DEVOIR MAISON NO. 2

Let k be an algebraically closed field.

Exercise 1. Let $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ be the multiplicative group of k and let it act on the affine plane $\mathbb{A}^2_k = \operatorname{Spec} k[x_1, x_2]$ by

$$t(x_1, x_2) = (tx_1, tx_2).$$

Let $f(x, y) := x_1 x_2$ and for $i = 1, 2$ let $U_i = \{(x_1, x_2) \in \mathbb{A}^2_k : x_i \neq 0\}$. Show the following facts:

(1) The quotient $\mathbb{A}^2/k^\times$ endowed with the quotient topology is not the topological space underlying any algebraic variety.
(2) The map f induces a bijection between the orbits of $\mathbb{G}_m$ contained in $U_1 \cap U_2$ and $k^\times$.

For $i = 1, 2$ let $Y_i = \mathbb{A}^1_k$ and $V_i = \mathbb{A}^1 \smallsetminus \{0\}$. Let Y be the "glueing" of Y_1 and Y_2 along the identity $\mathrm{id}\colon V_1 \to V_2$. The variety obtained in this way is the affine line with doubled origin.

(3) There exists a unique morphism of algebraic varieties $\tilde{f}\colon U_1 \cup U_2 \to Y$ such that $\tilde{f}_{|U_i} = f_{|U_i}$ for $i = 1, 2$.
(4) The morphism $\tilde{f}$ is $\mathbb{G}_m$-invariant and induces a bijection between $\mathbb{G}_m$-orbits in $U_1 \cup U_2$ and points of Y.
(5) A polynomial $f \in k[x_1, x_2]$ is invariant under the action of $\mathbb{G}_m$ if and only if $f(x_1, x_2) = a x_1^n x_2^n$ for some $a \in k$ and $n \in \mathbb{N}$.
(6) The map $f\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$ has the following property: for all affine variety Z and $\mathbb{G}_m$-invariant morphism $g\colon \mathbb{A}^2 \to Z$ there exists a unique morphism $\tilde{g}\colon \mathbb{A}^1_k \to Z$ such that $\tilde{g} \circ f = f$.[1]
(7) The map $\tilde{f}$ has the same property: for all affine variety Z and $\mathbb{G}_m$-invariant morphism $g\colon U_1 \cap U_2 \to Z$ there exists a unique morphism $\tilde{g}\colon \mathbb{A}^1_k \to Z$ such that $\tilde{g} \circ \tilde{f} = \tilde{f}$.
(Hint: show that such a g extends to a $\mathbb{G}_m$-invariant morphism $\mathbb{A}^2_k \to Z$.)

Exercise 2. For points $x = (x_0, x_1)$, $y = (y_0, y_1)$ in $\mathbb{A}^2_k$ let

$$x \wedge y := \det \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix}.$$

Let $\hat{X} := (\mathbb{A}^2_k)^4$ and, for $x = (x_1, \dots, x_4) \in \hat{X}$,

$$s_0(x) = x_1 \wedge x_2 \cdot x_3 \wedge x_4,$$
$$s_1(x) = x_1 \wedge x_3 \cdot x_2 \wedge x_4,$$
$$s_2(x) = x_1 \wedge x_4 \cdot x_3 \wedge x_2.$$

Define $s\colon \hat{X} \to \mathbb{A}^3_k$ as $s(x) = (s_0(x), s_1(x), s_2(x))$. Show the following facts:

(1) Let $x_1, \dots, x_4 \in \mathbb{A}^2_k$ be non-zero. Then $s(x) = 0$ if and only if there are three indices i, j, k such that $x_i = x_j = x_k$, that is, one of the points is repeated three times.
(2) For $\lambda_1, \dots, \lambda_4 \in k^\times$, we have $s(\lambda_1 x_1, \dots, \lambda_4 x_4) = \lambda_1 \cdots \lambda_4 s(x)$.

[1] Actually, this is true also when one discards the hypothesis of Z being affine. Can you prove this? For this reason the couple $(\mathbb{A}^1_k, f)$ is said to be categorical quotient of $\mathbb{A}^2_k$ by $\mathbb{G}_m$ in the category of algebraic varieties.

(3) Let $\mathrm{SL}_{2,k}$ act on $\hat{X}$ by

$$g(x_1,\ldots,x_4) = (gx_1,\ldots,gx_4).$$

Show that the map s is $\mathrm{SL}_{2,k}$-invariant.

Let $X := (\mathbb{P}^1_k)^4$ and U the open subset of X where none of the points is repeated more than twice.[2]

(4) The map s induces a well-defined $\mathrm{SL}_{2,k}$-invariant map $\bar{s}\colon U \to \mathbb{P}^2_k$.
(5) The image of $\bar{s}$ is a line $L \subset \mathbb{P}^2_k$. Find its equation.
(6) Show that the map $r\colon L \to \mathbb{P}^1_k$, $r([y_0 : y_1 : y_2]) = [y_0 : y_1]$ is well defined. The map $p := r \circ \bar{s}$ is the cross-ratio of the points $x_1,\ldots,x_4$.
(7) Let V the open subset of X where none of the points is repeated twice. Compute $W := p(V)$.
(8) For $z \in \mathbb{P}^1_k \setminus W$ compute $p^{-1}(z)$. What property do the 4-tuples belonging to these fibers have in common?
(9) The map p induces a bijection between $\mathrm{SL}_{2,k}$-orbits in V and points in W.
(10) $(*)$ Show that for every variety Z and every $\mathrm{SL}_{2,k}$-invariant morphism $g\colon U \to Z$ there exists a unique map $\tilde{g}\colon \mathbb{P}^1_k \to Z$ such that $g = \tilde{g} \circ p$.

[2]That is, given $x = (x_1,\ldots,x_4) \in (\mathbb{P}^1_k)^4$ and three indices i,j,k the set $\{x_i,x_j,x_k\}$ has cardinality at least 2.