# FEUILLE DE TD NO. 4

TANGENT SPACES, REGULAR AND SINGULAR POINTS

Let $k$ be an algebraically closed field.

**Definition 1.** Let $X$ be an irreducible algebraic variety over $k$. A point $x \in X$ is said to be *smooth* if $\dim X = \dim_k T_x X$. It is said *singular* otherwise.

The variety $X$ is said to be smooth if every point is *smooth*.

**Exercise 1.**

(1) Let $I \subset k[T_0, \dots, T_n]$ be a homogeneous ideal and $X = V(I)$ the associated closed subariety of $\mathbb{P}^n_k$. For $x \in X$ consider the linear subvariety

$$PT_xX = \left\{ [t_0 : \cdots : t_n] \in \mathbb{P}^n_k : \sum_{i=0}^n \frac{\partial F}{\partial T_i}(x) t_i = 0 \text{ for all } F \in I \right\}.$$

Show $\dim_k T_x X = \dim PT_x X$.

(2) Let $F(T_0, \dots, T_n)$ be a homogeneous polynomial of degree $d$. Then,

$$dF = \sum_{i=0}^n \frac{\partial F}{\partial T_i}.$$

In particular, if $\mathrm{char}(k)$ does not divide $d$, then a point $x \in \mathbb{P}^n_k$ belongs to $X = V(F)$ and is singular if and only if

$$\frac{\partial F}{\partial T_0}(x) = \cdots = \frac{\partial F}{\partial T_n}(x) = 0.$$

**Exercise 2.** Suppose $\mathrm{char}(k) \neq 2$. Let $F$ be a quadratic form in $n+1$ variables and $Q = V(F)$ in $\mathbb{P}^n_k$.

(1) Show that, up changing coordinates, $F$ can be written as

$$F(x_0, \dots, x_n) = x_0^2 + \cdots + x_r^2,$$

where $r+1$ is the rank of $F$.

(2) Show that $F$ is irreducible (resp. reduced) if and only $r \geq 2$ (resp. $r \geq 1$).

(3) Compute the tangent space $T_xQ$ for all point $x \in Q$.

(4) Show that $Q$ is smooth if and only if $r = n$.

**Exercise 3.** Suppose $\mathrm{char}(k) \neq 2$. Let $F$ be a non-degenerate quadratic form in 3 variables, $C = V(F)$ and $x \in C$. Let $L$ be a line passing through $x$ and different from $PT_xC$.

(1) Show that $L$ meets $C$ also in another point $y_L$ different from $x$.

(2) Show that the map $L \mapsto y_L$ induces an isomorphism $\mathbb{P}^1_k \to C$.

Suppose $\mathrm{char}(k) = 2$ and let $F$ be a quadratic form in 3 variables such that $C = V(F)$ is smooth. Then, $F$ can be put in the form $x^2 - yz$.

**Exercise 4.** Compute the singular points of the following curves in $\mathbb{P}^2_k$:

$C_1 : x_0x_2^2 - x_1^3,$ (courbe cuspidale)

$C_2 : x_0x_2^2 - x_1^2(x_1 + x_0),$ (courbe nodale)

$C_3 : (x_1^2 + x_2^2)^2 - 3x_1^2x_2x_0 - x_0x_1^3 = 0,$ (trifolium)

$C_4 : (x_1^2 + x_2^2)^3 - 4x_0^2x_1^2x_2^2 = 0,$ (quadrifolium)

$C_5 : x_0x_1^4 + x_1x_2^4 + x_2x_0^4 = 0,$

$C_6 : x_1^2x_2^3 + x_1^2x_0^3 + x_2^2x_0^3 = 0,$

$C_7 : x_0^n + x_1^n + x_2^n = 0,$ $n \geq 1,$

$C_8 : (x_1^2 - x_0x_2)^4 + x_1^3(x_2 - x_0) = 0,$

$C_9 : 2x_1^4 + x_2^4 - x_0x_2(3x_1^2 + 2x_2^2) + x_2^2x_0^2 = 0,$

$C_{10} : x_0^2x_2^2 + x_0^2x_1^2 + x_1^2x_2^2 - 2x_0x_1x_2(x_0 + x_1 + x_2) = 0.$

**Exercise 5.** Compute the singular points and the tangent spaces of the following subvarieties of $\mathbb{P}^3_k$:

$$X_1 : x_1^2 + x_2^2 - x_3^2 = 0,$$
$$X_2 : x_0x_1x_2 + x_1^3 + x_2^3 = 0,$$
$$X_3 : x_1x_2^2 - x_0x_3^2,$$
$$X_4 : x_0x_1x_2 + x_1x_2x_3 + x_0x_2x_3 + x_0x_1x_3 = 0,$$
$$X_5 : \begin{cases} x_0x_1 - x_2x_3 = 0, \\ x_2x_0^2 - x_3^3 = 0, \\ x_3x_1^2 - x_2^3 = 0. \end{cases}$$

**Exercise 6.** Let $X$ be an algebraic variety. Set up a "natural" bijection between morphisms of $k$-algebraic varieties

$$\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)) \longrightarrow X,$$

and couples $(x, v)$ with $x \in X$ and $v \in T_xX$.

**Exercise 7.** Compute the tangent space at the identity of the following algebraic groups:

$$\mathrm{GL}_n, \qquad \mathrm{SL}_n, \qquad \mathrm{O}(n), \qquad \mathrm{Sp}(2n), \qquad \mu_n = \mathrm{Spec}\, k[t]/(t^n - 1).$$