FEUILLE DE TD NO. 2

We work over an algebraically closed field k.

Exercise 1. (a) Show that the addition defines a regular, hence continuous map from $\mathbb{A}^2$ to $\mathbb{A}^1$. What if we consider the addition as an application from the topological space $k \times k$ with the product of the Zariski topologies to k with the Zariski topology: is it continuous?

(b) Same question for the multiplication seen as a map $\operatorname{Spec} k[x, x^{-1}, y, y^{-1}] \to \operatorname{Spec} k[t, t^{-1}]$, or as a map $k^* \times k^* \to k^*$ with the product topology on the domain.

(c) Find an example of two continuous functions from $\mathbb{A}^1$ to $\mathbb{A}^1$ whose sum is not continuous (for the Zariski topology). Hence $\mathcal{C}_X$, for X an algebraic variety over k, is not a sheaf of k-algebras, and not even an abelian sheaf![1] This means that being continuous for the Zariski topology is not a very interesting notion. We really care about regular functions and morphisms.

Exercise 2. Let X be a topological space, and let pt be a point (the terminal object in topological spaces).

(a) Why can we identify $\mathbf{Ab}_{\mathrm{pt}}$ with $\mathbf{Ab}$?

(b) Let $x \in X$ and let i_x be the fonction from pt to X with image $\{x\}$. With the identification of the previous question, how can we interpret $i_x^{-1}\mathcal{F}$, for $\mathcal{F} \in \mathbf{Ab}_X$?

(c) Prove that if $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then

$$(g \circ f)_* = g_* \circ f_*$$

and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

(d) Deduce that if $f : X \to Y$ is continuous, $\mathcal{G} \in \mathbf{Ab}_Y$, and $x \in X$, then $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$.

Exercise 3 ([Per95, Exercices III.A.2 et III.A.4])**.** Let X be an affine algebraic variety. We assume that $\mathcal{O}(X)$ is a UFD (a Unique Factorization Domain, or anneau factoriel). (For example, this is the case if $X = \mathbb{A}^n$.)

(a) Let $f_1, \ldots, f_m \in \mathcal{O}(X)$ be nonzero elements and let h be their gcd. Show that we have $X_{f_1} \cup \cdots \cup X_{f_m} \subset X_h$ and that the natural restriction morphism:

$$\rho : \Gamma(X_h, \mathcal{O}_X) \to \Gamma\left(\bigcup_{i=1}^{m} X_{f_i}, \mathcal{O}_X\right)$$

is an isomorphism.

(b) Deduce that if U is an open subvariety of X which is not contained in any strict principal open subset $X_f \subsetneq X$, then $\Gamma(U, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$.

Example: $X = \mathbb{A}^2$, $U = \mathbb{A}^2 - \{(0,0)\}$; more generally, $\Gamma(U, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$ as soon as $X - U$ is of codimension ≥ 2 in X.

(c) Conclude that $\mathbb{A}^2 - \{(0,0)\}$ is not an affine variety.

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[1] I thank Marina Masselot for asking many questions. So in the course, one should replace the sheaf of continuous functions by the sheaf of *all* functions with values in k. This is a sheaf in k-algebras!

Exercise 4 ([Per95, Exercice III.A.5])**.** Let $(x_n)_{n\geq 1}$ be a sequence of distinct points in $\mathbb{A}^1$. Show that the algebraic varieties $\mathbb{A}^1 - \{x_1, \dots, x_n\}$ (for $n \geq 0$) are pairwise non-isomorphic. (Consider the group of invertible elements in their function algebras.)

Exercise 5 ([Per95, Exercice III.A.3])**.** Let $Q = V(XY - ZT) \subset \mathbb{A}^4 = \operatorname{Spec} k[X, Y, Z, T]$. We consider the principal open subsets Q_Y and Q_Z and their union $U = Q_Y \cup Q_Z$.

(a) Show that the function $f : U \to \mathbb{A}^1$ defined by

$$f(x, y, z, t) = \begin{cases} x/z & \text{if } (x, y, z, t) \in Q_Z \\ t/y & \text{if } (x, y, z, t) \in Q_Y \end{cases}$$

is an element of $\Gamma(U, \mathcal{O}_Q)$.

(b*) Show that f is not the restriction to U of a quotient G/H with G, H in $k[X, Y, Z, T]$ and $H(P) \neq 0$ for all $P \in U$.

Exercise 6. (a) Let $\mathcal{C}$ be a category admitting finite products: in particular, it has a terminal object (empty product). What meaning could we give to the notion of 'group object' in $\mathcal{C}$?[2] This defines for you the notion of topological group, Lie group, group scheme, etc (take for $\mathcal{C}$ the category of topological spaces, differential manifolds, schemes…). A group in **Set** is just a group.

(b) What is a group object in the category of groups? (That one is fun!)

(c) Describe the notion of group object G in Aff_k in terms of its algebra $\mathcal{O}(G)$. You should reinvent the notion of Hopf algebra!

(d) Describe this structure in the case of the additive group $(k, +)$ denoted by $\mathbb{G}_a$ and in the case of the multiplicative group $(k^*, \times)$ denoted by $\mathbb{G}_m$. Generalize the last example to GL_n.

References

[Per95] Daniel Perrin, *Géométrie algébrique*, Savoirs Actuels, InterEditions, Paris ; CNRS Éditions, Paris, 1995.

[2] You will be able to think about this exercise when we see products in Var_k. It is deliberately open-ended, you may check 'group object' on Wikipedia before going further!