FEUILLE DE TD NO. 5

VECTOR BUNDLES

Let k be an algebraically closed field.

Exercise 1 ([Liu02, 1.2.8])**.** Let A be a Noetherian ring, M a finitely generated A-module and N an A-module. Let B a flat A-algebra and consider the canonical homomorphism

$$\rho\colon \operatorname{Hom}_A(M,N)\otimes_A B \longrightarrow \operatorname{Hom}_B(M\otimes_A B, N\otimes_A B).$$

(1) Show that ρ is an isomorphism if M is free of finite rank.
(2) Let $0\to M'\to M\to M''\to 0$ be a short exact sequence of A-modules. Show that it induces an exact sequence

$$0\longrightarrow \operatorname{Hom}_A(M'',N)\longrightarrow \operatorname{Hom}_A(M,N)\longrightarrow \operatorname{Hom}_A(M',N).$$

(3) Conclude that ρ is bijective.

Exercise 2 ([Liu02, 5.1.5-6])**.** Let X be a k-algebraic variety and let F, G be $\mathcal{O}_X$-modules.

(1) Show that the correspondence

$$U\longmapsto \operatorname{Hom}_{\mathcal{O}_U\text{-mod}}(F_{|U}, G_{|U}),$$

defines a $\mathcal{O}_X$-module, which is denoted $\mathcal{H}om_{\mathcal{O}_X}(F,G)$;
(2) Suppose $X=\operatorname{Spec} A$. Show that the natural map

$$\operatorname{Hom}_{\mathcal{O}_X}(F,G)\longrightarrow \operatorname{Hom}_A(F(X),G(X)),$$

is bijective if F is quasi-coherent.
(3) If F is coherent and G quasi-coherent, then $\mathcal{H}om_{\mathcal{O}_X}(F,G)$ is quasi-coherent.
(4) If F and G are coherent, then so is $\mathcal{H}om_{\mathcal{O}_X}(F,G)$.

Exercise 3 ([Har77, 5.1])**.** Let X be a k-algebraic variety and E a locally free $\mathcal{O}_X$-module of finite rank. The *dual* $E^\vee$ of E is the coherent $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(E,\mathcal{O}_X)$.

(1) Show that the canonical map $E\to (E^\vee)^\vee$ is an isomorphism.
(2) Let F be an $\mathcal{O}_X$-module. Then the canonical map

$$E^\vee\otimes_{\mathcal{O}_X} F\longrightarrow \mathcal{H}om_{\mathcal{O}_X}(E,F),$$

is an isomorphism.
(3) Let F, G be $\mathcal{O}_X$-modules. Then the canonical map

$$\operatorname{Hom}_{\mathcal{O}_X}(F,\mathcal{H}om_{\mathcal{O}_X}(E,G))\longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(E\otimes_{\mathcal{O}_X} F, G),$$

is an isomorphism.
(4) Let $f\colon Y\to X$ be a morphism of algebraic variety and F an $\mathcal{O}_Y$-module. Show that the canonical morphism

$$f_*(E\otimes_{\mathcal{O}_X} f^*E)\longrightarrow f_*E\otimes_{\mathcal{O}_Y} F,$$

is an isomorphism.

Exercise 4. Let A be a ring and M an A-module. Set

$$T^n(M)=\begin{cases} M^{\otimes n} & \text{if } n\geq 1,\\ A & \text{otherwise.}\end{cases}$$

The *tensor algebra* is the non commutative algebra $T_A^\bullet M = \bigoplus_{n\geq 0} T^n(M)$. The *symmetric algebra* $\operatorname{Sym}_A^\bullet M$ is the quotient of $T_A^\bullet M$ by the two-sided ideal generated by the elements of the form $m \otimes m' - m' \otimes m$ for all $m, m' \in M$.

(1) Suppose M free of rank n. Then $\operatorname{Sym}_A^\bullet M \simeq A[t_1, \ldots, t_n]$.

(2) A homomorphism $\varphi\colon M \to M'$ of A-modules induces a homomorphism of A-algebras
$$\operatorname{Sym}\varphi\colon \operatorname{Sym}_A^\bullet M \longrightarrow \operatorname{Sym}_A^\bullet M'.$$
Moreover if φ is surjective, then $\operatorname{Sym}\varphi$ is surjective.

(3) Let B be a A-algebra. The natural map
$$\begin{aligned} \operatorname{Hom}_{A\text{-alg}}(\operatorname{Sym}_A^\bullet M, B) &\longrightarrow \operatorname{Hom}_{A\text{-mod}}(M, B) \\ \varphi\colon \operatorname{Sym}_A^\bullet M \to B &\longmapsto \varphi_{|M}\colon M \to B, \end{aligned}$$
is bijective.

(4) Suppose M is finitely generated and locally free, that is the $\mathcal{O}_X$-module $\tilde{M}$ is locally free on $X = \operatorname{Spec} A$. Let $f\colon Y \to X$ be a morphism of k-algebraic varieties. Set $\mathbb{V}(M) = \operatorname{Spec}(\operatorname{Sym}^\bullet M^\vee)$ and $p\colon \mathbb{V}(M) \to X$ the map induced by the inclusion $A \to \operatorname{Sym}^\bullet M^\vee$. Set up a natural bijection between
$$\operatorname{Hom}_X(Y, \mathbb{V}(M)) = \{g\colon Y \to \mathbb{V}(M) : p \circ g = f\},$$
and $\Gamma(Y, f^*\tilde{M})$.

(5) Apply the preceding constructions with
   (a) $A = k$ and M a finite dimensional k-vector space;
   (b) $Y = U$ an open subset of X and $f\colon U \to X$ the inclusion.

Exercise 5. Let A be a PID and M a finitely generated A-module.

(1) Suppose A to be a DVR. Then the following conditions are equivalent:
   (a) M is free of finite rank;
   (b) M is torsion-free.

(2) Suppose M torsion-free. Show that the canonical map of $M \to (M^\vee)^\vee$ is an isomorphism.

(3) Show that every submodule M of A^n is free of rank $\leq n$.

(4) Deduce that every torsion-free finitely generated A-module M is free of finite rank.

Recall the following:

Theorem 1 (Elementary divisors)**.** *Let $K = \operatorname{Frac}(A)$ and $g \in \operatorname{GL}_n(K)$. Then there are $u, v \in \operatorname{GL}_n(A)$ and $a_1, \ldots, a_n \in K$ such that*
$$g = u \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} v.$$

(5) Let E be a vector bundle on $\mathbb{P}^1$ of rank n. Then there exist $d_1, \ldots, d_n \in \mathbb{Z}$ such that
$$E \simeq \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n).$$

References

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications. MR 1917232