## FEUILLE DE TD NO. 8

VECTOR BUNDLES III & COHERENT SHEAVES

Let $k$ be an algebraically closed field.

**Exercise 1.** Let $V$ be a $k$-vector space of dimension $n$, let $r \leq n$ be a non-negative integer and let $\mathrm{Gr}_r(V)$ be the set of $k$-vector subspaces of $V$ of dimension $r$. For a point $x \in \mathrm{Gr}_r(V)$ denote by $W_x$ the associated $k$-vector space. For a $k$-vector subspace $L$ of $V$ of dimension $n-r$ let

$$U_L = \{x \in \mathrm{Gr}_r(V) : W_x \cap L = 0\}.$$

(1) Let $Z_L$ be the subset of $\mathrm{Hom}(V, L)$ given by the linear maps $\varphi \colon V \to L$ such that $\varphi_{|L} = \mathrm{id}$. Show $Z_L$ is Zariski-closed, isomorphic to $\mathbb{A}^{r(n-r)}$ and that the map

$$\begin{array}{rcl} \theta_L \colon Z_L & \longrightarrow & U_L \\ \varphi & \longmapsto & \mathrm{Ker}\, \varphi, \end{array}$$

is a bijection.

(2) If $L'$ is a $k$-vector subspace of $V$ of dimension $n-r$, then $U_L \cap U_{L'}$ induces an open Zariski subset of $Z_L$.

(3) There is a unique structure of algebraic variety on $\mathrm{Gr}_r(V)$ such that $\theta_L$ is an isomorphism for all subspaces $L \subset V$ of dimension $n-r$.

(4) Let

$$S = \{(x, v) \in \mathrm{Gr}_r(V) \times V : v \in W_x\}.$$

Show that $S$ is subvector bundle of $\mathrm{Gr}_r(V) \times V$.

(5) Let $Q$ be the vector bundle $(\mathrm{Gr}_r(V) \times V)/S$ and $f \colon X \to \mathrm{Gr}_r(V)$ a morphism of algebraic varieties. Show that $f^*S$ is a subvector bundle of $X \times V$.

(6) A subvector bundle $E$ of $X \times V$ of rank $r$ induces a morphism of algebraic varieties $\beta_E \colon X \to \mathrm{Gr}_r(V)$.

(7) Show that the maps $f \mapsto \alpha_f$ and $E \mapsto \beta_E$ are mutually inverse and induce a bijection

$$\mathrm{Mor}(X, \mathrm{Gr}_r(V)) \longleftrightarrow \{\text{subvector bundles of } X \times V \text{ of rank } r\}.$$

(8) Show that the tangent bundle $TX$ is isomorphic to $\underline{\mathsf{Hom}}(S, Q)$.

**Definition 1.** Let $X$ be a $k$-algebraic variety and $L$ a line bundle.

(1) Let $x \in X$ and let $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ be the maximal ideal at $x$. Let $\mathcal{O}_x(L)$ be the $\mathcal{O}_{X,x}$-module of germs of sections of $L$ and $J_x(L) = \mathcal{O}_x(L)/\mathfrak{m}_x^2\mathcal{O}_x(L)$.

(2) A *linear system* is a finite dimensional subvector space $V \subset \Gamma(X, L)$. A linear system $V$ is said to
   - *globally generates* $L$ if for every $x \in X$ there is a global section $s \in V$ such that $s(x) \neq 0$;
   - *separate the points* if for all distinct points $x, x'$ there is a global section $s \in W$ such that $s(x) = 0$ and $s(x') \neq 0$;
   - *separate the tangent vectors* if for all $x \in X$ the linear map $V \to J_x(L)$, $s \mapsto j_x s$ is surjective.

**Exercise 2.** Let $X$ be an algebraic variety, $L$ a line on $X$ and $V$ a linear system.

(1) Suppose $V$ globally generates $L$. Show that there is map
$$f_V\colon X \longrightarrow \mathbb{P}(V^*)$$
such that $f_V^*\mathcal{O}(1) \simeq L$.
(2) Suppose $V$ separates the points. Then the map $f_V$ is injective.

The aim of the rest of the exercise is to prove that if $X$ is projective and $V$ separates the tangents vectors, then the map $f_V$ is an embedding.

(3) For $x \in X$ show that we have an exact sequence
$$0 \longrightarrow T_x^*X \otimes_k L_x \longrightarrow J_x(L) \longrightarrow L_x \longrightarrow 0.$$
(4) Let $s \in \Gamma(X, L)$ such that $s(x) = 0$. Then $j_x s = d_x s \in \mathrm{Hom}(T_x X, L_x)$.
(5) Show that the linear system $V$ separates the tangent vectors if and only if for all $x \in X$ the natural map
$$\begin{array}{ccc} T_x X & \longrightarrow & \mathrm{Hom}_k(H_x, L_x) \\ v & \longmapsto & [s \mapsto d_x s(v)] \end{array}$$
is injective, where $H_x = \{s \in V : s(x) = 0\}$.

From now on $X$ is projective and $V$ separates the tangent vectors. We admit that the map $f = f_V$ is finite. For $f$ to be an embedding it suffices to prove that for every $x \in X$ the tangent map $d_x f\colon T_x X \to \mathbb{P}(V^*)$ is injective (see next exercise).

It suffices to show that the tangent map $d_x f$ is the map $T_x X \to \mathrm{Hom}_k(H_x, L_x)$ defined above. Let $s_0 \in V$ be a global section not vanishing at $x$.

(6) Show that the map
$$\begin{array}{ccc} \varphi_0\colon U_0 := \{s_0 \neq 0\} & \longrightarrow & \mathrm{Hom}_k(H_x, L_x) \\ [\varphi] & \longmapsto & \left[v \mapsto \dfrac{\varphi(v)}{\varphi(s_0)} s_0(x)\right] \end{array}$$
is an isomorphism. The differential of $\varphi_0$ induces an isomorphism
$$d_x\varphi_0\colon T_x\mathbb{P}(V^*) \xrightarrow{\sim} \mathrm{Hom}_k(H_x, L_x).$$

Let $s_1, \dots, s_n$ be a basis of $H_x$ and $s_0^*, \dots, s_n^* \in V^*$ be the dual basis.

(7) Show that on $f^{-1}(U_0)$ the global section $s_i$ can be written as $s_i = f_i s_0$ for a regular function $f_i$ on $f^{-1}(U_0)$.
(8) Show that $f$ on $f^{-1}(U_0)$ is given by
$$f(x) = [s_0^* + f_1(x)s_1^* + \cdots + f_n(x)s_n^*]\,.$$
(9) Conclude.

**Exercise 3.** Let $f\colon X \to Y$ be a finite injective morphism of $k$-algebraic varieties such that the tangent map
$$d_x f\colon T_x X \longrightarrow T_{f(x)}Y$$
is injective for all $x \in X$. Then $f$ is an embedding.

**Exercise 4** ([Liu02, 1.2.8])**.** Let $A$ be a Noetherian ring, $M$ a finitely generated $A$-module and $N$ an $A$-module. Let $B$ a flat $A$-algebra and consider the canonical homomorphism
$$\rho\colon\ \mathrm{Hom}_A(M, N) \otimes_A B \longrightarrow \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B).$$

(1) Show that $\rho$ is an isomorphism if $M$ is free of finite rank.
(2) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $A$-modules. Show that it induces an exact sequence
$$0 \longrightarrow \mathrm{Hom}_A(M'', N) \longrightarrow \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A(M', N).$$
(3) Conclude that $\rho$ is bijective.

**Exercise 5** ([Liu02, 5.1.5-6])**.** Let $X$ be a $k$-algebraic variety and let $F, G$ be $\mathcal{O}_X$-modules.

(1) Show that the correspondence
$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_U\text{-mod}}(F_{|U}, G_{|U}),$$
defines a $\mathcal{O}_X$-module, which is denoted $\mathcal{H}om_{\mathcal{O}_X}(F, G)$;

(2) Suppose $X = \operatorname{Spec} A$. Show that the natural map
$$\mathrm{Hom}_{\mathcal{O}_X}(F, G) \longrightarrow \mathrm{Hom}_A(F(X), G(X)),$$
is bijective if $F$ is quasi-coherent.

(3) If $F$ is coherent and $G$ quasi-coherent, then $\mathcal{H}om_{\mathcal{O}_X}(F, G)$ is quasi-coherent.

(4) If $F$ and $G$ are coherent, then so is $\mathcal{H}om_{\mathcal{O}_X}(F, G)$.

**Exercise 6** ([Har77, 5.1])**.** Let $X$ be a $k$-algebraic variety and $E$ a locally free $\mathcal{O}_X$-module of finite rank. The *dual* $E^\vee$ of $E$ is the coherent $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$.

(1) Show that the canonical map $E \to (E^\vee)^\vee$ is an isomorphism.

(2) Let $F$ be an $\mathcal{O}_X$-module. Then the canonical map
$$E^\vee \otimes_{\mathcal{O}_X} F \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(E, F),$$
is an isomorphism.

(3) Let $F, G$ be $\mathcal{O}_X$-modules. Then the canonical map
$$\mathrm{Hom}_{\mathcal{O}_X}(F, \mathcal{H}om_{\mathcal{O}_X}(E, G)) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(E \otimes_{\mathcal{O}_X} F, G),$$
is an isomorphism.

(4) Let $f\colon Y \to X$ be a morphism of algebraic variety and $F$ an $\mathcal{O}_Y$-module. Show that the canonical morphism
$$f_*(E \otimes_{\mathcal{O}_X} f^*E) \longrightarrow f_*E \otimes_{\mathcal{O}_Y} F,$$
is an isomorphism.

**Exercise 7.** Let $A$ be a ring and $M$ an $A$-module. Set
$$T^n(M) = \begin{cases} M^{\otimes n} & \text{if } n \geq 1, \\ A & \text{otherwise.} \end{cases}$$

The *tensor algebra* is the non commutative algebra $T^\bullet_A M = \bigoplus_{n \geq 0} T^n(M)$. The *symmetric algebra* $\mathrm{Sym}^\bullet_A M$ is the quotient of $T^\bullet_A M$ by the two-sided ideal generated by the elements of the form $m \otimes m' - m' \otimes m$ for all $m, m' \in M$.

(1) Suppose $M$ free of rank $n$. Then $\mathrm{Sym}^\bullet_A M \simeq A[t_1, \ldots, t_n]$.

(2) A homomorphism $\varphi\colon M \to M'$ of $A$-modules induces a homomorphism of $A$-algebras
$$\mathrm{Sym}\,\varphi\colon \mathrm{Sym}^\bullet_A M \longrightarrow \mathrm{Sym}^\bullet_A M'.$$
Moreover if $\varphi$ is surjective, then $\mathrm{Sym}\,\varphi$ is surjective.

(3) Let $B$ be a $A$-algebra. The natural map
$$\begin{array}{rcl} \mathrm{Hom}_{A\text{-alg}}(\mathrm{Sym}^\bullet_A M, B) & \longrightarrow & \mathrm{Hom}_{A\text{-mod}}(M, B) \\ \varphi\colon \mathrm{Sym}^\bullet_A M \to B & \longmapsto & \varphi_{|M}\colon M \to B, \end{array}$$
is bijective.

(4) Suppose $M$ is finitely generated and locally free, that is the $\mathcal{O}_X$-module $\tilde{M}$ is locally free on $X = \operatorname{Spec} A$. Let $f\colon Y \to X$ be a morphism of $k$-algebraic varieties. Set $\mathbb{V}(M) = \mathrm{Spec}(\mathrm{Sym}^\bullet M^\vee)$ and $p\colon \mathbb{V}(M) \to X$ the map induced by the inclusion $A \to \mathrm{Sym}^\bullet M^\vee$. Set up a natural bijection between
$$\mathrm{Hom}_X(Y, \mathbb{V}(M)) = \{g\colon Y \to \mathbb{V}(M) : p \circ g = f\},$$

and $\Gamma(Y, f^*\tilde{M})$.

(5) Apply the preceding constructions with
  (a) $A = k$ and $M$ a finite dimensional $k$-vector space;
  (b) $Y = U$ an open subset of $X$ and $f\colon U \to X$ the inclusion.

**Exercise 8.** Let $A$ be a PID and $M$ a finitely generated $A$-module.

(1) Suppose $A$ to be a DVR. Then the following conditions are equivalent:
  (a) $M$ is free of finite rank;
  (b) $M$ is torsion-free.

(2) Suppose $M$ torsion-free. Show that the canonical map of $M \to (M^\vee)^\vee$ is an isomorphism.

(3) Show that every submodule $M$ of $A^n$ is free of rank $\leq n$.

(4) Deduce that every torsion-free finitely generated $A$-module $M$ is free of finite rank.

Recall the following:

**Theorem 2** (Elementary divisors)**.** *Let $K = \mathrm{Frac}(A)$ and $g \in \mathrm{GL}_n(K)$. Then there are $u, v \in \mathrm{GL}_n(A)$ and $a_1, \ldots, a_n \in K$ such that*

$$g = u \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} v.$$

(5) Let $E$ be a vector bundle on $\mathbb{P}^1$ of rank $n$. Then there exist $d_1, \ldots, d_n \in \mathbb{Z}$ such that

$$E \simeq \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n).$$

## References

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications. MR 1917232