FEUILLE DE TD NO. 3

PRODUCTS, PROJECTIVE VARIETIES, IRREDUCIBLE COMPONENTS

Let k be an algebraically closed field.

Exercise 1. Let V, W be finite dimensional k-vector spaces, $d, r \geq 0$ be non-negative integers with $r \leq \dim V$. The Grassmannian $\mathrm{Gr}_r(V)$ is the set of r-dimensional vector subspaces of V. Consider the following maps:

- The *Veronese embedding*:
$$\begin{array}{ccc} \mathbb{P}(V) & \longrightarrow & \mathbb{P}(\mathrm{Sym}^d V) \\ [v] & \longmapsto & [v^d] . \end{array}$$
- The *Plücker embedding*:
$$\begin{array}{ccc} \mathrm{Gr}_r(V) & \longrightarrow & \mathbb{P}(\bigwedge^d V) \\ T & \longmapsto & \bigwedge^d T. \end{array}$$
- The *Segre embedding*:
$$\begin{array}{ccc} \mathbb{P}(V) \times_k \mathbb{P}(W) & \longrightarrow & \mathbb{P}(V \otimes_k W) \\ [v], [w] & \longmapsto & [v \otimes w] . \end{array}$$

With these definitions:

(1) Write down in coordinates these maps.
(2) Show that Veronese embedding and the Segre embedding are closed embeddings.
(3) $(*)$ The Plücker embedding gives a bijection of $\mathrm{Gr}_r(d)$ with a closed variety of $\mathbb{P}(\bigwedge^d V)$. Write down equations.[1]
(4) Compute the dimension of $\mathrm{Gr}_r(V)$.
(5) For $r = 2$ and $\dim_k V = 4$ compute the equation of $\mathrm{Gr}_2(V)$ in $\mathbb{P}(\bigwedge^2 V)$: it is a smooth quadric, called *Klein's quadric.*
(6) Consider the triple Segre embedding of $\mathbb{P}^1_k$:
$$\begin{array}{ccc} \mathbb{P}^1_k & \longrightarrow & \mathbb{P}^3_k \\ [s : t] & \longmapsto & [s^3 : s^2t : st^2 : t^3] . \end{array}$$
Give a complete set of equations of the image C. Deduce that C is *not* a complete intersection, that is, one needs more equations than expected to define it. The curve C is called the *twisted cubic.*

Exercise 2. Consider the closed subvariety C of $\mathbb{P}^2_k \times \mathbb{P}^1_k$ given by the equation
$$\lambda_0(x_0x_1x_2 + x_0^3) + \lambda_1(x_1^3 + x_2^3) = 0,$$
where $x = [x_0 : x_1 : x_2] \in \mathbb{P}^2_k$ and $\lambda = [\lambda_0 : \lambda_1] \in \mathbb{P}^1_k$. For $\lambda \in \mathbb{P}^1_k$ set
$$C_\lambda := C \times_{\mathbb{P}^1} \{\lambda\},$$
where the morphism $C \to \mathbb{P}^1$ is the projection onto the first factor $(x, \lambda) \mapsto \lambda$. Show the following facts:

(1) If $\lambda \neq [1 : 0], [0 : 1]$, then C_λ is irreducible.

[1] An intrinsic structure of algebraic variety can be defined on $\mathrm{Gr}_r(V)$ in such a way that the Plücker embedding is a closed embedding.

(2) The curve $C_{[1:0]}$ has two irreducible components.
(3) If $\mathrm{char}(k) \neq 3$ then $C_{[0:1]}$ has two irreducible components.
(4) If $\mathrm{char}(k) = 3$ then $C_{[0:1]}$ is irreducible but not reduced.

Exercise 3 ([Har77, Ex. 1.3])**.** Consider the algebraic variety $V(x^2 - yz, xz - x)$ in $\mathbb{A}^3_k$. Compute its irreducible components and describe them.

Exercise 4. Consider the map $\nu \colon \mathbb{A}^2_k \to \mathbb{A}^2_k$,

$$\nu(x,t) := (x, tx).$$

Consider the curve $V(y^2 - x^3)$.

(1) Compute the irreducible components of $\nu^{-1}(C) := \mathbb{A}^2_k \times_{\mathbb{A}^2_k} C$ together with their multiplicities.

Let $f \in k[x,y]$ be a polynomial and $X = V(f)$. If $f = \sum_{i,j} a_{ij} x^i y^j$ with $a_{ij} \in k$, the multiplicity of f at $(0,0)$ is

$$m = \min\{i + j : a_{ij} \neq 0\}.$$

(2) Show that the multiplicity of the irreducible component $V(x)$ of in $\nu^{-1}(X)$ is m.

Exercise 5. Let $n \geq 2$ and the affine space $\mathrm{M}_n(k)$ of $n \times n$ matrices. For $A \in \mathrm{M}_n(k)$ let

$$P_A(t) = \det(t \cdot \mathrm{id}_n - A) = t^n - \sigma_1(A) t^{n-1} + \cdots + (-1)^n \sigma_n(A),$$

be the characteristic polynomial of A. Its coefficients $\sigma_1, \ldots, \sigma_n$ are polynomial in the coefficients of A. Consider the closed subset $N = V(\sigma_1, \ldots, \sigma_n)$ of matrices whose characteristic polynomial vanishes. As a set, it is the set of nilpotent matrices.

Show the following facts:

(1) N is irreducible.
(2) Given a non-zero nilpotent matrix $A \in N$, the matrix 0 belongs to the closure of $\mathrm{SL}_n(k)A$.
(3) Describe the $\mathrm{SL}_n(k)$-orbits contained in N.
(4) (∗∗) The condition $A^n = 0$ is expressed in terms of polynomials $f_1, \ldots, f_{n^2}$ in the coefficients of A. Show that the ideal $I = (f_1, \ldots, f_{n^2})$ is generated by polynomial expressions of $\sigma_1, \ldots, \sigma_n$.[2]

References

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[2]Already for $n = 2$ is not trivial. Show for instance that $\sigma_1^{n^2}$ belongs to N. See `https://webusers.imj-prg.fr/~julien.marche/GIT-exam.pdf`