chi \circ (e_0 \times id_Y))^*(\widetilde{B}) = \sum_{t\in T_1(k)} \psi_{\partial(t)}(\widetilde{B_{\partial(t)}}) \supseteq \psi_{\partial(\alpha)}(\widetilde{B_{\partial(\alpha)}})$$

where $\widetilde{B}$ is the image of B in $\mathrm{Br}_a(Z)$. This implies $i_\alpha^*(B) + \mathrm{Br}_0(\theta^{-1}(\alpha)) \supseteq B_{\partial(\alpha)}$ by Lemma 2.7 and

$$(\beta_v)^{-1} \cdot (z_v) \in \theta^{-1}(\alpha)(\mathbf{A}_k)^{i_\alpha^*(B)+[(i_\alpha^*\circ\rho^*)(A)]} \subseteq \theta^{-1}(\alpha)(\mathbf{A}_k)^{B_{\partial(\alpha)}+[(i_\alpha^*\circ\rho^*)(A)]}$$

as desired. □

The first part of the following result is also proved in Theorem 1.7 of [34].

Corollary 4.2. *Let X be a smooth and geometrically integral variety. If $f : Y \to X$ is a torsor under a torus G over a number field k, then*

$$X(\mathbf{A}_k)^{\mathrm{Br}_1(X)} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}_1(Y^\sigma)})$$

and

$$X(\mathbf{A}_k)^{\mathrm{Br}} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}_1(Y^\sigma)+f_\sigma^*(\mathrm{Br}(X))}).$$

Proof. Take $A = \mathrm{Br}_1(X)$ and $B_\sigma = \mathrm{Br}_1(Y^\sigma)$ for each $\sigma \in H^1(k,G)$ in Theorem 4.1 for the first part. Since $\mathrm{Pic}(G_{\bar{k}}) = 0$, one has

$$f^{*-1}\left(\sum_{\sigma \in H^1(k,G)} \psi_\sigma(\widetilde{B_\sigma})\right) \subseteq f^{*-1}(\mathrm{Br}_a(Y)) \subseteq \mathrm{Br}_1(X) = A$$

as required by Proposition 6.10 in [28].

The second part follows from Theorem 4.1 by taking $A = \mathrm{Br}(X)$ and $B_\sigma = \mathrm{Br}_1(Y^\sigma)$ for each $\sigma \in H^1(k,G)$. □

5. An application

In this section, we apply the previous results to study the necessary conditions for a connected linear algebraic group satisfying strong approximation with Brauer-Manin obstruction.

When X is affine, one has $X(k)$ is discrete in $X(\mathbf{A}_k)$ by the product formula. Therefore X satisfying strong approximation off S implies that $\prod_{v\in S} X(k_v)$ is not compact. However this necessary condition for strong approximation is no longer true for strong approximation with Brauer-Manin obstruction if $\mathrm{Br}(X)/\mathrm{Br}(k)$ is not finite. For example, a torus X always satisfies strong approximation with Brauer-Manin obstruction off ∞_k whenever X is anisotropic over k_∞ or not. When X is a semi-simple linear algebraic group, the necessary and sufficient condition for X satisfying strong approximation with Brauer-Manin obstruction is given by Proposition 6.1 in [4]. In this section, we'll extend this result to a general connected linear algebraic group.

The following lemma explains that strong approximation with Brauer-Manin obstruction for a general connected linear algebraic group can be reduced to the reductive case.

Lemma 5.1. *Let G be a connected linear algebraic group over a number field k and $G^{red} = G/R_u(G)$ be the reductive part of G where $R_u(G)$ is the unipotent radical of G. If $\pi : G \to G^{red}$ is the quotient map, then $G^{red}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{red})} = \pi(G(\mathbf{A}_k)^{\mathrm{Br}_1(G)})$.*

In particular, for any finite subset S of Ω_k, G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ off S if and only if G^{red} satisfies strong approximation with respect to $\mathrm{Br}_1(G^{red})$ off S.

Proof. By applying Lemma 2.1 for k and $\bar{k}$, one obtains that $\pi^*(\mathrm{Br}_1(G^{red})) = \mathrm{Br}_1(G)$. The first part follows from Theorem 3.4 and Proposition 6 of §2.1 of Chapter III in [29].

Suppose G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ off S. For any open subset

$$M = [\prod_{v\in S} G^{red}(k_v)] \times \prod_{v \notin S} M_v$$

of $G^{red}(\mathbf{A}_k)$ such that $M \cap [G^{red}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{red})}] \neq \emptyset$, one has that

$$\pi^{-1}(M) = [\prod_{v\in S} G(k_v)] \times \prod_{v\notin S} \pi^{-1}(M_v)$$

with $\pi^{-1}(M) \cap G(\mathbf{A}_k)^{\mathrm{Br}_1(G)} \neq \emptyset$ by the first part. Then there is $x \in G(k) \cap \pi^{-1}(M)$ by the assumption. This implies that $\pi(x) \in M \cap G^{red}(k)$ as required.

Conversely, suppose G^{red} satisfies strong approximation with respect to $\mathrm{Br}_1(G^{red})$ off S. For any open subset

$$N = [\prod_{v\in S} G(k_v)] \times \prod_{v\notin S} N_v$$

of $G(\mathbf{A}_k)$ such that $N \cap G(\mathbf{A}_k)^{\mathrm{Br}_1(G)} \neq \emptyset$, one has

$$\pi(N) = [\prod_{v\in S} G^{red}(k_v)] \times \prod_{v\notin S} \pi(N_v)$$

is an open subset of $G^{red}(\mathbf{A}_k)$ with $\pi(M) \cap [G^{red}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{red})}] \neq \emptyset$ by Proposition 6 of §2.1 of Chapter III in [29], Proposition 6.5 in Chapter 6 of [26] and the functoriality of Brauer-Manin pairing. Then there is $y \in G^{red}(k) \cap \pi(N)$ by the assumption. By Proposition 6 of §2.1 of Chapter III in [29] again, one concludes that $\pi^{-1}(y) \cong R_u(G)$ as algebraic varieties and satisfies strong approximation off S. Since

$$\pi^{-1}(y) \cap N = \prod_{v\in S} \pi^{-1}(y)(k_v) \times \prod_{v\notin S} (\pi^{-1}(y)(k_v) \cap N) \neq \emptyset,$$

there is $z \in \pi^{-1}(y)(k) \cap N \subset G(k) \cap N$ as desired. □

The main result of this section is the following result.

Theorem 5.2. *Let G be a connected linear algebraic group over a number field k and $G^{qs} = G/R(G)$ where $R(G)$ is the solvable radical of G. If $\pi : G \to G^{qs}$ is the quotient map, then*

$$G^{qs}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{qs})} = \pi(G(\mathbf{A}_k)^{\mathrm{Br}_1(G)}) \cdot G^{qs}(k).$$

In particular, if G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ *off a finite subset S of Ω_k, then G^{qs} satisfies strong approximation with respect to* $\mathrm{Br}_1(G^{qs})$ *off S.*

Proof. For the first part, one only needs to show

$$G^{qs}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{qs})} \subseteq \pi(G(\mathbf{A}_k)^{\mathrm{Br}_1(G)}) \cdot G^{qs}(k)$$

by functoriality of Brauer-Manin pairing. By Lemma 5.1, we can assume G is reductive. Then $R(G)$ is a torus contained in the center of G by Theorem 2.4 in Chapter 2 of [26] and $\pi : G \to G^{qs}$ is a torsor under $R(G)$. For any $(x_v) \in G^{qs}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{qs})}$, there are $\sigma \in H^1(k, R(G))$ and $(y_v) \in G^{\sigma}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{\sigma})}$ such that $(x_v) = \pi_\sigma((y_v))$ by Corollary 4.2. Since $G^\sigma(k) \neq \emptyset$ by Corollary 8.7 in [28] (see also Theorem 5.2.1[30]), there is $\gamma \in G^{qs}(k)$ such that $\partial(\gamma) = \sigma$ with the following exact sequence

$$1 \to R(G)(k) \to G(k) \to G^{qs}(k) \xrightarrow{\partial} H^1(k, R(G)) \to H^1(k, G)$$

by Galois cohomology. Moreover, one has the following isomorphism of left G-torsors

$$\begin{array}{ccc} G^\sigma & \xrightarrow{\cong} & G \\ \pi_\sigma \downarrow & & \downarrow \pi \\ G^{qs} & \xrightarrow{\cdot\, \gamma^{-1}} & G^{qs} \end{array}$$

by Example 2 of P.20 in [30]. This implies that

$$\pi_\sigma(G^\sigma(\mathbf{A}_k)^{\mathrm{Br}_1(G^\sigma)}) = \pi(G(\mathbf{A}_k)^{\mathrm{Br}_1(G)})\cdot\gamma$$

as desired.

Suppose G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ off S. For any open subset

$$M = [\prod_{v\in S} G^{qs}(k_v)] \times \prod_{v\notin S} M_v$$

of $G^{qs}(\mathbf{A}_k)$ such that $M\cap [G^{qs}(\mathbf{A}_k)^{\mathrm{Br}_1(G^{qs})}]\neq\emptyset$, there is $g\in G^{qs}(k)$ such that

$$\pi^{-1}(M\cdot g) = [\prod_{v\in S} G(k_v)] \times \prod_{v\notin S} \pi^{-1}(M_v\cdot g)$$

with $\pi^{-1}(M\cdot g)\cap G(\mathbf{A}_k)^{\mathrm{Br}_1(G)}\neq\emptyset$ by the first part. Then there is $x\in G(k)\cap\pi^{-1}(M\cdot g)$ by the assumption. This implies that $\pi(x)\cdot g^{-1}\in M\cap G^{qs}(k)$ as required. □

Corollary 5.3. *Let G be a connected linear algebraic group over a number field k and S be a finite subset of Ω_k containing ∞_k. Then G satisfies strong approximation with respect to $\mathrm{Br}_1(G)$ off S if and only if $\prod_{v\in S} G'(k_v)$ is not compact for any non-trivial simple factor G' of the semi-simple part G^{ss} of G.*

Proof. By Theorem 2.3 and Theorem 2.4 of Chapter 2 in [26], the quotient map

$$G^{red}\to G/R(G) = G^{qs}$$

induces an isogeny $G^{ss}\to G^{qs}$. One side follows from Corollary 3.20 in [15]. The other side follows from Theorem 5.2 and Proposition 6.1 in [4]. □

Remark 5.4. *All the results in this section with respect to $\mathrm{Br}_1(G)$ is also true with respect to $\mathrm{Br}(G)$ for a connected linear algebraic group G over a number field k. Indeed, since there is a sufficiently large subset S of Ω_k containing ∞_k such that $\prod_{v\in S} G'(k_v)$ is not compact for any non-trivial simple factor G' of G^{ss}, one concludes that*

$$G(\mathbf{A}_k)^{\mathrm{Br}_1(G)} = \overline{G(k)\cdot\rho(\prod_{v\in S} G^{scu}(k_v))} \subseteq G(\mathbf{A}_k)^{\mathrm{Br}(G)} \subseteq G(\mathbf{A}_k)^{\mathrm{Br}_1(G)}$$

by Corollary 3.20 in [15], Proposition 2.6 in [8] and the functoriality of Brauer-Manin pairing where $G^{scu} = G^{sc}\times_{G^{red}} G$ with the projection map $G^{scu}\xrightarrow{\rho} G$ and G^{sc} is the simply connected covering of G^{ss}.

6. Comparison I, $X(\mathbf{A}_k)^{\mathrm{desc}}\subseteq X(\mathbf{A}_k)^{\mathrm{et,Br}}$

Let $Y\xrightarrow{f} X$ be a left torsor under a linear algebraic group G over a number field k. The fundamental problem to define the descent obstruction for strong approximation with respect to $Y\xrightarrow{f} X$ is to decide if the set

$$X(\mathbf{A}_k)^f = \{(x_v)\in X(\mathbf{A}_k) : ([Y](x_v))\in \mathrm{Im}(H^1(k,G)\to\prod_v H^1(k_v,G))\} = \bigcup_{\sigma\in H^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k))$$

is closed or not in $X(\mathbf{A}_k)$. Indeed, this is true when G is connected or a group of multiplicative type by Theorem 3.4. For a general linear algebraic group G, this result is proved by Skorobogatov in Corollary 2.7 of [31] when X is proper over k. The proof depends on Proposition 5.3.2 in [32] or Proposition 4.4 in [21] which is not true for open varieties by the following example.

Example 6.1. *The short exact sequence of linear algebraic groups*

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{f} \mathbb{G}_m \to 1$$

with $f(x) = x^2$ *can be viewed as torsor under* μ_2. *For any* $\sigma \in H^2(k, \mu_2) = k^\times/(k^\times)^2$, *the twisted morphism* $f_\sigma : \mathbb{G}_m^\sigma \to \mathbb{G}_m$ *is finite etale. This implies that* $\mathbb{G}_m^\sigma$ *is affine and* $k(\mathbb{G}_m^\sigma)$ *is a quadratic extension of* $k(\mathbb{G}_m)$ *by* f_σ^*. *Write* $\mathbb{G}_m = Spec(k[t, t^{-1}])$. *By ramification consideration, one concludes that*

$$k[\mathbb{G}_m^\sigma] = k[y, y^{-1}] \quad \text{with} \quad y^2 = \sigma \cdot f_\sigma^*(t)$$

which is the integral closure of $k[t, t^{-1}]$ *inside* $k(\mathbb{G}_m^\sigma)$ *under* f_σ^*. *Therefore* $\mathbb{G}_m^\sigma \cong \mathbb{G}_m$ *as varieties over* k, *hence it always contains adelic points.*

The following definition is taken from [25].

Definition 6.2. *Let* X *be a variety over a number field* k *and* S *be a finite subset of* Ω_k *containing* ∞_k. *A faithful flat separated scheme* $\mathcal{X}_S$ *of finite type over* O_S *is called an integral model of* X *if* $\mathcal{X}_S \times_{O_S} k = X$.

The replacement for Proposition 5.3.2 in [32] or Proposition 4.4 in [21] is the following result.

Proposition 6.3. *Let* X *be a variety over a number field* k *and* S *be a finite subset of* Ω_k *containing* ∞_k. *Fix an integral model* $\mathcal{X}_S$ *of* X *over* O_S. *If* $Y \xrightarrow{f} X$ *is a left torsor under a linear algebraic group* G *over* k, *then the set*

$$\{[\sigma] \in H^1(k, G) : \ f_\sigma(Y^\sigma(\mathbf{A}_k)) \cap [\prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}_S(O_v)] \neq \emptyset\}$$

is finite.

Proof. It follows from the same argument as the proof of Proposition 4.4 in [21]. □

One can extend Corollary 2.7 in [31] to open varieties by using the above replacement for Proposition 4.4 in [21].

Proposition 6.4. *Let* X *be a (not necessarily proper) variety over a number field* k. *If* $Y \xrightarrow{f} X$ *is a left torsor under a linear algebraic group* G *over* k, *then the set* $X(\mathbf{A}_k)^f$ *is closed in* $X(\mathbf{A}_k)$.

Proof. Take an integral model $\mathcal{X}_{S_0}$ of X over O_{S_0} where S_0 is a finite subset of Ω_k containing ∞_k. Then

$$\left\{ \prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} \mathcal{X}_{S_0}(O_v) \right\}_S$$

is an open covering of $X(\mathbf{A}_k)$ (see Theorem 3.6 in [10]), where S runs all finite subsets of Ω_k containing S_0. Since

$$X(\mathbf{A}_k)^f \cap [\prod_{v\in S} X(k_v) \times \prod_{v\in\Omega_k\setminus S} \mathcal{X}_{S_0}(O_v)]$$

is closed in $[\prod_{v\in S} X(k_v) \times \prod_{v\in\Omega_k\setminus S} \mathcal{X}_{S_0}(O_v)]$ by Proposition 6.3 and Corollary 2.5 in [31], one concludes that $X(\mathbf{A}_k)^f$ is closed in $X(\mathbf{A}_k)$. □

Applying Proposition 6.3, one can also extend Lemma 2.2 and Theorem 1.1 in [31] to open varieties. For any variety over a number field k, as defined in [31], we write

$$X(\mathbf{A}_k)^{\mathrm{desc}} = \bigcap_{Y\xrightarrow{f}X} X(\mathbf{A}_k)^f$$

where $Y \xrightarrow{f} X$ runs all torsors under all linear algebraic groups over k.

Lemma 6.5. *Let X be a (not necessarily proper) variety and $Y \to X$ be a torsor over a number field k. For any $(P_v) \in X(\mathbf{A}_k)^{desc}$, there is a twist $Y' \to X$ of $Y \to X$ such that the following property holds.*

For any surjective X-torsor morphism $Z \to Y'$ (see Definition 2.1 in [31]), there is a twist $Z' \to Y'$ of $Z \to Y'$ such that (P_v) lies in the image of $Z'(\mathbf{A}_k)$.

Proof. Since there are a finite subset S_0 of Ω_k containing ∞_k and an integral model $\mathcal{X}_{S_0}$ over O_{S_0} such that

$$(P_v) \in [\prod_{v\in S_0} X(k_v) \times \prod_{v\in\Omega_k\setminus S_0} \mathcal{X}_{S_0}(O_v)]$$

by Theorem 3.6 in [10], one obtains that there are only finitely many twists of a given torsor over X such that (P_v) lifts. As pointed out in the proof of Lemma 2.2 in [31], the finite combinatorics in the first part of the proof of Proposition 5.17 in [33] are still valid. □

Proposition 6.6. *Let X be a (not necessarily proper) variety over a number field k. If $Y \xrightarrow{f} X$ is a left torsor under a finite group scheme F over k, then*

$$X(\mathbf{A}_k)^{desc} = \bigcup_{\sigma\in H^1(k,F)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{desc}).$$

Proof. One only needs to modify the proof of Theorem 1.1 in [31] by replacing Lemma 2.2 in [31] with Lemma 6.5, Corollary 2.7 in [31] with Proposition 6.4. Moreover, since f is finite, the induced map $Y(\mathbf{A}_k) \xrightarrow{f} X(\mathbf{A}_k)$ is topologically proper by Proposition 4.4 in [10]. This implies that $f^{-1}((P_v))$ is compact. □

For any variety X over a number field k, following [27], one can define

$$X(\mathbf{A}_k)^{\mathrm{et,Br}} = \bigcap_{Y\xrightarrow{f}X} \bigcup_{\sigma\in H^1(k,F)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}})$$

where $Y \xrightarrow{f} X$ runs over all torsors under all finite groups F over k. Since the induced map $Y(\mathbf{A}_k) \xrightarrow{f} X(\mathbf{A}_k)$ is topologically closed for any finite morphism $Y \xrightarrow{f} X$ by Proposition 4.4 in [10], one concludes that $X(\mathbf{A}_k)^{\text{et,Br}}$ is closed in $X(\mathbf{A}_k)$ by the same argument as Proposition 6.4.

Corollary 6.7. *If X is a smooth quasi-projective variety over a number field k, then*

$$X(\mathbf{A}_k)^{desc} \subseteq X(\mathbf{A}_k)^{\text{et,Br}} \subseteq X(\mathbf{A}_k)^{\text{Br}}.$$

Proof. One only needs to show that $X(\mathbf{A}_k)^{\text{desc}} \subseteq X(\mathbf{A}_k)^{\text{et,Br}}$. Indeed, for any torsor $Y \xrightarrow{f} X$ under a finite group scheme F, one has

$$X(\mathbf{A}_k)^{\text{desc}} = \bigcup_{\sigma \in H^1(k,F)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\text{desc}})$$

by Proposition 6.6. Since X is quasi-projective, Y^σ is quasi-projective as well. By a theorem of Gabber (see [11]), one has

$$Y^\sigma(\mathbf{A}_k)^{\text{desc}} \subseteq Y^\sigma(\mathbf{A}_k)^{\text{Br}}$$

(see the proof of Lemma 2.8 in [31]) and the result follows. □

7. Comparison II, $X(\mathbf{A}_k)^{\text{et,Br}} \subseteq X(\mathbf{A}_k)^{\text{desc}}$

In this section, we prove the other side containment for open varieties which implies Theorem 1.5. The strategy of proof is the same as in [12]. The second named author would like to thank Laurent Moret-Bailly warmly for finding a mistake and for suggesting the following alternative proof of Lemma 4 in [12] (which already appeared in [13]). The statement of this lemma is correct, but the proof in [12] uses a result of Stoll (see [33]) that is not. Note that in contrast with [12], all torsors (unles explicitely mentioned) are assumed to be left torsors.

Lemma 7.1. *Let X be a smooth geometrically connected k-variety. Let $(P_v) \in X(\mathbb{A}_k)^{\text{ét,Br}}$ and $Z \xrightarrow{g} X$ a torsor under a finite k-group F. Then there are a cocycle $\sigma \in Z^1(k, F)$ and a connected component X' of Z^σ over k such that the restriction of g_σ to X' gives a torsor $X' \to X$ under F', where F' is the stabilizer of X' under the action of F^σ, and the point (P_v) lifts to a point $(Q'_v) \in X'(\mathbb{A}_k)^{\text{Br}}$.*

In particular, X' is geometrically integral and the following diagram commutes

$$\begin{array}{ccc} X' \times_k F' & \xrightarrow{\psi \times p} & Z^\sigma \times_k F^\sigma \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\psi} & Z^\sigma \\ \downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array}$$

where $X' \xrightarrow{\psi} Z^\sigma$ and $F' \xrightarrow{p} F^\sigma$ are the natural inclusion maps and the two vertical maps of the upper square are given by the respective actions of F' and F^σ on X' and Z^σ.

Proof. By assumption, the point (P_v) lifts to some point $(Q_v) \in Z^\sigma(\mathbb{A}_k)^{\mathrm{Br}}$ for some cocycle σ with values in F. Since Z^σ is smooth, Z^σ is a disjoint union of connected components over k. By Proposition 3.3 in [25], there is a k-connected component X' of Z^σ such that $(Q_v)_{v\notin\Xi} \in P_\Xi(X'(\mathbb{A}_k)^{\mathrm{Br}})$, where Ξ is the set of all complex primes of k and P_Ξ is the projection of $X'(\mathbb{A}_k)$ to $X'(\mathbb{A}_k^\Xi)$ and $\mathbb{A}_k^\Xi$ is the adeles without Ξ-components. Since $Z^\sigma \times_k k_v$ is a trivial torsor under a constant group scheme $F^\sigma \times_k k_v$ for $v \in \Xi$, one concludes that $g_\sigma(X'(k_v)) = X(k_v)$ for $v \in \Xi$. One can choose $Q_v \in X'(k_v)$ for $v \in \Xi$ and obtains that $(Q_v) \in X'(\mathbb{A}_k)^{\mathrm{Br}}$.

Since X' is connected and $X'(\mathbb{A}_k) \neq \emptyset$, the proof of Lemma 5.5 in [33] implies that X' is geometrically connected. Eventually, X' being geometrically connected guarantees that the variety X' is an X-torsor under the stabilizer F' of X' in F^σ, and the required commutative diagram follows by construction. □

For a linear algebraic group G over k, one has the following short exact sequence of algebraic groups

$$1 \to H \to G \to F \to 1$$

over k, where H is the connected component of G and F is finite over k. This induces the following diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T & \longrightarrow & G' & \longrightarrow & F & \longrightarrow & 1 \end{array}$$

where T denotes the maximal toric quotient of H and G' is the quotient of G by the kernel of $H \to T$.

Let $Y \to X$ be a torsor under G and $Z \to X$ be the push-forward of $Y \to X$ by the morphism $G \to F$, which is a torsor under F. If $\sigma \in Z^1(k, F)$ is a 1-cocycle given by Lemma 7.1 applied to the torsor $Z \to X$ and the point (P_v), we wish to show that the cocycle $\sigma \in Z^1(k, F)$ lifts to a cocycle $\tau \in Z^1(k, G)$ as in Proposition 5 in [12]. The obstruction to lift σ in $Z^1(k, G)$ gives a natural cohomology class $\eta_\sigma \in H^2(k, \kappa_\sigma)$ by (5.1) in [16] (see also (7.7) in [1]), where κ_σ is a k-kernel on $H_{\bar{k}}$. Lemma 6 in [12] implies that there is a canonical map $H^2(k, \kappa_\sigma) \to H^2(k, T^\sigma)$ such that the class η_σ is neutral if and only if its image $\eta'_\sigma \in H^2(k, T^\sigma)$ is zero.

We further apply the open descent theory and the extended type developed by Harari and Skorobogatov in [23] to establish analogue of Lemma 7 in [12] for open varieties. As in the proof of [12], the torsor $Y \to Z$ under H induces a torsor $W \xrightarrow{\varpi} Z$ under T by the natural map $H^1(Z, H) \to H^1(Z, T)$. Instead of using the type of the torsor ϖ that was used in [12], we consider the so-called "extended type" of the torsor ϖ that was introduced by Harari and Skorobogatov (see Definition 8.2 in [23]). For variety Z over k, the complex of Galois modules $[\overline{k}(Z)^*/\overline{k}^* \to \mathrm{Div}(Z_{\bar{k}})]$ in the derived category of bounded complexes of étale sheaves over $\mathrm{Spec}(k)$ is denoted by $KD'(Z)$. One can associate to the torsor $W \xrightarrow{\varpi} Z$ under T a canonical morphism in this derived category

$$\lambda_W : \widehat{T} \to KD'(Z)$$

called the extended type of ϖ. This induces a morphism in the derived category of bounded complexes of abelian groups

$$\lambda_W^\sigma : \widehat{T}^\sigma \to KD'(Z^\sigma)$$

for the above $\sigma \in Z(k, F)$.

Lemma 7.2. *The morphism* $\lambda_W^\sigma : \widehat{T}^\sigma \to KD'(Z^\sigma)$ *is a morphism in the derived category of bounded complexes of étale sheaves over* $\mathrm{Spec}(k)$.

Proof. The natural left actions of F on both T and Z induces right actions of F on $\widehat{T}$ and on $KD'(Z)$.

We first prove that the morphism λ_W is F-equivariant for those actions. For any $f \in F(\bar{k})$, we denote by $f_Z : Z_{\bar{k}} \to Z_{\bar{k}}$ the morphism of $\bar{k}$-varieties defined by $z \mapsto f \cdot z$. This morphism induces a natural morphism in the derived category $f_Z^* : KD'(Z_{\bar{k}}) \to KD'(Z_{\bar{k}})$. Similarly, the element f defines a natural morphism of $\bar{k}$-tori $f_T : T_{\bar{k}} \to T_{\bar{k}}$ such that $f_T(t) := gtg^{-1}$, where $g \in G'(\bar{k})$ is any point lifting $f \in F(\bar{k})$. This morphism f_T induces a morphism of abelian groups $\widehat{f_T} : \widehat{T} \to \widehat{T}$ such that $\widehat{f_T}(\chi) := \chi \circ f_T$.

One needs to prove that the following diagram

$$\begin{array}{ccc} \widehat{T} & \xrightarrow{\lambda_{W_{\bar{k}}}} & KD'(Z_{\bar{k}}) \\ {\scriptstyle \widehat{f_T}}\downarrow & & \downarrow{\scriptstyle f_Z^*} \\ \widehat{T} & \xrightarrow[\lambda_{W_{\bar{k}}}]{} & KD'(Z_{\bar{k}}) \end{array}$$

is commutative.

Let $f_{T,*}W_{\bar{k}}$ be the push-forward of the torsor $W_{\bar{k}} \to Z_{\bar{k}}$ under $T_{\bar{k}}$ by the $\bar{k}$-morphism $T_{\bar{k}} \xrightarrow{f_T} T_{\bar{k}}$ and $f_Z^* W_{\bar{k}}$ be the pullback of the torsor $W_{\bar{k}} \to Z_{\bar{k}}$ under $T_{\bar{k}}$ by the $\bar{k}$-morphism $f_Z : Z_{\bar{k}} \to Z_{\bar{k}}$. Then we have

$$f_Z^* \circ \lambda_{W_{\bar{k}}} = \lambda_{f_Z^* W_{\bar{k}}} \quad \text{and} \quad \lambda_{f_{T,*}W_{\bar{k}}} = \lambda_{W_{\bar{k}}} \circ \widehat{f_T}$$

by functoriality of the extended type. To prove the required commutativity $f_Z^* \circ \lambda_{W_{\bar{k}}} = \lambda_{W_{\bar{k}}} \circ \widehat{f_T}$, it is enough to show that the torsors $f_Z^* W_{\bar{k}} \to Z_{\bar{k}}$ and $f_{T,*}W_{\bar{k}} \to Z_{\bar{k}}$ under $T_{\bar{k}}$ are isomorphic. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} T_{\bar{k}} \times W_{\bar{k}} & \xrightarrow{g} & W_{\bar{k}} \\ {\scriptstyle \varpi \circ p_W}\downarrow & & \downarrow{\scriptstyle \varpi} \\ Z_{\bar{k}} & \xrightarrow[f_Z]{} & Z_{\bar{k}}\,, \end{array}$$

where p_W denotes the projection on $W_{\bar{k}}$ and the morphism g is defined by $(t, w) \mapsto (tg) \cdot w$. This diagram induces a natural $Z_{\bar{k}}$-morphism $\phi : T_{\bar{k}} \times W_{\bar{k}} \to f_Z^* W_{\bar{k}}$. Consider now the right action of $T_{\bar{k}}$ on $T_{\bar{k}} \times W_{\bar{k}}$ defined by $(s, w) \cdot t := (s f_T(t), t^{-1} \cdot w) = (sgtg^{-1}, t^{-1} \cdot w)$. Then the morphism ϕ is $T_{\bar{k}}$-invariant under this action, hence it induces a $Z_{\bar{k}}$-morphism $\psi : f_{T,*}W_{\bar{k}} \to f_Z^* W_{\bar{k}}$. One can check by a simple computation that ψ is $T_{\bar{k}}$-equivariant, i.e. that ψ is a morphism of (left) torsors over $Z_{\bar{k}}$ under $T_{\bar{k}}$. It concludes the required commutativity, hence the morphism λ_W is F-equivariant.

By definition of the twists T^σ and Z^σ, the fact that λ_W is F-equivariant implies that the morphism λ_W^σ is Galois equivariant, i.e. that λ_W^σ is a morphism in the derived category of bounded complexes of étale sheaves over $\mathrm{Spec}(k)$. □

By Proposition 8.1 in [23], there is a natural exact sequence of abelian groups

$$H^1(k, T^\sigma) \to H^1(X', T^\sigma) \xrightarrow{\lambda} \mathrm{Hom}_k(\widehat{T^\sigma}, KD'(X')) \xrightarrow{\partial} H^2(k, T^\sigma)$$

where the map λ is the extended type. Let $\lambda'_\sigma = \psi^* \circ \lambda_W^\sigma$ with $KD'(Z^\sigma) \xrightarrow{\psi^*} KD'(X')$ by Lemma 7.1. The following lemma, which is an analogue of Lemma 8 in [12], is a crucial step for proving the main result of this section. We give here a more conceptual proof than that in [12], where a similar statement was proven by cocycle computations under the assumption that $\bar{k}[X]^\times = \bar{k}^\times$.

Lemma 7.3. *With the above notation, one has*

$$\partial(\lambda'_\sigma) = 0 \text{ if and only if } \eta'_\sigma = 0\,.$$

Proof. In the following proof, we work over the small étale site of $\mathrm{Spec}(k)$.

Given the cocycle $\sigma \in Z^1(k, F)$, one can associate a $\mathrm{Spec}(k)$-torsor U under F with a point $u_0 \in U(\bar{k})$. This torsor U is naturally a homogeneous space of the group G' with geometric stabiliser isomorphic to $T_{\bar{k}}$. Section IV.5.1 in [17] implies that $\eta'_\sigma \in H^2(k, T^\sigma)$ is the class of the $\mathrm{Spec}(k)$-gerbe $\mathcal{E}_\sigma$ banded by T^σ such that for all étale schemes S over $\mathrm{Spec}(k)$, one has the following category $\mathcal{E}_\sigma(S)$. The objects of $\mathcal{E}_\sigma(S)$ are triples (P, p, α) where $P \to S$ is a torsor under G', $p \in P(S_{\bar{k}})$ and $\alpha : P \to U_S$ is a G'-equivariant S-morphism. The morphisms of $\mathcal{E}_\sigma(S)$ between triples (P, p, α) and (P', p', α') are given by morphisms of torsors $P \to P'$ over S under G' that commute with α and α'.

Similarly, one can associate to the morphism λ'_σ a $\mathrm{Spec}(k)$-gerbe banded by T^σ that will be the obstruction for the morphism λ'_σ to be the extended type of a torsor over X' under T^σ. The morphism λ'_σ induces a morphism $\overline{\lambda'_\sigma} : \widehat{T^\sigma_{\bar{k}}} \to KD'(X'_{\bar{k}})$ in $D^b_{\text{ét}}(\bar{k})$. By construction, $\overline{\lambda'_\sigma}$ is the extended type of the torsor $Y_0 := W_{\bar{k}} \times_{Z_{\bar{k}}} X'_{\bar{k}}$ over $X'_{\bar{k}}$ under $T^\sigma_{\bar{k}} = T_{\bar{k}}$.

We now define $\mathcal{L}_\sigma$ to be the fibered category defined as follows : for all étale schemes S over $\mathrm{Spec}(k)$, the objects of the category $\mathcal{L}_\sigma(S)$ are pairs (V, φ), where $V \to X'_S$ is a torsor under T^σ_S of extended type λ_V compatible with λ'_σ and $\varphi : V_{\bar{k}} \to Y_0 \times_{\bar{k}} S_{\bar{k}}$ is an isomorphism of torsors over $X' \times_k S_{\bar{k}}$ under $T^\sigma_{S_{\bar{k}}}$. Given two such objects (V, φ) and (V', φ'), a morphism between (V, φ) and (V', φ') in the category $\mathcal{L}_\sigma(S)$ is a pair (α, t), where $\alpha : V \to V'$ is a morphism of torsors over X'_S under T^σ_S and $t \in T^\sigma(S_{\bar{k}})$ such that the diagram

$$\begin{array}{ccc} V_{\bar{k}} & \xrightarrow{\overline{\alpha}} & V'_{\bar{k}} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ Y_0 \times_{\bar{k}} S_{\bar{k}} & \xrightarrow[t]{} & Y_0 \times_{\bar{k}} S_{\bar{k}} \end{array}$$

commutes.

One can check that $\mathcal{L}_\sigma$ is a stack for the étale topology over $\mathrm{Spec}(k)$, and the fact that this is a gerbe is a consequence of the exact sequence of Proposition 8.1 in [23]

$$H^1(S,T^\sigma) \to H^1(X'_S,T^\sigma) \xrightarrow{\lambda} \mathrm{Hom}_S(\widehat{T^\sigma}, KD'(X'_S)) \xrightarrow{\partial} H^2(S,T^\sigma)$$

provided that S is integral regular and noetherian (which is enough since one works with the small étale site of $\mathrm{Spec}(k)$).

The band of this gerbe is the abelian band represented by T^σ.

In addition, it is clear that $\mathcal{L}_\sigma$ is neutral if and only if $\mathcal{L}_\sigma(k) \neq \emptyset$ if and only if there exists a torsor over X' under T^σ of type λ'_σ if and only if $\partial(\lambda'_\sigma) = 0$.

Let us now construct an equivalence of gerbes between $\mathcal{E}_\sigma$ and $\mathcal{L}_\sigma$.

For all étale $\mathrm{Spec}(k)$-schemes S, consider the functor

$$m_S : \mathcal{E}_\sigma(S) \to \mathcal{L}_\sigma(S)$$

that maps an object (P,p,α) to the object (V,φ), where V is defined to be the contracted product $V := (P \times_S^{G'} W_S) \times_{Z_S^\sigma} X'_S$ and $\varphi : V_{\bar{k}} \to Y_0 \times_{\bar{k}} S_{\bar{k}} = (W_{\bar{k}} \times_{Z_{\bar{k}}} X'_{\bar{k}}) \times_{\bar{k}} S_{\bar{k}}$ is induced by the point $p \in P(S_{\bar{k}})$. Indeed, by construction, we have a natural map $P \times_S^{G'} W_S \to U_S \times_S^F Z_S = Z_S^\sigma$, and a simple computation proves that this map is a torsor under T^σ of extended type compatible with λ_W^σ.

By definition, the functor m_S sends a morphism $\varphi : (P,p,\alpha) \to (P',p',\alpha')$ to the morphism $(\widetilde{\varphi}, t_0)$ such that $\widetilde{\varphi} : (P \times_S^{G'} W_S) \times_{Z_S^\sigma} X'_S \to (P' \times_S^{G'} W_S) \times_{Z_S^\sigma} X'_S$ is the morphism induced by the morphism of torsors $\varphi : P \to P'$, and $t_0 \in T^\sigma(S_{\bar{k}})$ is the element such that $p' = t_0 \cdot \varphi(p)$ as $S_{\bar{k}}$-points in $(P' \times_S^{G'} W_S) \times_{Z_S^\sigma} X'_S$.

Eventually, one checks that the collection of functors m_S defines a morphism of gerbes $m : \mathcal{E}_\sigma \to \mathcal{L}_\sigma$ banded by the identity of T^σ, which implies that $\eta'_\sigma := [\mathcal{E}_\sigma] = [\mathcal{L}_\sigma] \in H^2(k,T^\sigma)$.

Therefore, $\eta'_\sigma = 0$ if and only if $\mathcal{E}_\sigma(k) \neq \emptyset$ if and only if $\mathcal{L}_\sigma(k) \neq \emptyset$ if and only if $\partial(\lambda'_\sigma) = 0$. □

The immediate consequence of Lemma 7.3 is the following result which extends Proposition 5 in [12] to open cases.

Proposition 7.4. *Let X be a smooth geometrically integral k-variety. Let $(P_v) \in X(\mathbb{A}_k)^{\text{ét, Br}}$ and $Y \to X$ be a torsor under a linear k-group G. Let*

$$1 \to H \to G \to F \to 1$$

be an exact sequence of linear k-groups, where H is connected and F finite. Let $Z \to X$ be the push-forward of $Y \to X$ by the morphism $G \to F$, which is a torsor under F. Let $\sigma \in Z^1(k,F)$ be a 1-cocycle given by Lemma 7.1 applied to the torsor $Z \to X$ and the point (P_v).

Then the cocycle $\sigma \in Z^1(k,F)$ lifts to a cocycle $\tau \in Z^1(k,G)$.

Proof. By construction (5.1) in [16] (see also (7.7) in [1]), there is a class η_σ of $H^2(k,\kappa_\sigma)$ such that σ can be lifted to $Z^1(k,G)$ if and only if η_σ is neutral, where κ_σ is a k-kernel on $H_{\bar{k}}$. By (6.1.2) of [1] and Lemma 6 in [12], there is a canonical map $H^2(k,\kappa_\sigma) \to H^2(k,T^\sigma)$ such that the class η_σ is neutral if and only if its image $\eta'_\sigma \in H^2(k,T^\sigma)$ is zero. By Lemma 7.3, one only needs to show that $\partial(\lambda'_\sigma) = 0$ where $\lambda'_\sigma = \psi^* \circ \lambda_W^\sigma$, with $KD'(Z^\sigma) \xrightarrow{\psi^*} KD'(X')$ given by Lemma 7.1 and λ_W^σ defined by Lemma 7.2.

By Lemma 7.1, we know that $X'(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$. Therefore the map λ in the exact sequence (see Proposition 8.1 in [23])

$$H^1(X', T^\sigma) \xrightarrow{\lambda} \mathrm{Hom}_k(\widehat{T^\sigma}, KD'(X')) \xrightarrow{\partial} H^2(k, T^\sigma)$$

is surjective by Corollary 8.17 in [23]. Hence the map ∂ is the zero map and $\partial(\lambda'_\sigma) = 0$, which concludes the proof. □

The main result of this section is the following theorem.

Theorem 7.5. *If X is a smooth quasi-projective and geometrically integral variety over a number field k, then*

$$X(\mathbf{A}_k)^{desc} = X(\mathbf{A}_k)^{\mathrm{et,Br}}.$$

Proof. By Corollary 6.7, one only needs to prove that $X(\mathbf{A}_k)^{\mathrm{et,Br}} \subseteq X(\mathbf{A}_k)^{\mathrm{desc}}$. Since the statement 2 of Theorem 2 in [19] (which we apply to X') holds for any geometrically integral variety without assumption on $\bar{k}[X']^\times$, the result follows from the same argument that Proposition 5 implies Theorem 1 (p.244-245) in [12]. □

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