Manin's theorem of the kernel : a remark on a paper of C-L. Chai

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In a current work with A. Pillay [3], we use a result of Ching-Li Chai [5] about Manin's theorem of the kernel. In [5], Chai gave two proofs of his result. His second proof, based on Hodge theory, concerns a special case, say $(*)$, which is sufficient to establish Manin's theorem, and sufficient for [3] as well; I describe it in detail in Section 4 below, and give a dual presentation in Section 5. The first proof of [5] concerns a more general situation, but contains a gap. We here show that Chai's general result is nevertheless valid: this is deduced in Section 3 from Y. André's work [1] on mixed Hodge structures.

As pointed out by C. Simpson, it is possible to present these arguments in a unified way. See Remark 3.2 below for a brief sketch[1].

1 Setting

Let S be a smooth affine curve over $\mathbb{C}$, with field of rational functions $K = \mathbb{C}(S)$. A D-module will here means a vector bundle over S with an (integrable) connection. We denote by $\mathbf{1}$ the D-module $(\mathcal{O}_S, d)$. Let further $\pi : A \to S$ be an abelian scheme over S, and let $H^1_{dR}(A/S)$ be the D-module formed by the de Rham cohomology of A/S, with its Gauss-Manin connection $\nabla_{A/S}$. In his proof [13] of the Mordell conjecture over K, Y. Manin constructs a map

$$\mathcal{M}_K : A(S) \to Ext^1_{D-mod.}(H^1_{dR}(A/S), \mathbf{1}),$$

[1] I thank Y. André, C-L. Chai, C. Simpson and C. Voisin for their comments on this Note, and N. Katz for having shown me Deligne's counterexample [10].

and shows that its kernel is reduced to the divisible hull of the group of constant sections of the constant "part" of A. This aspect of Manin's theorem of the kernel is all right. But he needs to study a more elaborate map, and R. Coleman found a gap in his proof at this point (cf. [6], Remark after Prop. 2.1.2, and [13], middle of p. 214). In [6], Coleman provides a correction, and a full proof of the Mordell conjecture/Manin theorem.

Another way to correct Manin's proof was found by C-L. Chai [5]. For any D-submodule M of $H^1_{dR}(A/S)$, let i^*_M be the canonical map

$$i^*_M : Ext^1_{D-mod.}(H^1_{dR}(A/S), \mathbf{1}) \to Ext^1_{D-mod.}(M, \mathbf{1})$$

given by pull-back. In the special case where

Case $(*)$: $M = M_\Omega$ is the D-submodule of $H^1_{dR}(A/S)$ generated by the space $\Omega^1_{A/S}$ of invariant 1-forms,

we simply denote this map by $i^* = i^*_{M_\Omega}$. Coleman noticed that the whole of Manin's proof is all right if M_Ω fills up $H^1_{dR}(A/S)$, and that in order to correct it in general, it suffices to show that i^* is injective on the image of $\mathcal{M}_K$. To prove this, we may, and from now on, will assume that *the abelian variety* A_K *is geometrically simple*. Chai's result then is the following generalization of the above assertion on M_Ω.

Theorem 1.1. (Chai [5]) *Let* M *be* any *non-zero* D*-submodule of* $H^1_{dR}(A/S)$*. Then,* i^*_M *is injective on the image of* $\mathcal{M}_K$*.*

Let s be a point in $S(\mathbb{C})$ and let

$$H_{A,s} := H^1_B(A^{an}_s, \mathbb{Q})$$

be the $\mathbb{Q}$-vector space formed by the Betti cohomology of the fiber A_s. Via the local system $R^1\pi^{an}_*\mathbb{Q}$, $H_{A,s}$ provides a representation of the fundamental group $\pi_1(S^{an}, s)$. The first proof in [5] relies on the assumption that this representation is irreducible (over $\mathbb{Q}$) if A/S is not isoconstant. Deligne proves this in [8] under a set of hypotheses on the division algebra $\mathbb{Q} \otimes End(A/S)$, but has also shown that it is false in general: for an 8-dimensional counterexample, see [11], p. 338, and his recent letter to Katz [10]. His example also witnesses that M_Ω can be strictly contained in $H^1_{dR}(A/S)$. And Y. André has shown me examples of the latter phenomenon in all even dimensions $g = 2k \geq 4$: take a non constant abelian variety with $\mathbb{Q} \otimes End(A/S) =$ a CM field of degree g, with CM type $(r_1 = s_1 = 1, r_2 = ... = r_k = 2, s_2 = ... = s_k = 0)$.

2 Chai's method in the general case

In spite of this problem, let us now recall Chai's method, since the proof of Section 3 relies on similar arguments (a variation of the "Bashmakov-Ribet" method in the study of ℓ-adic representations).

We must first recall what $\mathcal{M}_K$ is. For the sake of brevity, I'll use the view point of smooth one-motives, as defined in [9], and briefly[2] studied in [4], Facts 2.2.2.1 and 2.2.2.2 :

• Let $\tilde{A}/S$ be the universal vectorial extension of A/S, and let $L\tilde{A}$ be the pull-back by the 0-section of its relative tangent bundle. Thus, $L\tilde{A} := T_{dR}(A) \simeq$ dual of $H^1_{dR}(A/S)$ (cf. [7]) is the de Rham realization of the pure S-one-motive associated to A. Its Betti realization $R_1\pi_*\mathbb{Q}$ is the $\mathbb{Q}_S$-dual of the local system $R^1\pi_*\mathbb{Q}$ (I drop the ^{an} exponents). The adjoint $\nabla^*_{A/S}$ of $\nabla_{A/S}$ provides $T_{dR}(A)$ with a structure of D-module, whose space of horizontal sections is locally generated over $\mathbb{C}_S$ by a $\mathbb{Q}_S$-local system $T_B(A) \simeq R_1\pi_*\mathbb{Q}$ (we will need to specify this isomorphism only for the last method, cf. p.10).

• A section $y \in A(S)$ defines a smooth one-motive $\mathbf{M}_y \in Ext_{S-1-mot.}(\mathbb{Z}, A)$, with no W_{-2} part, and with $W_{-1}(\mathbf{M}_y) = A$. Its de Rham realization $T_{dR}(\mathbf{M}_y)$ defines an element in $Ext_{D-mod.}(\mathbf{1}, T_{dR}(A))$, and the extension

$$\mathcal{M}_K(y) := H^1_{dR}(\mathbf{M}_y) \in Ext^1_{D-mod.}(H^1_{dR}(A/S), \mathbf{1})$$

can simply be described as the dual of $T_{dR}(\mathbf{M}_y)$. In particular, the local system of solutions of $\mathcal{M}_K(y)$ has a $\mathbb{Q}_S$-structure $H^1_B(\mathbf{M}_y)$, dual to the Betti realization $T_B(\mathbf{M}_y) \in Ext^1_{loc.syst}(\mathbb{Q}_S, T_B(A))$ of $\mathbf{M}_y$. The sections of the latter are the various continuous determinations of the logarithms of the division points of al multiples of y.

For other descriptions of $\mathcal{M}_K(y)$, see [6], based on [12], a letter of Katz to Ogus (which I have not seen), and [1], based on [15].

• Fix a point s in S. The fiber $H_{A,y,s}$ of $H^1_B(\mathbf{M}_y)$ at s defines a $\mathbb{Q}$-representation

$$0 \to \mathbb{Q} \to H_{A,y,s} \to H_{A,s} \to 0$$

of $\pi_1(S, s)$. Dually, the fiber $T_{A,y,s}$ of $T_B(\mathbf{M}_y)$ at s (resp. $T_{A,s} \simeq H^*_{A,s}$ of $T_B(A)$) define $\mathbb{Q}$-representations

$$0 \to T_{A,s} \to T_{A,y,s} \to \mathbb{Q} \to 0.$$

[2] perhaps too briefly. But the situation should change at some point, hopefully thanks to [2] and its authors.

Here, $\mathbb{Q}$ is the trivial representation. Since all our differential equations are fuchsian at the missing points of S, these extensions of monodromy representations split if and only if $\mathcal{M}_K(y)$ splits.

Chai's first proof [5] now goes as follows (up to a duality). We consider the following algebraic groups over $\mathbb{Q}$:

- $\tilde{G} \subset GL(T_{A,y,s})$ is the $\mathbb{Q}$-Zariski closure of the image of π_1 acting on $T_{A,y,s}$; this group depends on y;

- $G \subset GL(T_{A,s})$ is the $\mathbb{Q}$-Zariski closure of the image of π_1 acting on $T_{A,s}$; the (connected component G^0 of the) group G is a reductive group ([8]);

- N = kernel of the natural map $\tilde{G} \to G$; the construction below shows that N is abelian, hence acted upon naturally by $\tilde{G}/N = G$.

Fixing a point $\tilde{\lambda} \in T_{A,y,s}$ above $1 \in \mathbb{Q}$, and considering $g\tilde{\lambda} - \tilde{\lambda}$, we obtain a cocycle $\xi_y \in H^1(\tilde{G}, T_{A,s})$, whose restriction to N

$$\xi(y) : N \to T_{A,s}$$

is a G-equivariant injective morphism between vectorial groups over $\mathbb{Q}$. So, N identifies with a $\mathbb{Q}[G]$-submodule of $T_{A,s}$. Since G is reductive, $N = 0$ if and only if the above representations splits (indeed, $T_{A,y,s}$ becomes a representation of G if $N = 0$), i.e. by fuchsianity if and only if $\mathcal{M}_K(y) = 0$.

Let now M be a non-zero D-submodule of $H^1_{dR}(A/S)$. and assume that $i^*_M(\mathcal{M}_K(y)) = 0$. Equivalently, let M' be a strict D-submodule of $T_{dR}(A)$, and assume that the quotient $T_{dR}(\mathbf{M}_y)/M'$ splits as a D-module extension of $\mathbf{1}$ by $T_{dR}(A)/M'$. Since the $\mathbb{C}_S$-local system $T_{M'}$ of horizontal vectors of M' need not be generated by its intersection with the $\mathbb{Q}_S$-structure $T_B(A)$, we must now extend the scalar to $\mathbb{C}$. We do so and consider the projection of N to $(T_{A,s} \otimes \mathbb{C})/(T_{M'})_s$. Since $T_{dR}(\mathbf{M}_y)/M'$ splits, one easily checks that this projection vanishes. So, $N \otimes \mathbb{C} \subset (T_{M'})_s$ does not fill up $T_{A,s} \otimes \mathbb{C}$, and N must be a strict $\mathbb{Q}[G]$-submodule of $T_{A,s}$.

If $T_{A,s}$ is an irreducible $\mathbb{Q}[G]-$module (equivalently, if $H_{A,s}$ is an irreducible $\mathbb{Q}$-representation of $\pi_1(S,s)$), this implies that $N = 0$, hence $\mathcal{M}_K(y) = 0$, as was to be shown.

Remark 2.1.- We can summarize the method as follows. The semi-simplicity of $T_{A,s}$ allows us to speak of the smallest π_1-submodule $\mathcal{N}$ of $T_{A,s}$ such that the quotient $T_{A,y,s}/\mathcal{N}$ is a trivial extension of $\mathbb{Q}$ by $T_{A,s}/\mathcal{N}$ (notice that this

$\mathcal{N}$ is automatically defined over $\mathbb{Q}$). We have proved that $N = \mathcal{N}$ and the question reduces to showing that $\mathcal{N}$ is either $\{0\}$ or the full $T_{A,s}$.

Remark 2.2.- Dually, we can consider the largest π_1-submodule $\mathcal{P}$ of $H_{A,s}$ such that the extension $H_{A,s,y}$ of $H_{A,s}$ by $\mathbb{Q}$ splits over $\mathcal{P}$ (again because of semi-simplicity, and again defined over $\mathbb{Q}$). This $\mathcal{P}$ is the orthogonal of $\mathcal{N}$, and the question reduces to showing that $\mathcal{P}$ is either the full $H_{A,s}$ or $\{0\}$.

3 André's normality theorem

This concerns the monodromy group of smooth one-motives over S. We use the notation $A, y, \mathbf{M}_y$, ... of the previous paragraph, and for any $s \in S$, we denote by $MT_{A,s} \subset GL(T_{A,s})$ (resp. $MT_{A,y,s} \subset GL(T_{A,y,s})$) the Mumford-Tate group of the Hodge structure (resp. mixed HS) attached to A (resp. $\mathbf{M}_y$). These are connected algebraic groups over $\mathbb{Q}$. The following facts will be crucial.

• ([1], Lemma 4) : there is a meager subset of S whose complement S_0 is pathwise connected, and such that $MT_{A,y,s}$ (hence $MT_{A,s}$) is locally constant over S_0.

• ([1], Theorem 1) Let $\tilde{G}_s^0$ be the connected component of the group called $\tilde{G}$ in the previous paragraph (which was the $\mathbb{Q}$-Zariski closure of the monodromy group of $T_B(\mathbf{M}_y)$, based at s). Then, *for any $s \in S_0$, $\tilde{G}_s^0$ is a* **normal** *subgroup of $MT_{A,y,s}$.*

Actually, [1] further shows that $\tilde{G}_s^0$ is contained in the derived group of $MT_{A,y,s}$, but we will not need this sharpening. In the (more classical) analogous statements at the level G and $MT_{A,s}$, it is precisely this sharpening which is responsible for Deligne's counterexamples to irreducibility, as was pointed out to me by Chai. For another view-point, see [16].

After extension to a finite cover of S, we may assume that G, hence $\tilde{G}$ are already connected. We make this assumption from now on, and proceed to prove Chai's *complete* theorem along the lines of Proposition 1 of [1].

We fix a base point s in S_0, yielding the algebraic groups

- $\tilde{G}$ as above, normal in $\tilde{MT} := MT_{A,y,s}$;
- G as above, normal in $MT := MT_{A,s}$;

- N as above, contained in the kernel

$$NT = \{g \in \tilde{MT}, g(T_{A,y,s}) \subset W_{-1}(T_{A,y,s}) = T_{A,s}\}$$

of the natural map $\tilde{MT} \to MT$. Fixing a point $\tilde{\lambda} \in T_{A,y,s}$ above $1 \in \mathbb{Q}$, and considering $g\tilde{\lambda} - \tilde{\lambda}$, we obtain a cocycle $\Xi_y \in H^1(\tilde{MT}, T_{A,s})$, whose restriction $\Xi(y) : NT \to T_{A,s}$ to NT shows that NT is abelian (and is a $\mathbb{Q}[MT]$-submodule of $T_{A,s}$). Notice for later use that the restriction of $\Xi_y, \Xi(y)$, to $\tilde{G}, N$, coincide with the maps $\xi_y, \xi(y)$, of the previous paragraph.

By André's theorem, $\tilde{G}$ is normal in $\tilde{MT}$. We will now show that N too is normal in $\tilde{MT}$. Extending the scalars to $\mathbb{C}$, it suffices to show that $N_{\mathbb{C}}$ is normal in $\tilde{MT}_{\mathbb{C}}$. Since $G_{\mathbb{C}}$ is reductive and $N_{\mathbb{C}}$ is abelian, $N_{\mathbb{C}}$ is the unipotent radical of $\tilde{G}_{\mathbb{C}}$, i.e. the (unique) maximal connected unipotent normal subgroup of $\tilde{G}_{\mathbb{C}}$. Therefore, $N_{\mathbb{C}}$ is fixed under any automorphism of $G_{\mathbb{C}}$, and in particular, under all outer automorphisms $Int(g), g \in \tilde{MT}(\mathbb{C})$ of $\tilde{G}_{\mathbb{C}}$ that the normality of $\tilde{G}$ in $\tilde{MT}$ provides. So, $N_{\mathbb{C}}$ is indeed normal in $\tilde{MT}_{\mathbb{C}}$. And since the abelian group NT acts trivially on its subgroup N, the action of $\tilde{MT}$ on N by conjugation induces an action of $\tilde{MT}/NT = MT$.

We now see that the $\mathbb{Q}$-morphism

$$\xi(y) = (\Xi_y)_{|N} : N \to T_{A,s}.$$

is equivariant not only under G, but also under the full action of MT. So, N identifies with a MT-submodule of $T_{A,s}$. Now, $T_{A,s}$ **is** irreducible as a $\mathbb{Q}[MT]$-module, since our choice of s forces $End_{MT}(T_{A,s}) = End(A_s) = End(A/S)$, and we conclude that either $N = 0$ (implying $\mathcal{M}_K(y) = 0$ as before), or that $N = T_{A,s}$. As we already saw, the latter case prevents the existence of *any* non-zero D-submodule M such that $i_M^*(\mathcal{M}_K(y)) = 0$, unless $\mathcal{M}_K(y) = 0$.

Remark 3.1.- In a connected algebraic group G over a perfect fied k, there is a unique unipotent radical $R_u(G)$, defined over k. Checking the normality of $N = R_u(\tilde{G})$ in $\tilde{MT}$ therefore did not require extending the scalars to $\mathbb{C}$.

Remark 3.2.- As noticed by C. Simpson [14], one can hide the role of normality in this proof by working directly on the modules themselves. In the notations of Remark 2.2, the question reduces to showing that $\mathcal{P}$ *is the fiber at* s *of a sub-VHS of the variation of pure Hodge structures* $R^1\pi_*\mathbb{Q}$. As in [1], this follows from the theorem of the fixed part of [15], but no explicit appeal to Mumford-Tate groups is required. This approach provides a proof of the theorem closer in spirit to the "second proof" of Chai, which we now describe.

4 Chai's proof in Case $(*)$

From now on, we assume that $M = M_\Omega$ is the D-submodule of $H^1_{dR}(A/S)$ generated by $\Omega^1_{A/S}$. In the last paragraph of [5], Chai gives the following Hodge theoretic argument to check his result in this special case.

Fixing a point s in S, we recall the notations $H^1_B(A_s, \mathbb{Q}) := H_{A,s}, H_{A,y,s}$ of §2, and here denote by G the $\mathbb{Q}$-Zariski closure of the image of $\pi_1(S, s)$ acting on $H_{A,s}$. This is the same algebraic group as before, but we are looking at it via the contragredient of its initial representation $T_{A,s}$. Actually, in this paragraph, only the group $G_\mathbb{R}$ deduced from G by extension of scalars to $\mathbb{R}$ will play a role. It has a real representation $H_{A,s} \otimes \mathbb{R}$, which we extend by $\mathbb{C}$-linearity to the complex representation $H_{A,y} \otimes \mathbb{C}$. We further denote by $H_M \subset R^1\pi_*\mathbb{C}$ the $\mathbb{C}_S$-local system of horizontal sections of the D-module M. Its fiber $H_{M,s} \subset H_{A,s} \otimes \mathbb{C}$ at s is a complex representation of $G_\mathbb{R}$. In other words, the injection $i_s : H_{M,s} \hookrightarrow H_{A,s} \otimes \mathbb{C}$ is a $\mathbb{G}_\mathbb{R}$-morphism.

The local system $R^1\pi_*\mathbb{R}$ is a variation of real Hodge structures, with respect to which we can consider the complex conjugate $\overline{H_M}$ of H_M in $R^1\pi_*\mathbb{C}$. Then, $\overline{H_M}$ is again a $\mathbb{C}_S$-local system, and its fiber $\overline{H_{M,s}} \subset H_{A,s} \otimes \mathbb{C}$ provide another complex subrepresentation of $G_\mathbb{R}$, whose underlying $\mathbb{C}$-vector space is the complex conjugate of $H_{M,s}$ in $H_{A,s} \otimes \mathbb{C}$ with respect to $H_{A,s} \otimes \mathbb{R}$; in other words, the $\mathbb{C}$-linear injection $\bar{i}_s : \overline{H_{M,s}} \hookrightarrow H_{A,s} \otimes \mathbb{C}$ is a $\mathbb{G}_\mathbb{R}$-morphism. Denoting by c the antilinear involution of $H_{A,s} \otimes \mathbb{C}$ given by complex conjugation with respect to $H_{A,s} \otimes \mathbb{R}$, we have $\bar{i}_s = c \circ i_s \circ c$.

Since M contains $\Omega^1_{A/S}$, $H_{M,s}$ contains $H^{1,0}(A_s, \mathbb{C}) = F^1(H_{A,s} \otimes \mathbb{C})$, hence as $\mathbb{C}$-vector spaces:

$$H_{M,s} + \overline{H_{M,s}} = H_{A,s} \otimes \mathbb{C}.$$

This is compatible with the action of $G_\mathbb{R}$, since both factors on the left side are subrepresentations of the right side. Therefore, the complex representation $H_{A,s} \otimes \mathbb{C}$ is a quotient of $H_{M,s} \oplus \overline{H_{M,s}}$. Since $G_\mathbb{R}$ is a reductive group, we derive a $\mathbb{C}[G_\mathbb{R}]$-section $j_s : H_{A,s} \otimes \mathbb{C} \hookrightarrow H_{M,s} \oplus \overline{H_{M,s}}$ of the addition map.

We now consider the S-one-motive $\mathbf{M}_y$, recall the notation $H_{A,y,s}$ of §2, denote here by $\tilde{G}$ the $\mathbb{Q}$-Zariski closure of the image of $\pi_1(S, s)$ acting on $H_{A,y,s}$ (same algebraic group as in §2, but viewed via the representation contragredient to $T_{A,y,s}$), and consider the extension of real representations $0 \to \mathbb{R} \to H_{A,y,s} \otimes \mathbb{R} \to H_{A,s} \otimes \mathbb{R} \to 0$ of $\tilde{G}_\mathbb{R}$, and its complexification

$$0 \to \mathbb{C} \to H_{A,y,s} \otimes \mathbb{C} \to H_{A,s} \otimes \mathbb{C} \to 0,$$

for which we denote by C complex conjugation with respect to $H_{A,y,s} \otimes \mathbb{R}$. The hypothesis $i^*(\mathcal{M}_K(y)) = 0$ forces a splitting of the pull-back

$$0 \to \mathbb{C} \to i_s^*(H_{A,y,s} \otimes \mathbb{C}) \to H_{M,s} \to 0$$

of $H_{A,y,s} \otimes \mathbb{C}$ under $i_s : H_{M,s} \hookrightarrow H_{A,s} \otimes \mathbb{C}$, and we denote by

$$\sigma_s : H_{M,s} \to i_s^*(H_{A,y,s} \otimes \mathbb{C}) \subset H_{A,y,s} \otimes \mathbb{C}$$

a $\mathbb{C}$-linear $\tilde{G}_{\mathbb{R}}$-section. Similarly, we consider the pull-back

$$0 \to \mathbb{C} \to \bar{i}_s^*(H_{A,y,s} \otimes \mathbb{C}) \to \overline{H_{M,s}} \to 0$$

of $H_{A,y,s} \otimes \mathbb{C}$ under $\bar{i}_s$, and claim that this extension splits. Indeed,

$$\overline{\sigma}_s := C \circ \sigma_s \circ c : \overline{H_{M,s}} \to H_{A,y,s} \otimes \mathbb{C}.$$

is a $\mathbb{C}$-linear section of $\bar{i}_s^*(H_{A,y,s})$, since the action of $G_{\mathbb{R}}$ (resp. $\tilde{G}_{\mathbb{R}}$) commutes with c (resp. C).

Finally, recall the section $j_s : H_{A,s} \otimes \mathbb{C} \to H_{M,s} \oplus \overline{H_{M,s}}$ of the addition map, and consider the $\mathbb{C}[G_{\mathbb{R}}]$-morphism

$$\phi : H_{A,s} \otimes \mathbb{C} \to H_{A,y,s} \otimes \mathbb{C} : \lambda \mapsto \phi(\lambda) := (\sigma_s + \overline{\sigma}_s)(j_s(\lambda)).$$

This is a section of the extension $H_{A,y,s} \otimes \mathbb{C}$, whose vanishing implies, by fuchsianity, the vanishing of $\mathcal{M}_K(y)$, as required.

Remark 4.1. - (cf. [5]) Simpson has pointed out that the $\mathbb{C}_S$-local system $\overline{H}_M$ underlies a variation of complex Hodge structures, complex conjugate to that of H_M.

Remark 4.2. - The rational structure $H_{A,s}$ plays no role in this proof, which relies only on the real Hodge structure of $H_{A,s} \otimes \mathbb{R}$ and on the semisimplicity of the complex representation $H_{A,s} \otimes \mathbb{C}$. Furthermore, the hypothesis $\Omega^1 \subset M$ can be weakened, since we merely used its corollary $H_{M,s} + \overline{H_{M,s}} = H_{A,s} \otimes \mathbb{C}$. Assuming $rk(M) > dim(A/S)$, however, would not suffice (consider an A/S of RM type).

5 Same proof, viewed dually

In this paragraph, we again assume that $M = M_\Omega$, and translate the previous proof, viewed dually, into a statement on periods. So, we go back to the covariant view-point $T_{A,s}, T_{A,y,s}$ used in §§ 2 and 3, and in particular, to the de Rham realization $T_{dR}(A) = L\tilde{A}$ given by the relative Lie algebra of the universal extension $\tilde{A}/S$ of A/S. Thus, the S-one-motive $\mathbf{M}_y$ gives rise to an extension

$$T_{dR}(\mathbf{M}_y) := \mathcal{M}'_K(y) \in Ext^1_{D-mod.}(\mathbf{1}, L\tilde{A}),$$

dual to $\mathcal{M}_K(y)$, and as mentioned at the end of §2, Chai's general theorem reads as follows: let M' be a strict D–submodule of $L\tilde{A}$; if the pushout

$$p_*(\mathcal{M}'_K(y)) = \mathcal{M}'_K(y)/M' \in Ext^1_{D-mod.}(\mathbf{1}, L\tilde{A}/M')$$

of $\mathcal{M}'_K(y)$ by the projection $p : L\tilde{A} \to L\tilde{A}/M'$ splits, then $\mathcal{M}'_K(y)$ too splits. We now prove this under the assumption dual to $M = M_\Omega$.

To make the translation, recall that $\tilde{A}/S$ is an extension of A/S by a vectorial S-group $W_{A/S}$, whose associated vector bundle is canonically dual to $R^1\pi_*\mathcal{O}_{A/S}$. The dual of the condition $M = M_\Omega$ then becomes: *let M' be the maximal D-submodule of $L\tilde{A}$ contained in $W_{A/S}$.* We assume this from now on and proceed to prove that $p_*(\mathcal{M}'_K(y)) = 0 \Rightarrow \mathcal{M}'_K(y) = 0$.

Actually, it suffices to repeat almost all of §2, p. 4, where we defined the group N and deduced from the reductivity of the group G that

$$N = \{0\} \Leftrightarrow \mathcal{M}_K(y) = 0,$$

or equivalently, by duality, $\mathcal{M}'_K(y) = 0$. Recall that N is naturally embedded, via $\xi(y)$, in the $\mathbb{Q}$-structure $T_{A,s}$. With the specificity of our M' now in mind, the last but one paragraph reads as follows.

Let $T_{M'} \subset T_B(A) \otimes \mathbb{C}_S$ be the local system of horizontal sections of M', and let $(T_{M'})_s \subset T_{A,s} \otimes \mathbb{C}$ be its fiber above s. Since the extension $p_*(T_{dR}(\mathbf{M}_y))$ of $\mathbf{1}$ by $T_{dR}(A)/M'$ splits, the image of N under the projection to $(T_{A,s} \otimes \mathbb{C})/(T_{M'})_s$ vanishes, and $N \otimes \mathbb{C} \subset (T_{M'})_s$. Since $M' \subset W_{A/S}$, we therefore get

$$N \subset T_{A,s} \cap (W_{A/S})_s \subset T_{A,s} \otimes \mathbb{C} = (L\tilde{A})_s.$$

We will now show that $T_{A,s} \cap (W_{A/S})_s = \{0\}$, hence $N = 0$ and $\mathcal{M}'_K(y) = 0$.

In order to compute this intersection, we must identify the subgroup $T_{A,s}$ of $(L\tilde{A})_s$, or more precisely, describe the isomorphism $\iota_{\mathbb{Q}} : R_1\pi_*\mathbb{Q} \simeq T_B(A) \subset T_{dR}(A)$ mentioned anonymously on p. 3. The Betti realization $R_1\pi_*\mathbb{Q}$ of A/S is generated over $\mathbb{Q}_S$ by the kernel of the exact sequence of S^{an}-sheaves given by the exponential map :

$$0 \to R_1\pi_*\mathbb{Z} \to LA^{an} \to A^{an} \to 0,$$

where LA denotes the relative Lie algebra of the abelian scheme A/S. It is a variation of $\mathbb{Q}$-Hodge structures of weight -1, whose Hodge filtration is given by the kernel F^0_B of the natural map $R_1\pi_*\mathbb{Z} \otimes \mathcal{O}_{S^{an}} \to LA^{an}$. The de Rham realization $T_{dR}(A) = L\tilde{A}$ of A/S lies in the exact sequence

$$0 \to W_{A/S} \to L\tilde{A} \to LA \to 0,$$

whose Hodge filtration F^0_{dR} is given by $W_{A/S}$. The canonical isomorphism

$$\iota : R_1\pi_*\mathbb{Z} \otimes \mathcal{O}_{S^{an}} \simeq L\tilde{A}^{an}$$

described at the level of fibers in [9], 10.1.8, respects these Hodge filtrations. We set $T_B(A) := \iota(R_1\pi_*\mathbb{Q} \otimes 1) \subset L\tilde{A}^{an}$, and this defines $\iota_{\mathbb{Q}}$. By [9], 10.1.9, $T_B(A)_{\mathbb{Z}} = \iota(R_1\pi_*\mathbb{Z} \otimes 1)$ is the kernel of the exponential map on $\tilde{A}$:

$$0 \to T_B(A)_{\mathbb{Z}} \to L\tilde{A}^{an} \to \tilde{A}^{an} \to 0.$$

So, $T_B(A) \subset T_{dR}(A)$ is indeed horizontal for $\nabla^*_{A/S}$, as claimed in §2.

Now, $R_1\pi_*\mathbb{Q}$ injects in LA^{an}, while $\iota(F^0_B) = F^0_{dR}$. So, $T_B(A) \cap W_{A/S} = \iota(R_1\pi_*\mathbb{Q})) \cap \iota(F^0_B) = \{0\}$, and we do have, on the fiber above any $s \in S$:

$$T_{A,s} \cap (W_{A/S})_s = \{0\}.$$

Remark 5.1.- One can summarize the argument by saying that the periods of $\tilde{A}$ project bijectively onto the periods of A.

Remark 5.2.- In the final step, one can replace $R_1\pi_*\mathbb{Q}$ by $R_1\pi_*\mathbb{R}$. The argument then becomes the exact dual of Chai's proof from §4.

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