On the equivalence between geometric and functional versions of Brunn-Minkowski inequalities

Dario Cordero-Erausquin

Abstract

Motivated by a recent work of Malliaris, Melbourne, Roberto and Roysdon, we review some techniques, old and new, allowing to pass from geometric to functional inequalities of Brunn-Minkowski type.

Functional inequalities in Brunn-Minkowski theory have a long story, with early, sometimes implicit, contributions by Busemann, Berwald, Henstock-Macbeath, Knothe. After Prékopa and Leindler, a somehow definitive form appears in the works of Borell and Brascamp-Lieb, although such form was partly anticipated by Dinghas[1]

Whereas functional versions generalize geometric ones, the converse implication is less direct and was often treated by ad-hoc methods depending on the inequality. The standard procedure is of course to apply the geometric inequality to super-levels sets of functions. However, until now this led often to some change in the inequality under study. Recently, Malliaris, Melbourne, Roberto and Roysdon[2] stated a general equivalence. So we have now a very handy reference for this question.

This question regained interest when it was realized that measures other than the Lebesgue measure can verify a dimensional Brunn-Minkowski inequality. However, by a celebrated classification due to C. Borell, such inequality cannot hold for all sets, but only for a class of set. In the sequel, we fix some class $\mathcal{C}$ of measurable sets of $\mathbb{R}^n$; we have in mind the following two situations: the class of all measurable sets, and the class of all symmetric convex sets. The natural question is whether a measure satisfying a Brunn-Minkwoski inequality satisfy a Borell-Brascamp-Lieb inequality. To this aim, we recall the following classical notation: we fix $\lambda \in (0,1)$ and for $p \in \mathbb{R}$ we set

$$\mathcal{M}_p(a,b) = \big((1-\lambda)a^p + \lambda b^p\big)^{1/p} \quad \text{when } a>0,\ b>0,$$

and $\mathcal{M}_p(a,b) = 0$ otherwise. The limit cases are given for $a,b>0$ by $\mathcal{M}_{-\infty}(a,b) = \min(a,b)$, $\mathcal{M}_0(a,b) = a^{1-\lambda}b^\lambda$ and $\mathcal{M}_\infty(a,b) = \max(a,b)$.

It is important to set the value 0 when one of the number is zero, as there is no hope to lower bound the measure of $A + \emptyset = \emptyset$ by the measure of A; for instance $\mathcal{M}_1$ is not exactly the convex combination. In the functional statements, this property is crucial as it allows to restrict ourselves to the support of the functions. That being said, these $\mathcal{M}_p$ means satisfy the same Hölder inequality as the corresponding L^p-norms (on a two-point space) and homogeneity.

The next statement is a particular case of Malliaris, Melbourne, Roberto and Roysdon's result.

Theorem 1. *Let $p \geq -1$ and let ν be a Borel measure on a finite dimensional Euclidean space E which satisfies the following Brunn-Minkowski inequality:*

$$\nu((1-\lambda)A + \lambda B) \geq \mathcal{M}_p(\nu(A), \nu(B))$$

for all A, B belonging to some class $\mathcal{C}$ of measurable sets of E.

Then ν verifies the following BBL inequality. For any $q \geq -p$ and $f,g,h : E \to \mathbb{R}^+$, with f and g having their super-level sets in the class $\mathcal{C}$, if we have

$$\forall x,y \in E \qquad h((1-\lambda)x + \lambda y) \geq \mathcal{M}_q(f(x), g(y)),$$

then $\int h\, d\nu \geq \mathcal{M}_{\frac{qp}{q+p}}(\int f\, d\nu, \int g\, d\nu)$.

[1] *Über eine Klasse superadditiver Mengenfunkionale von Brunn-Minkowski-Lusternikschen Typus*, Math. Z. 68 (1957), pp. 111-129.

[2] *Functional Liftings of Restricted Geometric Inequalities*, arXiv:2508.15247, 2025.

The classical (dimensional) Brunn-Minkowski inequality on $\mathbb{R}^n$ corresponds to $p = \frac{1}{n}$, which therefore imply the usual BBL inequality. Note however that the previous statement is dimension free, and should not be confused with the Borell (Brascamp-Lieb) correspondence between properties of the measure and properties of the density of the measure. As we will see, the previous result is somehow formal and does not encode any geometry.

Note that if the class $\mathcal{C}$ consists of (symmetric) convex sets, we are asking that the functions f, g are (even) quasi-concave; this includes (even) log-concave functions.

The goal of this note is two-fold: first we explain how this result of Malliaris, Melbourne, Roberto and Roysdon can be obtained by a simple, old, method, and then we try to provide a different point of view on their work.

1 First proof: the Borell-Ball approach

In his p.h.d. dissertation, Keith Ball[3] presented a way to reduce functional inequalities to some one dimensional inequality for the measure of the level sets. He was considering a rather particular case: the Prékopa-Leindler inequality and the Lebesgue measure. Let us see how to adapt this to the Theorem above.

So we are given a measure ν as in the Theorem. As is well known, it obviously suffices to treat the case $q = -p$, as this case imply the result for all the other $q > -p$, by Hölder's inequality. So we assume we are given three functions f, g, h as in the theorem with $q = -p$. The conclusion on the integrals is for the mean $\mathcal{M}_{\frac{qp}{q+p}} = \mathcal{M}_{-\infty} = \min$.

Introduce the notation $\nu_u(t) := \nu(\{u \geq t\})$ for a nonnegative function u and $t \geq 0$, so that $\int u\, d\nu = \int_0^\infty \nu_u(t)\, dt$.

By assumption we have for $s, t > 0$

$$\{h \geq \mathcal{M}_{-p}(s,t)\} \supset (1-\lambda)\{f \geq s\} + \lambda\{g \geq t\}$$

and so according to the geometric inequality we have

$$\forall s, t > 0, \qquad \nu_h(\mathcal{M}_{-p}(s,t)) \geq \mathcal{M}_p(\nu_f(s), \nu_g(t)).$$

We then readily conclude to $\int h\, d\nu \geq \min(\int f\, d\nu, \int g\, d\nu)$ thanks to the following Fact:

Fact 2 (Borell). *Let* $p \geq -1$, $u, v, w :]0, \infty[\to \mathbb{R}^+$ *such that*

$$\forall s, t > 0, \qquad w(\mathcal{M}_{-p}(a,b)) \geq \mathcal{M}_p(u(s), v(t)).$$

Then $\int_0^\infty w \geq \min(\int_0^\infty u, \int_0^\infty v)$.

For the proof of this fact, one has three possibilities: one can try to extract it from the general statement of Borell[4], Theorem 2.1 (with the Lebesgue measure on $\mathbb{R}^+$), or else prove it with the standard and simple (in dimension one) monotone transport method, or finally deduce it from the usual Borell-Brascamp-Lieb inequality in

[3] See also: K. Ball, *Some remarks on the geometry of convex sets.*, GAFA seminar (1986/87), Springer Lecture Notes in Math., Vol. 1317, (1988), 224-231.

[4] *Convex set functions in d-space*, Periodica Mathematica Hungarica VoL 6 (2), (1975), pp. 111–136

dimension one for the Lebesgue measure. Let us detail this last approach, which is reminiscent of Ball's argument. Actually, the case $p = 0$ corresponds to the case treated by Ball, so we will assume that $p \neq 0$.

Let us be given three functions as in the Fact, and let us agree on the notation

$$\tilde{z}(s) = \frac{1}{|p|} z(s^{-\frac{1}{p}}) s^{-\frac{1}{p}-1}, \ s > 0,$$

for any given function $z :]0, \infty[\to \mathbb{R}^+$ so that $\int_0^\infty \tilde{z} = \int_0^\infty z$. We have, for all $s, t > 0$,

$$\begin{aligned} \tilde{w}((1-\lambda)s + \lambda t) &\geq \frac{1}{|p|} \mathcal{M}_p(u(s^{-1/p}), v(t^{-1/p}))\, \mathcal{M}_{-\frac{p}{p+1}}(s^{-\frac{1}{p}-1}, t^{-\frac{1}{p}-1}) \\ &\geq M_{-1}(\tilde{u}(s), \tilde{v}(t)), \end{aligned}$$

where the first inequality follows from the condition on the functions, and the second from Hölder's inequality (noting that $p - \frac{p}{p+1} \geq 0$), together with homogeneity. But we can now call upon the classical Borell-Brascamp-Lieb inequality in dimension 1, on $\mathbb{R}^+$ with the Lebesgue measure.

2 Second proof: rewriting Malliaris, Melbourne, Roberto and Roysdon

Malliaris, Melbourne, Roberto and Roysdon do not explicitely reduce the problem to a one-dimensional functional inequality, although their argument is reminiscent of a one dimensional transport argument of such one dimensional inequality. Let us rewrite their argument in a somehow simpler or more 'conceptual' way.

For the sake of variety, let us illustrate this in the particular case $\lambda = \frac{1}{2}$ and $p = 0$, corresponding to the Prékopa-Leindler or Prékopa inequality; it is clear that the same method works for more general $\mathcal{M}_p$ means as in the previous Theorem.

Theorem 3. *Let ν be a Borel measure on a finite dimensional Euclidean space E such that*

$$\nu(\frac{A+B}{2}) \geq \sqrt{\nu(A)\nu(B)}$$

for all A, B belonging to some class $\mathcal{C}$ of measurable sets of E.

Then, if $f, g, h : E \to \mathbb{R}^+$, with f and g having their super-level sets in the class $\mathcal{C}$, satisfy

$$\forall x, y \in E, \qquad h(\frac{x+y}{2}) \geq \sqrt{f(x)g(y)},$$

it holds that $\int h\, d\nu \geq \sqrt{\int f\, d\nu \int g\, d\nu}$.

For any nonnegative Borel function u on E, denote $\Lambda_u(t) = \{u \geq t\}$ and as before $\nu_u(t) = \nu(\Lambda_u(t))$, so that $u = \int_0^\infty 1_{\Lambda_u(t)}\, dt$ and $\int u\, d\nu = \int_0^\infty \nu_u(t)\, dt$, by Fubini. The idea in Malliaris, Melbourne, Roberto and Roysdon's work is to rearrange these level sets so that their measures 'match'. So let us be given three functions f, g, h as in the

Theorem above. By homogeneity, we can assume that $\int f = \int g = 1$. Let $T : \mathbb{R}^+ \to \mathbb{R}^+$ be the increasing continuous function such that for all $s \geq 0$,

$$\int_0^s \nu_f(t)\, dt = \int_0^{T(s)} \nu_g(t)\, dt$$

Equivalently, we have $T = R_g^{-1} \circ R_f$ with the notation $R_u(t) = \int_0^t \nu_u(s)\, ds$ and $R^{-1}(y) = \max\{x \; ; \; R(x) \leq y\}$. The function T is almost-everywhere differentiable with

$$\nu_f(t) = T'(t)\, \nu_g(T(t)).$$

Besides this pointwise equation, we will use that

$$g \geq \int_0^\infty 1_{\Lambda_g(T(s))}\, T'(s)\, ds;$$

A slightly deeper analysis could probably ensure equality there, but the inequality, which is a direct consequence of $\int_a^b T' \leq T(b) - T(a)$, is sufficient for our purposes.

Let us fix $z \in E$. Since we have an non-increasing family $\frac{\Lambda_f(s)+\Lambda_g(T(s))}{2}$ of sets, there is some $t = t(z)$ such that

$$\forall s < t, \quad z \in \frac{\Lambda_f(s) + \Lambda_g(T(s))}{2}, \quad \text{and} \quad \forall s > t, \quad z \notin \frac{\Lambda_f(s) + \Lambda_g(T(s))}{2}.$$

Fix an arbitrary $t_0 < t$ (this would not be necessary if we assume the functions are lower-semi-continuous, for instance, for then one can take $t_0 = t$). Then $z \in \frac{\Lambda_f(t_0)+\Lambda_g(T(t_0))}{2}$ so there exists $x \in \Lambda_f(t_0)$, $y \in \Lambda_g(T(t_0))$ such that $z = \frac{x+y}{2}$ and accordingly, again by monotonicity of the sets, we have

$$\forall s < t_0, \quad 1 = 1_{\frac{\Lambda_f(s)+\Lambda_g(T(s))}{2}}(z) = \sqrt{1_{\Lambda_f(s)}(x) 1_{\Lambda_g(T(s))}(y)}.$$

Therefore

$$\begin{aligned}
\int_0^{t_0} 1_{\frac{\Lambda_f(s)+\Lambda_g(T(s))}{2}}(z)\, \sqrt{T'(s)}\, ds &= \int_0^{t_0} \sqrt{1_{\Lambda_f(s)}(x) 1_{\Lambda_g(T(s))}(y)} \sqrt{T'(s)}\, ds \\
&\leq \sqrt{\int_0^{t_0} 1_{\Lambda_f(s)}(x)\, ds \int_0^{t_0} 1_{\Lambda_g(T(s))}(y) T'(s)\, ds} \\
&\leq \sqrt{\int_0^{\infty} 1_{\Lambda_f(s)}(x)\, ds \int_0^{\infty} 1_{\Lambda_g(T(s))}(y) T'(s)\, ds} \\
&\leq \sqrt{f(x)\, g(y)} \\
&\leq h(z)
\end{aligned}$$

We can let $t_0 \to t$ in the obtained inequality, and by the definition of t this implies that, for every $z \in \mathbb{R}^n$,

$$\int_0^\infty 1_{\frac{\Lambda_f(s)+\Lambda_g(T(s))}{2}}(z)\, \sqrt{T'(s)}\, ds \leq h(z).$$

Integrating and using the geometric inequality we find

$$\int h\, d\nu \geq \int_0^\infty \nu(\frac{\Lambda_f(s) + \Lambda_g(T(s))}{2})\, \sqrt{T'(s)}\, ds$$
$$\geq \int_0^\infty \sqrt{\nu_f(s)\nu_g(T(s))T'(s)}\, ds = \int_0^\infty \nu_f(s)\, ds = 1,$$

as wanted.