Transcendence problems related to heights.

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Heights and Applications to Unlikely Intersections

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References

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[B-Edixhoven] Pink's conjecture, Poincaré biextensions and generalised Jacobians; in prep. (see also ArXiv 1104.5178v1).

(*) Comments added after talk : see last slide.

I. Motivations

• Serre's question [1989] on the Néron-Tate pairing over an ell. c. $E/\mathbb{Q}$:

$$x, \eta \in E(\mathbb{Q}), < x, \eta > = 0 \Rightarrow x \textit{ or } \eta \textit{ torsion ?}$$

NB : can't hope this for a n.f. $k \neq \mathbb{Q}$, nor on an ab. var. $A/\mathbb{Q}$ with $g > 1$.

• Zilber-Pink for a curve in the (4-dim'l) Poincaré bi-extension $\mathcal{P}^{\times}$ of the Legendre curve $\mathcal{E}/S$ (cf. [B 2013], [B.-Edixhoven]).

Reduces to two "mixed" RMM problems :

P1 (recently solved by F. Barroero): let $(x, \eta) \in \mathcal{E} \times_S \mathcal{E}(S)$. *If* x_t, η_t *are* $End(\mathcal{E}_t)$*-lin. dep. for inft'ly many CM* $t \in S$*, then* x, η *are* $(\mathbb{Z})$*-lin. dep.*

P2: let $E/\bar{\mathbb{Q}}$ with CM, and let s be section of a non constant semi-ab. scheme $\mathcal{G} \in Ext_S(E, \mathbb{G}_m) \rightsquigarrow \eta \in \hat{E}(S) \setminus \hat{E}(\mathbb{C})$: *if* s_t *is a Ribet point of* $\mathcal{G}_t$ *for inft'ly many* $t \in S(\bar{\mathbb{Q}})$*, is then* s *a Ribet section ?*

NB : let x be the projection of s to $E(S)$. Then,

s_t Ribet $\Rightarrow x_t, \eta_t$ are $End^{antisym}(E)$-related $\Rightarrow < x_t, \eta_t > = 0$.

Isotropic abelian subvarieties

For A/k, k a n.f., L symmetric ample, and $\mathcal{P}$ the Poincaré bundle on $A \times \hat{A}$

$$< x, \eta > := h_{\mathcal{P}}(x, \eta) = - < x, y >_L, \text{ where } \eta = \phi_L(y).$$

$< \sigma x, \sigma\eta > = < x, \eta >$ for $\sigma \in Gal(\bar{\mathbb{Q}}/k)$, and $< x, f(y) > = < y, \hat{f}(x) >$, so orthogonality occurs as soon as $\eta = f(x)$ with $f \in Hom^{antisym}(A, \hat{A})$, or because of relations on conjugates.

More generally, *let* $B = B_{x,\eta}$ *be the abelian variety generated by* (x, η) *in* $A \times \hat{A}$. Then, $\mathcal{P}_{|B}$ torsion ($\Leftrightarrow c_1(\mathcal{P}_{|B}) = 0$) implies that $h_{\mathcal{P}}(x, \eta) = 0$.

Conjecture [B 1994] : *let* $A/\mathbb{Q}$ *and* $(x, \eta) \in (A \times \hat{A})(\mathbb{Q})$. *Then*

$$h_{\mathcal{P}}(x, \eta) = 0 \Rightarrow c_1(\mathcal{P}_{|B}) = 0 \ ?$$

NB 1: rigidifying $\mathcal{P}^{\times}$ above $A \times 0$, the relation $\mathcal{P}_{|B} = 0$ provides a canon'l point s_R above (x, η) (a). We call s_R the **Ribet point** of $G_\eta \in Ext(A, \mathbb{G}_m)$ above x. Ditto for its orbit under $(\mathbb{G}_m)_{tors}$.

NB 2 : $c_1(\mathcal{P}_{|B}) = 0 \Leftrightarrow \exists F \in Hom^{as}(A \times \hat{A}, \hat{A} \times A)$ and $N \in \mathbb{N}$ such that $N.(\eta, x) = F(x, \eta)$. If x generates A, $\Leftrightarrow \exists f \in Hom^{as}(A, \hat{A})$ s.t. $N\eta = f(x)$.

One archimedean place

Bloch's construction of $< x, \eta > = h_\eta(x)$. By the product formula, the (absolute, logarithmic) normalized height on $\mathbb{G}_m(k)$ is

$$h(\alpha) = \Sigma_{v \in \mathcal{M}_k} \frac{[k_v:\mathbb{Q}_p]}{[k:\mathbb{Q}]} \, |\ell og(|\alpha|_v)|.$$

For $G = G_\eta$ and $v \in \mathcal{M}_k$, there is a unique extension of $\ell og|.|_v$ to $\lambda_v = \lambda_v^{(\eta)} : G(k_v) \to \mathbb{R}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k_v^* & \longrightarrow & G(k_v) & \xrightarrow{\pi} & A(k_v) & \longrightarrow & 0 \\ & & \downarrow \ell og|.|_v & & \downarrow (\lambda_v, \pi) & & \| & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} \times A(k_v) & \xrightarrow{\pi} & A(k_v) & \longrightarrow & 0 \end{array}$$

Then, $ker(\lambda_v) =$ maximal compact subgroup $G(k_v)^c$ of $G(k_v)$, and for any $s \in G_\eta(k)$ above x :

$$< x, \eta > = \Sigma_{v \in \mathcal{M}_k} \frac{[k_v:\mathbb{Q}_p]}{[k:\mathbb{Q}]} \, \lambda_v(s)$$

We may choose s with all finite λ_v's vanishing, so if k has just **one inf'te** place, $< x, \eta > = 0 \Leftrightarrow$ this s lies in $G(k_v)^c$ for *all* $v \in \mathcal{M}_k$.
NB : if $c_1(\mathcal{P}_{|B}) = 0$, the Ribet point $s_R \in G_\eta(k)$ above x satisfies this property for *any* number field k.

$$\text{Set } \mathcal{L} = \ell og(\bar{\mathbb{Q}}^\times) \supset \mathbb{Q}.\ell og(\mathbb{Q}^\times).$$

Still assume that k has only one infinite place ∞, but take any $s \in G_\eta(k)$ above x. Then,

$$< x, \eta >= 0 \Rightarrow \exists \alpha \in k^\times, s + \alpha \in G(k_\infty)^c \Rightarrow \lambda_\infty(s) \in \mathcal{L}.$$

To turn this into an amenable transcendence problem, we'd rather have a *complex analytic* expression for λ_∞, which happens if ∞ is real. This leads to :

Question : assume that the n.f. k has *at least* one real place w, and that $\lambda_w(s) \in \mathcal{L}$. Then, $c_1(\mathcal{P}_{|B}) = 0$? (If so, s will lie in the $\mathbb{G}_m$-orbit of s_R.)

This may be too bold, so let's go back to an elliptic curve E, firstly over $\mathbb{C}$, with $\wp, \zeta, \sigma, \omega_i, \eta_i$ as usual, $u = log_E(x), v = log_E(\eta)$, and

$$\kappa_v(\omega_i) = \zeta(v)\omega_i - \eta_i v, \ \ (i = 1, 2).$$

These are the basic periods of the standard logarithmic form ξ_η on E with residue divisor $-1.(0) + 1.(-\eta)$.

Let $G = G_\eta$ (maybe $G_{-\eta}$?). Then, $G \xrightarrow{\pi} E$ admits a rational section $\rho : E \dashrightarrow G$, with $(\rho) = (-\eta) - (0)$, and the exponential map of G is

$$LG(\mathbb{C}) \ni \begin{pmatrix} t \\ z \end{pmatrix} \mapsto \begin{pmatrix} f_v(z)\ e^t \\ (\wp, \wp')(z) \end{pmatrix} \in G(\mathbb{C})$$

where $f_v(z) = \frac{\sigma(v+z)}{\sigma(v)\sigma(z)} e^{-\zeta(v)z}$, so $\frac{df_v}{f_v} = \frac{1}{2}\frac{\wp'(z)-\wp'(v)}{\wp(z)-\wp(v)} dz = exp_E^*(\xi_\eta)$.

Over $k \subset \mathbb{C}$, a point $s \in G(k)$ above $x \in E(k)$ is given by

$$s = \begin{pmatrix} \delta_s \\ x \end{pmatrix},\ log_G(s) = \begin{pmatrix} -g(u,v) + \zeta(v)u + \ell_s \\ u \end{pmatrix}$$

where $\delta_s := s - \rho(x) \in k^\times$, $\ell_s = \ell og(\delta_s)$, and

$$g(u,v) = \ell og \frac{\sigma(u+v)}{\sigma(v)\sigma(u)}$$

is the "Green function" for the divisor $\Delta^\pm - E \times 0 - 0 \times E$ on $E \times E$.

$G(\mathbb{C}) \simeq \mathbb{C}^2/\Omega_G$, where $\Omega_G = \mathbb{Z}\varpi_0 \oplus \mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2$ with

$$\varpi_0(\lambda) = \begin{pmatrix} 2\pi i \\ 0 \end{pmatrix}, \varpi_1 = \begin{pmatrix} \kappa_v(\omega_1) \\ \omega_1 \end{pmatrix}, \varpi_2 = \begin{pmatrix} \kappa_v(\omega_2) \\ \omega_2 \end{pmatrix}.$$

Assume now that ∞ is a **real** place of k. Then, $E(\mathbb{R})^0 = \mathbb{R}\omega_1/\mathbb{Z}\omega_1$ and $G(\mathbb{R})^c = \mathbb{R}\varpi_1/\mathbb{Z}\varpi_1$ is 1-dim'l (while $\dim_{\mathbb{R}}(G(\mathbb{C})^c) = 3$). So,

$$s \in G(\mathbb{R})^c \Leftrightarrow \det \begin{pmatrix} -g(u,v) + \zeta(v)u + \ell_s & \zeta(v)\omega_1 - \eta_1 v \\ u & \omega_1 \end{pmatrix} = 0$$

$$\Leftrightarrow g(u,v) - \frac{\eta_1}{\omega_1} uv = \ell_s \ (= \ell og(\delta_s) \in \mathcal{L}).$$

Not a surprise : this is the restriction to $\mathbb{R}$ of the log of the "polar form" of the Klein form $\mathfrak{k}(u) = \sigma(u) exp(-\frac{1}{2}\eta(u)u)$.
We can now forget about the reality assumption and consider any $k \subset \mathbb{C}$.

Conjecture (b) : *assume E, x, η defined over $\bar{\mathbb{Q}}$, $u, v, u+v \notin \Omega_E$. Then,*

$$g(u,v) - \frac{\eta_1}{\omega_1} uv \in \mathcal{L} \Rightarrow x \text{ or } \eta \text{ is a torsion point}. \ (\Rightarrow \text{yes to Serre})$$

Known : 1) *if $g(u,v) - \zeta(v)u \in \mathcal{L}$, then η is torsion.*
2) in the CM case, let $s_2 \in \bar{\mathbb{Q}}$, given by the Hecke form of weight 2.
If $\exists f \in End^{as}(E), \eta = f(x)$, then $g(u,v) - s_2 uv \in \mathcal{L}$ (c).

But for a **complex** place, $s \in G(k) \cap G(\mathbb{C})^c$ does **not** imply $c_1(\mathcal{P}_{|B}) = 0$.

p-adic interlude

Kim Minhyong has given an "anabelian-Chabauty" proof of Siegel's theorem on $E(\mathbb{Z})$, when $E/\mathbb{Q}$ has rank 1 (i.e. just above the analogue of Chabauty's condition). The idea is that $E(\mathbb{Z})$ is contained in the set of zeroes of a non trivial *p-adic analytic* function on $E(\mathbb{Z}_p)$.

Take p ordinary, so there is a p-adic height h_p on $E(\mathbb{Q})$, which is the sum of the p-adic log of a rational number and of $log_p(\sigma(u)) - \kappa u^2$ (with $\kappa = \frac{1}{2}s_2$ in the CM case).

For $x \in E(\mathbb{Z})$, the first term vanishes, so $h_p(x) = log_p(\sigma(u)) - \kappa u^2$. Now, $\frac{h_p(x)}{u^2}$ is a constant C since h_p is quadratic and $rk(E(\mathbb{Q})) = 1$. Therefore $E(\mathbb{Z})$ is contained in the set of zeroes of the p-adic analytic function $log_p(\sigma(z)) - (\kappa + C)z^2$, non trivial since (say by Ax-Schanuel on G_x) $\log(\sigma(z))$ and z are algebraically independent over $\mathbb{C}$. Done !

It's anabelian because $\log(\sigma(u))$ is an iterated integral $\int_0^x \omega(\int_0 \eta)$, which Kim relates to $\pi_1^{unip}(E(\mathbb{C}) \setminus 0)$.

Additive interlude

For an ell. c. E over $k \subset \mathbb{R}$, let $\tilde{E} \in Ext(E, \mathbb{G}_a)$ be its universal extension. Its maximal compact subgroup $\tilde{E}(\mathbb{R})^c$ is $\mathbb{R}\tilde{\omega}_1/\mathbb{Z}\tilde{\omega}_1$ for the real period $\tilde{\omega}_1$ of $\tilde{E}$. Let $\tilde{x} \in \tilde{E}(k)$, above $x \in E(k)$; then (cf. [B 1998]),

$$\tilde{x} \in \tilde{E}(\mathbb{R})^c \Leftrightarrow \kappa_u(\omega_1)/\omega_1 \in k \Leftrightarrow \tilde{x} \in \tilde{E}_{tors}.$$

Indeed, $log_{\tilde{E}}(\tilde{x}) = \begin{pmatrix} \zeta(u) - \alpha \\ u \end{pmatrix}$ (for some $\alpha \in k$) and $\tilde{\omega}_1 = \begin{pmatrix} \eta_1 \\ \omega_1 \end{pmatrix}$ are $\mathbb{R}$-lin. dep. iff $\zeta(u)\omega_1 - \eta_1 u = \alpha\omega_1 \Rightarrow x \in E_{tor} \Rightarrow \tilde{x} \in \tilde{E}_{tor}$.

But much better : let $k \in \mathbb{C}$ be **any** n.f , let $A' = \hat{A} \simeq Pic^0(A)$ be an ab. var., with universal extension $\tilde{A}' \in Ext(A', \Omega^1_A)$, and let $\tilde{\eta} \in \tilde{A}'(k)$, above $\eta \in A'(k)$. Then [Bost-Künnemann 2009]:

$$\tilde{\eta} \in \tilde{A}'(\mathbb{C})^c \Leftrightarrow \tilde{\eta} \in \tilde{A}'_{tor}.$$

Idea : $\tilde{\eta} \in \tilde{A}' \leftrightsquigarrow G_\eta$, plus a connection on the line bundle $(G_\eta \cup 0)/A \leftrightsquigarrow$ a character χ_α of $\pi_1(A)$, for some $\alpha = \alpha(\tilde{\eta}) \in Hom(LieA, Lie\mathbb{G}_m) \simeq \Omega^1_A$, and $\tilde{\eta} \in \tilde{A}'(\mathbb{C})^c$ iff χ_α is unitary ($\Rightarrow \pm 1$ over $\mathbb{R}$). E.g. on an elliptic curve : $|\chi_\alpha(\gamma))| = 1 \Leftrightarrow \kappa_v(\omega) - \alpha\omega \in i\mathbb{R}$.

Unlikely intersections

Let S be a curve over $\bar{\mathbb{Q}}$, $\mathcal{E}/S$ an elliptic scheme, x, η two sections. On $\mathcal{E}(S)$ (and $\mathcal{E}(S')$ for $S' \to S$), we have the Néron-Tate height at the generic point and its polar form $< x, \eta >$, non degenerate on $\mathcal{E}(S)/\mathcal{E}^\sharp$, where $\mathcal{E}^\sharp$ is the Manin kernel ($=$ torsion $+$ constant parts).

• Assume that there are infin'ly many CM points $t \in S(\bar{\mathbb{Q}})$ such that $< x_t, \eta_t > = 0$ in $\mathcal{E}_t(\bar{\mathbb{Q}})$. If $\mathcal{E}/S$ not isoconstant, then (Silverman) $h_\eta(x) = < x, \eta > = 0$. Requires conditions on S to go further.

• In Problem **P2** on Zilber-Pink for $\mathcal{P}^\times$, over an E with CM, all x_t, η_t are $End(E)$-dep., so $h_S(t)$ is bounded (d). How to use $End(E)^{as}$?

• On G/k, the *relative height* $h_{G,rel}(s) = \Sigma_{v \in \mathcal{M}_k} \frac{[k_v:\mathbb{Q}_p]}{[k:\mathbb{Q}]} \; |\lambda_v(s)|$ is "linear" and vanishes on Ribet points. Under suitable conditions on $\mathcal{G}/S$, it too satisfies $\lim_{h_S(t)\to\infty} \frac{h_{G_t,rel}(s_t)}{h_S(t)} = h_{\mathcal{G},rel}(s)$. Does (not) lead to study sections s in $\mathcal{G}(S)$ with $h_{\mathcal{G},rel}(s) = 0$.

III. Functional transcendence

Geometric heights have no "transcendental" parts, but the following alg. indep. results may help for an o-minimal approach to **P2**.

Let $S/\mathbb{C}, K = \mathbb{C}(S) \subset F$, embedded in some diff'l field of meromorphic functions, let $E/K, G/F \in Ext_F(E, \mathbb{G}_m) \rightsquigarrow y \in E(F)$, $x \in E(F)$, and let E_0, G_0 be the constant parts. The universal extension $\tilde{G} = G \times_E \tilde{E}$ of G has dimension 3, and carries differential operators $\nabla_{L\tilde{G}} : L\tilde{G} \to L\tilde{G}$, $\partial ln_{\tilde{G}} := \nabla_{L\tilde{G}} \circ log_{\tilde{G}} : \tilde{G} \to L\tilde{G}$. Ditto with $\tilde{E}$. Their solutions generate the Picard-Vessiot extensions $K^{\sharp}_{L\tilde{E}} = K(\omega_{1,2}, \eta_{1,2})$ of $K = K^{\sharp}_{\tilde{E}}$ and $F^{\sharp}_{L\tilde{G}} = F(\omega_{1,2}, \eta_{1,2}, \kappa_v(\omega_{1,2}))$ of F, while $F^{\sharp}_{\tilde{G}}/F$ is still mysterious. Finally, let $u = log_E(x), v = log_E(y)$.

Ax-Schanuel (on G_0): *if E and G are constant (so $v := v_0 \in E_0(\mathbb{C})$), y not torsion, x not constant and $\ell \in F$ arbitrary, then*

$$tr.deg_K K(u, \zeta(u), \wp(u), \ell, e^{\ell} \exp(g(u, v_0))) \geq 3.$$

For instance, if $x \in E_0(K) \setminus E_0(\mathbb{C}) : tr.deg_K(u, \zeta(u), g(u, v_0) - \frac{\eta_1}{\omega_1} u v_0) = 3$.

Assume now that $y \in E(K)$, so G/K. For $s \in G(F)$, set $log_G(s) = U$.

Theorem

(**Exponential Ax = L-W**) [B.-Pillay 2016] *Let* $U \in LG(K)$, *projecting to* $u \in LE(K)$, *such that* $\forall H \neq G, U \notin LH + LG_0(\mathbb{C})$. *Let* $\tilde{U} \in L\tilde{G}(K)$ *be any lift of* U, *and let* $\tilde{s} = exp_{\tilde{G}}(\tilde{U}) \in \tilde{G}$. *Then,*

$$tr.deg.(K^\sharp_{\tilde{G}}(\tilde{s})/K^\sharp_{\tilde{G}}) = \begin{cases} 3 & in\ general,\ except \\ 1 & if\ u \in LE_0(\mathbb{C}). \end{cases}$$

(**Logarithmic Ax**) [B.-Masser-Pillay-Zannier 2016] *Let* $s \in G(K)$, *proj. to* $x \in E(K)$, *such that* $\forall H \neq G, s \notin H + G_0(\mathbb{C})$. *Let* $\tilde{s} \in \tilde{G}(K)$ *be any lift of* s, *and let* $\tilde{U} = ln_{\tilde{G}}(\tilde{s}) \in L\tilde{G}$. *Then,*

$$tr.deg.(K^\sharp_{L\tilde{G}}(\tilde{U})/K^\sharp_{L\tilde{G}}) = \begin{cases} 3 & in\ general,\ except \\ 1 & if\ N\ x \in End(E)y\ (mod.\ E_0(\mathbb{C})),\ except \\ 0 & if\ s\ is\ Ribet\ (mod.\ G_0(\mathbb{C})). \end{cases}$$

In particular, assume that $y \in E(K) \setminus E^\sharp$ where $E^\sharp = E_0(\mathbb{C}) + E_{tors}$, i.e. G/K is not isoconsant nor isotrivial, and that $x \in E(F) \setminus E^\sharp$. Then,

Exponential Ax : if $u \in K$ and $\ell \in K^\times$,

$$tr.deg_K K(\wp(u), \zeta(u), \frac{\sigma(u+v)}{\sigma(u)\sigma(v)} e^{\ell - \zeta(v)u}) = 3.$$

Logarithmic Ax : if $x \in E(K) \setminus E^\sharp$ and $\ell = \ell og(\alpha)$, where $\alpha \in K^\times$,

$$tr.deg_{K(\omega_{1,2},\eta_{1,2})}(u, v, \zeta(u), \zeta(v), g(u,v)) - \ell)$$

is equal to

- 5 , in general, e.g. if E is not constant and x, y are lin. indep. over $\mathbb{Z}$;
- 3 , if $E = E_0$ and x, y are lin. dep. over $End(E_0)$ mod $E_0(\mathbb{C})$,

unless x, y are $End^{as}(E_0)$-related mod $E_0(\mathbb{C})$, in which case $\exists \ell \in log(K^\times)$ such that it is equal to

- 2 , and indeed $g(u,v) - s_2 uv$ then lies in $log(K^\times) := \mathfrak{L}$.

Corollary (e) : *let $x, y \in E(K)$, not both constant if $E = E_0$. Then*

$$g(u,v) - \frac{\eta_1}{\omega_1} uv \in \mathfrak{L} \Rightarrow x \text{ or } y \text{ is torsion.}$$

So, the functional version of the Conjecture holds true (but to no avail...).

Further comments

(a) That is, if (x, η) itself lies in B. In the general case, s_R is defined only up to addition of a root of unity.

(b) (answering a question of B. Zilber) This conjecture would follow from Grothendieck's period conjecture, applied to the 1-motive $[\mathbb{Z} \to G_\eta \times \mathbb{G}_m, 1 \mapsto (s, \alpha)]$ with $\alpha \in \bar{\mathbb{Q}}^\times$.

(c) In fact, $g(u, v) - s_2 uv \in \mathcal{L} \Leftrightarrow \exists N \in \mathbb{N}, f \in End^{as}(E), N\eta = f(x)$, unless x or η is torsion. See Springer LN 1068, p. 19-22, Corollaire 3.

(d) assuming that x and η are $End(E)$-linearly independent modulo $E(\bar{\mathbb{Q}})$.

(e) This corollary also follows from Ayoub's theorem on the functional analogue of Grothendieck's conjecture.