

Special points and Poincaré bi-extensions

by Daniel Bertrand, with an *Appendix* by Bas Edixhoven

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The context is the following :

i) in a joint project with D. Masser, A. Pillay and U. Zannier [7], we aim at extending to semi-abelian schemes the Masser-Zannier approach [11] to Conjecture 6.2 of R. Pink's preprint [13]; this conjecture also goes under the name "*Relative Manin-Mumford*". Inspired by Anand Pillay's suggestion that the semi-constant extensions of [6] may bring trouble, I found a counter-example, which is described in Section 1 below.

ii) at a meeting in Pisa end of March, Bas Edixhoven found a more concrete way of presenting the counter-example, with the additional advantage that the order of the involved torsion points can be controlled in a precise way : this is the topic of the Appendix.

iii) finally, I realized that when rephrased in the context of mixed Shimura varieties, the construction, far from providing a counter-example, actually *supports* Pink's general Conjecture 1.3 of [13]; a sketch of this view-point is given in Section 2.

1 A counter-example to relative Manin-Mumford ...

This counterexample is provided by a "Ribet section" on a semi-abelian scheme B/X of relative dimension 2 over a base curve X . Roughly speaking, given an elliptic curve E_0/\mathbb{C} with complex multiplications and using an idea of L. Breen, K. Ribet constructed a non-torsion point β_0 with strange divisibility properties on any given non isotrivial extension B_0 of E_0 by \mathbb{G}_m , cf. [10]. In the relative situation B/X , the very same construction yields :

Theorem 1. *Let B/X be a non constant (hence non isotrivial) extension of $E_0 \times X$ by \mathbb{G}_m . There exists a section $\beta : X \rightarrow B$ which does not factor through any proper closed subgroup scheme of B/X , but whose image $Y := \beta(X)$ meets the torsion points of the various fibers of B/X infinitely often (so, Zariski-densely, since X is a curve).*

More precisely, let X be a smooth connected affine curve defined over (say) \mathbb{C} , with function field $K := \mathbb{C}(X)$. We may have to delete some points of X , or consider finite covers of X , but will still denote by X the resulting curve. We write x for the generic point

of X , i.e. $K = \mathbb{C}(x)$, and $\xi \in X(\mathbb{C})$ for its closed points. We start with a CM elliptic curve E_0/\mathbb{C} , denote by $\hat{E}_0 \simeq E_0$ its dual, and fix an antisymmetric isogeny

$$\varphi : \hat{E}_0 \rightarrow E_0.$$

This means that its transpose $\hat{\varphi} : \hat{E}_0 \rightarrow E_0$ is equal to $-\varphi$, i.e., in the identification $\hat{E}_0 \simeq E_0$, that φ is a totally imaginary complex multiplication.

We consider the constant elliptic schemes $E = E_0 \times_{\mathbb{C}} X$, $\hat{E} = \hat{E}_0 \times_{\mathbb{C}} X$, and fix a non constant section $q : X \rightarrow \hat{E}$. In particular, q is not a torsion section, and the semi-abelian scheme B/X attached to q is a non constant¹, hence non isotrivial, extension of E/X by \mathbb{G}_m ; conversely, any such B is of this type. Let $\pi : B \rightarrow E$ be the corresponding X -morphism. Extending the construction of [10], we will attach to these data (q, φ) a section β of B/X such that the section $p := \pi \circ \beta$ of E/X satisfies $p = 2 \varphi \circ q$. Furthermore, β will have the following “*lifting property*” : *for any $\xi \in X(\mathbb{C})$ such that $p(\xi)$ is a torsion point on the fiber $E_\xi \simeq E_0$ of $E \rightarrow X$, its lift $\beta(\xi)$ is automatically a torsion point on the fiber B_ξ of $B \rightarrow X$.*

Before describing this construction, let us show that such a section β does satisfy the conditions announced in Theorem 1 :

- on the one hand, since $q \in \hat{E}(X)$ has infinite order, the only proper closed subgroup schemes of B projecting onto X are contained in finite unions of translates of $\mathbb{G}_m \times_{\mathbb{C}} X$ and of fibers of the projection; and the section $p = 2 \varphi \circ q \in E(X)$ too has infinite order. Therefore, β , which projects to p on E , cannot factor through any proper closed subgroup scheme of B/X .
- on the other hand, since q is not a constant section of \hat{E}/X , neither is $p \in E(X)$, and the set $X_{tor}^E = \{\xi \in X(\mathbb{C}), p(\xi) \text{ is a torsion point on } E_\xi \simeq E_0\}$ is infinite. But by the lifting property, this set coincide the set $X_{tor}^B = \{\xi \in X(\mathbb{C}), \beta(\xi) \text{ is a torsion point on } B_\xi\}$. Therefore, the curve $Y = \beta(X) \subset B$ meets the set of torsion points of the various fibers of $B \rightarrow X$ Zariski-densely.

To perform the construction of β , we go to the generic fiber $B_x := B \otimes_S K := B_K$ of B/S , consider the non-constant point $q(x) := q_K \in \hat{E}_K = \hat{E}_0 \otimes_{\mathbb{C}} K$, and recall the construction of the Ribet point $\beta_K \in B_K(K)$ attached to q_K and to the antisymmetric isogeny $\varphi : \hat{E}_K \rightarrow E_K$, with transpose $\hat{\varphi} = -\varphi$. There are two ways to describe β_K :

- (i) The first one [10] goes as follows : consider the pullback

$$\varphi^* B_K \in \text{Ext}(\hat{E}_K, \mathbb{G}_m) \text{ of } B_K \in \text{Ext}(E_K, \mathbb{G}_m)$$

under φ , and denote again by $\varphi : \varphi^* B_K \rightarrow B_K$ the natural extension of φ to $\varphi^* B_K$. Since B_K is parametrized by q_K , $\varphi^* B_K$ is parametrized by the point $\hat{\varphi}(q_K)$ of the dual E_K of \hat{E}_K . Now, choose an arbitrary K -rational point t_K in the fiber above q_K of the extension

¹ In particular, B/X is a semi-constant semi-abelian variety in the sense of [6]. However, the counter-example to Lindemann-Weierstrass given there is of a different nature; see also Remark 1.(ii) below.

φ^*B_K ; in particular, its image $t_K^1 := \varphi(t_K) \in B_K(K)$ satisfies $\pi_K(t_K^1) = \varphi(q_K)$, where $\pi_K = \pi(x) : B_K \rightarrow E_K$. The point t_K defines a one-motive $M_K : \mathbb{Z} \rightarrow \varphi^*B_K$, whose Cartier dual $\hat{M}_K : \mathbb{Z} \rightarrow B_K$ is given by a point $t_K^2 \in B_K(K)$ projecting to $\hat{\varphi}(q_K) \in E(K)$. Finally, set $\beta_K = t_K^1 - t_K^2$: this point of $B_K(K)$ is *independent of the choice of the auxiliary point* t_K above q_K (this is clear on the symmetric definition of duals given in Hodge III), and its image under π_K is the point $\varphi(q_K) - \hat{\varphi}(q_K) = 2\varphi(q_K) := p_K$ of $E_K(K)$.

(ii) The second one [4] is more geometric (and will actually not be used here): consider the Poincaré bundle \mathcal{P}_K on $\hat{E}_K \times E_K$, rigidified above $(0, 0)$. Since φ is antisymmetric, the square of its restriction to the graph $\Phi_K \subset \hat{E}_K \times E_K$ of φ is trivial. Up to a 2- isogeny, we therefore get a unique non-zero K -regular section $\sigma_\varphi : \Phi_K \rightarrow \mathcal{P}_K|_{\Phi_K}$. Now B_K (plus a zero section) identifies with the restriction of \mathcal{P}_K to $\{q_K\} \times E_K \simeq E_K$, and its fiber over $p'_K := \varphi(q_K)$ with the fiber of \mathcal{P}_K over $(q_K, p'_K) \in \Phi_K(K)$. We then set $\beta'_K = \sigma_\varphi((q_K, p'_K))$, and view β'_K as a point of $B_K(K)$ above p'_K . Up to multiplication by 2, this is the same point as the β_K above.

Restricting X if necessary, we can extend this point β_K to a section $\beta : X \rightarrow B$, which may be called the Ribet section attached to φ of the semi-abelian scheme B/X defined by the non-constant section q of \hat{E}/X we had started with. More precisely, we can extend the auxiliary point t_K of the first construction to a section t of φ^*B/X , and repeat the whole process over X (minus some points), getting in particular a smooth one-motive M/X , sections t^1, t^2 over X , etc. By definition, $\pi \circ \beta$ is the section $p = 2\varphi \circ q$ of E/X extending p_K over X , and it remains to show that β satisfies the “lifting property”.

So, let $\xi \in X(\mathbb{C})$ be a point such that $p(\xi)$ is a torsion point on the fiber $E_\xi (\simeq E_0)$ of $E \rightarrow X$. We must show that $\beta(\xi)$ is a torsion point on the fiber B_ξ of $B \rightarrow X$. By the relation $p(\xi) = 2\varphi(q(\xi))$, $q(\xi)$ too is a torsion point on \hat{E}_ξ (in passing, this shows that B_ξ is an isotrivial extension). So, among the points which lie on the fiber $(\varphi^*B)_\xi$ of φ^*B/X above ξ (which is the pull-back

$$\varphi^*B_\xi \in \text{Ext}(\hat{E}_\xi, \mathbb{G}_m) \text{ of } B_\xi \in \text{Ext}(E_\xi, \mathbb{G}_m) ,$$

and which project to $q(\xi) \in (\hat{E}_\xi)_{\text{tor}}$, we now have not only the value $t(\xi)$ of the section $t : X \rightarrow \varphi^*B$ at ξ , but also plenty of torsion points of the complex semi-abelian variety φ^*B_ξ . Choose one of them, and call it \tilde{t}_ξ . Since \tilde{t}_ξ and $t(\xi)$ differ by an element of \mathbb{G}_m , the first construction, whether applied to $t(\xi)$ or to \tilde{t}_ξ , will yield the *same* point $\beta_\xi \in B_\xi$, with $\pi(\beta_\xi) = p(\xi)$. Using $t(\xi)$, we see that $\beta_\xi = \beta(\xi)$; using the torsion point \tilde{t}_ξ and the fact that $\hat{\varphi}(q(\xi))$ is a torsion point, we see that the weight filtrations of the corresponding complex one-motive \tilde{M}_ξ , hence of its dual $\hat{\tilde{M}}_\xi$, split up completely up to isogeny. Consequently, the points $\tilde{t}_\xi^1, \tilde{t}_\xi^2$ and β_ξ associated to \tilde{t}_ξ by the first construction are all torsion points, and $\beta(\xi)$ is indeed a torsion point of B_ξ .

Remark 1 : i) (from X to \hat{E}_0) : let \mathcal{B} be the “universal” extension of E_0 by \mathbb{G}_m , viewed as a group scheme over $\underline{\text{Ext}}^1(E_0, \mathbb{G}_m) \simeq \text{Pic}_{E_0/\mathbb{C}}^0 = \hat{E}_0$. The extension B attached to the section $q : X \rightarrow \hat{E}_0$ is the pull-back of \mathcal{B} under q . Choosing $X = \hat{E}_0$, and $q =$ the identity

map, so that $K = \mathbb{C}(\hat{E}_0)$, we can therefore restrict to the case where $K = \mathbb{C}(\hat{E}_0)$ and q_K is the generic point of $\hat{E}_0(K)$. The Appendix - and most of §2 - concerns this generic case

$$X = \hat{E}_0, q = id, B = \mathcal{B}.$$

ii) (*when $g > 1$*) : Ribet sections β can be defined over any abelian scheme A/X , of relative dimension g , which admits an antisymmetric isogeny $\varphi : \hat{A} \rightarrow A$. If $g = 1$, this forces A to be iso-constant, hence the E_0/\mathbb{C} above. But as soon as $g > 1$, there are examples of simple non constant A/X with such a $\varphi = -\hat{\varphi}$. The section β attached to φ and to a section $q \in \hat{A}(X)$ will again satisfy the “lifting property”. However, in order to ensure that the set $X_{tor}^A = \{\xi \in X(\mathbb{C}), \pi \circ \beta(\xi) \text{ is a torsion point on } A_\xi\}$ be infinite, one must in general insist that $\dim(X) \geq g$. So, the counterexample does not extend to extensions by \mathbb{G}_m of higher dimensional abelian schemes *over curves*.

2 ... in support of Pink’s general conjecture.

In [13], R. Pink mentions the similarity of Conjecture 6.2 with Y. André’s result on special points on elliptic pencils [2], III, p. 9. Viewing the scheme B/X above as a “semi-abelian pencil” over the fixed elliptic curve E_0 , one could define its special points as the torsion points lying on a fiber which is an isotrivial extension. As mentioned in passing during the proof of the lifting property, the curve $Y = \beta(X)$ even contains infinitely many special points in this sense.

Going further in this direction, we will now construct a mixed Shimura variety $S(\varphi)$ into which the image $Y = \beta(X)$ of the Ribet section β can be mapped in a natural way. Denoting this map by $i : Y \rightarrow S(\varphi)$, we have under the hypotheses of §1 (or more generally, of Footnote ⁽²⁾ below) :

Theorem 2. *The algebraic subvariety $Z = i(Y)$ of the mixed Shimura variety $S(\varphi)$ passes through a Zariski-dense set of special points of $S(\varphi)$, and is indeed a special subvariety of $S(\varphi)$.*

This, of course, is in full concordance with the prediction of the general Conjecture 1.3 of [13] (more specifically, of the case $d = 0$ of Conjecture 1.1).

Here, I will merely give a set-theoretic description of the construction of $S(\varphi)$. We fix an integer $g \geq 1$ and a totally imaginary quadratic integer $\alpha = -\bar{\alpha}$, and denote by S_0 (a component of) the pure Shimura variety parametrizing abelian varieties A endowed with a principal polarization $\psi : \hat{A} \rightarrow A$ (in particular, $\psi = \hat{\psi}$), with some level structure, and with an embedding $j : \mathbb{Z}[\alpha] \rightarrow \text{End}(A)$ such that $\psi \circ \widehat{j(\alpha)} \circ \psi^{-1} = j(\bar{\alpha})$. Let

$$(\mathcal{A}, \psi, \varphi := j(\alpha) \circ \psi)$$

be the corresponding universal abelian scheme over S_0 (in particular, $\varphi = -\hat{\varphi} : \hat{\mathcal{A}} \rightarrow \mathcal{A}$ is antisymmetric). Then, $S_1 := \mathcal{A} \times_{S_0} \hat{\mathcal{A}}$ is a mixed Shimura variety parametrizing one-motives of the shape $M : \mathbb{Z} \rightarrow A \times \hat{A}$, i.e. couples of points $(p, q) \in A \times \hat{A}$, with $\{A\} \in S_0$,

and we can view

$$S_1(\varphi) = \{(A, p, q) \in S_1, p = 2\varphi(q)\}$$

as a mixed Shimura subvariety of S_1 . Finally, consider the Poincaré bi-extension

$$\varpi = (\varpi_1, \varpi_2) : \mathcal{P}^* \rightarrow \mathcal{A} \times_{S_0} \hat{\mathcal{A}}.$$

This is a \mathbb{G}_m -torsor over $S_1 = \mathcal{A} \times_{S_0} \hat{\mathcal{A}}$, which can again be viewed as a mixed Shimura variety, now parametrizing one-motives $M : \mathbb{Z} \rightarrow B$ of constant and toric ranks equal to 1, where on denoting by $(A, p := \varpi_1(M), q := \varpi_2(M))$ the point defined by $\varpi(M)$ in S_1 , $B = B_q$ is the extension of A by \mathbb{G}_m attached to q , while the image b of $1 \in \mathbb{Z}$ is a point on B projecting to $p \in A$. Breaking the symmetry between \mathcal{A} and $\hat{\mathcal{A}}$, we can alternatively consider $\hat{\mathcal{A}} \simeq \underline{\text{Ext}}_{S_0}^1(\mathcal{A}, \mathbb{G}_m)$ as a mixed Shimura variety, and view $\varpi_2 : \mathcal{P}^* \rightarrow \hat{\mathcal{A}}$ as the “universal” extension \mathcal{B} of \mathcal{A} by \mathbb{G}_m , over its parameter space $\hat{\mathcal{A}}$. We at last define

$$S(\varphi) := \varpi^{-1}(S_1(\varphi))$$

as the mixed Shimura subvariety of $\mathcal{P}^* \simeq \mathcal{B}$ whose points parametrize one-motives $M : \mathbb{Z} \rightarrow B$ such that $\varpi_1(M) = 2\varphi(\varpi_2(M))$.

We now consider an abelian scheme A/X of the type parametrized by S_0 , over some irreducible algebraic variety X/\mathbb{C} . There then exists a unique morphism $i_0 : X \rightarrow S_0$ such that A/X is the pull-back of \mathcal{A} under i_0 . We fix a section $q : X \rightarrow \hat{\mathcal{A}}$, corresponding to a semi-abelian scheme $\pi : B \rightarrow A$ over X , and perform Ribet’s construction², yielding sections $p = 2\varphi \circ q : X \rightarrow A$, $\beta : X \rightarrow B$, with $p = \pi \circ \beta$. By the universal property of $S(\varphi)$, there exists a unique morphism

$$i : X \rightarrow S(\varphi)$$

above i_0 such that the smooth X -one-motive $M : \mathbb{Z} \rightarrow B$ defined by β is the pull-back under i of the universal one-motive $\mathcal{M} : \mathbb{Z} \rightarrow \mathcal{B}$. We again denote by $i : B \rightarrow S(\varphi) \subset \mathcal{B}$ the extension of i to B/X . The image $Z := i(Y) \subset S(\varphi)$ of $Y = \beta(X) \subset B$ is the algebraic subvariety of $S(\varphi)$ to be studied for Theorem 2.

We first check that Z contains a Zariski-dense set of special points of $S(\varphi)$. By definition, these points represent complex one-motives such that the underlying abelian variety is CM, and whose weight filtrations are totally split up to isogeny. By the first hypothesis

² When $g = 1$ as in the first Section, S_0 is reduced to a CM point $\{E_0\}$. Taking into account Remark 1.(ii), we are now proving Theorem 2 for any $g \geq 1$. However, in the case $g > 1$, we must add the following hypotheses to its statement : the base X/\mathbb{C} is an irreducible variety of dimension $\geq g$, and

- (a) the image $i_0(X)$ meets the set of CM points of S_0 Zariski-densely.
- (b) $q(X)$ meets the set of torsion points of the various CM fibers of $\hat{\mathcal{A}}$ Zariski-densely;

For the sake of simplicity, we will *assume*, in what follows, that $i_0(X)$ is a Shimura subvariety of S_0 (more or less \Leftrightarrow (a) under André-Oort), and that $q : X \rightarrow \hat{\mathcal{A}}$ dominates the “generic case” $\text{id} : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ (\Rightarrow (b)). Furthermore, the latter hypothesis implies the useful (but not necessary) property that :

- (c) q factors through no proper closed subscheme of $\hat{\mathcal{A}}/X$.

made in Footnote ⁽²⁾, the projection $i_0(X)$ of Z to S_0 passes through a Zariski-dense set of CM points. In the fiber of any such point, the projection of Z to $S_1(\varphi)$ passes through a Zariski-dense set of torsion points $(p = 2\varphi(q), q)$, because of the second hypothesis made in this foot-note. The lifting property established in §1 (more accurately, the sharper version mentioned in passing) now shows that Z does meet the set of special points of $S(\varphi)$ Zariski-densely.

It remains to show that Z is a special subvariety of $S(\varphi)$. We will check this Hodge-theoretically. Denote by G , resp. P , the generic Mumford-Tate group of the Shimura variety $i_0(X)$, resp. of the mixed Shimura subvariety of $S(\varphi)$ lying above $i_0(X)$. Then, P is the semi-direct product of its unipotent radical $W_{-1}(P)$ by G , and $W_{-1}(P)$ is the semi-direct product of $W_{-2}(P) \subset \mathbb{G}_a$ by the vectorial group $Gr_{-1}(P) \subset \mathbb{G}_a^{2g}$: notice that $Gr_{-1}(P)$ is the unipotent radical of the generic Mumford-Tate group of $S_1(\varphi)$, and the inclusion $Gr_{-1}(P) \subset \mathbb{G}_a^{2g}$ follows from the linear dependence relation $p = 2\varphi(q)$. Similarly, let $P_z \subset P$ denote the Mumford-Tate group of a sufficiently general point z in Z ; then, P_z is the semi-direct product of its unipotent radical $W_{-1}(P_z)$ by G , and $W_{-1}(P_z)$ is the semi-direct product of $W_{-2}(P_z) \subset W_{-2}(P)$ by the vectorial group $Gr_{-1}(P_z) \subset Gr_{-1}(P)$.

Now, by the second hypothesis of the footnote, $\mathbb{Z}.q(X)$ is Zariski-dense in \hat{A} , and Proposition 1 of [1] shows that $Gr_{-1}(P_z) = Gr_{-1}(P) = \mathbb{G}_a^{2g}$. On the other hand, Theorem 1 of [4] shows that $W_{-2}(P) = \mathbb{G}_a$, while $W_{-2}(P_z) = \{0\}$, and more precisely, that *for any point $s \in S(\varphi)(\mathbb{C})$ whose Mumford-Tate group P_s satisfies $Gr_{-1}(P_s) = \mathbb{G}_a^{2g}$, we have*

$$W_{-2}(P_s) = \{0\} \Leftrightarrow \exists \tilde{s} \in [s], \tilde{s} \in Z,$$

where $[s]$ denotes the Hecke orbit of s . In other words, up to isogenies, the points of Z are characterized by the existence of an exceptional Hodge tensor in their Betti realization, which does not exist at the generic point of $S(\varphi)$. So, Z is indeed a special subvariety of $S(\varphi)$.

Remark 2 : (i) This section shows that the special subvarieties of the mixed Shimura variety \mathcal{B} do not necessarily correspond to families of semi-abelian subvarieties, so that Theorem 5.7 and 6.3 of [13] must be modified. Similarly, the mixed Shimura varieties of Hodge type $\{W_{-2}(P) = 0\}$ are not necessarily Kuga fiber varieties (compare [12], Example 1.10, and [4], Remark (v)).

(ii) The strange divisibility properties of Ribet points alluded to in the introduction of §1 are precisely reflected by the ℓ -adic analogue of the vanishing of $W_{-2}(P_z)$, $z \in S(\varphi)(\overline{\mathbb{Q}})$; cf. [4], Theorem 1.(ii). Actually, K. Ribet gives in [14] an explicit description of the exceptional Galois invariant tensor occurring in the ℓ -adic cohomology of these one-motives, which applies to all their realizations.

iii) We refer to [3] for a complete description of the unipotent radical $W_{-1}(P)$ of the Mumford-Tate group of one-motives of higher toric or constant ranks. When both ranks are equal to 1, the fact that $W_{-2}(P)$ vanishes in the case of antisymmetrically self-dual one-motives is an exercise in group theory, cf. [5], Lemme 6.

3 Appendix, by Bas Edixhoven

We go back to the setting and the notations $E_0, X, q, \varphi = -\hat{\varphi}, \pi : B \rightarrow E$ of §1, but to make later comparisons easier, we henceforth consider the extension

$$B = B_{2q}$$

of E by \mathbb{G}_m given by $2q \in \hat{E}(X)$, and denote by β_R in $B_{2q}(X)$ the Ribet section which the construction of §1 attaches to the data $\{2q, \varphi\}$; in particular, the projection $p := \pi \circ \beta_R = \varphi(2q) - \hat{\varphi}(2q) = 2\varphi(2q)$ of β_R on $E(X)$ is now twice the section considered in §1.

Interpreting B_{2q}/X as the generalized jacobian $\text{Pic}_{C/X}^0$ of a singular curve C/X with normalization $\hat{E} = \text{Pic}_{E/X}^0 = \hat{E}_0 \times X/X$, we will here construct³ a concrete section β_J (with “J” for “Jacobian”) of B_{2q}/X , which enjoys all the properties⁴ of the Ribet section β_R , and thereby provides a new proof of Theorem 1, but for which the “lifting property” takes the following sharper form :

Theorem 3. *For any $\xi \in X$, the image of $\beta_J(\xi)$ under $\pi : B_\xi \rightarrow E_\xi \simeq E_0$ satisfies $\pi(\beta_J(\xi)) = \pi(\beta_R(\xi)) := p(\xi)$. And if $p(\xi)$ is a torsion point of E_0 of order n , with n prime to $2 \deg(\varphi) \deg(\varphi + \psi) \deg(\varphi - \psi)$, then $\beta_J(\xi)$ is a torsion point of B_ξ , of order dividing n^2 .*

Here, $\psi = \hat{\psi} : \hat{E} \rightarrow E$ denotes the standard principal polarization: its inverse sends a point P to the class of the divisor $(P) - (0)$. Without loss of generality, we assume that we are in the generic case $X = \hat{E}_0, q = \text{id}$, with $K := \mathbb{C}(X) = \mathbb{C}(\hat{E}_0)$, but we keep to the notation X to indicate that some points of X may have to be removed in the constructions which follow. We denote by $Q = \psi \circ q : X \rightarrow E$ the section of E/X such that $q \in \hat{E}(X) = \text{Pic}^0(E/X)/\text{Pic}(X)$ is represented by the divisor $(Q) - (0)$ on E . By biduality, the section Q in $E(X) = \text{Pic}^0(\hat{E}/X)/\text{Pic}(X)$ is represented by the divisor $(q) - (0)$ on \hat{E} .

Let C/X be the singular curve over X obtained by identifying the disjoint sections q and $-q$ of \hat{E} (we remove $\hat{E}_0[2]$ from X). As a set, it is the quotient of \hat{E} by the equivalence relation generated by $q(\xi) \sim -q(\xi)$ with ξ ranging over X . The topology on C is the finest one for which the quotient map $\text{quot} : \hat{E} \rightarrow C$ is continuous: a subset U of C is open if and only if $\text{quot}^{-1}U$ is open in \hat{E} . The regular functions on an open set U of C are the regular functions f on $\text{quot}^{-1}U$ such that $f(q(\xi)) = f(-q(\xi))$ whenever $\text{quot}(q(\xi))$ is in U . It is proved in Thm. 5.4 of [9] that this topological space with sheaf of \mathbb{C} -valued functions is indeed an algebraic variety over \mathbb{C} . In categorical terms, $\text{quot} : \hat{E} \rightarrow C$ is the equalizer of the pair of morphisms $(q, -q)$ from X to \hat{E} .

The curve $C \rightarrow X$ is a family of singular curves, each with an ordinary double point; it is semi-stable of genus two (see [8, 9.2/6]). Its normalization is $\text{quot} : \hat{E} \rightarrow C$. Its

³ I thank Lenny Taelman for the suggestion to pay more attention to the symmetry between the points q and $-q$ to be identified and the divisor β_a that gives the section β_J .

⁴ In particular, β_J does not factor through any proper closed subgroup scheme of B . For a discussion relating β_J and β_R , see Remark 3.ii below.

generalized jacobian $B := \text{Pic}_{C/X}^0$ is described in [8], 8.1/4, 8.2/7, 9.2/1, 9.3/1. As $C \rightarrow X$ has a section (for example $\bar{0} := \text{quot} \circ 0$ and $\bar{q} := \text{quot} \circ q$), we have, for every $T \rightarrow X$, that $B(T) = \text{Pic}^0(T \times_X C/T)/\text{Pic}(T)$, where $\text{Pic}^0(T \times_X C/T)$ is the group of isomorphism classes of line bundles on $T \times_X C$ that have degree zero on the fibres of $T \times_X C \rightarrow T$. The group $\text{Pic}(T)$ is contained as direct summand in $\text{Pic}^0(T \times_X C/T)$ via pullback by the projection $T \times_X C \rightarrow X$ and a chosen section. In particular, a divisor D on C that is finite over X , disjoint from $\bar{q}(X)$ and of degree zero after restriction to the fibers of $C \rightarrow X$ gives the invertible \mathcal{O}_C -module $\mathcal{O}_C(D)$ that has degree zero on the fibers and therefore gives an element denoted $[D]$ in $B(X)$.

For ξ in X , the fiber B_ξ is, as abelian group, the group $\text{Pic}^0(C_\xi)$. In terms of divisors this is the quotient of the group $\text{Div}^0(C_\xi)$ of degree zero divisors with support outside $\{\bar{q}(\xi)\}$ by the subgroup of principal divisors $\text{div}(f)$ for nonzero rational functions f in $\mathbb{C}(C_\xi)^\times$ that are regular at $\bar{q}(\xi)$. As $C_\xi - \{\bar{q}(\xi)\}$ is the same as $\hat{E}_0 - \{q(\xi), -q(\xi)\}$, $\text{Div}^0(C_\xi)$ is the group of degree zero divisors on \hat{E}_0 with support outside $\{q(\xi), -q(\xi)\}$. An element f of $\mathbb{C}(C_\xi)^\times$ that is regular at $\bar{q}(\xi)$ is an element of $\mathbb{C}(\hat{E}_0)^\times$ that is regular at $q(\xi)$ and $-q(\xi)$ and satisfies $f(q(\xi)) = f(-q(\xi))$. This gives us a useful description of B_ξ .

The normalization map $\text{quot}: \hat{E} \rightarrow C$ induces a morphism of group schemes over X

$$\pi: B = \text{Pic}_{C/X}^0 \rightarrow \text{Pic}_{\hat{E}/X}^0 = E/X,$$

and identifies B with the extension of E by \mathbb{G}_m given by the section $2q \in \hat{E}(X)$. For ξ in X , the class $[\delta]$ in B_ξ of a divisor $\delta \in \text{Div}^0(C_\xi)$ lies in the kernel \mathbb{C}^\times of π_ξ if and only if there exists $f \in \mathbb{C}(\hat{E}_0)^\times$ such that $\delta = \text{div}(f)$ on \hat{E}_0 , and it is then a torsion point in \mathbb{C}^\times if and only if the quotient $f(q(\xi))/f(-q(\xi)) \in \mathbb{C}^\times$, which does not depend on the choice of f , is a root of unity.

We recall that for an elliptic curve \mathcal{E} over an algebraically closed field k , $\psi: \hat{\mathcal{E}} \rightarrow \mathcal{E}$ the standard polarization and u in $\text{End}(\mathcal{E})$, the pullback map u^* on $\text{Div}(\mathcal{E})$ induces \hat{u} in $\text{End}(\hat{\mathcal{E}})$, the dual of u , and then $\bar{u} := \psi \hat{u} \psi^{-1}$ in $\text{End}(\mathcal{E})$ is called the Rosati-dual of u ; it is characterized by the property that in $\text{End}(\mathcal{E})$ we have $\bar{u}u = \text{deg}(u)$ and $u + \bar{u} \in \mathbb{Z}$. Also, the pushforward map u_* on $\text{Div}(\mathcal{E})$ induces an element still denoted u_* in $\text{End}(\hat{\mathcal{E}})$ such that $\psi^{-1}u = u_*\psi^{-1}$ and u_*u^* is multiplication by $\text{deg}(u)$ in $\text{End}(\hat{\mathcal{E}})$. Hence u_* is the Rosati dual of u^* . We have $\hat{u} = u$, $\bar{\bar{u}} = u$ and $\bar{u}_* = u^*$ in $\text{End}(\hat{\mathcal{E}})$. For f a nonzero rational function on \mathcal{E} and $u \neq 0$ we have $u^*\text{div}(f) = \text{div}(f \circ u)$, and $u_*\text{div}(f) = \text{div}(\text{Norm}_u(f))$, where $\text{Norm}_u: k(\mathcal{E})^\times \rightarrow k(\mathcal{E})^\times$ is the norm map along u . Of course, all this applies to E_0 and \hat{E}_0 over \mathbb{C} , and to E and \hat{E} over the algebraic closure of the function field K of X .

We will use Weil reciprocity: for f and g nonzero rational functions on a nonsingular irreducible projective curve \mathcal{E} over an algebraically closed field k such that $\text{div}(f)$ and $\text{div}(g)$ have disjoint supports, one has $f(\text{div}(g)) = g(\text{div}(f))$, where for $D = \sum_P D(P) \cdot P$ a divisor on \mathcal{E} one defines $f(D) = \prod_P f(P)^{D(P)}$, cf. [15], III, Prop. 7. In Remark 3.(i) after the proof, we will also use the Weil pairing on \mathcal{E} . For n a positive integer and P and Q in $\text{Pic}^0(\mathcal{E})[n]$ the element $e_n(P, Q)$ in $\mu_n(k)$ is defined as follows. Let D_P and D_Q in $\text{Div}^0(\mathcal{E})$ be disjoint divisors representing P and Q . Let f and g be in $k(\mathcal{E})^\times$ such that

$nD_P = \text{div}(f)$ and $nD_Q = \text{div}(g)$. Then $e_n(P, Q) = f(D_Q)/g(D_P)$. For n invertible in k this pairing e_n is a perfect alternating pairing.

To define $\beta_J = \beta_J(a)$, we let $a \in \text{End}(\hat{E}_0)$ be *any* endomorphism such that $a^3 - a \neq 0$. We set $\alpha := \hat{a}$ in $\text{End}(E_0)$ and $\varphi := \alpha \circ \psi: \hat{E}_0 \rightarrow E_0$. Just as in [10], we will not need that $\hat{\varphi} = -\varphi$. However, if $\hat{\varphi} = \varphi$ then $\pi\beta_J$ will be zero in $\hat{E}(X)$, so we will insist that $\hat{\varphi} - \varphi \neq 0$. For s in $\hat{E}(X)$ we let (s) denote the relative divisor that it gives on \hat{E} , and, if (s) is disjoint from the singular locus $\bar{q}(X)$ of C , also the relative divisor on C that it gives.

We set:

$$\beta_a := a^*((q) - (-q)) - a_*((q) - (-q)) \quad \text{in } \text{Div}^0(\hat{E}/X). \quad (1)$$

The reader should note the antisymmetric use of both a^* and a_* in the definition of β_a . To get its support disjoint from $\bar{q}(X)$ we remove from X the finite set $\ker(a^2 - 1)$: note that, for $\xi \in X$, $\beta_a(\xi)$ and $(q(\xi)) - (-q(\xi))$ are not disjoint if and only if $aq(\xi) = q(\xi)$ or $aq(\xi) = -q(\xi)$. We can now also view β_a as element of $\text{Div}^0(C/X)$, and we set:

$$\beta_J := [\beta_a] \quad \text{in } B(X).$$

The image of $\pi\beta_J$ of β_J in $E(X)$ is the class of the divisor β_a on \hat{E} , hence we have, on denoting by \simeq linear equivalence on $\text{Div}^0(\hat{E}/X)$:

$$\begin{aligned} \pi\beta_J &= \bar{a}_*((q) - (-q)) - a_*((q) - (-q)) \\ &\simeq (\bar{a}q) - (\bar{a}(-q)) - ((aq) - (a(-q))) \\ &\simeq (2(\bar{a} - a)q) - (0) = 2(\alpha - \bar{\alpha})Q \quad \text{in } E(X). \end{aligned}$$

In particular, since $\bar{a} - a$ is nonzero, then $\pi\beta_J$ is not torsion in $E(X)$, and in fact

$$\pi\beta_J = ((2\bar{a}q) - (0)) - ((2aq) - (0)) = \varphi(2q) - \hat{\varphi}(2q) = p = \pi\beta_R,$$

as was to be checked for the first part of Theorem 3.

Now we start the proof of the second part of Theorem 3. Let n be a positive integer prime to $2 \deg(\varphi) \deg(\varphi + \psi) \deg(\varphi - \psi) = 2 \deg(a(a^2 - 1))$, and let $\xi \in X$ be a point such that $p(\xi) \in E_\xi = E_0$ is a torsion point of order n . Actually, we will *assume that $q(\xi)$ is a point of order n in $\hat{E}_\xi = \hat{E}_0$* : in the antisymmetric case considered in Theorem 3, we have $p(\xi) = 4\varphi(q(\xi))$, and the two conditions are equivalent by the primality assumption. To ease notations, we now drop the mention of ξ , writing E, B, q, \dots instead of $E_\xi, B_\xi, q(\xi), \dots$

As $nq = 0$ in \hat{E}_0 , we have $n\pi\beta_J = 0$ in E_0 . This means that $n\beta_a$ is a principal divisor on \hat{E}_0 . Let $f \in \mathbb{C}(\hat{E}_0)^\times$ be such that $\text{div}(f) = n(q) - n(-q)$ in $\text{Div}(\hat{E}_0)$. Then we have, on \hat{E}_0 :

$$\begin{aligned} \text{div}(f \circ a) &= a^* \text{div}(f) = a^*(n(q) - n(-q)), \\ \text{div}(\text{Norm}_a(f)) &= a_* \text{div}(f) = a_*(n(q) - n(-q)). \end{aligned}$$

We define:

$$g_a := (f \circ a) / \text{Norm}_a(f) \quad \text{in } \mathbb{C}(\hat{E}_0)^\times.$$

Then we have:

$$n\beta_a = \text{div}(f \circ a) - \text{div}(\text{Norm}_a(f)) = \text{div}(g_a) \quad \text{on } \hat{E}_0.$$

This means that $n[\beta_J]$ in B is the element $g_a(q)/g_a(-q)$ of \mathbb{C}^\times . As the divisor of f has support disjoint from that of g_a and of $a^*\text{div}(f)$ and $a_*\text{div}(f)$, Weil reciprocity gives us:

$$\begin{aligned} \left(\frac{g_a(q)}{g_a(-q)} \right)^n &= g_a(\text{div}(f)) = f(\text{div}(g_a)) = f(\text{div}(f \circ a) - \text{div}(\text{Norm}_a(f))) \\ &= \frac{f(\text{div}(f \circ a))}{f(\text{div}(\text{Norm}_a(f)))} = \frac{(f \circ a)(\text{div}(f))}{f(a_*\text{div}(f))} = \frac{f(a_*\text{div}(f))}{f(a_*\text{div}(f))} = 1. \end{aligned}$$

This finishes the proof of Theorem 3.

Remark 3 : i) We have shown that for ξ torsion in $X \subset \hat{E}_0$ and n its order, we have $n^2\beta_J(\xi) = 0$ in B_ξ . We will show that for $n > 1$ odd and prime to $\deg((a^2 - 1)(\bar{a} - a))$ there exist ξ in X with $p(\xi)$ of order n in E_0 such that the order of $\beta_J(\xi)$ equals n^2 .

Let n be such an integer. Recall the notation $Q = \psi(q)$, and let ξ in \hat{E}_0 be of order n such that $e_n(2(\alpha - \bar{\alpha})Q(\xi), 2Q(\xi))$ is of order n in \mathbb{C}^\times . Such ξ exist because $\alpha - \bar{\alpha}$ is an automorphism of $E_0[n]$ that is not scalar multiplication by an element of $\mathbb{Z}/n\mathbb{Z}$. And such a ξ is in X because n is prime to $2 \deg(a^2 - 1)$. By construction, $p(\xi) = 2(\alpha - \bar{\alpha})Q(\xi)$ is represented by the divisor $\beta_a(\xi)$, and $2Q(\xi)$ is represented by $(q(\xi)) - (-q(\xi))$. Again, let us drop the ξ 's from our notation. Then $n\beta_a = \text{div}(g_a)$, and $n((q) - (-q)) = \text{div}(f)$. We first compute:

$$\begin{aligned} f(\beta_a) &= f(a^*((q) - (-q)) - a_*((q) - (-q))) = \frac{(\text{Norm}_a f)((q) - (-q))}{(f \circ a)((q) - (-q))} \\ &= g_a^{-1}((q) - (-q)) = \frac{g_a(-q)}{g_a(q)}. \end{aligned}$$

Hence:

$$e_n(2(\alpha - \bar{\alpha})Q, 2Q) = \frac{g_a((q) - (-q))}{f(\beta_a)} = \frac{g_a(q)}{g_a(-q)f(\beta_a)} = \left(\frac{g_a(q)}{g_a(-q)} \right)^2.$$

ii) The Ribet section β_R and the present section β_J of B/X differ by an element of $\mathbb{G}_m(X)$, and it is natural to ask whether they are actually equal. In this respect, notice that by Theorem 3, $\beta_J(X)$ passes through a Zariski-dense set of special points of the mixed Shimura variety $S(\varphi)$ of §2. If we assume Pink's general Conjecture 1.3 of [13], $\beta_J(X)$ must then be a component of the Hecke orbit of the special subvariety Z defined by $\beta_R(X)$. Hence, at least conjecturally, the section $\beta_R - \beta_J$ of B/X is a root of unity.

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