Modular sheaves and representations

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Contents

1 Introduction

I work at the interface between algebra (representation theory), geometry (study of the singularities of some algebraic varieties), and topology (sheaf cohomology, perverse sheaves).

Geometric representation theory establishes links between certain categories of representations and certain geometric categories (for example perverse sheaves, D-modules or coherent sheaves) on some related variety. This approach was highly successful in representation theory.

Most results obtained in geometric representation theory applied to categories of representations over a field of characteristic zero. However, more recently some relations between modular perverse sheaves, with coefficients in a field of characteristic $\ell > 0$, have been discovered. In the survey article [8], we explain those relations (state of the art in 2008), and give some examples of concrete computations. This can be a good starting point to see the differences between characteristic zero and characteristic ℓ coefficients in the theory of perverse sheaves, and also to see what happens with the torsion over $\mathbb{Z}$.

In the introduction, I will give some background on modular representations of finite groups and reductive algebraic groups, on intersection cohomology methods in representation theory, and on the Springer correspondence, including my thesis work in the modular case. In Part 2, I will present the work that I did from 2008 up to now; a substantial part of it is about modular sheaves and representations. Finally, in Part 3, I will give some perspectives. An outline is given at the end of this introduction.

1.1 Modular representations of finite groups

I refer to [Ser, Partie 3] for an introduction. Brauer developed the modular representation theory of finite groups from the early 1940s. Throughout, $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ will denote an ℓ-modular system: $\mathbb{K}$ is a finite extension of $\mathbb{Q}_\ell$ (which we will always assume to be "sufficiently large" for all the finite groups involved) with ring of integers $\mathbb{O}$ and residue field $\mathbb{F}$. In particular, Brauer defined modular characters, now called Brauer characters, for representations of $\mathbb{F}H$, where H is a finite group. The Brauer character of a modular representation V is not computed just by taking the trace of the elements on V. Since $\mathbb{F}$ has characteristic ℓ, this would lose information: an ℓ-dimensional trivial representation would have zero character. Instead, one restricts to ℓ-regular elements (of order prime to ℓ); their eigenvalues are (of course) roots of unity in $\mathbb{F}$; after choosing a bijection between the roots of unity in $\mathbb{F}$ and the roots of unity of order prime to ℓ in $\mathbb{C}$, one can define the value of the Brauer character of V on an ℓ-regular element h to be the sum of the lifts of its eigenvalues. The Brauer characters of irreducible representations of H, called irreducible Brauer characters, form a basis of the space of central (i.e. conjugation invariant) functions on the set of ℓ-regular elements of H; this space can be identified with the Grothendieck group of $\mathbb{F}H$, just as the space of central functions on H can be identified with the Grothendieck group of $\mathbb{K}H$.

For an abelian category $\mathcal{A}$, we denote by $\operatorname{Irr}\mathcal{A}$ the set of isomorphisms classes of simple objects in $\mathcal{A}$. If $\mathcal{A}$ is the category of A-modules for a ring A, we write $\operatorname{Irr} A$. If $E \in \operatorname{Irr} \mathbb{K}H$ and $F \in \operatorname{Irr} \mathbb{F}H$, then the decomposition number $d^H_{E,F}$ is defined as the composition multiplicity of F in $\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}}$, where $E_{\mathbb{O}}$ is any integral form of E (the modular reduction of E is not well defined up to isomorphism, but its class in the Grothendieck group is, hence so are the multiplicities of the simple modules). The matrix $D^H := (d^H_{E,F})$ is called the *decomposition matrix*. The Brauer character of the modular reduction of E is just the restriction of the character of E to the ℓ-regular elements in H. If $\ell \nmid |H|$ then Maschke's theorem implies that $\mathbb{F}H$ is semisimple and D^H is the identity.

It is still an open problem to determine the decomposition matrices of important classes of groups like the symmetric groups. The situation is even worse than it seemed: it used to be believed that when $\ell > \sqrt{n}$, the decomposition matrix for $\mathfrak{S}_n$ was the same as a decomposition matrix defined similarly for the Hecke algebra at an ℓ-th root of unity in $\mathbb{C}$, which can be computed by the Lascoux–Leclerc–Thibon algorithm, as proved by Ariki. But Williamson found counterexamples in [Wil], which come from the counterexamples he found to Lusztig's conjecture about modular representations of reductive algebraic groups.

1.2 Modular representations of reductive algebraic groups

The book [Jan] is an excellent reference for this topic. Let G be a connected reductive algebraic group over an algebraically closed field k, e.g. GL_n, SO_n, $\mathrm{Sp}_{2n} \dots$ We choose a maximal torus $T \subset G$. It is isomorphic to the product of r copies of the multiplicative group $\mathbb{G}_m$, where r is the rank of G. We denote by $\mathcal{X}^*(T) = \operatorname{Hom}(T, \mathbb{G}_m) \simeq \mathbb{Z}^r$ (resp. $\mathcal{X}_*(T) = \operatorname{Hom}(\mathbb{G}_m, T) \simeq \mathbb{Z}^r$) the group of characters (resp. cocharacters) of T. The Weyl group $W = N_G(T)/T$ acts on characters and cocharacters.

For each $\lambda \in \mathcal{X}^*(T)$, we have a one dimensional T-module k_λ, which is the vector space k on which T acts by λ. Any T-module M is completely reducible, and decomposes as a direct sum of weight spaces

$$M = \bigoplus_{\lambda \in \mathcal{X}^*(T)} M_\lambda.$$

If M is such that each M_λ is finite dimensional, and only finitely many of them are non-zero, then the formal character of M is

$$\operatorname{ch} M = \sum_{\lambda \in \mathcal{X}^*(T)} \dim(M_\lambda) e^\lambda,$$

an element of the group algebra $\mathbb{Z}[\mathcal{X}^*(T)] = \bigoplus_{\lambda \in \mathcal{X}^*(T)} \mathbb{Z} e^\lambda$. It is a central problem to compute the characters of simple G-modules (that is, the characters of their restrictions to T). Note that the character of a G-module is W-invariant.

By differentiating at 1 the action of conjugation of G on itself, we obtain the adjoint representation of G on its Lie algebra $\mathfrak{g}$. The non-zero T-weights in $\mathfrak{g}$

form a root system $\Phi \subset \mathcal{X}^*(T)$, and we have a dual root system $\Phi^\vee \subset \mathcal{X}_*(T)$. Both are stable by W. The quadruple $(\mathcal{X}^*(T), \Phi, \mathcal{X}_*(T), \Phi^\vee)$ is called the root datum of (G, T). It determines G up to isomorphism (making this isomorphism unique requires the notion of "pinning"). The roots (resp. coroots) span a subgroup $\mathbb{Z}\Phi \subset \mathcal{X}^*(T)$ (resp. $\mathbb{Z}\Phi^\vee \subset \mathcal{X}_*(T)$), called the root (resp. coroot) lattice. The rank of the root lattice is the semisimple rank of G. The weight (resp. coweight) lattice consists of the elements of $\mathcal{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\mathcal{X}_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$) which take integral values on the coroots (resp. on the roots).

To each root α corresponds a coroot $\alpha^\vee$, which satisfies $\langle \alpha, \alpha^\vee \rangle = 2$. We also have a reflection $s_\alpha \in W$ which acts on characters by:

$$\begin{array}{rcl} s_\alpha : \mathcal{X}^*(T) & \longrightarrow & \mathcal{X}^*(T) \\ \lambda & \longmapsto & \lambda - \langle \lambda, \alpha^\vee \rangle \alpha. \end{array}$$

The Weyl group W of (G, T) is the group generated by the reflections s_α, $\alpha \in \Phi$. It is a Coxeter group: it has a presentation with set of generators $S = \{s_\alpha, \alpha \in \Delta\}$, and relations $(st)^{m_{s,t}} = 1$, where $m_{s,t}$ denotes the order of st (for $s, t \in S$). The Coxeter matrix $(m_{s,t})$ is symmetric and satisfies $m_{s,s} = 1$, while $m_{s,t} \geq 2$ otherwise. If $s = s_\alpha$, we also set $\alpha = \alpha_s$. If $w \in W$, we denote by $l(w)$ the length of w with respect to S, i.e. the minimal possible length for a word in the generators S whose product is w.

If $I \subset S$, we denote by W_I the reflection subgroup generated by I, called a standard parabolic subgroup of W (a general parabolic subgroup of W is conjugate to a standard parabolic subgroup). The parabolic subgroups of W are also all the possible centralizers of subsets $X \subset \mathcal{X}^*(T)$: the centralizer of X is the subgroup generated by the reflections fixing X.

On the other hand, a parabolic subgroup $P \subset G$ is a closed subgroup such that G/P is a complete variety (it is actually projective). A Borel subgroup $B \subset G$ is a minimal parabolic subgroup, or equivalently a maximal connected solvable subgroup.

Now fix a Borel subgroup $B \subset G$ containing T. Since all Borel subgroups are conjugate and a Borel subgroup is self-normalizing, the flag variety $\mathcal{B} := G/B$ parametrizes the Borel subgroups of G, or equivalently the Borel subalgebras of $\mathfrak{g}$ (maximal solvable Lie subalgebras). A general parabolic subgroup of G is conjugate to a standard parabolic subgroup $P_I := BW_IB$ (this makes sense because T normalizes B). In particular, for each simple root α we have a (next to) minimal parabolic subgroup $P_\alpha := BW_{\{s_\alpha\}}B$.

In the case $G = \mathrm{GL}_n$, the Borel subgroup B can be taken to be the upper triangular matrices, so in that case $\mathcal{B}$ also parametrizes complete flags in k^n (hence the name). Standard parabolic subgroups are then groups of upper block triangular matrices (for various configurations of block sizes), and the varieties G/P_I then parametrize the different kinds of partial flags in k^n. For other classical groups, there are similar interpretations in terms of complete or partial isotropic flags.

Say that the T-weights on the Lie algebra $\mathfrak{b}$ of B are the negative roots. Their opposites form the set Φ^+ of positive roots. For $\lambda, \mu \in \mathcal{X}^*(T)$, say $\lambda \geq \mu$

if $\lambda - \mu$ is a positive linear combination of positive roots (this is the dominance order). The positive roots that are minimal in Φ^+ with respect to the dominance order form the set Δ of simple roots. Then Δ is a basis for the root lattice. The set of dominant weights is

$$\mathfrak{X}^*(T)^+ := \{\lambda \in \mathfrak{X}^*(T) \mid \forall \alpha^\vee \in \Phi_+^\vee,\ \langle \lambda, \alpha^\vee \rangle \geq 0\}.$$

We have similarly simple and positive coroots $\Delta^\vee \subset \Phi_+^\vee \subset \mathfrak{X}_*(T)$.

Let k be a field. For $\lambda \in \mathfrak{X}^*(T)$, consider $\mathcal{L}(\lambda) := G \times^B k_\lambda$: lift the T-module k_λ to B, then take the quotient of $G \times k_\lambda$ by the middle B-action (inverse multiplication on the right on G, action given by λ on k_λ). This is a line bundle on the flag variety G/B, which is a projective G-variety. We will denote similarly the corresponding locally free sheaf. Hence we can consider the cohomology groups $H^i(\lambda) := H^i(G/B, \mathcal{L}(\lambda))$. They are G-modules. The module $H^0(\lambda)$ is non-zero if and only if λ is dominant. Moreover, in that case its socle $L(\lambda)$ is then simple, and one gets in this way a complete collection of non-isomorphic simple G-modules.

Kempf's vanishing theorem says that, for λ dominant, all higher $H^i(\lambda)$ (with $i > 0$) vanish. This is a powerful result because then $H^0(\lambda)$ is equal to the Euler characteristic, which is always much easier to compute than individual cohomology groups (it is additive on short exact sequences, etc). From a study of the rank 1 situation, one can deduce that $\operatorname{ch} H^0(\lambda)$ is given by the famous Weyl Character Formula (this approach, used in [Jan], is due to Donkin).

If $k = \mathbb{C}$, then all G-modules are completely reducible (i.e. semisimple), hence $L(\lambda) = H^0(\lambda)$, and by the previous paragraph this gives the character of simple modules in this case.

If k is of characteristic p, one is led to consider the composition multiplicities $d^G_{\lambda,\mu} := [H^0(\lambda) : L(\mu)]$. It is easy to see that $d^G_{\lambda,\lambda} = 1$, and that $d^G_{\lambda,\mu} \neq 0$ implies $\mu \leq \lambda$. But the linkage principle gives more a priori constraints. Let ρ be the half-sum of the positive roots. The affine Weyl group $W_p := p\mathbb{Z}\Phi \rtimes W$ acts on $\mathfrak{X}^*(T)$ by the dot action, defined as follows: p times a character acts by translation, and the action of W is shifted so that the fixed point is $-\rho$ instead of 0, that is, $w \cdot \lambda := w(\lambda + \rho) - \rho$. Then $d^G_{\lambda,\mu} = 0$ unless $\mu \in W_a \cdot \lambda$. We actually have a decomposition of the category of G-modules into a direct sum of categories, one for each dot action orbit of the affine Weyl group. A further reduction (the translation principle), implies that, when p is at least equal to the Coxeter number h of G (which is equal to n for $G = SL_n$), it is enough to consider the orbit of the zero weight. Moreover, Steinberg's tensor product theorem shows that we can obtain all simple modules by taking tensor products of iterated Frobenius twists of a finite number of them, namely those with highest weight λ is in the fundamental box $\mathfrak{X}_1^*(T) := \{\lambda \in \mathfrak{X}^*(T) \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < p, \forall \alpha \in \Delta\}$. Hence the problem becomes to compute the multiplicities $d^G_{x,y} := [H^0(x \cdot 0) : L(y \cdot 0)]$ where $x, y \in W_a$ are such that $x \cdot 0 \geq y \cdot 0$ are in the fundamental box. Lusztig proposed a conjecture for those multiplicities, in terms of Kazhdan–Lusztig polynomials. To put it into perspective, we will first give some background about intersection cohomology methods in representation theory.

1.3 Intersection cohomology and representation theory

Historically, the first application of intersection cohomology to representation theory was the proof of the Kazhdan–Lusztig conjecture about the characters of simple modules for complex semisimple Lie algebras. Let us keep the notation of the previous section, but let us work over $\mathbb{C}$. The Lie algebra $\mathfrak{g}$ has a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, which is the quotient of the tensor algebra on $\mathfrak{g}$ by the relations $xy - yx - [x, y]$, for $x, y \in \mathfrak{g}$. A $\mathfrak{g}$-module is the same thing as a $\mathcal{U}(\mathfrak{g})$-module. Now, by the Poincaré-Birkhoff-Witt theorem, the algebra $\mathcal{U}(\mathfrak{g})$ as a triangular decomposition: as vector spaces, $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{t}) \otimes \mathcal{U}(\mathfrak{n})$, where (keeping the notations of the previous section) $\mathfrak{t}$ is the Lie algebra of the maximal torus T, while $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is the direct sum of the negative (resp. positive) root spaces. The Lie algebra of B is $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$. We consider the category $\mathcal{O}$ of $\mathfrak{g}$, which consists of (possibly infinite dimensional) $\mathfrak{g}$-modules which are finitely generated, semisimple for $\mathfrak{t}$, and locally finite for $\mathfrak{n}$. Although we work over $\mathbb{C}$, this category is not semisimple. The character of a module in category $\mathcal{O}$ can be defined in a certain completion of the group algebra of $\mathfrak{t}^*$. Indeed, all weight spaces are finite dimensional, and although infinitely many weights may be involved, because of the $\mathfrak{n}$-local finiteness, they only go to infinity in one direction — hence one can still define the product of two characters. For each $\lambda \in \mathfrak{t}^*$, there is a Verma module $M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one dimensional module $\mathbb{C}$ with $\mathfrak{b}$ acting by λ via the projection to $\mathfrak{t}$. It belongs to category $\mathcal{O}$. As a $\mathfrak{t}$-module, it is isomorphic to $\mathcal{U}(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$, so its character is easy. Moreover, M_λ contains a unique maximal submodule, and the quotient $L(\lambda)$ is the unique simple $\mathfrak{g}$-module in category $\mathcal{O}$ with highest weight λ. Again, there are several reductions (notably using central characters), so that to understand the character of all simple modules, it is enough to compute the composition multiplicities $[M(x \cdot 0) : L(y \cdot 0)]$, where $x, y \in W$ are such that $x \cdot 0 \geq y \cdot 0$.

The free $\mathbb{Z}[v, v^{-1}]$-module with basis $(T_x)_{x \in W}$ admits a structure of $\mathbb{Z}[v, v^{-1}]$-algebra uniquely determined by the properties that $T_x T_y = T_{xy}$ whenever $l(x) + l(y) = l(xy)$ and $(T_s + 1)(T_s - v^{-2}) = 0$ [Bou, IV, §2, Exercice 23]. It is the Hecke algebra of (W, S). Following [Soe1], we set $H_s = vT_s$. Then $\mathcal{H}$ has a presentation with generators H_s satisfying the quadratic relations $(H_s + v)(H_s - v^{-1}) = 0$ and the braid relations of (W, S), i.e. $H_s H_t H_s \ldots = H_t H_s H_t \ldots$ ($m_{s,t}$ terms on both sides) for $s \neq t$. More generally, we set $H_w = v^{l(w)} T_w$ for $w \in W$. We call $(H_w)_{w \in W}$ the standard basis. In [KL1], Kazhdan and Lusztig define a new basis $(\underline{H}_w)_{w \in W}$ of $\mathcal{H}$, as follows (here we use Soergel's normalization). The Hecke algebra is endowed with a ring involution $a \mapsto \overline{a}$, mapping H_w to $H_{w^{-1}}^{-1}$ and v to v^{-1}. The elements $\underline{H}_w$ are characterized by two properties: they are fixed by the involution, and $\underline{H}_w = H_w \mod \bigoplus_{y \in W} v\mathbb{Z}[v] H_y$. The Kazhdan–Lusztig polynomials (in short, KL polynomials) $h_{y,x} \in \mathbb{Z}[v, v^{-1}]$, for $x, y \in W$, are defined as the coefficients of the transition matrix between the two bases:

$$\underline{H}_x = \sum_{y \in W} h_{y,x} H_y.$$

Kazhdan and Lusztig conjectured that their values at 1 give the answer to the multiplicity problem:

$$\operatorname{ch} L(xw_0 \cdot 0) = \sum_{yw_0 \cdot 0 \le xw_0 \cdot 0} (-1)^{l(y)+l(x)} h_{y,x}(1) \operatorname{ch} M(yw_0 \cdot 0),$$

where w_0 is the longest element in W. (Comparing with [KL1], note that $xw_0 \cdot 0 = x \cdot (-2\rho) = -w\rho - \rho$.) Note that the definition of KL polynomials is valid for any Coxeter system. The Kazhdan–Lusztig positivity conjecture states that KL polynomials should have coefficients in $\mathbb{N}$, for any Coxeter system.

The flag variety admits an affine paving called the Bruhat decomposition:

$$G/B = \bigsqcup_{w \in W} BwB/B, \qquad BwB/B \simeq \mathbb{A}^{l(w)}.$$

For $w \in W$, the closure $X_w := \overline{BwB}/B$ is called a Schubert variety. It may have singularities. We have $X_y \subset X_x$ if and only if $y \le x$ for the Bruhat order, which means that y can be obtained by deleting some simple reflections from a reduced expression for x. In [KP1], Kazhdan and Lusztig noticed that the polynomial $h_{y,x}$ was a kind of measure of the failure of Poincaré duality of X_x.

Fortunately, the ideal tools to deal with the Kazhdan–Lusztig conjecture appeared at the right time. Namely, M. Goresky and R. MacPherson introduced intersection (co)homology in [GM1, GM2], which was a new cohomology theory well adapted to singular spaces: Poincaré duality is restored. In [GM2], the authors follow a suggestion of P. Deligne and J.-L. Verdier and upgrade their theory to the setting of the bounded constructible derived category. This is the full subcategory of the derived category of sheaves of vector spaces consisting of complexes $\mathcal{F}$ whose cohomology sheaves $\mathcal{H}^n\mathcal{F}$ restrict to local systems on each stratum of some stratification, and vanish for $|n|$ large.

In our situation, for each $x \in W$, there is a complex of sheaves $\mathcal{IC}(X_x)$ supported on X_x which extends the constant sheaf on the open cell BxB/B, and is self-dual (up to shift) with respect to Grothendieck–Verdier duality. The global intersection cohomology is then by definition $\mathrm{IH}^\bullet(X_x) = \mathsf{H}^\bullet(X_x, \mathcal{IC}(X_x))$. Deligne's construction of the $\mathcal{IC}$ complex uses only the usual sheaf operations (direct images by open immersions) and truncations in the derived category; hence it makes sense even in the setting of varieties over $\mathbb{F}_q$ with the étale topology. Using purity arguments, Kazhdan and Lusztig prove in [KL2] that the KL polynomial $h_{y,x}$ encodes the graded dimension of the local intersection cohomology of X_x at y, that is, the stalk $\mathcal{IC}(X_x)_y$. This shows the positivity conjecture in the case of Weyl groups.

Intersection cohomology complexes were later interpreted as the simple objects in the category of perverse sheaves [BBD], which is a full abelian subcategory of the constructible derived category. Working over $\mathbb{F}_q$, Beilinson, Bernstein, Deligne and Gabber exploited the powerful formalism of weights (which Deligne had used before to prove the last of the Weil conjectures) to derive a very strong result called the decomposition theorem. Part of its significance lies in the fact that it makes concrete calculations of local intersection cohomology feasible in many cases. For example, using the decomposition theorem it is very easy to prove that KL polynomials encode local intersection cohomology of Schubert varieties. The two conditions characterizing KL polynomials correspond to two conditions characterizing $\mathcal{IC}$ sheaves: self-duality and bounds on the cohomology degrees where the stalks can be non-zero. In the case of the flag variety with the Bruhat decomposition, only constant local systems had to be considered, but in general more general coefficients can appear. The simple perverse sheaves are of the form $\mathcal{IC}(\overline{S}, \mathcal{L})$, where S is a smooth irreducible locally closed subvariety, and $\mathcal{L}$ is a simple local system on $\mathcal{L}$. They are very interesting singularity invariants.

Very soon afterwards, the Kazhdan–Lusztig conjecture was proved independently by Beilinson–Bernstein and Brylinski–Kashiwara. Both proofs go from $\mathfrak{g}$-representations to D-modules on the flag variety, and then use the Riemann–Hilbert correspondence to go to perverse sheaves. An account is given in [HTT]. This success was followed by many others in geometric representation theory. For an excellent survey, see [Lu5].

Let us mention that the KL positivity conjecture has been proved recently in the full generality of an arbitrary Coxeter system [EW1]. W. Soergel defined a category of bimodules which categorifies the Hecke algebra of any Coxeter system, and in the case of a Weyl group provides an algebraic model for mixed perverse sheaves on the flag variety [Soe3]. He conjectured that the indecomposable Soergel bimodules have a "character" given by the KL basis. Some kind of algebraic analogue of the decomposition theorem was required to complete this program. In their wonderful proof, B. Elias and G. Williamson import Hodge theoretic techniques (coming from the proof of the decomposition theorem by de Cataldo and Migliorini) to the world of Soergel bimodules. This is also the first algebraic proof of the KL conjecture.

Now let us come back to the problem of modular representations of reductive algebraic groups. The KL polynomials for the Coxeter group W_a are called affine KL polynomials. They encode the local intersection cohomology of Schubert varieties in affine Kac–Moody flag varieties. Lusztig conjectured that their values at 1 give the multiplicities $[H^0(x \cdot 0) : L(y \cdot 0)]$, in much the same way as the Kazhdan–Lusztig conjecture. He also outlined a program for a proof, which was achieved in the 1990's, starting from representations for affine Kac–Moody Lie algebras (Kashiwara–Tanisaki, Casian), from there to quantum groups at roots of unity (Kazdhan–Lusztig), and then the reduction from characteristic 0 to reductive algebraic groups in characteristic p (Andersen–Jantzen–Soergel). However, the proof was only for p bigger than an unknown bound, depending on the root system. The conjecture was also reproved by P. Fiebig using moment graph techniques. He was able to prove the conjecture with the expected bound $p \geq h$ in the multiplicity one case. He could give an explicit but absolutely enormous bound in the general case. Actually the bound $p \geq h$ was very optimistic, as we shall see in §3.1.

1.4 Springer correspondence

Another very early application of perverse sheaves to representation theory was the reinterpretation [Lu1, BM] of Springer correspondence [Spr]. Here G will be a connected complex reductive group again (although it is possible and useful to consider the case of a base field of characteristic $p > 0$ and the étale topology), and we keep the notation from §1.2. In particular, recall that $\mathcal{B} = G/B$ denotes the flag variety. The variety $\mathcal{N}_G$ of nilpotent elements in $\mathfrak{g}$ consists of finitely many adjoint orbits, called *nilpotent orbits*. The Springer correspondence makes a link between the Weyl group W of G and the nilpotent orbits in $\mathcal{N}_G$. In my thesis, I studied a modular analogue. I will describe both cases now. We will need an ℓ-modular system of coefficients $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ as in §1.1.

Let $\tilde{\mathfrak{g}} = \{(x, gB) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathrm{Ad}(g)(\mathfrak{b})\}$. The first projection $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is known as Grothendieck's simultaneous resolution of the singularities of the fibers of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{g}/\!/G \simeq \mathfrak{t}/W$ (the last isomorphism being Chevalley's restriction theorem), in the sense that in the following commutative diagram

$$\begin{tikzcd}
\tilde{\mathfrak{g}} \arrow[r, "\pi"] \arrow[d, "\theta"'] & \mathfrak{g} \arrow[d, "\chi"] \\
\mathfrak{t} \arrow[r, "\phi"'] & \mathfrak{t}/W
\end{tikzcd}$$

the morphism π is proper (because it is projective) and surjective, the morphism $\theta : (x, gB) \mapsto \mathrm{Ad}(g^{-1})(x) + [\mathfrak{b}, \mathfrak{b}] \in \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \simeq \mathfrak{t}$ is smooth and surjective, the quotient morphism $\phi : x \to \overline{x}$ is finite and surjective, while χ is flat and for each $x \in \mathfrak{t}$, the morphism π induces a resolution of singularities $\theta^{-1}(x) \to \chi^{-1}(\overline{x})$. The morphism χ can be interpreted as the map sending x to the adjoint orbit of its semisimple part (the points of $\mathfrak{g}/\!/G$ correspond to the closed adjoint orbits, which are the semisimple ones, and each such orbit intersects $\mathfrak{t}$ in a W-orbit); in the case $G = \mathrm{SL}_n$, it can also be interpreted as the map giving the characteristic polynomial. In the case $x = 0$, the restriction of π gives Springer's resolution of the nilpotent cone $\pi_{\mathcal{N}} : \widetilde{\mathcal{N}}_G \to \mathcal{N}_G$.

Springer (and later Lusztig, Borho–MacPherson...) constructed an action of W on the cohomology of the fibers $\mathcal{B}_x := \pi^{-1}(x)$, for $x \in \mathfrak{g}$. Moreover, he showed that one can find all irreducible representations of W inside the top cohomology groups $H^{\mathrm{top}}(\mathcal{B}_x)$ of the Springer fibers, with $x \in \mathcal{N}_G$. However, this action is not easy to define, as it does not come in general from an action on $\mathcal{B}_x$ itself. We will explain Lusztig's construction of the W-action, which uses the machinery of intersection cohomology [Lu1]. The idea is to consider all the cohomology groups of all fibers $\mathcal{B}_x$, $x \in \mathfrak{g}$, in a family: by the proper base change theorem, the constructible complex $\mathrm{Groth} := \pi_* \underline{\mathbb{E}}$ (where $\mathbb{E} = \mathbb{K}$ or $\mathbb{F}$) satisfies

$$\mathcal{H}^i(\mathrm{Groth})_x = H^i(\mathcal{B}_x, \mathbb{E}).$$

So one can extract from this single object in $D^b_c(\mathfrak{g}, \mathbb{E})$ all cohomology groups of all Springer fibers, just taking the cohomology of its stalks. Thus to get a W-action on those cohomology groups, it is enough to have an action on Groth. Now the nice thing is that π is generically a W-torsor, namely above the open dense subvariety $\mathfrak{g}_{\mathrm{rs}}$ of regular semisimple elements: if $x \in \mathfrak{g}_{\mathrm{rs}}$ then it is contained in a unique Cartan subalgebra, and the Borel subalgebras containing it are in bijection with the Weyl chambers, hence with W. One can understand this easily for $G = \mathrm{SL}_n$: being regular semisimple means having n distinct eigenvalues, and the stabilized flags are in bijection with all possible permutations of the n one-dimensional eigenspaces. It follows that for $x \in \mathfrak{g}_{\mathrm{rs}}$, the fiber $\mathcal{B}_x$ consists of a copy of W, on which W acts by left translation. Better, if $\pi_{\mathrm{rs}} : \tilde{\mathfrak{g}}_{\mathrm{rs}} \to \mathfrak{g}_{\mathrm{rs}}$ is the restriction of π above $\mathfrak{g}_{\mathrm{rs}}$, then W acts on $\tilde{\mathfrak{g}}_{\mathrm{rs}}$, and the direct image $\pi_{\mathrm{rs},*}\underline{\mathbb{E}}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}}$ is the local system $\mathrm{Groth}_{\mathrm{rs}}$ corresponding to the regular representation of W (the fundamental group of $\mathfrak{g}_{\mathrm{rs}}$ is the braid group B_W, and there is a short exact sequence $1 \to P_W \to B_W \to W \to 1$ where P_W is the pure braid group; the monodromy of $\mathrm{Groth}_{\mathrm{rs}}$ factors though W).

Now comes the magic of intersection cohomology. Because π is proper and *small* (this condition is a bound on the dimensions of the fibers $\mathcal{B}_x$ with respect to some adapted stratification), the direct image Groth is actually determined by its restriction $\mathrm{Groth}_{\mathrm{rs}}$ to $\mathfrak{g}_{\mathrm{rs}}$ (where it is étale): it is the intersection cohomology complex of $\mathfrak{g}$ with coefficients in the local system $\mathrm{Groth}_{\mathrm{rs}}$ on $\mathfrak{g}_{\mathrm{rs}}$. In other words, it is obtained from $\mathrm{Groth}_{\mathrm{rs}}$ by the functor of intermediate extension $j_{\mathrm{rs},!*}$. Already from this we see that W acts on Groth since it acts on $\mathrm{Groth}_{\mathrm{rs}}$, and hence on all cohomology groups $H^i(\mathcal{B}_x)$. But, even better, the intermediate extension functor is fully faithful, so we have isomorphisms $\mathbb{E}W = \mathrm{End}_{\mathbb{E}W}(\mathbb{E}W) = \mathrm{End}(\mathrm{Groth}_{\mathrm{rs}}) = \mathrm{End}(\mathrm{Groth})$.

$$\begin{tikzcd}
\tilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[r, hook, "\tilde{j}_{\mathrm{rs}}"] \arrow[d, "\pi_{\mathrm{rs}}"'] & \tilde{\mathfrak{g}} \arrow[d, "\pi"'] & \widetilde{\mathcal{N}}_G \arrow[l, hook', "i_{\mathcal{N}}"'] \arrow[d, "\pi_{\mathcal{N}}"'] \\
\mathfrak{g}_{\mathrm{rs}} \arrow[r, hook, "j_{\mathrm{rs}}"'] & \mathfrak{g} & \mathcal{N}_G \arrow[l, hook', "i_{\mathcal{N}}"]
\end{tikzcd}$$

Now we want to go on to the nilpotent cone. The complex $\mathrm{Spr} := \pi_{\mathcal{N},*}\underline{\mathbb{E}}_{\widetilde{\mathcal{N}}_G}$ is actually a perverse sheaf (up to shift) because $\pi_{\mathcal{N}}$ is proper and *semi-small* (a condition a bit laxer than small). The obvious way is to just use the restriction functor to the nilpotent cone, since $i_{\mathcal{N}}^*\mathrm{Groth} = \mathrm{Spr}$ by the proper base change theorem. This shows that W acts on Spr. But more is true: the morphism

$$\mathrm{Res} := i_{\mathcal{N}}^* : \mathrm{End}(\mathrm{Groth}) \longrightarrow \mathrm{End}(\mathrm{Spr})$$

is an isomorphism. This was proved by Borho and MacPherson when $\mathbb{E} = \mathbb{K}$ in two steps [BM]. First, prove injectivity by identifying the zero stalk Spr_0, which is the cohomology of G/B with the Springer W-action, with the cohomology of G/T with the classical W-action coming from the right W-action on G/T; it is well known that this is the regular representation of W; in particular it is faithful and we have the injectivity. Secondly, compare the dimensions using results of Steinberg.

In the modular case, this method does not work directly: the cohomology of the zero stalk Spr_0 is not the regular representation; it is not even faithful in general (see the example of $G = \mathrm{SL}_2$ in Subsection 3.3). This difficulty was avoided in [1] by considering the other known method (due to Hotta–Kashiwara in the context of D-modules, and to Brylinski in the context of ℓ-adic sheaves) to go from Groth to Spr, that is, a Fourier transform $\mathbb{T}_{\mathfrak{g}}$ (the Fourier–Sato transform in the present context, or the Fourier-Deligne transform in the étale case). It turns out that this transform just exchanges Groth and Spr (with appropriate shifts). Because it is a self-equivalence, $\mathbb{T}_{\mathfrak{g}}$ induces an isomorphism

$$\mathbb{T}_{\mathfrak{g}} : \mathrm{End}(\mathrm{Groth}) \longrightarrow \mathrm{End}(\mathrm{Spr}).$$

I was thus able to define a modular Springer correspondence using the Fourier transform in [PhD]. For a nilpotent orbit $\mathcal{O}$, we denote by $\mathrm{Loc}_G(\mathcal{O}, \mathbb{E})$ the category of G-equivariant $\mathbb{E}$-local systems on $\mathcal{O}$. The category $\mathbf{P}(\mathcal{N}_G, \mathbb{E})$ of G-equivariant perverse sheaves on $\mathcal{N}_G$ has for simple objects the intersection cohomology complexes $\mathcal{IC}(\overline{\mathcal{O}}, \mathcal{L})$, where $(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}_{G,\mathbb{E}} := \{(\mathcal{O}, \mathcal{L}) \mid \mathcal{O} \subset \mathcal{N}_G,\ \mathcal{L} \in \mathrm{Irr}\,\mathrm{Loc}_G(\mathcal{O})\}$. A simple G-equivariant local system $\mathcal{L}$ on the orbit $\mathcal{O}$ may be seen as a simple $\mathbb{E}A_G(\mathcal{O})$-module, where $A_G(\mathcal{O}) := C_G(x_{\mathcal{O}})/C_G(x_{\mathcal{O}})^0$ for some choice of representative $x_{\mathcal{O}} \in \mathcal{O}$.

Theorem 1. *The intermediate extension functor $j_{\mathrm{rs},!*}$ and the Fourier transform $\mathbb{T}_{\mathfrak{g}}$ allow to define an injection:*

$$\psi_{G,\mathbb{E}} : \mathrm{Irr}\,\mathbb{E}W \hookrightarrow \mathrm{Irr}\,\mathbf{P}_G(\mathcal{N}_G, \mathbb{E}) \simeq \mathfrak{P}_{G,\mathbb{E}}$$

Moreover, the Springer correspondence preserves decomposition numbers. Recall from §1.1 that for $E \in \mathrm{Irr}\,\mathbb{K}W$ and $F \in \mathrm{Irr}\,\mathbb{F}W$, we have a decomposition number $d^W_{E,F}$. Similarly, for $\mathcal{E} \in \mathrm{Irr}\,\mathbf{P}_G(\mathcal{N}_G, \mathbb{K})$ and $\mathcal{F} \in \mathrm{Irr}\,\mathbf{P}_G(\mathcal{N}_G, \mathbb{F})$, one can define a decomposition number $d^{\mathcal{N}}_{\mathcal{E},\mathcal{F}}$.

Theorem 2. *For $E \in \mathrm{Irr}\,\mathbb{K}W$ and $F \in \mathrm{Irr}\,\mathbb{F}W$, we have $d^W_{E,F} = d^{\mathcal{N}}_{\psi_{G,\mathbb{K}}(E),\psi_{G,\mathbb{F}}(F)}$.*

The decomposition matrix for perverse sheaves is easily seen to be triangular, by considering supports. Using this fact and a similar known property for the decomposition matrix of the symmetric group, I could determine the modular Springer correspondence for $G = \mathrm{GL}_n$ thanks to Theorem 2.

To be more precise, let us recall James's classification of simple $\mathbb{F}\mathfrak{S}_n$-modules. First, the simple $\mathbb{K}\mathfrak{S}_n$-modules are the Specht modules S^λ, where λ belongs to the set $\mathrm{Part}(n)$ of partitions of n; they are defined over $\mathbb{Z}$ and have a natural symmetric bilinear form, defined over $\mathbb{Z}$ and positive definite when tensored with $\mathbb{Q}$; when tensored with $\mathbb{F}$, the radical of the form is either a maximal submodule, or the whole module; hence the quotient by the radical is either a simple module or zero. It is non-zero if and only if λ belongs to the set $\mathrm{Part}_\ell(n)$ of ℓ-regular partitions of n, i.e. all parts of λ should occur with multiplicity $< \ell$. It is then denoted by D^λ. The D^λ for $\lambda \in \mathrm{Part}_\ell(n)$ form a complete set of non-isomorphic simple $\mathbb{F}\mathfrak{S}_n$-modules. Note that this labelling implicitly defines an injection $\beta_J : \mathrm{Irr}\,\mathbb{F}W \hookrightarrow \mathrm{Irr}\,\mathbb{K}W$, namely $D^\lambda \mapsto S^\lambda$. James showed that decomposition numbers $d^{\mathfrak{S}_n}_{S^\lambda, D^\mu}$ satisfy a unitriangularity property: they can be non-zero only for $\mu \geq \lambda$ (with respect to the dominance order), and are equal to 1 when $\lambda = \mu$.

On the other hand, nilpotent orbits for GL_n are also parametrized by $\mathrm{Part}(n)$, by the Jordan canonical form. We will denote them by $\mathcal{O}_\lambda$, with $\lambda \in \mathrm{Part}(n)$. The closure inclusion order on the orbits is again given by the dominance order of partitions: by a theorem of Gerstenhaber, $\mathcal{O}_\mu \leq \mathcal{O}_\lambda$ if and only if $\mu \leq \lambda$. Hence the decomposition numbers for perverse sheaves on the nilpotent cone satisfy the same unitriangularity property as those for the symmetric group, but with the opposite order. It turns out that the ordinary Springer correspondence via Fourier transform sends S^λ to $\mathcal{IC}(\overline{\mathcal{O}}_{\lambda^{\mathrm{t}}}, \mathbb{K})$, where λ^{t} is the transposed partition. The transposition reverses the dominance order. Using those properties and Theorem 2, we see that there is only one possibility for the modular Springer correspondence: $\psi_{G,\mathbb{F}}(D^\lambda) = \mathcal{IC}(\overline{\mathcal{O}}_{\lambda^{\mathrm{t}}}, \mathbb{F})$.

It follows that James's row and column removal rule for decomposition numbers of symmetric groups can be seen as a consequence of an equivalence of nilpotent singularities proved by Kraft and Procesi [KP1].

Apart from those results, my thesis contained generalities about modular perverse sheaves and torsion (which gave rise to the article [9]); explicit geometric calculations of decomposition numbers involving the regular and subregular nilpotent orbits on the one hand (using the results of Brieskorn and Slodowy showing that the singularity is a rational double point with some symmetries), and the minimal and trivial nilpotent orbits on the other hand (this required to compute the integral cohomology of the minimal nilpotent orbit, see [10]); and tables for low ranks. The article [1] contains both the main results of my thesis about the modular Springer correspondence, and new results which will be described below.

1.5 Outline of the research report

Complements to the modular Springer correspondence. In [7], we compare the "Fourier transform" and "restriction to the nilpotent cone" approaches in the modular case, and deduce some consequences. Completing the results of my thesis, the notion of Springer basic set [1, 16] gives a geometric proof that the decomposition matrices of Weyl groups are unitriangular, and provides a way to deduce the determination of the modular Springer correspondence from that of the ordinary Springer correspondence. Tables for exceptional groups are given in [1]. We determine the modular Springer correspondence for classical types in [16] which is joint work with C. Lecouvey and K. Sorlin (in the case $\ell \neq 2$). This requires to compare a coarse dominance order on bipartitions with the order induced by the Springer correspondence (via closure inclusions of nilpotent orbits).

Modular generalized Springer correspondence. In a series of articles with P. Achar, A. Henderson and S. Riche [4, 2, 13], we develop a modular version of Lusztig's generalized Springer correspondence, in the hope of potential applications to the modular representation theory of finite reductive groups in non-defining characteristic. In [12], we focus on the case of coefficients in "rather good" characteristic, where we can be more precise on the relation with the characteristic zero theory. See [11] for an overview.

Geometry of nilpotent cones. The work on the Springer correspondence led me to study in more detail the geometry of nilpotent cones. In joint work with B. Fu, P. Levy and E. Sommers [15], we study in particular all minimal degenerations in the nilpotent cones of simple Lie algebras of exceptional type (this problem was solved in classical types by H. Kraft and C. Procesi in the 1980's).

Parity sheaves. In joint work with C. Mautner and G. Williamson [6], we introduce the notion of parity sheaves, which is particularly well adapted for the geometric theory of modular representations, and which opened new perspectives. They satisfy a weak form of the decomposition theorem, and their stalks define a p-canonical basis, generalizing the Kazhdan–Lusztig basis, which can be computed by computer (up to a certain point). In [3], we prove that parity sheaves on the affine Grassmannian correspond via the geometric Satake equivalence to tilting modules for the dual reductive group (under mild conditions on the characteristic). This gives a nice geometric explanation for the stability of tilting modules by tensor product and by restriction to a Levi subgroup.

Kumar's criterion modulo p. In this joint work with G. Williamson [5], we give in particular an algorithm, in terms of equivariant multiplicities, to determine the p-smooth locus of Schubert varieties (building on previous work of S. Kumar, A. Arabia and M. Brion on the smooth and rationally smooth loci).

Rational Cherednik algebras. In joint work with S. Griffeth, A. Gusenbauer and M. Lanini [14], we introduce a new tool called parabolic degeneration, and we deduce two necessary criteria for two important problems: understanding which simple modules are finite dimensional, and when there can be a non-zero morphism between two standard modules.

1.6 Outline of the research project

Modular representation theory of reductive algebraic groups. The big aim is to understand the category of representations of a reductive algebraic group in characteristic p. We would like to understand the characters of simple modules, but it makes sense to consider the a priori harder problem of determining the characters of tilting modules. The framework of S. Riche and

G. Williamson is very appealing: they have a conjecture giving those characters in terms of a p-canonical basis (which is related to parity sheaves), and they proved it for GL_n (see [RW]). But this is not the end of the story: now one has to understand that p-canonical basis. The characters of simple modules would be a consequence.

Modular characters sheaves and representations of finite reductive groups in transverse characteristic. Building on the modular generalized Springer correspondence, we want to develop a theory of modular character sheaves in relation with the modular representation theory of finite groups of Lie type. It would be nice to see the interaction with modular Deligne–Lusztig theory.

A W-equivariant model for G/T. The problem came from the study of the structure of the 0-stalk of the Springer sheaf with integer coefficients. In the end, we want to understand G/T as a CW-complex, or as a simplicial set, but equivariantly with respect to the Weyl group.

Geometry of special nilpotent orbits. We have work in progress (joint with B. Fu, P. Levy and E. Sommers) about the singularities of certain unions of nilpotent orbits involving the Lusztig–Spaltenstein theory of special orbits and duality. It is a continuation of [15]. Here we see the appearance of configuration spaces like $\mathfrak{h} \oplus \mathfrak{h}^*/W$, where $\mathfrak{h}$ is the reflection representation of a Coxeter group (usually a symmetric group). Those are very well-known symplectic singularities. Lusztig has conjectured that the so-called special pieces are quotients of unknown smooth varieties by precise finite groups. We are able to prove that they are locally the product of a configuration space (sometimes with a higher number of copies of the reflection representation) by a smooth space. Also, we see many relations between parts of nilpotent cones of different types.

Motives, periods, and biarrangements. In this project in collaboration with C. Dupont, we apply sheaf-theoretic techniques to develop further his theory of motives of bi-arrangements of hypersurfaces. This is motivated by the study of periods like multiple zeta values.

2 Research report

2.1 Complements to the modular Springer correspondence

Results obtained after my thesis allowed in particular to determine the modular Springer correspondence explicitly in all types.

2.1.1 Fourier transform versus restriction

In [7] (joint work with P. Achar, A. Henderson and S. Riche), we show that two Weyl group actions on the Springer sheaf with arbitrary coefficients, one defined by Fourier transform and one by restriction, agree up to a twist by the sign character. This generalizes a familiar result from the setting of ℓ-adic cohomology, making it applicable to modular representation theory. We use the Weyl group actions to define a Springer correspondence in this generality, and identify the zero weight spaces of small representations in terms of this Springer correspondence.

2.1.2 Springer basic sets

This idea, developed in [1] and [16], is to try to adapt the proof of the determination of the modular Springer correspondence for GL_n for other reductive groups: unitriangularity of decomposition matrices both for Weyl groups and perverse sheaves should determine the correspondence.

A basic set for W is a basis of the space of central functions on the ℓ-regular elements, which can be seen as the Grothendieck group of $\mathbb{F}W$-modules. It is an ordinary basic set if it consists of restrictions of ordinary characters. We will say that we have a unitriangular ordinary basic set if there is a partial order $\leq$ on $\operatorname{Irr}\mathbb{K}W$ and an injection $\beta : \operatorname{Irr}\mathbb{F}W \hookrightarrow \operatorname{Irr}\mathbb{K}W$ such that:

$$\forall F \in \operatorname{Irr}\mathbb{F}W,\ d^W_{\beta(F),F} = 1, \tag{1}$$

$$\forall E \in \operatorname{Irr}\mathbb{K}W,\ \forall F \in \operatorname{Irr}\mathbb{F}W,\ d^W_{E,F} \neq 0 \Longrightarrow E \leq \beta(F). \tag{2}$$

We will see that we can obtain a triangular ordinary basic set for W using the Springer correspondence. As a preliminary, let us note that since we deal with the non-generalized Springer correspondence, we may assume that G is simple and adjoint. Then the components groups $A_G(\mathcal{O}) = C_G(x_\mathcal{O})/C_G(x_\mathcal{O})^0$ are either elementary abelian 2-groups or symmetric groups $\mathfrak{S}_k$ (with $k \leq 5$); hence we can choose injections $\beta_\mathcal{O} : \operatorname{Irr}\mathbb{F}A_G(\mathcal{O}) \hookrightarrow \operatorname{Irr}\mathbb{K}A_G(\mathcal{O})$ and an order on $\operatorname{Irr}\mathbb{K}A_G(\mathcal{O})$ defining a unitriangular basic set for $A_G(\mathcal{O})$. Putting them all together, this defines an injection $\beta_\mathcal{N} : \mathfrak{P}_{G,\mathbb{F}} \hookrightarrow \mathfrak{P}_{G,\mathbb{K}}$. Moreover, the ordinary Springer correspondence $\psi_{G,\mathbb{K}}$ induces a partial order on $\operatorname{Irr}\mathbb{K}W$ by first comparing orbits, then local systems on the same orbit.

Theorem 3. *There exists an injection $\beta : \operatorname{Irr} \mathbb{F}W \hookrightarrow \operatorname{Irr} \mathbb{K}W$ making the following diagram commutative:*

$$\begin{tikzcd}
\operatorname{Irr} \mathbb{F}W \arrow[r, hook, "\psi_{G,\mathbb{F}}"] \arrow[d, hook, dashed, "\beta"] & \mathfrak{P}_{G,\mathbb{F}} \arrow[d, hook, "\beta_{\mathcal{N}}"] \\
\operatorname{Irr} \mathbb{K}W \arrow[r, hook, "\psi_{G,\mathbb{K}}"] & \mathfrak{P}_{G,\mathbb{K}}
\end{tikzcd}$$

It satisfies (1) *and* (2)*, hence defines a unitriangular ordinary basic set, which we call Springer basic set.*

The proof relies on Theorem 2 and the following proposition.

Proposition 4. *For $(x, \rho) \in \mathfrak{P}_{G,\mathbb{K}}$ and $(y, \sigma) \in \mathfrak{P}_{G,\mathbb{F}}$, if $(x, \rho) \notin \operatorname{Im} \psi_{G,\mathbb{K}}$ and $d^{\mathcal{N}}_{(x,\rho),(y,\sigma)} \neq 0$ then $(y, \sigma) \notin \operatorname{Im} \psi_{G,\mathbb{F}}$.*

Theorem 3 was used in [1] to determine the modular Springer correspondence explicitly in all exceptional types. Let us give the example of type G_2 for $\ell = 2$. Here is the part of the character table of W restricted to the 2-regular classes, with the ordinary characters ordered by the Springer correspondence.

0	$\chi_{1,0}$	1	1
A_1	$\chi'_{1,3}$	1	1
$\tilde{A}_1$	$\chi_{2,2}$	2	-1
$G_2(a_1), 3$	$\chi_{2,1}$	2	-1
$G_2(a_1), 21$	$\chi''_{1,3}$	1	1
G_2	$\chi_{1,6}$	1	1

The Springer basic set corresponds to the lines which are not in the linear span of the preceding lines, namely $\{\chi_{1,0}, \chi_{2,2}\}$ in this example. Moreover, we deduce that the modular Springer correspondence associates 0 to the trivial Brauer character, and $\tilde{A}_1$ to the other Brauer character.

Of course, if the decomposition matrix is known (which is the case for all exceptional Weyl groups, see [Kho84, KM85]), one can read off the correspondence even more directly. For G_2, it turns out that the character $\chi_{2,2}$ remains irreducible modulo 2, hence the decomposition matrix is as follows:

0	$\chi_{1,0}$	1	.
A_1	$\chi'_{1,3}$	1	.
$\tilde{A}_1$	$\chi_{2,2}$	.	1
$G_2(a_1), 3$	$\chi_{2,1}$	.	1
$G_2(a_1), 21$	$\chi''_{1,3}$	1	.
G_2	$\chi_{1,6}$	1	.

The Springer basic set corresponds to the top 1's in each column. The modular Springer correspondence follows. Actually this can be done blockwise. We have the principal block, of defect two:

0	$\chi_{1,0}$	1
A_1	$\chi'_{1,3}$	1
$G_2(a_1), 21$	$\chi''_{1,3}$	1
G_2	$\chi_{1,6}$	1

and another block of defect one:

$\tilde{A}_1$	$\chi_{2,2}$	1
$G_2(a_1), 3$	$\chi_{2,1}$	1

The G_2 example is small anyway, but for larger groups it is much more convenient to work blockwise to keep the matrices to a reasonable size (see [1]).

2.1.3 Modular Springer correspondence for classical types

With C. Lecouvey and K. Sorlin, we determined the modular Springer correspondence for classical types, when ℓ is odd [16]. Since these are infinite series, the fact that there is an algorithm for any given type is not enough: we need a general proof, like for GL_n, using just some known information about the decomposition matrix. We define the notion of *basic set datum* for W as a pair $(\leq, \beta)$ satisfying (1) and (2) and observe that if $(\leq_1, \beta_1)$ and $(\leq_2, \beta_2)$ are two basic set data with $\leq_1 \subset \leq_2$, then $\beta_1 = \beta_2$. Thus we are reduced to compare a known basic set (due to Dipper–James) with the Springer basic set.

For simplicity, let us assume here that we are in type B_n or C_n, so the Weyl group is $W = W_n = \{\pm 1\}^n \rtimes \mathfrak{S}_n$. Then $\operatorname{Irr} \mathbb{K} W_n = \{S^{(\lambda^1,\lambda^2)}\}$ is parametrized by bipartitions of n, while $\operatorname{Irr} \mathbb{F} W_n = \{D^{(\mu^1,\mu^2)}\}$ is parametrized by ℓ-regular bipartitions of n: both μ^1 and μ^2 have to be ℓ-regular. We write $(\mu^1, \mu^2) \geq (\lambda^1, \lambda^2)$ if $\left|\mu^i\right| = |\lambda^i|$ and $\mu^i \geq \lambda^i$, for $i = 1, 2$. Then the injection $D^{(\mu^1,\mu^2)} \mapsto S^{(\mu^1,\mu^2)}$ and the dominance order define a basic set datum for W_n (see [DJ]). We prove that the dominance order is compatible with the Springer correspondence, which implies that the Springer basic set is the same thing as the Dipper–James basic set. Hence the image of $D^{(\mu^1,\mu^2)}$ by the modular Springer correspondence is "the same" as the image of $S^{(\mu^1,\mu^2)}$ by the ordinary Springer correspondence, where we identify local systems over $\mathbb{K}$ and over $\mathbb{F}$ because all component groups are 2-groups and $\ell \neq 2$. The combinatorics for type D_n are similar but one should take unordered pairs, and add $\pm$ decorations where needed. The case $\ell = 2$ is clearly very different, but in that case the full modular generalized Springer correspondence is described in [2].

2.2 Modular generalized Springer correspondence

In this joint project with P. Achar, A. Henderson and S. Riche [4, 2, 13], we develop a modular version (for a complex reductive Lie algebra, using a Fourier transform) of Lusztig's generalized Springer correspondence [Lu2, Lu4]. In [12], we focus on the case where ℓ is *rather good* for G, that is, ℓ is good for G and does not divide the order of the component group of the centre of G: in that case there is a more precise relationship between the ordinary and modular cases.

2.2.1 Main theorem

The theory is parallel to Harish-Chandra theory for finite reductive groups. For $L \subset P$ a Levi factor in a parabolic subgroup, there is an induction functor $\mathbf{I}_{L\subset P}^G : D^b_L(\mathcal{N}_L, \mathbb{E}) \to D^b_G(\mathcal{N}_G, \mathbb{E})$ with left and right adjoint functors ${}'\mathbf{R}_{L\subset P}^G, \mathbf{R}_{L\subset P}^G : D^b_G(\mathcal{N}_G, \mathbb{E}) \to D^b_L(\mathcal{N}_L, \mathbb{E})$. They are all t-exact: they preserve perverse sheaves. A simple perverse sheaf $\mathcal{F} \in \mathbf{P}(\mathcal{N}_G, \mathbb{E})$ is said to be cuspidal if $\mathbf{R}_{L\subset P}^G \mathcal{F} = 0$ for all $L \subset P \subsetneq G$. By a result of Braden on hyperbolic localization, we have ${}'\mathbf{R}_{L\subset P}^G = \mathbf{R}_{L\subset P^-}^G$, where P^- is the opposite parabolic subgroup with respect to L. Hence both versions of restriction give the same notion of cuspidality. Moreover, by adjunction, $\mathcal{F}$ is cuspidal if and only if it is not a quotient (or, equivalently, a socle) of a non-trivially induced perverse sheaf. Note that in [Lu2], Lusztig gives another definition of cuspidality. It turns out to be equivalent to the one we use for $\mathbb{E} = \mathbb{K}$, however in the modular case Lusztig's notion implies ours but not conversely.

A cuspidal datum for G a triple $(L, \mathcal{O}_L, \mathcal{E}_L)$ where L is a Levi subgroup and $(\mathcal{O}_L, \mathcal{E}_L)$ is cuspidal for L. It gives rise to the induction series $\mathfrak{N}_{G,\mathbb{E}}^{(L,\mathcal{O}_L,\mathcal{E}_L)}$, the set of isomorphism classes of simple quotients of $\mathbf{I}_{L\subset P}^G \,\mathcal{IC}(\overline{\mathcal{O}}, \mathcal{L})$. We denote by $\mathfrak{M}_{G,\mathbb{E}}$ the set of cuspidal data for G.

We say that $\mathbb{E}$ is big enough for G if for every Levi subgroup L of G and every pair $(\mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{N}_{L,\mathbb{E}}$, the irreducible L-equivariant local system $\mathcal{E}_L$ is absolutely irreducible. We set $W_G(L) := N_G(L)/L$.

Theorem 5. *Assume $\mathbb{E}$ is big enough for G. Then we have a disjoint union*

$$\mathfrak{N}_{G,\mathbb{E}} = \bigsqcup_{(L,\mathcal{O}_L,\mathcal{E}_L)\in\mathfrak{M}_{G,\mathbb{E}}} \mathfrak{N}_{G,\mathbb{E}}^{(L,\mathcal{O}_L,\mathcal{E}_L)}, \tag{3}$$

and for any $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,\mathbb{E}}$ we have a canonical bijection

$$\psi_{G,\mathbb{E}}^{(L,\mathcal{O}_L,\mathcal{E}_L)} \colon \operatorname{Irr} \mathbb{E}W_G(L) \xrightarrow{\sim} \mathfrak{N}_{G,\mathbb{E}}^{(L,\mathcal{O}_L,\mathcal{E}_L)}, \tag{4}$$

Hence we obtain a bijection

$$\Psi_{G,\mathbb{E}} := \bigsqcup \psi_{G,\mathbb{E}}^{(L,\mathcal{O}_L,\mathcal{E}_L)} \colon \bigsqcup_{(L,\mathcal{O}_L,\mathcal{E}_L)\in\mathfrak{M}_{G,\mathbb{E}}} \operatorname{Irr} \mathbb{E}W_G(L) \xrightarrow{\sim} \mathfrak{N}_{G,\mathbb{E}}. \tag{5}$$

For $\mathbb{E} = \mathbb{F}$, we call it the *modular generalized Springer correspondence*. For $\mathbb{E} = \mathbb{K}$ (in the étale topology setting), the generalized Springer correspondence was developed by Lusztig in the group case (using restriction rather than Fourier transform) [Lu2]; later he studied the Fourier transform on the Lie algebra [Lu4]. The classical (ordinary or modular) Springer correspondence $\psi_{G,\mathbb{E}}$ is $\psi_{G,\mathbb{E}}^{(T,0,\mathbb{E})}$.

It is not hard to see that every simple perverse sheaf on $\mathcal{N}_G$ is a quotient of the induction of some cuspidal perverse sheaf; the point of (3) is the disjointness of the induction series. In [4, 2] we proved it inductively simultaneously with the classification of cuspidal perverse sheaves for general linear groups first, then for all classical groups: by the classification of the cuspidal perverse sheaves for proper Levi subgroups (see below), we observe that if we have two non-conjugate cuspidal data $(L, \mathcal{O}_L, \mathcal{E}_L)$ and $(M, \mathcal{O}_M, \mathcal{E}_M)$ on smaller Levi subgroups, then either $(L, \mathcal{O}_L)$ and $(M, \mathcal{O}_M)$ are already non-conjugate, or the local systems $\mathcal{E}_L$ and $\mathcal{E}_M$ afford distinct central characters. This is enough to ensure the disjunction. In turn, once the theorem is established for G, we can use it to count the number of cuspidal perverse sheaves for G, which in those cases allow us to classify them. This knowledge can then be used to prove the theorem for bigger groups where G appears as a Levi subgroup.

However, this strategy failed for exceptional groups in bad characteristic, because we did not have enough information to classify all cuspidal perverse sheaves. This was not so bad for G_2 or F_4 because they do not appear as Levi subgroups in bigger simple groups. But the ambiguity for E_7, $\ell = 2$ prevented us from proving disjunction in type E_8, for $\ell = 2$. Finally, we gave a general proof in [13], adapting the argument of [MS] to the modular case. We still have a Mackey type formula, but with a filtration instead of a direct sum.

2.2.2 Classification of modular cuspidal perverse sheaves

The first remark to be made is that characteristic zero cuspidal perverse sheaves give rise to modular cuspidal perverse sheaves by modular reduction.

Proposition 6. *If $\mathcal{F}$ is an ordinary cuspidal perverse sheaf, then all composition factors of its modular reduction are cuspidal.*

The main tool for the classification of cuspidal pairs is, as usual, an inductive count, using (5) in Theorem 5: by induction, one can determine the number of cuspidal data involving Levi subgroups strictly smaller than G; the number of modular pairs in each of the corresponding induction series is equal to the number of modular irreducible representations of the "relative Weyl group"; then one just has to substract from the total number of modular pairs the cardinal of the disjoint union of those induction series. This strategy is crucial already in [Lu2]. In the most favorable case, when doing the count we see that there are no more modular perverse sheaves than those given by Proposition 6.

Theorem 7. *Suppose that G is semisimple and simply connected and that ℓ is good for G. Then all modular cuspidal perverse sheaves are modular reductions of ordinary cuspidal perverse sheaves.*

Note that the classification for an arbitrary reductive group G can be reduced to the case where G is semisimple and simply connected. However the statement in Theorem 7 is false for general G, as we will see shortly. Let us explain the count for GL_n (see [4]). This group has several particularly nice features: all $A_G(\mathcal{O})$ are trivial, the Levi subgroups are products of smaller general linear groups, and similarly for the Weyl group $W = \mathfrak{S}_n$. Moreover, we use the result for GL_n when we deal with other groups.

The Levi subgroups are of the form $L_\nu := \mathrm{GL}_1^{m_1(\nu)} \times \mathrm{GL}_2^{m_2(\nu)} \times \cdots$, where $\nu = (1^{m_1(\nu)}, 2^{m_2(\nu)}, \dots) \in \mathrm{Part}(n)$ (using a standard notation to denote partitions in terms of the multiplicities of their parts). We have $W_\nu := N_G(L_\nu)/L_\nu \simeq \mathfrak{S}_{m_1(\nu)} \times \mathfrak{S}_{m_2(\nu)} \times \cdots$. Then $\mathrm{Irr}\,\mathbb{K}W_\nu$ is parametrized by the set of multipartitions of the composition $\mathbf{m}(\nu) = (m_1(\nu), m_2(\nu), \dots)$,

$$\mathbf{Part}(\mathbf{m}(\nu)) := \{\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \dots) \mid \forall i, \lambda^i \in \mathrm{Part}(m_i(\nu))\};$$

while $\mathrm{Irr}\,\mathbb{F}W_\nu$ is parametrized by the set of ℓ-regular multipartitions of $\mathbf{m}(\nu)$,

$$\mathbf{Part}_\ell(\mathbf{m}(\nu)) := \{\boldsymbol{\mu} = (\mu^1, \mu^2, \dots) \mid \forall i, \mu^i \in \mathrm{Part}_\ell(m_i(\nu))\}.$$

We will denote the simple modules by $\mathbf{S}^{\boldsymbol{\lambda}}$ and $\mathbf{D}^{\boldsymbol{\mu}}$ respectively.

Theorem 8. *The group $G = \mathrm{GL}_n$ has a modular cuspidal pair if and only n is a power of ℓ. In that case, this cuspidal pair is $(\mathcal{O}, \mathbb{F})$.*

(In contrast, over $\mathbb{K}$ we have a cuspidal pair only for $n = 1$.) In [4], we still did not have the general proof of the disjunction, so we proved the classification and the disjunction simultaneously, by induction. Indeed, by induction the smaller Levi subgroups supporting a cuspidal pair are of the form L_ν, where ν belongs to the set $\mathrm{Part}(n, \ell^{\mathbb{N}})$ of partitions of n into powers of ℓ; and for each such ν, there is just one cuspidal pair, $(\mathcal{O}_{\mathrm{reg}}^{L_\nu}, \mathbb{F})$. The disjunction follows from the fact that those Levi subgroups are distinct (hence the Fourier transforms of the simple perverse sheaves belonging to different induction series have distinct supports, they are closures of distinct strata of Lusztig's stratification). For $\nu = (1^{m_1(\nu)}, \ell^{m_\ell(\nu)}, (\ell^2)^{m_{\ell^2}(\nu)}, \ldots) \in \mathrm{Part}(n, \ell^{\mathbb{N}})$, we set $\mathbf{m}_\ell(\nu) := (m_1(\nu), m_\ell(\nu), m_{\ell^2}(\nu), \ldots)$. Then, to make the counting argument, we need the following combinatorial preliminary (see Table 2 in §2.2.3 for an example).

Lemma 9. *We have a bijection*

$$\begin{array}{rcrcl} \Psi^{\mathrm{co}} = \bigsqcup \psi_\nu^{\mathrm{co}} & : & \displaystyle\bigsqcup_{\nu \in \mathrm{Part}(n, \ell^{\mathbb{N}})} \mathbf{Part}_\ell(\mathbf{m}_\ell(\nu)) & \xrightarrow{\sim} & \mathrm{Part}(n) \\ & & \boldsymbol{\lambda} = (\lambda^1, \lambda^\ell, \lambda^{\ell^2}, \ldots) & \longmapsto & \sum_{i \geq 0} \ell^i \cdot (\lambda^{\ell^i})^{\mathrm{t}}, \end{array} \tag{6}$$

where the sum and scalar multiplication of partitions are defined componentwise.

The target set $\mathrm{Part}(n)$ always parametrizes $\mathfrak{P}_{G,\mathbb{F}}$. By induction, we know that the $\boldsymbol{\lambda} \in \mathbf{Part}_\ell(\mathbf{m}_\ell(\nu))$ with $\nu \neq (n)$ are in bjiection with the union of the induction series corresponding to cuspidal data for Levi subgroups $L < G$. If n is not a power of ℓ, then $(n) \notin \mathrm{Part}(n, \ell^{\mathbb{N}})$ so there cannot be any cuspidal pair. On the other hand, if $n = \ell^m$, then there is exactly one cuspidal pair, and it has to be $(\mathcal{O}_{(n)}, \mathbb{F})$ by Proposition 6, using modular reduction of the SL_n-equivariant cuspidal perverse sheaf $\mathcal{IC}(OC_{(n)}, \mathbb{K}_\zeta)$ where the local system $\mathbb{K}_\zeta$ corresponds to a primitive n-th root of unity; indeed, the root of unity becomes trivial modulo ℓ, thus the local system also becomes trivial, and hence PGL_n-equivariant, modulo ℓ.

The following result, proved by Lusztig in the $\mathbb{E} = \mathbb{K}$ case, restricts the possibilities for the cuspidal perverse sheaves. It is consistent with the result for GL_n, since in that case $\mathcal{O}_{\mathrm{reg}}$ is the only distinguished nilpotent orbit.

Proposition 10. *If $(\mathcal{O}, \mathcal{E})$ is a cuspidal pair for G, then $\mathcal{O}$ is a distinguished nilpotent orbit.*

For classical groups when $\ell = 2$, there are many more cuspidal pairs. Since all $A_G(\mathcal{O})$ are 2-groups for classical types, all the irreducible G-equivariant modular local systems $\mathcal{E}$ must be trivial. The count shows that there are as many modular cuspidal pairs as there can possibly be.

Theorem 11. *If G is of BCD type and $\ell = 2$, then the modular cuspidal perverse sheaves for G are all the $\mathcal{IC}(\overline{\mathcal{O}}, \mathbb{F})$ with $\mathcal{O}$ distinguished.*

For exceptional types in bad characteristic, we only have partial information. See Table 1 for the number of cuspidal pairs for quasi-simple, simply connected groups of exceptional type (according to central characters). Note that, as observed by Lusztig, for $\mathbb{E} = \mathbb{K}$ (hence also for $\mathbb{E} = \mathbb{F}$ if ℓ is big enough, and it turns out that good is big enough) there is at most one cuspidal pair for a fixed central character. But there can be many more for bad ℓ, like for classical types.

Apart from the modular reduction of ordinary cuspidal pairs, we have the following criterion to determine the induction series of $\mathcal{IC}(\overline{\mathcal{O}}_{\mathrm{reg}}, \mathbb{F})$. In particular, it tells us when it is cuspidal. Most of the time, those two cuspidal pairs are the only ones that we know for sure for bad ℓ in exceptional types. This is enough to settle the case of G_2.

Theorem 12. *The perverse sheaf $\mathcal{IC}(\overline{\mathcal{O}}_{\mathrm{reg}}, \mathbb{F})$ belongs to the induction series corresponding to the cuspidal datum $(L, \mathcal{O}^L_{\mathrm{reg}}, \mathbb{F})$, where L denotes the smallest Levi subgroup of G (well defined up to conjugacy) such that W_L contains a Sylow ℓ-subgroup of W.*

In other words, $\mathbf{R}^G_{L \subset P}\, \mathcal{IC}(\overline{\mathcal{O}}_{\mathrm{reg}}, \mathbb{F}) = 0$ if and only if $|W : W_L| = 0$ modulo ℓ. The proof is geometric, it involves the geometric Ringel duality of Achar and Mautner. In the end, it reduces to the question: when does the skyscraper sheaf $\underline{\mathbb{F}}_{\{0\}}$ appear as a direct summand of $\pi^P_* \underline{\mathbb{F}}$, where $\pi^P : T^*(G/P) \to \mathcal{N}$, and the answer is given by the self-intersection of the null-section of $T^*(G/P)$, which is up to sign the Euler characteristic of G/P, i.e. $|W : W_L|$. So the multiplicity of $\underline{\mathbb{F}}_{\{0\}}$ as a direct summand is one if $\ell \nmid |W : W_L|$, but zero if $\ell \mid |W : W_L|$.

There is an entirely analogous criterion [GHM, Theorem 4.2] for the Harish–Chandra series of the Steinberg module of a finite reductive group G^F in non-defining characteristic, with $|G^F : L^F|$ replacing $|W : W_L|$. Note that, up to a power of q which is irrelevant since $\ell \nmid q$, this is given by the following q-analogue:

$$|W : W_L|_q = \frac{\sum_{w \in W} q^{l(w)}}{\sum_{w \in W_L} q^{l(w)}}$$

where $l(w)$ is the length of w. By the Bruhat decomposition, this can also be seen as the Poincaré polynomial of G/P. Now the question is, when is $|W : W_L|_q$ zero modulo ℓ.

Stunningly, the criterion in [Eti] for the support of the spherical module of a rational Cherednik algebra of a Coxeter group (at $t = 1$) is still exactly similar! This time, the variable q is the exponential of the parameter c, hence lives in $\mathbb{C}$ (and in the case of unequal parameters, there is a version with one variable for each conjugacy class of reflections). Since $|W : W_L|_q$ is a product of cyclotomic polynomials, special behavior can happen at roots of unity.

The parallel between those three criteria is still mysterious.

2.2.3 Making the bijection explicit

Case of GL_n. We prove that the combinatorial bijection Ψ^{co} (see Lemma 9) that we used to count the number of cuspidal pairs is actually the canonical

	$\ell = 2$	$\ell = 3$	$\ell = 5$	$\ell \geq 7$
G_2	2	2	1	1
F_4	4	3	1	1
$E_6, \chi = 1$	0	3	0	0
$\chi \neq 1$	2	–	1	1
$E_7, \chi = 1$	6	0	0	0
$\chi \neq 1$	–	3	1	1
E_8	10	8	5	1

Table 1: Count of cuspidal pairs for exceptional types

bijection $\Psi_{\mathrm{GL}_n,\mathbb{F}}$ defined by the modular generalized Springer correspondence (see Theorem 5).

Theorem 13. *Up to identifying objects with their natural labelings, we have* $\Psi_{\mathrm{GL}_n,\mathbb{F}} = \Psi^{\mathrm{co}}$.

Let us just make some comments on the proof. Since both maps are bijections between the same sets, and we have a natural order on the target, it suffices to prove one inequality: $\Psi_{G,\mathbb{F}} \leq \Psi^{\mathrm{co}}$.

For $\boldsymbol{\lambda} \in \mathbf{Part}_\ell(\mathbf{m}_\ell(\nu))$, we can use the intermediate Levi subgroup $M := L_\sigma$, where $\sigma = (m_1(\nu), \ell m_\ell(\nu), \ell^2 m_{\ell^2}(\nu), \ldots)$. It satisfies $W_M(L_\nu) \simeq W_G(L_\nu)$. If $\sigma \neq (n)$, then $M < G$ and we can use induction (in both senses of the word), and the fact that the support of $\mathbf{I}_{M \subset Q}^G(\mathcal{IC}(\mathcal{O}_{(\lambda^1)^{\mathrm{t}}}) \boxtimes \mathcal{IC}(\mathcal{O}_{(\ell\lambda^\ell)^{\mathrm{t}}}) \boxtimes \cdots)$ is included in the closure of the induced nilpotent orbit, which turns out to be $\mathcal{O}_{\psi_\nu^{\mathrm{co}}(\boldsymbol{\lambda})}$. It is crucial to work on the Fourier transform side. There are compatibilities to check. See [4, Lemma 3.11] for the details.

There remains the case where $\sigma = (n)$, i.e. $\nu = (\ell^i, \ldots, \ell^i)$. Then, similarly to the determination of the non-generalized modular Springer correspondence for GL_n, we consider the modular reduction of a cuspidal perverse sheaf for SL_n. Using the known correspondence for SL_n, the fact that the Fourier transform commutes with modular reduction, and the fact that the composition factors of a modular reduction can only have smaller support, we get the desired inequality. Again, one has to work on the Fourier transform side. See [4, Lemma 3.10] for the details.

Comments on the other cases. In [2], we determine the modular generalized Springer correspondence for SL_n (and all ℓ), and for classical groups when $\ell = 2$. The answer of SL_n is what one would guess from the knowledge of the case of GL_n combined with the characteristic zero story for SL_n.

Also for classical groups, Levi subgroups are of the form: a smaller classical group times a product of general linear groups, so for the classification of cuspidal pairs we use induction and the result for the general linear groups. The bijections we use (which combine the bijections in [Lu2] with the bijections for

L_ν	GL_1^6	$\mathrm{GL}_1^4 \times \mathrm{GL}_2$	$\mathrm{GL}_1^2 \times \mathrm{GL}_2^2$	GL_2^3	$\mathrm{GL}_1^2 \times \mathrm{GL}_4$	$\mathrm{GL}_2 \times \mathrm{GL}_4$
W_ν	$\mathfrak{S}_6$	$\mathfrak{S}_4$	$\mathfrak{S}_2 \times \mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_2$	
6						1
51					2	
42				21		
41^2		31				
3^2			2×2			
321	321					
31^3		4				
2^3				3		
2^21^2	42					
21^4	51					
1^6	6					

Table 2: Modular generalized Springer correspondence for GL_6, $\ell = 2$

general linear groups) should describe the modular generalized Springer correspondence, but we are not able to show this conjecture for ℓ odd. The reason is that, even is we start with a simply connected group, there can be Levi subgroups L such that $L/Z(L)^0$ is not simply-connected, and we cannot lift the cuspidal data to characteristic zero. New ideas would be needed.

In any case, the principal series is completely determined for all reductive groups and all ℓ, since the only case missing after [1, 16] was that of classical groups when $\ell = 2$. In [13], we determine some other series for exceptional groups, where the cuspidal datum arises as the modular reduction of a characteristic zero cuspidal datum. This uses decomposition matrices.

If $(L, \mathcal{O}_L, \mathcal{E}_L)$ is a characteristic zero cuspidal datum, Lusztig proved [Lu2] that the lower and upper bounds for the corresponding induction series, namely the nilpotent G-orbit $G \cdot \mathcal{O}_L$ containing $\mathcal{O}_L$ and the induced nilpotent orbit, both appear in the series, and correspond to the trivial and sign representations. Considering those two orbits allows him to prove a priori that $W_G(L)$ has to be a reflection group. One difficulty in the modular case is that we do not have a modular analogue of Lusztig's result: the bounds are not necessarily attained, and actually in a few cases $W_G(L)$ fails to be a reflection group, so the sign representation does not even make sense in general. This makes the determination of the correspondence particularly difficult in the modular case: Spaltenstein says that Lusztig's result is the starting point to apply a compatibility result on parabolic restriction in a non-trivial way.

2.2.4 Rather good characteristic

Recall that ℓ is rather good for G if it is good for G and does not divide the order of Z_G/Z_G^0. This property is inherited by Levi subgroups, and is equivalent to: ℓ does not divide the orders of the groups $A_G(\mathcal{O})$. Thus for ℓ rather good, we can identify $\mathfrak{P}_{G,\mathbb{F}}$ with $\mathfrak{P}_{G,\mathbb{K}}$. This gives a meaning to the following result [12].

Theorem 14. *If ℓ is rather good for G, then the ℓ-modular induction series refine the ordinary induction series.*

A very important feature of characteristic zero cuspidal perverse sheaves is the *cleanness* property: for $(\mathcal{O},\mathcal{E})$ cuspidal, we have $\mathcal{IC}(\overline{\mathcal{O}},\mathcal{E})_{|\overline{\mathcal{O}}-\mathcal{O}} = 0$. This fails in the modular case, already for GL_2 in characteristic 2. However, we expect (and it was conjectured by C. Mautner) that for the modular reduction of a characteristic 0 cuspidal perverse sheaf, it holds as soon as ℓ is rather good. Apart from the trivial case $\ell \nmid |W|$, we can prove it for G of type A or of exceptional type, and in the first non-trivial example of each classical series (B_4, C_3, D_5). We also proved that the conjecture implies an orthogonal decomposition of $D^b_G(\mathcal{N},\mathbb{F})$ according to the 0-series. It would also explain what would happen if we considered supercuspidal perverse sheaves instead of cuspidal ones (those simple perverse sheaves which are not a subquotient of a properly induced perverse sheaf): the supercuspidal perverse sheaves would be the 0-cuspidal ones, and the corresponding superseries (consisting of all the composition factors of the induction of a supercuspidal object) would just be the 0-series (hence disjoint).

2.3 Geometry of nilpotent cones

The general aim of this joint project with B. Fu, P. Levy and E. Sommers is to understand better the geometry of nilpotent cones, namely: describe the minimal degenerations between nilpotent orbits, and also non-minimal ones if we can (in particular between special orbits); understand what happens geometrically with Lusztig-Spaltenstein duality; and approach Lusztig's conjecture on special pieces. We have different motivations: the modular generalized Springer correspondence, the normality problem for nilpotent orbit closures, symplectic singularities, the Mori program in the example of nilpotent orbit closures…

2.3.1 Background and known results

We say that two pointed complex varieties (X,x) and (Y,y) (or their germs) are smoothly equivalent, and write $\mathrm{Sing}(X,x) = \mathrm{Sing}(Y,y)$, if there exists (Z,z) and two morphisms $\varphi : (Z,z) \to (X,x)$, $\psi : (Z,z) \to (Y,y)$, both smooth at z. If G acts on X, then $\mathrm{Sing}(X,x)$ only depends on the G-orbit $\mathcal{O} := G\cdot x$, and we write $\mathrm{Sing}(X,\mathcal{O}) := \mathrm{Sing}(X,x)$. If $\dim_x X = \dim_y Y + r$, with $r \geq 0$, then we have

$$\mathrm{Sing}(X,x) = \mathrm{Sing}(Y,y) \quad \Longleftrightarrow \quad \widehat{\mathcal{O}}_{X,x} \simeq \widehat{\mathcal{O}}_{Y,y}[[T_1,\dots,T_r]].$$

The local intersection cohomology is an invariant: if $\mathrm{Sing}(X,x) = \mathrm{Sing}(Y,y)$ then we have an isomorphism of graded $\mathbb{E}$-modules

$$\mathcal{IC}(X,\mathbb{E})_x \simeq \mathcal{IC}(Y,\mathbb{E})_y$$

(but one also wants to keep track of local systems).

For each nilpotent orbit $\mathcal{O}$, we want to describe the singularities $\mathrm{Sing}(\overline{\mathcal{O}}, \mathcal{O}')$ for all orbits $\mathcal{O}'$ open in the boundary $\overline{\mathcal{O}} - \mathcal{O}$ (and possibly also for lower orbits). This problem was already solved in classical types [KP1, KP2] and G_2 [Kr]. In [15], we deal with all exceptional types.

Let $e \in \mathcal{N}$. By the Jacobson-Morozov theorem, one can find an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$, i.e. find h and f such that

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

They generate an $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}$. We denote by $\mathfrak{g}^f$ the centralizer of f in $\mathfrak{g}$. One can show that the affine subspace

$$S_e := e + \mathfrak{g}^f,$$

known as a Slodowy slice, is transverse to the orbit $\mathcal{O}'$ of e in $\mathfrak{g}$. If $\mathcal{O}$ is another nilpotent orbit whose closure contains $\mathcal{O}'$, we are interested in describing the singularity of the nilpotent slice $\mathcal{S}_{\mathcal{O},e} := \overline{\mathcal{O}} \cap \mathcal{S}_e$. Indeed, $\mathrm{Sing}(\overline{\mathcal{O}}, \mathcal{O}') = \mathrm{Sing}(\mathcal{S}_{\mathcal{O},e}, e)$. But if we can describe $\mathcal{S}_{\mathcal{O},e}$ algebraically, rather than up to smooth equivalence, this is even better. We will write $\mathcal{S}^G_{\mathcal{O},e}$ if we wish to emphasize the ambient group.

The description of the generic singularities of the whole nilpotent cone $\mathcal{N}_G$ goes back to 1970. There is a regular nilpotent orbit $\mathcal{O}_{\mathrm{reg}}$, which is dense in $\mathcal{N}_G$. Let us assume that $\mathfrak{g}$ is simple. Then there is a unique nilpotent orbit $\mathcal{O}_{\mathrm{subreg}}$, which is open in $\mathcal{N}_G - \mathcal{O}_{\mathrm{reg}}$.

Theorem 15 (Brieskorn)**.** *Suppose that $\mathfrak{g}$ is of ADE type X_n. Then*

$$\mathcal{S}_{\mathcal{O}_{\mathrm{reg}}, \mathcal{O}_{\mathrm{subreg}}} = X_n$$

is a simple surface singularity of the same type, and Grothendieck's simultaneous resolution restricted to the slice gives a versal deformation.

A simple surface singularity is of the form $\mathrm{Sing}(\mathbb{C}^2/\Gamma, 0)$, where Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Such groups Γ are classified up to conjugacy by ADE diagrams. The diagram describes the intersection pattern of the projective lines in the exceptional fiber of the minimal resolution of the simple surface singularity. We denote those singularities just by the same symbol as the Dynkin type: A_n, D_n or E_n.

Slodowy explained what happens non simply-laced types: to each inhomogeneous Dynkin diagram X_n, he associates a homogeneous diagram $\widehat{X}_n$ and a group of symmetries $A(X_n)$ of this diagram (induced by another finite subgroup $\Gamma' \supset \Gamma$ of SL_2 normalizing Γ). Then each diagram automorphism in $A(X_n)$ can be lifted to an automorphism of the simple singularity $\widehat{X}_n$ which induces the corresponding permutation of the projective lines in the exceptional divisor of the minimal resolution. We denote by X_n the simple singularity of type $\widehat{X}_n$

endowed with the symmetry group $A(X_n)$, as follows:

$$\begin{array}{lcl} B_n = A_{2n-1}^+ & := & A_{2n-1} \text{ with } \mathfrak{S}_2\text{-action;} \\ C_n = D_{n+1}^+ & := & D_{n+1} \text{ with } \mathfrak{S}_2\text{-action;} \\ F_4 = E_6^+ & := & E_6 \text{ with } \mathfrak{S}_2\text{-action;} \\ G_2 = D_4^{++} & := & D_4 \text{ with } \mathfrak{S}_3\text{-action.} \end{array}$$

It turns out that $A(X_n) = A_G(\mathcal{O}_{\mathrm{subreg}})$, assuming that G is adjoint. Since the subregular orbit is distinguished in types $BCFG$ (actually in all types but type A), we have $A_G(\mathcal{O}_{\mathrm{subreg}}) = C(\mathfrak{s})$, hence this group acts on the slice $\mathcal{S}_{\mathcal{O}_{\mathrm{reg}},\mathcal{O}_{\mathrm{subreg}}}$. Then the singularity endowed with its $A_G(\mathcal{O}_{\mathrm{subreg}})$ action is as in the above list.

We remark that the symmetry of the Dynkin diagram A_{2n} does not appear here. Actually it cannot be lifted to an involution of the simple singularity, but only to an order four automorphism (induced by the subgroup $\Gamma' \subset \mathrm{SL}_2$ corresponding to D_{2n+3}). We will denote this singularity with symmetry by A_{2n}^+. We will see occurrences of A_2^+ and A_4^+ with in exceptional types.

On the other extreme, one can consider the two orbits of lowest dimension: the trivial orbit $\{0\}$ and the minimal nilpotent orbit $\mathcal{O}_{\min}$ (which is unique if we still assume that $\mathfrak{g}$ is simple). The minimal nilpotent orbit is the orbit of a highest weight vector for the adjoint representation of G, whose highest weight is the highest root. For example, for $\mathfrak{g} = \mathfrak{sl}_n$ we have $\{0\} = \mathcal{O}_{(1^n)}$, and $\mathcal{O}_{\min} = \mathcal{O}_{(2,1^{n-2})}$ consists of the nilpotent elements of rank 1. For any G, the singularity $\overline{\mathcal{O}}_{\min}$ is obtained by collapsing the null section of a line bundle (corresponding to the highest root) on a partial flag variety G/P (where P is the parabolic subgroup stabilizing the highest weight line). We denote those singularities by lower-case letters a_{n-1}, b_n, $\ldots$, g_2. Note that $a_1 = A_1$. They can also come with symmetries.

In [KP1], H. Kraft and C. Procesi show a row and column removal rule for nilpotent orbit closure singularities in the case of general linear groups.

Theorem 16 (Kraft–Procesi). *Consider a degeneration $\overline{\mathcal{O}}_\lambda \supset \mathcal{O}_\mu$ between nilpotent orbits for GL_n. If the r first lines and the s first columns of λ and μ are identical, and if λ' and μ' are the partitions obtained by removing those common lines and columns, then*

$$\mathrm{Sing}(\overline{\mathcal{O}}_\lambda, \mathcal{O}_\mu) = \mathrm{Sing}(\overline{\mathcal{O}}_{\lambda'}, \mathcal{O}_{\mu'})$$

It follows that, for $\mathfrak{g} = \mathfrak{sl}_n$, all minimal degenerations are either simple singularities A_k (if the codimension if 2), or minimal singularities a_k (if the codimension is $2k > 2$). Both cases are exchanged by the "duality" which consists in the transposition of partitions.

H. Kraft and and C. Procesi also found a row and column removal rule for nilpotent singularities in nilpotent cones of classical types. As a consequence, they found that a minimal degeneration in that case is either a minimal singularity (if the codimension is > 2), a simple singularity for a classical Lie algebra of smaller rank, or the union of two simple singularities of type A_{2k-1} meeting at the singular point, which we will denote by $2A_{2k-1}$. The latter situation is the only case for non-normality: in classical types, the normality of a nilpotent orbit closure $\overline{\mathcal{O}}$ can be detected in codimension 2, and in this case there are two branches. We will see that many more phenomena can occur in exceptional types.

Levasseur and Smith have shown that the nilpotent orbit closure $\widetilde{A}_1$ in G_2 is not normal, but the normalization is a homeomorphism [LS]. Actually the surface singularity $m := \mathcal{S}^{G_2}_{\widetilde{A}_1, A_1}$ is already non-normal. It is the image of the morphism $\mathbb{C}^2 \to \mathbb{C}^7$, $(u,v) \mapsto (u^2, uv, v^2, u^3, u^2v, uv^2, v^3)$. The normalization of m is $\mathbb{C}^2$. It turns out that this singularity appears many times in the other exceptional types.

2.3.2 Results for exceptional types

Let us now describe our results. First, looking at tables for Green functions for exceptional types, we observe the following (it follows from [KP2] that it also holds for classical types).

Proposition 17. *When $\mathcal{O}' \ni e$ is a minimal degeneration of $\mathcal{O}$ in an exceptional Lie algebra, the action of $A_G(\mathcal{O}')$ on the set of irreducible components of the nilpotent Slodowy slice $\mathcal{S}_{\mathcal{O},e}$ is transitive. In particular, the irreducible components of $S_{\mathcal{O},e}$ are mutually isomorphic.*

Just as for classical groups, the main dichotomy is whether the codimension is 2 or ≥ 4.

Surface cases. For codimension 2 singularities, we find mostly simple singularities and the non-normal surface m above.

Proposition 18. *Let $\mathcal{O}'$ be a minimal degeneration of $\mathcal{O}$ of codimension 2. Then there exists a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$ such that the normalization $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ of $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to a disjoint union of k copies of X where $X = \mathbb{C}^2/\Gamma$.*

To determine Γ, we use the results of [Fu2]: either the normalization $\widetilde{\mathcal{O}}$ is $\mathbb{Q}$-factorial, or every $\mathbb{Q}$-factorial terminalization is given by the normalization of a generalized Springer map such as in the theory of Lusztig–Spaltenstein induction. Since a normal variety with only terminal singularities is smooth in codimension 2, by restricting this morphism to the slice, we get a symplectic, hence minimal, resolution of the surface singularity $\mathcal{S}_{\mathcal{O},e}$. Moreover, a formula of Borho and MacPherson gives the number of projective lines in the exceptional fiber, with the action of $A_G(\mathcal{O}')$. This is enough to determine Γ, except in three cases (between the subregular and subsubregular orbits in E_n types); but then, an argument using orbital varieties can resolve the ambiguities.

In most cases, we can show that the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are normal (either because the whole nilpotent orbit closure is known to be normal, or by reducing to a smaller Lie algebra). The only two cases where we know that this fails are: the case where $\Gamma = 1$ and $\mathcal{S}_{\mathcal{O},e} = m$, and the minimal degeneration $(D_7(a_1), E_8(b_6))$ in E_8: then $\Gamma = \mu_4$, the normalization is $A_3 = \operatorname{Spec} \mathbb{C}[st, s^4, t^4]$

(with an order 2 symmetry), and $\mathcal{S}_{\mathcal{O},e} = \operatorname{Spec}\mathbb{C}[(st)^2,(st)^3,s^4,t^4,s^5t,st^5] =: \mu$. There are ten cases (in type E_8) where we know the answer only up to normalization.

Study of nilpotent Slodowy slices. Before we deal with the slice method, which can be used for codimension ≥ 4 degenerations, and for surface cases with $|\Gamma| = 1$ or 2, we need some facts about S-varieties [VP]. Let H be a reductive group with a fixed maximal torus contained in a fixed Borel subgroup. If $(\lambda_1,\dots,\lambda_r)$ are dominant weights for H, the associated S-variety is $X := X(\lambda_1,\dots,\lambda_r) = \overline{Hv} \subseteq V = \bigoplus V(\lambda_i)$, where $v = (v_i)$ consists of highest weight vectors in the Weyl modules $V(\lambda_i)$. Those are the affine H-varieties with a dense H-orbit such that the stabilizer of a point in that orbit contains a maximal unipotent subgroup of H. We will only consider cases where the $\lambda_i = b_i\lambda$ are multiples of a fixed dominant weight λ. Then the results of [VP] imply: $Hv = X - \{0\}$; as an H-variety, X is determined by the submonoid of $\mathbb{N}$ generated by $b_1, \dots, b_r$; the normalization of X is $X(d\lambda)$, where $d = \gcd(b_1,\dots,b_r)$; and if b_1, $\dots$, b_s (with $s < r$) generate the same monoid as $b_1, \dots, b_r$, then the projection from X to $X(b_1\lambda,\dots,b_s\lambda)$ is an isomorphism. If V factors through $Z \subseteq H$ with Lie algebra $\mathfrak{sl}_2$, then we also write $X(b_1,\dots,b_r)$ instead of $X(b_1\varpi,\dots,b_r\varpi)$, where ϖ is the fundamental weight for $\mathfrak{sl}_2$.

Let $\mathcal{O}'$ be a minimal degeneration of $\mathcal{O}$, and $e \in \mathcal{O}'$. We choose an $\mathfrak{sl}_2$-subalgebra $\mathfrak{s} = \langle e,h,f\rangle$ containing e. Recall that $\mathcal{S}_{\mathcal{O},e} = e + \mathfrak{g}^f$. We have a non-positive grading $\mathfrak{g}^f = \bigoplus \mathfrak{g}^f(i)$ induced by $\operatorname{ad} h$. The reductive part of the centralizer G^e is $C(\mathfrak{s})$, with Lie algebra $\mathfrak{c}(\mathfrak{s}) = \mathfrak{g}^f(0)$. Let $\pi_0 : \mathcal{S}_{\mathcal{O},e} \to \mathfrak{c}(\mathfrak{s}) = \mathfrak{g}^f(0)$ and $\pi_{0,1} : \mathcal{S}_{\mathcal{O},e} \to \mathfrak{g}^f(0) \oplus \mathfrak{g}^f(-1)$ denote the $C(\mathfrak{s}) \times \mathbb{C}^*$-equivariant projections.

Proposition 19. *Let $\mathcal{O}'$ be a minimal degeneration of $\mathcal{O}$ of codimension at least four (other than the three normal cases in Remark 24 below), or of codimension two with $|\Gamma| = 1$ or 2.*

Then there exists $J = \{i_1,\dots,i_r\} \subset \mathbb{N}$ so that for each $i \in J$ there exists a highest weight vector $x_i \in \mathfrak{g}^f(-i)$ for the action of $C(\mathfrak{s})$, and there exists $x_0 \in \mathfrak{c}(\mathfrak{s})$ minimal nilpotent, such that

- $x := e + x_0 + \sum_{i\in J} x_i \in \mathcal{O} \cap \mathcal{S}_e$,
- *if the weight of x_0 is λ, then the weight of x_i equals $\frac{i+2}{2}\lambda$,*
- $\mathcal{S}_{\mathcal{O},e} = e + X$, *for the corresponding S-variety for $C(\mathfrak{s})$*

$$X := X\left(\lambda, \frac{i_1+2}{2}\lambda, \frac{i_2+2}{2}\lambda, \dots, \frac{i_r+2}{2}\lambda\right) \subseteq \mathfrak{g}^f$$

through the vector $x_0 + \sum_{i\in J} x_i$.

Moreover, there are two cases:

1. *All $i \in J$ are even. Then π_0 gives an isomorphism $\mathcal{S}_{\mathcal{O},e} \cong X(\lambda)$. In particular, each irreducible component of $\mathcal{S}_{\mathcal{O},e}$ is a minimal singularity, being isomorphic to the closure of the $C(\mathfrak{s})^\circ$-orbit through the minimal nilpotent element $x_0 \in V(\lambda) \subset \mathfrak{c}(\mathfrak{s})$. Hence each component is a normal variety.*

2. *We have $1 \in J$. This case occurs only if $\mathfrak{c}(\mathfrak{s})$ contains a simple factor $\mathfrak{z}$ of type $\mathfrak{sl}_2(\mathbb{C})$ or $\mathfrak{sp}_4(\mathbb{C})$ and $\mathfrak{z} = V(\lambda)$; moreover, $\lambda = 2\varpi$, where $V(\varpi)$ is the defining representation of $\mathfrak{z}$. Then $\pi_{0,1}$ gives an isomorphism $\mathcal{S}_{\mathcal{O},e} \cong X(2\varpi, 3\varpi)$. In the $\mathfrak{z} = \mathfrak{sl}_2(\mathbb{C})$ case, $\mathcal{S}_{\mathcal{O},e} \cong m$ and in the $\mathfrak{z} = \mathfrak{sp}_4(\mathbb{C})$ case, we call the resulting singularity m'. In both cases $\mathcal{S}_{\mathcal{O},e}$ is irreducible and non-normal.*

Remark 20. In case (1) of Proposition 19, when the codimension is at least four, we find that $J = \emptyset$ except for the two minimal degenerations ending in the orbit $D_4(a_1) + A_2$ in E_8, where $J = \{2\}$. On the other hand, when the codimension is two in case (1), then $\mathcal{S}_{\mathcal{O},e} \cong kA_1$. If $k > 1$, then always $J = \emptyset$. If $k = 1$, then J can be either $\emptyset, \{2\}$, or $\{2, 4\}$.

Let us explain some of the techniques involved. First, we note the following easy reduction, adapted from [KP1].

Lemma 21. *Let $H \subseteq G$ be a closed reductive subgroup with Lie algebra $\mathfrak{h}$. Let $x, e \in \mathcal{N}_H$ such that $He \subseteq \overline{Hx}$. We choose an $\mathfrak{sl}_2$-subalgebra $\mathfrak{s} = \langle e, h, f \rangle \subseteq \mathfrak{h}$. We have $\mathcal{S}^H_{Hx,e} \subseteq \mathcal{S}^G_{Gx,e}$. Assume that* $\operatorname{codim}_{\overline{Hx}}(He) = \operatorname{codim}_{\overline{Gx}}(Ge)$ *and $\mathcal{S}^H_{Hx,e}$ is equidimensional. Then $\mathcal{S}^H_{Hx,e}$ is a union of irreducible components of $\mathcal{S}^G_{Gx,e}$.*

Moreover, if $\overline{Gx}$ is unibranch at e, or if the number of branches of $\overline{Gx}$ at e equals the number of branches of $\overline{Hx}$ at e, then $\mathcal{S}^H_{Hx,e} = \mathcal{S}^G_{Gx,e}$.

We used this lemma for surface cases where we do not have the normality of the nilpotent orbit closure: normality may be available in a smaller Lie algebra. But also, we adapted this technique systematically for the study of nilpotent Slodowy slices. Let us come back to the previous notation. For

$$x = e + x_0 + \sum_{i>0} x_i,$$

with $x_0 \in \mathfrak{c}(\mathfrak{s})$ and $x_i \in \mathfrak{g}^f(-i)$, we set

$$x_+ = \sum_{i \geq 0} x_i \qquad \text{and} \qquad X = \overline{C(\mathfrak{s})x_+}.$$

The variety $\overline{C(\mathfrak{s})x} = e + X \simeq X$ is equidimensional, and its irreducible components are permuted transitively by $C(\mathfrak{s})/C(\mathfrak{s})^0 \simeq A_G(\mathcal{O}')$.

Lemma 22. *We have* $\dim(C(\mathfrak{s})x) = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}')$ *if and only if x_0 is nilpotent and* $\dim(C(\mathfrak{s})x_0) = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}')$. *When this holds, $e+X$ is a union of irreducible components of $\mathcal{S}_{\mathcal{O},e}$. Moreover, if the number of branches of $\overline{\mathcal{O}}$ at e equals the number of irreducible components of X, then $e + X = \mathcal{S}_{\mathcal{O},e}$.*

Thus we want to know when the hypothesis on the equality of codimensions holds. The next result clarifies the situation when all $x_i = 0$ for $i > 0$, which applies in many cases.

Proposition 23. *Let $\mathfrak{s}_0 = \langle e_0, h_0, f_0 \rangle$ be an $\mathfrak{sl}_2$-subalgebra of $\mathfrak{c}(\mathfrak{s})$; set $x = e + e_0$ and let $\mathcal{O} = Gx$. Let*

$$\mathfrak{g} = \bigoplus_{i=1}^{N} V^{(i)}_{m_i,n_i} \tag{7}$$

be a decomposition into irreducible subrepresentations $V^{(i)}_{m_i,n_i} \cong V_{m_i,n_i}$ for the action of $\mathfrak{s} \oplus \mathfrak{s}_0$. Then $\dim C(\mathfrak{s})e_0 = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}')$ if and only if

$$m_i \geq n_i \textit{ whenever } m_i > 0. \tag{8}$$

In particular, this always holds if e_0 is minimal nilpotent in $\mathfrak{g}$, or if e_0 is of height 2 and e is even.

Remark 24. The exceptional cases mentioned in Proposition 19 are:
— the minimal degeneration $(2A_2 + A_1, A_2 + 2A_1)$ in E_6, which is of the form $\tau := \mathbb{C}^4/\mu_3$;
— the minimal degeneration $(A_4 + A_1, A_3 + A_2 + A_1)$ in E_7, which isomorphic to $a_2/\mathfrak{S}_2$;
— the minimal degeneration $(A_4 + A_3, A_4 + A_2 + A_1)$ in E_8, denoted by χ, which is isomorphic to the blow-up at the singular locus of the quotient singularity $\mathbb{C}^4/\Gamma$, where Γ is a dihedral group of order 10 (according to G. Bellamy, this is the unique $\mathbb{Q}$-factorial terminalization of this symplectic quotient).

Zoology. We will give a list of all singularities with symmetries which appear as minimal degenerations in exceptional types. But first, let us explain what we mean exactly by describing the slice $\mathcal{S}_{\mathcal{O},e}$ with its $A_G(\mathcal{O}') = C(\mathfrak{s})/C(\mathfrak{s})^0$ symmetry. Clearly, $C(\mathfrak{s})$ acts on $\mathcal{S}_{\mathcal{O},e}$. We prove that, with four exceptions, the group $C(\mathfrak{s})$ splits as a semi-direct product $C(\mathfrak{s})^0 \rtimes H$, with of course $H \simeq A_G(\mathcal{O}')$. The splitting is not unique up to conjugacy in $C(\mathfrak{s})$ in general, but the image of H in $\operatorname{Aut}(\mathfrak{c}(\mathfrak{s}))$ is unique up to conjugacy in $\operatorname{Aut}(\mathfrak{c}(\mathfrak{s}))$ if we impose that H should act by diagram automorphisms on the semisimple part of $\mathfrak{c}(\mathfrak{s})$. We use H to define the "intrinsic symmetry of $A_G(\mathcal{O}')$".

The orbits $\mathcal{O}'$ appearing in the exceptions are: $A_4 + A_1$ in E_7; and $A_4 + A_1$, $E_6(a_1) + A_1$ and $D_7(a_2)$ in E_8. In those cases, the component group $A_G(\mathcal{O}') = \mathfrak{S}_2$, but we only have $C(\mathfrak{s}) = C(\mathfrak{s})^\circ \cdot H$ with H cyclic of order four [Som, §3.4]. The singularity with symmetry is then denoted by A_2^+ or A_4^+. Interestingly, the first three orbits correspond to the exceptional characters of the Weyl groups via the Springer correspondence. Among the A_2^+ degenerations, we have $(A_4 + A_2, A_4 + A_1)$ in E_7 and $(E_8(b_6), E_6(a_1) + A_1)$ in E_8. In both cases, the larger orbit is even Richardson, hence admits a symplectic resolution by [Fu1]. We obtain a symplectic resolution of the slice by restriction. As far as we know, those are the first examples of symplectic contractions to an affine variety whose generic positive-dimensional fiber is of type A_2 and with a non-trivial monodromy action. This disproves Conjecture 4.2 in [AW].

Similarly to the surface cases, for minimal singularities x_n we denote a non-trivial $\mathfrak{S}_2$-action by x_n^+, and a non-trivial $\mathfrak{S}_3$-action by x_n^{++}.

Theorem 25. *Let $\mathcal{O}'$ be a minimal degeneration of $\mathcal{O}$ in a simple Lie algebra of exceptional type. Let $e \in \mathcal{O}'$. Taking into consideration the intrinsic symmetry of $A_G(\mathcal{O}')$, we have*

(a) If the codimension of $\mathcal{O}'$ in $\overline{\mathcal{O}}$ is two, then, with one exception, $\mathcal{S}_{\mathcal{O},e}$ is isomorphic either to a simple surface singularity of type A – G or to one of the following

$$A_2^+, A_4^+, 2A_1, 3A_1, 3C_2, 3C_3, 3C_5, 4G_2, 5G_2, 10G_2, \text{ or } m,$$

up to normalization for ten cases in E_8. Here, kX_n denotes k copies of X_n meeting pairwise transversally at the common singular point. In the one remaining case, the singularity is smoothly equivalent to μ.

(b) If the codimension is greater than two, then, with three exceptions, $\mathcal{S}_{\mathcal{O},e}$ is isomorphic either to a minimal singularity of type $a - g$ or to one of the following types:

$$a_2^+, a_3^+, a_4^+, a_5^+, 2a_2, d_4^{++}, e_6^+, 2g_2, \text{ or } m',$$

where the branched cases $2a_2$ and $2g_2$ denote two minimal singularities meeting transversally at the common singular point. The singularities for the three remaining cases are smoothly equivalent to τ, $a_2/\mathfrak{S}_2$ and χ, respectively.

See [15, §6.2] for a complete statement of the intrinsic symmetry action. We tried to give suggestive names for each situation. For example, $10G_2$ occurs only once in E_8: then the component group $\mathfrak{S}_5$ and permutes the irreducible components transitively with isotropy group $\mathfrak{S}_3 \times \mathfrak{S}_2$; each irreducible component is a simple singularity of type D_4, with the $\mathfrak{S}_2$ factor acting trivially and the $\mathfrak{S}_3$ factor acting by diagram automorphism as in the G_2 case.

2.4 Parity sheaves

2.4.1 Generalities

In [6], which is joint work with C. Mautner and G. Williamson, we show the existence and uniqueness of *parity sheaves*, a new class of constructible complexes on stratified varieties satisfying certain conditions which are typically satisfied in representation theory (like Schubert varieties, toric varieties or nilpotent cones). They are complexes of sheaves whose stalks and costalks vanish according to parity of the cohomological degrees (the choice on the parity may depend on the stratum). This notion is crucial in the case of modular coefficients, because then the simple perverse sheaves (the intersection cohomology complexes) are extremely difficult to compute. The main reason is the failure of the decomposition theorem when the coefficients are in characteristic $p > 0$. In contrast, parity sheaves satisfy a weak form of the decomposition theorem: they are stable by pushforward by a special class of morphisms, the proper even morphisms, and the decomposition of the direct image is controlled by the ranks of some bilinear forms, namely the intersection forms which appear in the work of de Cataldo and Migliorini (in their Hodge theoretic proof of the decomposition theorem). Thus the main problem became to calculate those bilinear forms and their ranks. The idea to consider indecomposable direct summands of direct images by resolutions appears in [Soe2], in the case of Schubert varieties. The complexes he considers there are parity sheaves.

In the case of Schubert varieties, the intersection forms can now be algorithmically computed thanks to the Soergel calculus [EW2]. The algorithm is implemented, the only limitation is the computational power of the computer. The stalks of parity sheaves on Schubert varieties define an p-canonical basis for Hecke algebras, which satisfies positivity properties analogous to those of the Kazhdan–Lusztig basis (which is the 0-canonical basis). In the classical situation (coefficients in characteristic zero), the parity sheaves are the intersection cohomology complexes, but by varying the parity according to the dimension of the strata, we also recover as a particular case the tilting sheaves studied by Beilinson, Bezrukavnikov and Mirkovic.

2.4.2 Existence and uniqueness

To be more precise, let us introduce some notation. Let X be a complex algebraic variety equipped with a fixed algebraic stratification (in the sense of [CG, Definition 3.2.23])

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

into smooth connected locally closed subvarieties. For each $\lambda \in \Lambda$ we denote by $i_\lambda : X_\lambda \to X$ the inclusion and by d_λ the complex dimension of X_λ.

We denote by $D(X)$, or $D(X, \mathbb{E})$ if we wish to emphasize the coefficients, the bounded Λ-constructible derived category of $\mathbb{E}$-sheaves on X with respect to the given stratification. It is also possible to work with the G-equivariant derived category if some algebraic group G acts on X and the stratification is G-stable.

For simplicity, we will assume here that $\mathbb{O} = \mathbb{K}$ or $\mathbb{F}$, but most results hold over $\mathbb{O}$ or more generally a complete local principal ideal domain.

A *pariversity* is a function $\dagger : \Lambda \to \mathbb{Z}/2 = \{\overline{0}, \overline{1}\}$. For example, we have the constant pariversity $\natural : \lambda \mapsto \overline{0}$ and the dimension pariversity $\Diamond : \lambda \mapsto \overline{d_\lambda}$.

Definition 26. Fix a pariversity $\dagger$. In the following $? \in \{*, !\}$. A complex $\mathcal{F} \in D(X)$ is
– $(\dagger, ?)$-*even* if, for all $\lambda \in \Lambda$ and $n \in \mathbb{Z}$ not of the parity $\dagger(\lambda)$, the cohomology

sheaf $\mathcal{H}^n(i_\lambda^? \mathcal{F})$ vanishes;[1]
– $(\dagger, ?)$-*odd* if its shift by 1 is $(\dagger, ?)$-even;
– $(\dagger, ?)$-*parity* if it is either $(\dagger, ?)$-even or $(\dagger, ?)$-odd;
– $\dagger$-*even* (resp. $\dagger$-*odd*) if it is both $(\dagger, *)$- and $(\dagger, !)$-even (resp. odd);
– $\dagger$-*parity* if it splits as the direct sum of a $\dagger$-even complex and a $\dagger$-odd complex.

We assume that for all $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$, the cohomology $\mathsf{H}^\bullet(X_\lambda, \mathcal{L})$ vanishes in odd degrees.[2] Then we have the following uniqueness result [6, Theorem 2.12].

Theorem 27. *Let $\mathcal{F}$ be an indecomposable parity complex on X. Then*

1. *the support of $\mathcal{F}$ is irreducible, hence of the form $\overline{X}_\lambda$, for some $\lambda \in \Lambda$;*
2. *the restriction $i_\lambda^* \mathcal{F}$ is isomorphic to $\mathcal{L}[m]$, for some $\mathcal{L} \in \mathrm{Irr}\,\mathrm{Loc}(X_\lambda)$ and some integer m;*[3]
3. *any indecomposable parity complex supported on $\overline{X}_\lambda$ and extending $\mathcal{L}[m]$ is isomorphic to $\mathcal{F}$.*

In the situation of the theorem, if $m = d_\lambda$ we call $\mathcal{F}$ an indecomposable parity sheaf, and we denote it by $\mathcal{E}^\dagger(\lambda, \mathcal{L})$ or $\mathcal{E}^\dagger(\overline{X}_\lambda, \mathcal{L})$. Note that $\mathbb{D}\mathcal{E}(\lambda, \mathcal{L}) \simeq \mathcal{E}(\lambda, \mathcal{L}^\vee)$. Most of the time we will omit $\dagger$ from the notation. We extend the notation to a decomposable $\mathcal{L}$ by linearity.

In [6, Corollary 2.28], we give the following existence statement in a special situation.

Proposition 28. *Let $\dagger$ be a pariversity. Let X be stratified by contractible strata and $\mathbb{E}$ be a field. Then for every stratum, there exists a parity sheaf $\mathcal{E}^\dagger(\lambda, \mathbb{E}) \in D(X)$.*

But the main tool we use to prove existence is based on taking direct images by "even maps". First recall the following definition from [GM3, 1.6].

Definition 29. Let $X = \sqcup_{\lambda \in \Lambda_X} X_\lambda$ and $Y = \sqcup_{\mu \in \Lambda_Y} Y_\mu$ be stratified varieties. A morphism $\pi : X \to Y$ is *stratified* if

1. for all $\mu \in \Lambda_Y$, the inverse image $\pi^{-1}(Y_\mu)$ is a union of strata;
2. for each X_λ above Y_μ, the induced morphism $\pi_{\lambda,\mu} : X_\lambda \to Y_\mu$ is a submersion with smooth fibre $F_{\lambda,\mu} = \pi_{\lambda,\mu}^{-1}(y_\mu)$, where y_μ is some chosen base point in Y_μ.

We now introduce the notion of *even* morphism.

Definition 30. A stratified morphism π is said to be *even* if for all λ, μ as above, and for any local system $\mathcal{L}$ in $\mathrm{Loc}(X_\lambda)$, the cohomology of the fibre $F_{\lambda,\mu}$ with coefficients in $\mathcal{L}_{|F_{\lambda,\mu}}$ is (torsion free and) concentrated in even degrees.

[1] In the case $\mathbb{E} = \mathbb{O}$, we also require the stalks and costalks to be torsion-free.
[2] In the case $\mathbb{E} = \mathbb{O}$, one should take torsion-free local systems for the category $\mathrm{Loc}(X_\lambda)$, and we should require that the even cohomology groups are free.
[3] In the case $\mathbb{E} = \mathbb{O}$, the local system $\mathcal{L}$ should be indecomposable rather than simple.

Proposition 31. *The direct image of a $(\natural, ?)$-even (resp. odd) complex under a proper, even morphism is again $(\natural, ?)$-even (resp. odd). The direct image of a $\natural$-parity complex under such a map is $\natural$-parity.*

This proposition allows us to produce many parity complexes, and although we may not know the precise decomposition into indecomposable objects, it still allows us to prove the existence of many of them.

2.4.3 A weak decomposition theorem

Let $f : \widetilde{X} \to X$ be a proper, surjective and even morphism, with $\widetilde{X}$ smooth. Thus $f_*\mathbb{E}_{\widetilde{X}}[d_{\widetilde{X}}]$ is a parity complex, and hence may be decomposed into a direct sum of parity sheaves

$$f_*\mathbb{E}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus_{\lambda,n} \mathcal{E}(\lambda, \mathcal{L})[-n]^{\oplus m_n(\lambda,\mathcal{L})}.$$

We want to make the integers $m_n(\lambda, \mathcal{L})$ explicit. If X_λ is an open stratum, then $\mathcal{E}(\lambda, \mathbb{E})$ certainly appears (and maybe also $\mathcal{E}(\lambda, \mathcal{L})$ for other local systems $\mathcal{L}$), proving its existence. Assuming for simplicity the strata are simply connected so that that only constant local systems occur, if we can compute the other multiplicities, then we can compute the stalks of $\mathcal{E}(\lambda, \mathbb{E})$ (assuming by induction that we know the stalks of $\mathcal{E}(\mu, \mathbb{E})$ for $\mu < \lambda$); and many important representation theoretic problems are encoded in these stalks. The problem of computing the multiplicities can be challenging but the situation is much better than for $\mathcal{IC}(\overline{X}_\lambda, \mathcal{L})$. At least parity sheaves occur as direct summands rather than subquotients. This is what makes modular $\mathcal{IC}$ sheaves so difficult to compute.

The answer to the multiplicity problem is provided, at least theoretically, by some intersection forms appearing in the work of de Cataldo and Migliorini. Let $F_\lambda := f^{-1}(x_\lambda)$, where $x_\lambda \in X_\lambda$. As in [6, §3.3], we set

$$\mathcal{L}_\lambda^\bullet := i_\lambda^! f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \simeq \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_\lambda^n[-n],$$

where $\mathcal{L}_\lambda^n := \mathcal{H}^n(\mathcal{L}_\lambda^\bullet)$ is a local system on X_λ with stalks isomorphic to the Borel–Moore homology of the fiber, $H^{BM}_{d_{\widetilde{X}}-n-2d_\lambda}(F_\lambda)$ (the decomposition holds because f is even). We have

$$\mathbb{D}\mathcal{L}_\lambda^\bullet \simeq i_\lambda^* f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \simeq \bigoplus_{n\in\mathbb{Z}} (\mathcal{L}_\lambda^n)^\vee[2d_\lambda + n] \simeq \bigoplus_{n\in\mathbb{Z}} (\mathcal{L}_\lambda^{-n-2d_\lambda})^\vee[-n].$$

The canonical morphism

$$\mathcal{L}_\lambda^\bullet = i^! f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \xrightarrow{D_\lambda} i^* f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \mathbb{D}\mathcal{L}_\lambda^\bullet.$$

decomposes as a direct sum of morphisms

$$D_\lambda^n := \mathcal{H}^n(D_\lambda) : \mathcal{L}_\lambda^n \to (\mathcal{L}_\lambda^{-2d_\lambda - n})^\vee,$$

which can be interpreted as the local system version of the intersection forms

$$H^{BM}_{(d_{\widetilde{X}}-d_\lambda)-(n+d_\lambda)}(F_\lambda) \otimes H^{BM}_{(d_{\widetilde{X}}-d_\lambda)+(n+d_\lambda)}(F_\lambda) \to k.$$

Theorem 32. *We have an isomorphism*

$$f_*\mathbb{E}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus_{\lambda\in\Lambda;n\in\mathbb{Z}} \mathcal{E}(\lambda, \mathcal{L}^n_\lambda/\ker D^n_\lambda)[-n-d_\lambda].$$

In particular, the multiplicity $m_n(\lambda, \mathcal{L})$ of an indecomposable parity sheaf $\mathcal{E}(\lambda, \mathcal{L})[-n]$ as a direct summand of $f_*\mathbb{E}_{\widetilde{X}}[d_{\widetilde{X}}]$ is equal to the multiplicity of $\mathcal{L}$ in $\mathcal{L}^{n-d_\lambda}_\lambda/\ker D^{n-d_\lambda}_\lambda$.

Note that if f is semi-small then the direct image is perverse, and $\mathcal{L}^n_\lambda = 0$ unless $n = -d_\lambda$.

2.4.4 Parity sheaves and tilting modules

Let G be a reductive algebraic group over an algebraically closed field k of characteristic p, and let us use the notation of §1.2. The category of G-modules is a highest weight category on the poset $\mathcal{X}^*(T)$, with costandard objects the "induced modules" $\Delta(\lambda) := H^0(\lambda)$ and standard objects the "Weyl modules" $\nabla(\lambda) := H^0(-w_0\lambda)^*$ (they can be obtained by some procedure of reduction modulo p). A G-module is *tilting* if it has a filtration with successive quotients isomorphic to standard objects, and another filtration with successive quotients costandard objects. Indecomposable tilting modules are also parametrized by their highest weights (see [Don2]), and we denote them by $T(\lambda)$. The class of tilting modules is known to be stable by tensor product and by restriction to a Levi subgroup [Wan, Don1, Mat].

Let us now explain the geometric Satake equivalence [MV]. Let $\check{G} \supset \check{T}$ be the connected complex reductive group and maximal torus with the dual root datum $(\mathcal{X}_*(T), \Phi^\vee, \mathcal{X}^*(T), \Phi)$, i.e. the Langlands dual group. The affine Grassmannian $\mathcal{G}r$ for $\check{G}$ is an ind–$\check{G}(\mathcal{O})$-variety whose complex points form the set $\check{G}(\mathcal{K})/\check{G}(\mathcal{O})$, where $\mathcal{K} = \mathbb{C}(\!(t)\!)$ and $\mathcal{O} = \mathbb{C}[[t]]$. It is a Schubert variety for the affine Kac–Moody group associated to $\check{G}$. For each $\lambda \in \Lambda^+$, we have a point $t^\lambda \in \mathcal{G}r$. The $\check{G}(\mathcal{O})$-orbits are the $\mathcal{G}r^\lambda := \check{G}(\mathcal{O})t^\lambda$. Let $\mathbf{P}(\mathcal{G}r)$ denote the category of $\check{G}(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{G}r$, with coefficients in k. It is equipped with a convolution product $*$ which makes it a tensor category (the most subtle point being the commutativity constraint, which was worked out by Beilinson and Drinfeld). The global cohomology functor provides a fiber functor. It induces an equivalence of tensor categories between $(\mathbf{P}(\mathcal{G}r), *)$ and (G-mod, $\otimes$), called the geometric Satake equivalence.

In [3], we show that (with some conditions on p) parity sheaves on $\mathcal{G}r$ are perverse and correspond to tilting modules for G . This gives a geometric explanation to the fact that tilting sheaves are stable by tensor product (parity sheaves are stable by convolution) and by restriction to a Levi subgroup (parity sheaves are stable by hyperbolic localization: in the end, this is a consequence of a Bialynicki-Birula decomposition). We know that parity sheaves on $\mathcal{G}r$ may fail to be perverse for p bad. The parity = tilting theorem has been extended to the case of arbitrary good characteristic by C. Mautner and S. Riche [MR].

2.4.5 Some applications

Parity sheaves have had a considerable impact in the field: let us mention the proof by P. Achar and L. Rider [ARd] of the Mirkovic–Vilonen conjecture on intersection cohomology complexes with integral coefficients (with the same conditions on p as ours), the counterexample to Lusztig's conjecture by G. Williamson [Wil], and the work by P. Achar and S. Riche [AR1, AR2, AR3], where they define a mixed modular derived category (endowed with a perverse t-structure) as the homotopy category of parity sheaves. Also, S. Riche and G. Williamson have a very nice conjecture about the characters of tilting modules for reductive groups in characteristic p, which can be interpreted in terms of parity sheaves; they proved it for GL_n [RW]. Because of their result, it is a major problem to determine the p-canonical basis of the anti-spherical module for the affine Hecke algebra (which is the module induced by the the "sign" representation of the finite Hecke algebra).

2.5 Kumar's criterion modulo p

Let k be a commutative ring. A complex irreducible algebraic variety X of dimension n is k-smooth if for all $x \in X$, the local cohomology at x is just k in degree $2n$:

$$\mathsf{H}^\bullet(X, X \setminus \{x\}; k) \cong \mathsf{H}^\bullet(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\}; k) = k[-2n].$$

A point $x \in X$ is said to be k-smooth if it has an open neighborhood which is k-smooth. Finally, the k-smooth locus of X is the open subvariety of k-smooth points. Using the universal coefficient theorem, one can see easily that understanding the $\mathbb{Q}$-smooth locus and the $\mathbb{F}_p$-smooth locus is enough to understand the k-smooth locus for any field: we will call them the rationally smooth locus and the p-smooth locus respectively; the $\mathbb{Z}_p$-smooth locus coincides with the p-smooth locus; the $\mathbb{Z}$-smooth locus is the intersection of the p-smooth loci for all p; and we have inclusions

$$\begin{array}{c}\text{smooth}\\\text{locus}\end{array} \subset \begin{array}{c}\mathbb{Z}\text{-smooth}\\\text{locus}\end{array} \subset \begin{array}{c}p\text{-smooth}\\\text{locus}\end{array} \subset \begin{array}{c}\text{rationally}\\\text{smooth locus}\end{array}$$

all of which are strict in general, as can be seen in the example of simple surface singularities.

S. Kumar gave a criterion to determine the smooth and rationally smooth loci of Schubert varieties, using the torus action, in terms of equivariant multiplicities which can be defined by localizing the equivariant fundamental class in Borel–Moore homology. This was later generalized by A. Arabia and M. Brion to varieties endowed with a torus action satisfying certain conditions. In [5], G. Williamson and I build on these results to give an inductive criterion for p-smoothness. Let us first give some background on equivariant multiplicities.

2.5.1 Equivariant multiplicities

Let $T \simeq (\mathbb{C}^\times)^r$ be an algebraic torus. First recall that $\mathsf{H}^\bullet_T(\mathrm{pt}, k) = \mathsf{H}^\bullet(B_T, k)$ is a polynomial ring, namely $S := k \otimes_{\mathbb{Z}} S(\mathcal{X}^*(T))$, where $\mathcal{X}^*(T)$ is the character group of T. Now let X be a complex irreducible algebraic T-variety admitting a covering by T-stable affine varieties (by a theorem of Sumihiro, this is the case if X is normal). For any complex $\mathcal{F}$ in the equivariant derived category $D^b_T(X, k)$, the equivariant cohomology $\mathsf{H}^\bullet(X, \mathcal{F})$ is a module over S. There is an equivariant fundamental class $\mu_X \in H_T^{-2d}(X, \omega_X)$, where ω_X is the equivariant dualizing complex (this equivariant cohomology group can be interpreted as the top equivariant Borel–Moore homology). We assume that the fixed point set X^T is finite. Let $Q = \operatorname{Frac} S$. It is well known that the inclusion $i : X^T \hookrightarrow X$ induces an isomorphism

$$i_* : \bigoplus_{x \in X^T} Q = \mathsf{H}^\bullet_T(X^T) \otimes_S Q \xrightarrow{\sim} \mathsf{H}^\bullet_T(X, \omega_X) \otimes_S Q.$$

We define the rational functions $e_x X$, for $x \in X^T$, by

$$(e_x X)_{x \in X^T} = i_*^{-1}(\mu_X \otimes 1). \tag{9}$$

One can write $e_x X$ as a reduced fraction

$$e_x X = \frac{f_x}{\chi_1 \cdots \chi_m} \tag{10}$$

where $f_x \in S$ and $\chi_1, \dots, \chi_m$ are characters of T. This rational function is homogenous of degree $-2n$ (where $n = \dim_{\mathbb{C}} X$ and we see $\mathcal{X}^*(T)$ in degree 2).

Those equivariant multiplicities can be computed. First, if X is smooth at $x \in X^T$, then we have

$$e_x X = \frac{1}{\det T_x X} = \frac{1}{\chi_1 \dots \chi_n},$$

where $\chi_1, \dots, \chi_n$ are the weights of T on $T_x X$ (see [Bri, Corollary 15]).

Next, if $\pi : Y \to X$ be a proper surjective T-equivariant morphism of finite degree m with Y^T is finite, then for $x \in X^T$, we have [Bri, Lemma 16]

$$e_x X = \frac{1}{m} \sum_{\substack{y \in Y^T \\ \pi(y) = x}} e_y Y. \tag{11}$$

Applying this formula inductively to Bott–Samelson resolutions, one obtains an explicit formula in the case of Schubert varieties. Let $G \supset B \supset T$ be a complex algebraic reductive group with a Borel subgroup and a maximal torus, and choose the system of simple roots so that the weights of T in $\mathfrak{b}$ are negative. The fixed points of T on the flag variety G/B are parametrized by the Weyl group W. For $w \in W$, we denote by X_w the Schubert variety $\overline{BwB/B}$. If $w = s_{\alpha_1} \dots s_{\alpha_m}$ is a reduced expression, then [Bri, pp. 33–34]

$$e_x X_w = \sum_{s_1,\dots,s_m} \prod_{j=1}^{m} s_1 \dots s_j(\alpha_j^{-1}).$$

where the sum runs over all sequences $(s_1, \dots, s_m)$ such that $s_j = s_{\alpha_j}$ or 1, and that $s_1 \dots s_m = x$.

For example, for two distinct simple roots α and β, we have

$$e_1 X_{s_\alpha s_\beta} = \frac{1}{\alpha\beta},$$

and

$$e_1 X_{s_\alpha s_\beta s_\alpha} = \frac{1}{\alpha^2\beta} - \frac{1}{\alpha^2 s_\alpha(\beta)} = \frac{s_\alpha(\beta) - \beta}{\alpha^2 \beta s_\alpha(\beta)} = \frac{-\langle \beta, \alpha^\vee \rangle}{\alpha\beta s_\alpha(\beta)}.$$

The two terms correspond to the subexpressions $(1, 1, 1)$ and $(s_\alpha, 1, s_\alpha)$. As we will see, these calculations illustrate the fact that Schubert varieties are always smooth in codimension 2, and rationally smooth in codimension 3; but they may be singular in codimension 3 in non-simply-laced types: the last example is rationally smooth but not p-smooth for p dividing $\langle \beta, \alpha^\vee \rangle$. See [5, §8] for more examples.

2.5.2 The criteria

Let us state Brion's criterion for rational smoothness, which generalizes previous results by Kumar and Arabia.

Theorem 33 (Brion)**.** *Let X be a T-variety with an attractive fixed point x such that a punctured neighborhood of x in X is rationally smooth. Then the following conditions are equivalent:*

1. *The point x is rationally smooth.*
2. *For any subtorus $T' \subset T$ of codimension 1, the point x is rationally smooth in $X^{T'}$, and there exists a positive rational number c such that*

$$e_x X = c \prod_{T'} e_x(X^{T'}) \tag{12}$$

   (product over all subtori of codimension 1). If moreover each $X^{T'}$ is smooth, then c is an integer.
3. *For any subtorus $T' \subset T$ of codimension 1, the point x is rationally smooth in $X^{T'}$ and we have $\dim X = \sum_{T'} \dim X^{T'}$ (sum over all subtori of codimension 1).*

We deduce that, when the attractive fixed point x is rationally smooth in X, its equivariant multiplicity can be written f_x/π, where f_x is an integer and π is a product of n characters: this holds for T of rank 1 since $e_x X$ is homogeneous, and the formula (12) implies the general case. In fact, examining the proof we can be more precise: if there is a finite number of 1-dimensional orbits through x, then there are exactly n of them, and π is the product of the associated characters (up to a rational multiple, which may be needed to keep the fraction reduced); in general, π is a product of n characters which occur in $T_x X$.

Moreover, the Kumar–Arabia–Brion criterion for smoothness is: $f_x = 1$. In [5], we give the following inductive criterion for p-smoothness, refining the criterion for rational smoothness.

Theorem 34. *Let $T \simeq (\mathbb{C}^\times)^r$ be a complex torus, X a complex affine irreducible T-variety with an attractive (hence unique) fixed point x, and $U = X \setminus \{x\}$. Let p be a prime number. We assume that:*

1. *U is p-smooth (hence rationally smooth);*
2. *X is rationally smooth at x. By assumption 1 and the Kumar-Arabia-Brion criterion, the numerator f_x of the equivariant multiplicity $e_x X$ in (10) is an integer;*
3. *the T-equivariant cohomology $\mathbb{H}^\bullet_T(U;\mathbb{Z})$ is free of p-torsion.*

Then $x \in X$ is p-smooth if and only if p does not divide f_x.

It does not do any harm to assume X affine: for X not necessarily affine but normal, one can find a covering by T-stable affine open subvarieties. The assumptions 1 and 2 are natural from an inductive point of view, since x needs to have a p-smooth punctured neighborhood, and to be rationally smooth itself, to have any chance of being p-smooth. On the other hand, the assumption 3 is a priori hard to check in practice. Fortunately, it is always satisfied for normal slices in Schubert varieties [FW] (the proof uses parity sheaves). Hence we have an effective algorithm to compute the p-smooth locus of Schubert varieties. Combining this result with results by Dyer, we deduce

Corollary 35. *Let X be a Schubert variety in a (finite) flag variety G/B for a semi-simple algebraic group G. Then its p-smooth locus is the same as its rationally smooth locus for the following primes p: for all p if G is simply-laced; for $p > 2$ if G does not contain a component of type G_2; and for $p > 3$ in general.*

This corollary has been obtained independently in [FW] using moment graph techniques. We also deduce from our theorem that for Schubert varieties, the smooth and $\mathbb{Z}$-smooth loci coincide.

2.6 Parabolic degenerations of rational Cherednik algebras

Rational Cherednik algebras, also known as doubly degenerate double affine Hecke algebras, can be associated to any complex reflection group G and depend on a parameter c, a function on the conjugacy classes of reflections of G (there is also another parameter t, with essentially two distinct cases: $t = 0$ or $t = 1$; we work with $t = 1$).

The prepublication [14], which is joint work with S. Griffeth, A. Gusenbauer and M. Lanini, introduces a new tool for their study, *the parabolic degeneration*, which is then applied to give necessary conditions for two central questions in the theory: whether there exists non-zero maps between standard modules, and whether a given simple module is finite dimensional. Both criteria are valid for arbitrary complex reflection groups and arbitrary parameter c ("unequal parameters" are allowed), and they are expressed in terms of representations of the reflection group and some of its subgroups; both have a weak version where one considers only restrictions to (maximal) parabolic subgroups, and a refined version where one considers restrictions to normalizers of parabolic subgroups; for the finite dimensionality criterion, one can also use restriction to more general subgroups (stabilizers of lines which are not necessarily an intersection of reflecting hyperplanes). The criterion for morphisms between standard modules can be iterated, and this leads to a new combinatorics for arbitrary reflection groups, which generalizes the notions of standard tableaux, cores and dominance order for (multi)partitions, which play a central role for the symmetric groups and their wreath products.

To see what the criteria say concretely in each type of complex reflection group, we are thus led to consider a huge number of elementary conditions involving only the representation theory of complex reflection groups and their subgroups; this task is particularly adapted to be automatized. We implemented the criteria in GAP3 using the development version of the CHEVIE package [GAP3, Chev, Mic]. Taken together, all those little conditions give pretty nice results.

In this overview, we will first recall some background on the representation theory of rational Cherednik algebras (see [GGOR]). Then we explain the parabolic degeneration, and the two criteria.

2.6.1 Representations of rational Cherednik algebras

Let V be a finite dimensional complex vector space, and $G = \langle R \rangle \subseteq GL(V)$ a (finite) complex reflection group with set of reflections

$$R := \{s \in G \mid \operatorname{codim}(V^s) = 1\}\,.$$

For each $r \in R$, choose $\alpha_r \in V^*$ such that $\ker(\alpha_r) = V^r$. We fix a parameter c, which is a G-conjugation invariant function $R \to \mathbb{C}$, $r \mapsto c_r$. The *rational Cherednik algebra* $H_c(G, V)$ is the subalgebra of $\operatorname{End}_{\mathbb{C}}(\mathbb{C}[V])$ generated by $\mathbb{C}[V]$ (acting by multiplication), the group G, and the elements $y \in V$ acting via the Dunkl operators:

$$y(f) = \partial_y(f) - \sum_{r \in R} c_r \langle \alpha_r, y \rangle \frac{f - r(f)}{\alpha_r} \qquad \text{for } f \in \mathbb{C}[V].$$

Remarkably, the Dunkl operators commute, and generate a subalgebra isomorphic to $\mathbb{C}[V^*]$.

This algebra has a Poincaré-Birkhoff-Witt kind of decomposition:

$$\mathbb{C}[V] \otimes \mathbb{C}G \otimes \mathbb{C}[V^*] \cong H_c(G, V)$$

as vector spaces. Hence one can study "highest weight modules" like in Lie theory, with the group algebra playing the role of the Cartan subalgebra. The category $\mathcal{O}_c = \mathcal{O}_c(G, V)$ is the full subcategory of $H_c(G, V)$–mod consisting of modules that are locally nilpotent for the action of the Dunkl operators V and finitely generated over the polynomial subalgebra $\mathbb{C}[V] = S(V^*)$. The analogue of a Verma module is a standard module: for $E \in \operatorname{Irr} \mathbb{C}G$, it is defined by

$$\Delta_c(E) := \operatorname{Ind}_{\mathbb{C}[V^*] \rtimes G}^{H_c(G,V)} E.$$

The algebra $H_c(G, V)$ has a grading with $\deg V^* = 1$, $\deg \mathbb{C}G = 0$, and $\deg V = -1$. This grading is induced by the *Euler element*:

$$\mathbf{eu} = \sum_{1 \le j \le n} x_j \frac{\partial}{\partial x_j} = \sum_{1 \le j \le n} x_j y_j + \sum_{r \in R} c_r(1 - r) \in H_c(G, V),$$

where $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are dual bases of V^* and V. It induces a decomposition into finite dimensional generalized eigenspaces $M = \oplus M^d$ on each $M \in \mathcal{O}_c$.

The element $z := \sum_{r \in R} c_r(1 - r)$ is central in $\mathbb{C}G$, hence acts by a scalar c_E on each $E \in \operatorname{Irr} \mathbb{C}G$. Then $\mathbf{eu}$ acts on $\Delta_c(E)_d = S_d(V^*) \otimes E$ by $d + c_E$. The standard module $\Delta_c(E)$ has a unique simple quotient $L_c(E)$, and all simple modules in $\mathcal{O}_c$ arise in this way. We have a short exact sequence

$$0 \longrightarrow \operatorname{Rad} \Delta_c(E) \longrightarrow \Delta_c(E) \longrightarrow L_c(E) \longrightarrow 0,$$

and $\operatorname{Rad} \Delta_c(E) \in \mathcal{O}_c^{> c_E} := \{M \in \mathcal{O}_c \mid M^d \neq 0 \Rightarrow d \in c_E + \mathbb{Z}_{>0}\}$.

Theorem 36 (Ginzburg-Guay-Opdam-Rouquier)**.** *The category $\mathcal{O}_c$ is a highest weight category on the poset* $(\operatorname{Irr} \mathbb{C}G, \le_c)$*, where*

$$E >_c F \Longleftrightarrow c_E - c_F \in \mathbb{Z}_{>0},$$

with standard objects $\Delta_c(E)$*.*

In particular,

$$[\Delta_c(E) : L_c(F)] \neq 0 \Longrightarrow F \ge_c E.$$

and

$$\operatorname{Hom}(\Delta_c(E), \Delta_c(F)) \neq 0 \Longrightarrow E \ge_c F.$$

Actually the conditions imposed by morphisms between standard objects generate the coarsest order with respect to which $\mathcal{O}_c$ with the $\Delta_c(E)$'s is highest weight.

Consider $G = \mathfrak{S}_n$. The Specht module S^λ, for $\lambda \in \mathrm{Part}(n)$, has c-function

$$c_\lambda = c\left(\binom{n}{2} - \sum_{b\in[\lambda]} \mathrm{ct}(b)\right)$$

where $[\lambda]$ is the digram of λ, and the content of the box $b = (i,j)$ is $\mathrm{ct}(b) = j-i$.

2.6.2 Parabolic degeneration

First, let us say that two points $p, q \in V$ are equivalent if they have the same stabilizer in G. The equivalence classes are called *strata*. For S a stratum, we consider $G_S := C_G(S)$ and $N_S := N_G(S) = N_G(G_S)$. Note that G-orbits of strata are in bijection with conjugacy classes of parabolic subgroups of G. If S is one dimensional, then N_S/G_S is cyclic; let n_S denote its order. Moreover, N_S splits as a semi-direct product $G_S \rtimes \langle h\rangle$ [How, Mur]. For later use, let L denote the 1-dimensional representation of N_S factorizing through N_S/G_S, with $h \mapsto e^{2i\pi/n_S}$.

Fix a stratum S and let

$$V_S = V - \bigcup_{\substack{r\in R\\ V^r \not\supseteq S}} V^r.$$

There is a localization functor

$$\begin{array}{rcl} H_c(G,V)\text{–mod} & \longrightarrow & H_c(N_S, V_S)\text{–mod} \\ M & \longmapsto & \mathbb{C}[V_S] \otimes_{\mathbb{C}[V]} M =: M_S \end{array}$$

where $N_S = \{g \in G \mid g(S) \subset S\} \trianglerighteq G_S = C_G(S)$, and $H_c(N_S, V_S)$ is a subalgebra of $\mathrm{End}_{\mathbb{C}}(V_S)$ generated by $\mathbb{C}[V_S]$, the group N_S, and Dunkl operators y_S, for $y \in V$ (defined in the same way as for $H_c(G,V)$). The standard module $\Delta_c(E)$ goes to $\mathbb{C}[V_S] \otimes_{\mathbb{C}} E$ as a $\mathbb{C}[V_S] \rtimes N_S$–module. There is an explicit formula for the action of y_S on $\Delta_c(E)_S$.

There are two ways to focus more closely on S:

1. complete at $I_S = \{f \in \mathbb{C}[V_S] \mid f_{|S} = 0\}$ [BE];

2. take the associated graded gr_{I_S} for the filtration induced by I_S.

Let $M \in \mathcal{O}_c(G,V)$ and consider its localization $M_S \in H_c(N_S, V_S)$–mod. The associated graded algebra $\mathrm{gr}_{I_S} H_c(N_S, V_S)$, hence also $H_c(G_S, S^\perp)$ and $D(S) \rtimes N_S$, act on $\mathrm{gr}_{I_S} M_S$. The $\mathbb{C}[S]$-modules $\mathrm{gr}^j_{I_S} M_S$ are finitely generated for all j, and each is a $D(S) \rtimes N_S$-module: thus these are global sections of vector bundles on S with N_S-equivariant connections. For a point $p \in S$ and a finitely generated $D(S) \rtimes N_S$-module A (the space of sections of a vector bundle that we will also denote by A) we write $\mathrm{Sol}_p(A)$ for the space of germs of holomorphic sections f of A near p satisfying $yf = 0$ for all $y \in \overline{S}$.

Definition 37. Given $M \in \mathcal{O}_c$ and $p \in S$, the parabolic degeneration of M is defined by:

$$\operatorname{Deg}_p M = \bigoplus_j \operatorname{Sol}_p(\operatorname{gr}^j_{I_S} M_S) \in H_c(G_S, S^\perp)\text{–mod}.$$

The ambiguity in the choice of the point $p \in S$ implies that there is an action of the "stratum braid group" B_S which is generated by G_S and a "monodromy operator" T which satisfies $TgT^{-1} = hgh^{-1}$ for $g \in G_S$ (recall that $N_S = G_S \rtimes \langle h \rangle$).

Theorem 38. *If $\dim_{\mathbb{C}} S = 1$ and $V^G = 0$, then we have a direct sum of $H_c(G_S, S^\perp)$ and $\pi_1(S/N_S)$-modules*

$$\operatorname{Deg}_p \Delta_c(E) = \bigoplus_{F \in \operatorname{Irr} \mathbb{C} N_S} \Delta_c(F)^{[\operatorname{Res}^G_{N_S} E : F]} z^{c_F - c_E}$$

where $\Delta_c(F)$ is seen as a $H_c(G, S^\perp)$-module and $z^{c_F - c_E}$ means that T acts by $e^{2\pi i (c_F - c_E)/n_S} h$.

2.6.3 Morphisms between standard modules

Let $G_\bullet = (G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = 1)$ be a maximal chain of parabolic subgroups, and let $R_j = R \cap G_j$ denote the set of reflections in G_j. For $E \in \operatorname{Irr} \mathbb{C} G$, we define a set of "standard Young tableaux"

$$\operatorname{SYT}_{G_\bullet}(E) := \left\{ (E = E_0, E_1, \ldots, E_n) \in \prod_{j=0}^{n} \operatorname{Irr} \mathbb{C} G_j \;\middle|\; \langle E_{j+1}, \operatorname{Res}^{G_j}_{G_{j+1}} E_j \rangle \neq 0 \right\}$$

Given $T \in \operatorname{SYT}_{G_\bullet}(E)$, let

$$c_{T,n+1-j} = \text{scalar by which } \sum_{r \in R_j - R_{j-1}} c_r r \text{ acts on } E_j$$

The weak form of the criterion involves maximal parabolic subgroups and can be iterated, leading to maximal chains of parabolic subgroups.

Theorem 39. *If there is a nonzero map $\Delta_c(E) \to \Delta_c(F)$, then for each chain $G_\bullet$ as above,*

$$\exists T \in \operatorname{SYT}_{G_\bullet}(E),\ \exists U \in \operatorname{SYT}_{G_\bullet}(F),\ c_{U,j} - c_{T,j} \in \mathbb{Z}_{\geq 0} \text{ for } j = 1, \ldots, n$$

The relation between T and U in the statement of the weak form of the criterion could be stated: U dominates T. It generalizes the notion of dominance for multipartitions: for $G = G(r, 1, n)$, take the chain of parabolic subgroups $G_j = G(r, 1, n-j)$. The strong form, where one looks at normalizers N_S instead of maximal parabolic subgroups G_S, adds congruence type conditions (modulo n_S): this would be the analogue of requiring that two multipartitions be in the same core in order to be in the same block. We give the strong form without iteration.

Theorem 40. *If there is a nonzero map $\Delta_c(E) \to \Delta_c(F)$, then for each one-dimensional stratum S, there are simple constituents E_1 and $F_1 \in \operatorname{Irr} \mathbb{C}N_S$ of $\operatorname{Res}^G_{N_S} E$ and $\operatorname{Res}^G_{N_S} F$ respectively, such that*

$$d := (c_E - c_F) - (c_{E_1} - c_{F_1}) \in \mathbb{Z}_{\geq 0},$$

and there exists a nonzero map of $H_c(N_S, S^\perp)$–modules

$$\Delta_c(E_1 \otimes L^{\otimes d}) \to \Delta_c(F_1).$$

To illustrate the criterion, let $G = \mathfrak{S}_8$. We fix the parameter $c = \frac{1}{4}$, and consider $F = S^\mu$, where $\mu = (8) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \; & \; & \; & \; \\ \hline \end{array}$. What are the possible λ with $\operatorname{Hom}(\Delta_c(S^\lambda), \Delta_c(S^\mu)) \neq 0$? With the standard chain $S_8 \supset S_7 \supset \cdots \supset S_1$, we get the usual notion of standard tableau. There is no choice for U:

$$U = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline \end{array}$$

The possibilities for T are:

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 8 \\ \cline{1-1} \end{array} \qquad \text{or} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} \end{array} \qquad \text{or} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 8 \\ \hline 4 & 5 & 6 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$

Only the first two exist by [Dunkl]. We can get rid of the last one using the chain $S_8 \supset S_6 \times S_2 \supset S_6 \supset S_5 \ldots$.

2.6.4 Finite dimensionality

We want a criterion for the finite dimensionality of a simple $H_c(G, v)$-module $L_c(E)$. More generally, we would like to know its support. Being finite dimensional means having $\{0\}$ support. Since the support is stable by multiplication by a scalar, it is sufficient to ask, for each line in V, whether or not it is included in the support. So let $0 \neq v \in V$, and $S := \mathbb{C}^\times v$ (not necessarily a stratum). We consider the centralizer $G_S := C_G(v)$, a parabolic subgroup of G. We will first state the weak form of the criterion. For a collection (M_i) of objects in an abelian category $\mathcal{A}$, we denote by $\langle M_i \rangle_{\text{Serre}}$ the Serre subcategory (full subcategory stable by subobjects, quotients and extensions) generated by the M_i.

Theorem 41. *Let $E \in \operatorname{Irr} \mathbb{C}G$. If $S \not\subseteq \operatorname{supp} L_c(E)$, then $\operatorname{Res}^G_{G_S} E$ belongs to*

$$\left\langle \operatorname{Res}^G_{G_S} F \;\middle|\; F \in \operatorname{Irr} \mathbb{C}G,\ c_F - c_E \in \mathbb{Z}_{>0} \right\rangle_{\text{Serre}} \subseteq \mathbb{C}G_S\text{–mod}.$$

The refined criterion takes into account the normalizer of the line, $N_S := N_G(S)$; even in this more general setting than in the previous section (here S is not necessarily a stratum), N_S still splits as a semi-direct product with a cyclic group $G_S \rtimes \langle h \rangle$. We still denote by n_S the order of h, and by L the 1-dimensional representation of N_S factorizing through N_S/G_S, with $h \mapsto e^{2i\pi/n_S}$.

Theorem 42. *Let $E \in \operatorname{Irr} \mathbb{C}G$. If $S \nsubseteq \operatorname{supp} L_c(E)$, then $\operatorname{Res}^G_{N_S} E$ belongs to*

$$\left\langle L^{\otimes(c_F - c_E)} \otimes \operatorname{Res}^G_{N_S} F \;\middle|\; F \in \operatorname{Irr} \mathbb{C}G,\ c_F - c_E \in \mathbb{Z}_{>0} \right\rangle_{\text{Serre}} \subseteq \mathbb{C}N_S\text{-mod}.$$

As an illustration, Tables 3–5 display the list of potential finite dimensional simple modules for rational Cherednik algebras of type E allowed by our necessary criterion. The first column lists the corresponding characters of W, the second column lists the conditions on the parameter c assumed to be > 0, while the third column shows a funny observation: in almost all cases, the pair associated to the character of the Weyl group via the Springer correspondence (via restriction rather than Fourier transform) involves a distinguished nilpotent orbit! We have no explanation for this observation. We also give the local system in the fourth column; since it corresponds to a character of a symmetric group, we denote it by a partition.

We see that our criterion is quite good to eliminate a priori most characters of the Weyl group: there remain only 3 (resp.7, 16) candidates out of 25 (resp. 60, 122) irreducible characters. The weak form of the criterion would give slightly laxer conditions on c, but the list of possible characters would be almost the same, except for $\phi_{1575,10}$ in type E_8, which can be only be discarded by the strong form of the criterion. Comparing with the results of E. Norton [Nor] who computed supports of simple modules and decomposition matrices for $c \notin \mathbb{Z} + \frac{1}{2}$, we see that the list of characters is accurate but the conditions on the parameter c are quite good but not always sharp. For the lines whose condition is $2c + 1 \in 2\mathbb{Z}$, however, we cannot compare with her results, since she excludes the denominator 2. It would be very interesting to know the exact answer in those cases, in particular to see if we would get rid of the only two non-distinguished orbits which appear in Table 5.

$\phi_{1,0}$	$12c \in \mathbb{Z}$ or $9c \in \mathbb{Z}$	E_6	1
$\phi_{6,1}$	$6c \in \mathbb{Z}$	$E_6(a_1)$	1
$\phi_{15,5}$	$3c \in \mathbb{Z}$	$E_6(a_3)$	11

Table 3: Potential finite dimensional simple modules in type E_6, $c > 0$

$\phi_{1,0}$	$18c + 1 \in 2\mathbb{Z}$ or $14c \in \mathbb{Z}$	E_7	1
$\phi_{7,1}$	$6c \in \mathbb{Z}$ or $10c + 1 \in 2\mathbb{Z}$	$E_7(a_1)$	1
$\phi_{27,2}$	$2c + 1 \in 2\mathbb{Z}$	$E_7(a_2)$	1
$\phi_{21,6}$	$6c \in \mathbb{Z}$	$E_7(a_3)$	11
$\phi_{189,5}$	$2c + 1 \in 2\mathbb{Z}$	$E_7(a_4)$	2
$\phi_{15,7}$	$2c \in \mathbb{Z}$ or $6c + 1 \in 2\mathbb{Z}$	$E_7(a_4)$	11
$\phi_{35,13}$	$2c + 1 \in 2\mathbb{Z}$	$E_7(a_5)$	111

Table 4: Potential finite dimensional simple modules in type E_7, $c > 0$

$\phi_{1,0}$	$60c \in \mathbb{Z}$ or $24c \in \mathbb{Z}$	E_8	1
$\phi_{8,1}$	$30c \in$ or $18c \in 2\mathbb{Z} + 1$	$E_8(a_1)$	1
$\phi_{28,8}$	$20c \in 2\mathbb{Z}$ or $18c \in 2\mathbb{Z} + 1$ or $12c \in \mathbb{Z}$	$E_8(a_3)$	11
$\phi_{35,2}$	$12c \in \mathbb{Z}$	$E_8(a_2)$	1
$\phi_{50,8}$	$12c \in \mathbb{Z}$	$E_8(b_4)$	11
$\phi_{56,19}$	$6c + 1 \in 2\mathbb{Z}$ or $5c \in \mathbb{Z}$	$E_8(b_5)$	111
$\phi_{160,7}$	$8c \in \mathbb{Z}$ or $6c \in \mathbb{Z}$	$E_8(a_4)$	11
$\phi_{175,12}$	$6c \in \mathbb{Z}$	$E_8(b_6)$	21
$\phi_{300,8}$	$6c \in \mathbb{Z}$	$E_8(a_5)$	11
$\phi_{210,4}$	$4c \in \mathbb{Z}$ or $6c \in 2\mathbb{Z} + 1$	$E_8(a_4)$	2
$\phi_{350,14}$	$4c \in 2\mathbb{Z} + 1$	$E_8(a_6)$	111
$\phi_{840,13}$	$3c \in \mathbb{Z}$	$E_8(b_6)$	111
$\phi_{560,5}$	$2c \in 2\mathbb{Z} + 1$	$E_8(b_4)$	2
$\phi_{840,14}$	$2c \in 2\mathbb{Z} + 1$	$D_5 + A_2$	11
$\phi_{1050,10}$	$2c \in 2\mathbb{Z} + 1$	$D_7(a_1)$	11
$\phi_{1400,8}$	$2c \in 2\mathbb{Z} + 1$	$E_8(a_6)$	3

Table 5: Potential finite dimensional simple modules in type E_8, $c > 0$

3 Research project

3.1 Modular representation theory of reductive algebraic groups

G. Williamson's counter-examples to Lusztig's conjecture [Wil] change radically the perspective in the modular representation theory of reductive groups. Recall that it has been known since the 1990's that Lusztig's conjecture gives the right formula for the characters of simple modules when the characteristic p is very large. However, the expected bound for its validity (p greater than h, the Coxeter number, which is equal to n for $G = \mathrm{SL}_n$) turned out to be way too optimistic, since there is a family of counter-examples where p grows exponentially with the rank. As a consequence, there is now no conjecture for the characters of simple modules when p has a reasonable size. Hence there is a whole new world to explore.

Instead of trying to understand simple modules directly, one may try to deal with an a priori harder problem: determining the characters of indecomposable tilting modules. From this one could deduce the characters of simple modules, and the solution for all general linear groups would also give the characters of simple modules of all symmetric groups, another central problem in modular representation theory. The characters of tilting modules are not even known for GL_3 (this would give decomposition for symmetric groups corresponding to three-part partitions). However, the recent preprint [RW] shows that they can be computed for GL_n, and conjecturally for any G, by the p-canonical basis of the anti-spherical module for the affine Hecke algebra. This can be interpreted in terms of parity sheaves [RW, Part 3]. Now the problem becomes to understand the p-canonical basis. By the results of [6], there is a kind of Kazhdan-Lusztig algorithm to do that, but where at each step one has to compute some bilinear forms and their ranks modulo p. Those forms can be explicitly computed in terms of the Soergel calculus of Elias–Williamson [EW2]. G. Williamson has implemented this algorithm, so we can see many examples of p-canonical basis elements (we are only limited by the computational power of the machine). This will be a precious tool to formulate conjectures, that one will then have to prove.

I think that significant advances in our understanding are possible in the next few years, after a period of relative stagnation partly due to the excessive faith in the Coxeter bound for Lusztig's conjecture before 2013.

3.2 Modular characters sheaves and representations of finite reductive groups in transverse characteristic

One can also consider finite reductive groups G^F, fixed points of an algebraic group G under a Frobenius endomorphism F. If p is the characteristic of definition of the group, then the study of representations of G^F in characteristic p can be reduced to that of the algebraic representations of G, which we discussed in the last paragraph (simply by restriction, using a theorem by Steinberg). But one can also consider representations of G^F in characteristic $\ell \neq p$. Then very different methods are needed.

The study of representations of G^F in characteristic 0 already involves a wide array of techniques and results, due in a large part to Lusztig : Deligne–Lusztig theory (which uses the ℓ-adic cohomology of Deligne–Lusztig varieties), generalized Springer correspondence, character sheaves… Besides, we still do not have a complete answer for some groups like the Spin groups.

Thanks to our series of articles [4, 2, 13, 12], we have a modular version of the generalized Springer correspondence. There are still some open problems in this theory: classify cuspidal perverse sheaves for exceptional groups when ℓ is bad; determine explicitly the correspondence (we can do it for SL_n, classical groups when $\ell = 2$, and G_2; for classical groups in odd characteristic we have a conjecture but are not able to prove it); determine whether cuspidal perverse sheaves are their own Fourier transform even in bad characteristic, etc.

But more crucially, it would be very interesting to make a link with the modular representation theory of finite reductive groups. An immediate difficulty is that, in our setting, everything becomes trivial as soon as ℓ does not divide the order of the Weyl group (i.e. in that case everything is like in characteristic 0). However, one would like to have a different behavior for all ℓ dividing the order of G^F, which depends on q, the order of the finite field (it is the order of q modulo ℓ which should really matter). Thus one should take into account the Frobenius endomorphism in a more subtle way than the classical characteristic 0 theory where one just keeps the character sheaves which are isomorphic to their pull-back by F. One should consider an additional structure related to the Frobenius, rather than a property. Based on some encouraging calculations for $G = \mathrm{SL}_2$, Rouquier suggests to consider pairs with a perverse sheaf and a *homotopy equivalence* with its pull-back by F.

This is a great field to explore: find (or confirm) the right way to take Frobenius into account, relate the different definitions of character sheaves in the modular setting (one of them appears in the generalized Springer correspondence under the name “admissible complexes”), make a link with modular Deligne–Lusztig theory… Part of this program could be suitable for a PhD thesis.

A qualitative result which one could reasonably hope to obtain with those methods would be to show that decomposition numbers for a given type are bounded independently of ℓ and q (a question raised by M. Geck).

3.3 A W-equivariant model for G/T

Let us come back to the problem in the Borho-MacPherson approach in the modular case, in the example of $G = \mathrm{SL}_2$, $\ell = 2$. In this case $\mathcal{B} = \mathbb{P}^1$ and $W = \mathfrak{S}_2$; if $\ell \neq 2$ then the complex of W-modules Spr_0 (well defined in the homotopy category) splits as the trivial module in degree 0 plus the sign module in degree 2; but if $\ell = 2$, then both H^0 and H^2 of Spr_0 are trivial. Hence we do not get a faithful representation on the cohomology. Let us consider Spr_0 as a complex instead. It is really $\mathrm{R}\Gamma(G/T, \mathbb{E})$ and the action of $\mathfrak{S}_2$ comes from the right action on G/T. Since this action is free, the complex is perfect; since there are moreover only two non-zero cohomology groups, there is only one possibility up to isomorphism:

$$(\mathbb{E}\mathfrak{S}_2 \xrightarrow{1-s} \mathbb{E}\mathfrak{S}_2 \xrightarrow{1+s} \mathbb{E}\mathfrak{S}_2)$$

where s is the unique simple reflection, and the complex is in degrees between 0 and 2. This description is valid for any coefficients $\mathbb{E}$, even over $\mathbb{Z}$. But for $\ell = 2$, $\mathbb{F}\mathfrak{S}_2 = \mathbb{F}[t]/t^2$ (where $t = 1 - s = 1 + s$), and the complex becomes $(\mathbb{F}[t]/t^2 \xrightarrow{t} \mathbb{F}[t]/t^2 \xrightarrow{t} \mathbb{F}[t]/t^2)$. We can see that the endomorphism algebra is still the group algebra. One could hope that this is true in general type. Of course, it would be enough to understand what happens with $\mathbb{Z}$ coefficients.

Problem 1. Describe the complex $\mathrm{R}\Gamma(G/T, \mathbb{Z})$ explicitly, as a perfect complex of $\mathbb{Z}W$-modules in the homotopy category. Better, give a model of G/T endowed with its W-action as W-equivariant simplicial set.

Even though the initial motivation (adapting the argument of Borho and MacPherson to the modular case) is not so relevant now since it can be bypassed to get a modular Springer correspondence, I still think this complex is a very fundamental object in Lie theory which is worth studying in its own right. It should have a uniform description in terms of combinatorics of reflection groups and root systems (with the same flavor as the Salvetti complex). Moreover, there are related problems to think about: find a W-equivariant simplicial set model for T with its W-action, for the classifying spaces B_T and B_N (where $N = N_G(T)$), etc.

3.4 Geometry of special nilpotent orbits

B. Fu, P. Levy, E. Sommers and I have two papers in preparation following [15]. Both are concerned with special nilpotent orbits, a notion introduced by Lusztig and Spaltenstein. The point of view of Spaltenstein is the following: in type A, the transposition of partitions induces an order reversing involution of the set of nilpotent orbits, while in other types the poset of nilpotent orbits is not symmetric. However, he was able to define an order reversing map from the set of nilpotent orbits to itself, which generalizes the type A map and satisfies some properties (for example, related to Richardson orbits, and orbits of nilpotent elements which are regular in a Levi subalgebra), and notably $d^3 = d$: so d is an involution on its image. The orbits in the image of d are the special orbits. Alternatively, Lusztig defined the notion of special representations of a Weyl group, and it turns out that they are exactly those corresponding to $(\mathcal{O}, \mathbb{K})$ with $\mathcal{O}$ special, via the Springer correspondence. We call the map d the Lusztig–Spaltenstein duality. Apart from three exceptional cases, on the side of Weyl group representations, it corresponds to tensoring with the sign character.

The first paper is concerned with minimal special degenerations and the duality. Inside the poset of all nilpotent orbits, we consider the subposet of special nilpotent orbits, which is symmetric. Then for each edge in the Hasse diagram, we want to describe the singularity of the degeneration (which is not a minimal degeneration in general: there may be non-special orbits in between); and then, we want to compare the singularities corresponding to edges exchanged by the duality.

The regular and subregular orbits are always special, so on the top of the poset we still have a simple singularity, with non-trivial symmetry in non-simply-laced types. On the bottom of the poset, the zero orbit is always special, but the minimal nilpotent orbit is special only in simply-laced types: in types B_n, C_n and F_4, the minimal special orbit is not A_1 but $\widetilde{A}_1$, the orbit of a short root vector; in type G_2, the minimal special orbit is the subregular orbit $G_2(a_1)$. It turns out that the minimal special orbit closures are $b_n^{\mathrm{sp}} := d_{n+1}/\mathfrak{S}_2$, $c_n^{\mathrm{sp}} := a_{2n-1}/\mathfrak{S}_2$, $f_4^{\mathrm{sp}} := e_6/\mathfrak{S}_2$ and $g_2^{\mathrm{sp}} := d_4/\mathfrak{S}_3$ (this was observed by R. Brylinski and B. Kostant). Note that this is dual to the unfolding of Slodowy for the subregular slice. So it is more natural to see the duality as relating special nilpotent orbits of $\mathfrak{g}$ and ${}^L\mathfrak{g}$, where ${}^L\mathfrak{g}$ is the Langlands dual reductive Lie algebra (note that the posets of special orbits for B_n and C_n are isomorphic).

For $X = \overline{\mathcal{O}}_{\min} \subseteq \mathfrak{g}$ (where $\mathfrak{g}$ will be simply-laced), we consider the group $\operatorname{Aut}(X)$ of automorphisms of X induced by automorphisms of $\mathfrak{g}$, and the normal subgroup $\operatorname{Inn}(X)$ of the automorphisms of X induced by inner automorphisms of $\mathfrak{g}$; we denote by $\operatorname{Out}(X)$ the quotient group. It is the automorphism group of the Dynkin diagram of $\mathfrak{g}$.

Similarly, for a simple singularity $X = \mathbb{C}^2/\Gamma$, we consider the group $\operatorname{Out}(X) = \operatorname{Aut}(X)/\operatorname{Inn}(X)$ of automorphisms of X induced from $N_{\mathrm{GL}_2}(\Gamma)$ modulo those which act trivially on the set of projective lines in the exceptional fiber of the minimal resolution. Again, it is the group of graph automorphisms of the corresponding simply-laced Lie algebra $\mathfrak{g}$.

Now let $\mathcal{S}$ be a nilpotent Slodowy slice between two special nilpotent orbits $\mathcal{O}_1 \subseteq \overline{\mathcal{O}}_2$, passing through $e \in \mathcal{O}_1$, with $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}$. Either each irreducible component of $\mathcal{S}$ is a simple singularity, or $\mathcal{S}$ contains a $C(\mathfrak{s})$-orbit whose closure has irreducible components which are minimal special nilpotent orbit closures. In any case, $C(\mathfrak{s})$ permutes transitively the components. We choose one of them and denote it by X. We consider the permutation representation of $A_G(\mathcal{O}_1)$ on $\operatorname{Irr}\mathcal{S}$, and denote by $J(X)$ the subgroup stabilizing X.

Let $K(\mathcal{O}_1) \subseteq A_G(\mathcal{O}_1)$ be the kernel of Lusztig's canonical quotient (which he defined in terms of Springer rerpesentations). We observe that $K(\mathcal{O}_1) \subseteq J(X)$. Finally, let $H(X)$ denote the image of $J(X)$ in $\operatorname{Out}(X)$. We prove the following generalization of the observation of Kraft and Procesi in type A.

Theorem 43. *Let $\mathcal{S}$ be the nilpotent Slodowy slice between two special nilpotent orbits $\mathcal{O}_1 \subset \overline{\mathcal{O}}_2$ in $\mathfrak{g}$, and let $d(\mathcal{S})$ be the nilpotent Slodowy slice between their dual orbits $d(\mathcal{O}_2) \subset \overline{d(\mathcal{O}_1)}$ in ${}^L\mathfrak{g}$. Then either $\mathcal{S}$ or $d(\mathcal{S})$ is of dimension two. Assume without loss of generality that $\mathcal{S}$ is dimension two.*

Let X be an irreducible component of $\mathcal{S}$ and Y an irreducible component of $d(\mathcal{S})$. Suppose that the images of $K(X)$ in $\operatorname{Out}(X)$ and of $K(Y)$ in $\operatorname{Out}(Y)$ are trivial. If neither $\mathcal{O}_1$ nor $d(\mathcal{O}_2)$ belongs to the list of three exceptional nilpotent orbits: $A_4 + A_1$ in E_7, $A_4 + A_1$ or $E_6(a_1) + A_1$ in E_8, then the singularity with symmetry

$$(X, H(X))$$

corresponds to the slice of the nilcone at a subregular nilpotent orbit in a simple Lie algebra $\mathfrak{m}$ *and*

$$Y/H(Y)$$

corresponds to the closure of minimal non-zero special nilpotent orbit in the Langlands dual Lie algebra ${}^L\mathfrak{m}$.

On the other hand, if $\mathcal{O}_1$ *or* $d(\mathcal{O}_2)$ *is exceptional in the above sense, then* $(X, H(X))$ *is of type* $A_2^+ = (A_2, \mathfrak{S}_2)$ *or* $A_4^+ = (A_4, \mathfrak{S}_2)$*, while* $Y/H(Y)$ *is of dual type* $a_2/\mathfrak{S}_2$ *or* $a_4/\mathfrak{S}_2$*, respectively.*

In most cases, the hypothesis on K-groups is satisfied; we also have some observations and conjectures in the case where the cases where it fails. We are still working on those cases. This is also interesting for classical groups.

The second paper in preparation is about the geometry of special pieces. Spaltenstein observed that the special pieces $\widehat{\mathcal{O}}$, consisting of a special orbit $\mathcal{O}$ and those orbits in its closure not contained in the closure of a smaller special orbit, form a partition of the nilpotent cone. Lusztig remarked that special pieces in exceptional types are always rationally smooth, and conjectured that it is true also for classical types; this was later proved by Kraft and Procesi: in classical types, a special piece $\widehat{\mathcal{O}}$ is the quotient of a smooth variety by an elementary abelian 2-group related to $A_G(\mathcal{O})$. Then Lusztig conjectured that every special piece $\widehat{\mathcal{O}}$ is the quotient of a smooth variety by a precise finite group of the canonical quotient of $A_G(\mathcal{O})$. We were able to prove a closely related result. Namely, in a special piece $\widehat{\mathcal{O}}$, there is always a minimal orbit $\mathcal{O}'$. Then, we can show that for all special pieces in exceptional types, the slice between $\mathcal{O}$ and $\mathcal{O}'$ is of the form

$$(\mathfrak{h}_n \oplus \mathfrak{h}_n^*)^k/\mathfrak{S}_{n+1} \tag{13}$$

where $\mathfrak{h}_n$ is the reflection representation of $\mathfrak{S}_{n+1}$. The case $n = 1$ is $c_k = \mathbb{C}^{2k}/\{\pm 1\}$ (including the cases $c_1 = a_1$ and $c_2 = b_2$); apart from this case, most of the time $k = 1$. So we can say that Lusztig's conjecture is at least locally true. In particular, special pieces are normal: so this gives some information on the normal locus of special nilpotent orbit closures. We remark that the description by Kraft of Procesi of special pieces in classical types implies that for those types, the slice is isomorphic to a product of c_k singularities, so we have a statement for general G if we allow products of singularities like (13).

If $\widehat{\mathcal{O}}$ is a special piece containing at least three orbits, then except for $D_4(a_1) + A_1$ in E_8, the orbit $\mathcal{O}$ is even, hence $\overline{\mathcal{O}}$ admits a Springer resolution, which when restricted to the transverse slice gives a realization of the Hilbert–Chow resolution of $(\mathfrak{h}_n \oplus \mathfrak{h}_n^*)/\mathfrak{S}_{n+1}$.

3.5 Motives, periods, and biarrangements

In [Dup1], C. Dupont (whom I met during my stay at the Max Planck Institute for Mathematics in Bonn, during the Spring 2015) explains how to compute the mixed Tate motive associated to a bi-arrangement of hyperplanes, i.e.

an arrangement of hyperplanes where the strata are colored in blue or in red. Roughly, it is (the motivic version of) the relative cohomology of the complement of the blue strata, relatively to the red strata (more precisely, one should blow up some strata, and color the exceptional divisors according to the colors of those strata). To this end, he introduced the Orlik–Solomon bicomplex associated with a bi-arrangement, generalizing the Orlik–Solomon complex associated with a hyperplane arrangement (which is the case where all strata are blue). Under some exactness condition, he showed that this bicomplex allows to compute the sought for motive; and under an additional combinatorial condition, which he called tameness, he gave a presentation by generators, relations and corelations. This theory has a global version, where hyperplanes are replaced by hypersurfaces.

We have started a collaboration where we reinterpret and extend his results in terms of perverse sheaves. One can start with the constant sheaf on the complement of the bi-arrangement (with a suitable shift), and perform direct images stratum by stratum, choosing the j_* version for blue strata and the $j_!$ version for red strata. The relative cohomology group can be seen as the hypercohomology of this complex of sheaves (this was considered, in a more restrictive setting, by Goncharov). We have already proved that this complex does not depend on the chosen order on the strata, that it satisfies some Künneth property, and that it is compatible with blow-ups. Moreover, we observed that the exactness condition above corresponds to the case where this complex is a perverse sheaf. In that case, the structure of this perverse sheaf is controlled by the combinatorics of the Orlik–Solomon bicomplex. We intend to implement Orlik–Solomon bicomplexes using the GAP4 `homalg` package.

We have also started a theory going outside the framework of bi-arrangements: strata of any codimensions are allowed, as long as the closures are smooth (we might get rid of that condition by considering intersection cohomology); in that case, the complex is very far from being perverse, but we can describe all the perverse cohomology sheaves via a combinatorics of spectral sequences including a "perverse dimension".

Since we only use Grothendieck's 6 operations, by J. Ayoub's thesis all our constructions are motivic, and we can take any realization, for example mixed Hodge modules.

The motivation is arithmetic and belongs the motivic program to understand periods and particularly multizetas, developed notably by F. Brown. In the case of mixed Tate motives over the integers, it is known by theoretical reasons that for each odd $n \geq 3$, there exists a relative cohomology group which is a non-trivial extension of $\mathbb{Q}(-n)$ by $\mathbb{Q}(0)$; but we do not know an explicit geometric model. Since the associated period is $\zeta(n)$, such a geometric model would give many expressions of $\zeta(n)$ as an integral, maybe enough to prove its irrationality. Let us note that C. Dupont has already found a geometric construction (using a simple bi-arrangement) which allows to isolate the odd zeta values [Dup2], shedding new light on the results of Ball and Rivoal. It is thus desirable to develop tools to compute mixed Tate motives, and bi-arrangements already provide a surprisingly rich source of examples to explore.

4 Bibliography

PhD thesis

[PhD]

Title	Modular Springer correspondence and decomposition matrices
Directors	Cédric Bonnafé and Raphaël Rouquier
Referees	Ngô Bao Châu and Wolfgang Soergel
Defense	11 December, 2007 at Université Paris 7 Denis Diderot

The members of the jury were: Cédric Bonnafé, Michel Brion, Michel Broué, Bernard Leclerc, Jean Michel, Raphaël Rouquier, Wolfgang Soergel and Tonny Springer.

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