FEUILLE DE TD NO. 6

VECTOR BUNDLES II

Let k be an algebraically closed field.

Exercise 1. Let $n \geq 1$ be an integer. For a point $x \in \mathbb{P}^n$ let $L_x \subset k^{n+1}$ be the corresponding vector line. The *tautological bundle* is

$$\mathcal{O}(-1) = \{(x, v) \in \mathbb{P}^n \times k^n : v \in L_x\}.$$

For $d \in \mathbb{Z}$ let

$$\mathcal{O}(d) := \begin{cases} \underline{\mathsf{Hom}}(\mathcal{O}(-1), X \times k)^{\otimes d} & \text{if } d \geq 0, \\ \mathcal{O}(-1)^{\otimes d} & \text{if } d < 0. \end{cases}$$

so that $S(1) = S$. Let $x_0, \ldots, x_n$ be the homogeneous coordinates on $\mathbb{P}^n$ and for $i = 0, \ldots, n$ let $U_i = \{x_i \neq 0\}$.

(1) Show that $\mathcal{O}(d)$ is trivial on U_i.
(2) Fix trivializations on U_i and U_j and compute the transition maps of $\mathcal{O}(d)$ with respect to these trivializations.
(3) Compute the global sections of $\mathcal{O}(d)$.

Exercise 2. Let V be a k-vector space of dimension n. The cotangent bundle $T^*\mathbb{P}(V)$ is the dual of the tangent bundle TX:

$$T^*X := \underline{\mathsf{Hom}}(TX, X \times k).$$

(1) Show that $K_{\mathbb{P}(V)} = \mathcal{O}(-n-1)$.
(2) Let f be a homogeneous polynomial of degree d. Show that the associated global section s of $\mathcal{O}(d)$ is transversal to the zero section if and only if the differential $d_x f$ (as a regular function on V) does not vanish for all non-zero $x \in V(f)$.
(3) Let X be a smooth hypersurface of degree d of $\mathbb{P}(V)$. Then,

$$K_X \simeq \mathcal{O}(d - n - 1)_{|X}.$$

Exercise 3. Let V be a k-vector space of dimension n, let $r \leq n$ be a non-negative integer and let $\mathrm{Gr}_r(V)$ be the set of k-vector subspaces of V of dimension r. For a point $x \in \mathrm{Gr}_r(V)$ denote by W_x the associated k-vector space. For a k-vector subspace L of V of dimension $n - r$ let

$$U_L = \{x \in \mathrm{Gr}_r(V) : W_x \cap L = 0\}.$$

(1) Let Z_L be the subset of $\mathrm{Hom}(V, L)$ given by the linear maps $\varphi \colon V \to L$ such that $\varphi_{|L} = \mathrm{id}$. Show Z_L is Zariski-closed, isomorphic to $\mathbb{A}^{r(n-r)}$ and that the map

$$\begin{aligned} \theta_L \colon Z_L &\longrightarrow U_L \\ \varphi &\longmapsto \operatorname{Ker} \varphi, \end{aligned}$$

is a bijection.
(2) If L' is a k-vector subspace of V of dimension $n - r$, then $U_L \cap U_{L'}$ induces an open Zariski subset of Z_L.
(3) There is a unique structure of algebraic variety on $\mathrm{Gr}_r(V)$ such that θ_L is an isomorphism for all subspaces $L \subset V$ of dimension $n - r$.

(4) Let
$$S = \{(x, v) \in \mathrm{Gr}_r(V) \times V : v \in W_x\}.$$
Show that S is subvector bundle of $\mathrm{Gr}_r(V) \times V$.

(5) Let Q be the vector bundle $(\mathrm{Gr}_r(V) \times V)/S$ and $f\colon X \to \mathrm{Gr}_r(V)$ a morphism of algebraic varieties. Show that f^*S is a subvector bundle of $X \times V$.

(6) A subvector bundle E of $X \times V$ of rank r induces a morphism of algebraic varieties $\beta_E\colon X \to \mathrm{Gr}_r(V)$.

(7) Show that the maps $f \mapsto \alpha_f$ and $E \mapsto \beta_E$ are mutually inverse and induce a bijection
$$\mathrm{Mor}(X, \mathrm{Gr}_r(V)) \longleftrightarrow \{\text{subvector bundles of } X \times V \text{ of rank } r\}.$$

(8) Show that the tangent bundle TX is isomorphic to $\underline{\mathsf{Hom}}(S, Q)$.

Exercise 4. Let X be an algebraic variety. A line bundle L on X is said to be *globally generated* if the natural map of vector bundles
$$\begin{aligned} X \times \Gamma(X, L) &\longrightarrow L \\ (x, s) &\longmapsto (x, s(x)), \end{aligned}$$
is surjective. Suppose $\Gamma(X, L)$ is finite dimensional and L globally generated. Show that there exists a morphism $f\colon X \to \mathbb{P}(V)$ such that $f^*\mathcal{O}(1) \simeq L$.