Several results regarding the (B)-conjecture

Dario Cordero-Erausquin*, Liran Rotem†

Abstract

In the first half of this note we construct Gaussian measures on $\mathbb{R}^n$ which does not satisfy a strong version of the (B)-property. In the second half we discuss equivalent functional formulations of the (B)-conjecture.

1 Introduction

By a convex body in $\mathbb{R}^n$ we mean a set $K \subseteq \mathbb{R}^n$ which is convex, compact and has non-empty interior. Our convex bodies will always be symmetric, in the sense that $K = -K$. We will denote the standard Gaussian measure on $\mathbb{R}^n$ by γ_n, or simply by γ if there is no possibility of confusion. The density of γ_n is

$$\frac{\mathrm{d}\gamma_n}{\mathrm{d}x} = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-|x|^2/2},$$

where $|\cdot|$ denotes the standard Euclidean norm.

Banaszczyk asked the following question, which was popularized by Latała ([?]): Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body and fix $a, b > 0$ and $0 \leq \lambda \leq 1$. Is it true that

$$\gamma\left(a^{1-\lambda} b^{\lambda} K\right) \geq \gamma(aK)^{1-\lambda} \gamma(bK)^{\lambda}? \tag{1}$$

Recall that a nonnegative function f on $\mathbb{R}^n$ is called log-concave if $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$ for all $x, y \in \mathbb{R}^n$ and all $0 \leq \lambda \leq 1$, and that a Borel measure μ on $\mathbb{R}^n$ is said to be log-concave if

$$\mu\left((1-\lambda)A + \lambda B\right) \geq \mu(A)^{1-\lambda} \mu(B)^{\lambda}$$

for all Borel sets $A, B \subseteq \mathbb{R}^n$ and all $0 \leq \lambda \leq 1$. The addition of sets in the above definition is the Minkowski addition, defined by $A + B = \{a + b : \ a \in A, \ b \in B\}$. Borell proved the following relation between log-concave functions and measures ([?], [?]): Let μ be a Borel measure on $\mathbb{R}^n$ which is not supported on any affine hyperplane. Then μ is log-concave if and only if μ has a log-concave density $f = \frac{\mathrm{d}\mu}{\mathrm{d}x}$.

In particular, the Gaussian measure γ is log-concave. Choosing $A = aK$ and $B = bK$ we see that

$$\gamma\left(\left((1-\lambda)a + \lambda b\right) K\right) \geq \gamma(aK)^{1-\lambda} \gamma(bK)^{\lambda}. \tag{2}$$

However, this inequality is strictly weaker than (1). Moreover, inequality (2) holds for any convex body K, symmetric or not, while the symmetry of K has to be used in order to prove (1) – Nayar and Tkocz ([?]) constructed a counter-example if one replaces the assumption that K is symmetric with the weaker assumption that $0 \in K$.

In [?], Cordero-Erausquin, Fradelizi and Maurey answered Banaszczyk question positively. In fact, they proved the following stronger result. For $t_1, t_2 \ldots, t_n \in \mathbb{R}$ let $\Delta\left(t_1, t_2, \ldots, t_n\right)$ denote the $n \times n$ diagonal matrix with $t_1, t_2, \ldots t_n$ on its diagonal.

*Institut de Mathématiques de Jussieu, Sorbonne Université, dario.cordero@imj-prg.fr

†University of Minnesota, lrotem@umn.edu

Theorem 1 ([?])**.** *Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body. Then the map*

$$(t_1, t_2, \dots, t_n) \mapsto \gamma\left(e^{\Delta(t_1,t_2,\dots,t_n)}K\right)$$

is log-concave.

Banaszczyk's original question is answered by restricting the above function to the line $t_1 = t_2 = \cdots = t_n$.

The main goal of this paper is to discuss extensions of Theorem 1 to other log-concave measures. Let us make the following definitions:

Definition 2. *Let μ be a Borel measure on $\mathbb{R}^n$.*

1. *We say that μ satisfies the* (B)-property *if the function $t \mapsto \mu(e^t K)$ is log-concave for every symmetric convex body K.*
2. *We say that μ satisfies the* strong (B)-property *if the function $(t_1, t_2, \dots, t_n) \mapsto \mu\left(e^{\Delta(t_1,t_2,\dots,t_n)}K\right)$ is log-concave for every symmetric convex body K.*

Theorem 1 states that the standard Gaussian measure has the strong (B)-property. Not many other examples are known in dimension $n \geq 3$. In [?], Eskenazis, Nayar and Tkocz proved that certain Gaussian mixtures satisfy the strong (B)-property. In particular, their result covers the case where μ has density $\frac{\mathrm{d}\mu}{\mathrm{d}x} = e^{-|x|^p}$ for $0 < p \leq 1$ (note that unless $p = 1$ these measures are not log-concave).

In dimension $n = 2$ much more is known. Livne Bar-on showed ([?]) that if $T \subseteq \mathbb{R}^2$ is a symmetric convex body and μ is the uniform measure on T then μ satisfies the (B)-property. Saroglou later showed ([?]) that every even log-concave measure μ on $\mathbb{R}^2$ satisfies the (B)-property.

Saroglou's proof uses the relation between the (B)-property and the log-Brunn-Minkowksi conjecture. In fact, at lot of the interest in the (B)-property comes from this relation. Recall that the support function h_K of a convex body K is defined by

$$h_K(y) = \sup_{x \in K} \langle x, y \rangle .$$

The λ-logarithmic mean of two convex bodies K and T is then defined by

$$L_\lambda(K,T) = \left\{x \in \mathbb{R}^n : \ \langle x, y \rangle \leq h_K(y)^{1-\lambda} h_T(y)^\lambda \text{ for all } y \in \mathbb{R}^n \right\}.$$

In other words, $L = L_\lambda(K,T)$ is the largest convex body such that $h_L \leq h_K^{1-\lambda} h_T^\lambda$. We can also write

$$L_\lambda(K,T) = \bigcap_{s>0} \left((1-\lambda)s^{1/(1-\lambda)}K + \lambda s^{-1/\lambda}T\right).$$

since the support function of an intersection of convex bodies is the infimum of the support functions and $a^{1-\lambda}b^\lambda = \inf_{s>0}\{(1-\lambda)s^{1/(1-\lambda)}a + \lambda s^{-1/\lambda}b\}$ for $a, b \geq 0$. In [?], Böröczky, Lutwak, Yang and Zhang made the following conjecture:

Conjecture 3 (The log-Brunn-Minkowski conjecture)**.** *Let $K, T \subseteq \mathbb{R}^n$ be symmetric convex bodies. Then for every $0 \leq \lambda \leq 1$ we have*

$$|L_\lambda(K,T)| \geq |K|^{1-\lambda} |T|^\lambda$$

where $|\cdot|$ denotes the Lebesgue volume.

Since $L_\lambda(K,T) \subseteq (1-\lambda)K + \lambda T$, the log-Brunn-Minkowski conjecture is a strengthening of the Brunn-Minkowski inequality. Again, this strengthening can only hold under the extra assumption that K and T are convex and symmetric.

A considerable amount of work was done on the log-Brunn-Minkowski conjecture. In the original paper [?] it was proved by Böröczky, Lutwak, Yang and Zhang in dimension $n = 2$. Saroglou ([?]) proved the conjecture when K, T are unconditional, i.e. symmetric with respect to reflections by the coordinate hyperplanes. In [?] it was observed that a more general theorem from [?] implies the conjecture when K and T are unit balls of a complex normed space. Colesanti, Livshyts and Marsiglietti ([?]) proved the conjecture when K and T are small C^2-perturbations of the Euclidean ball. Kolesnikov and Milman ([?]) proved a local form of the closely related L^p–Brunn–Minkowski inequalities for p close enough to 1. Based on their result Chen, Huang, Li and Liu ([?]) then proved the full L^p–Brunn–Minkowski inequality for the same values of p.

Saroglou also proved several connections between the log-Brunn-Minkowski inequality and the (B)-property. In one direction, he proved the following:

Theorem 4 ([?]). *Assume the log-Brunn-Minkowski conjecture holds in dimension n. Then*

$$\mu\left(L_\lambda(K,T)\right) \geq \mu(K)^{1-\lambda}\mu(T)^\lambda$$

for every symmetric $K, T \subseteq \mathbb{R}^n$, every $0 \leq \lambda \leq 1$, and every even log-concave measure μ on $\mathbb{R}^n$. In particular, by choosing K and T to be dilates of each other one could conclude that every *even log-concave measure μ on $\mathbb{R}^n$ has the (B)-property.*

This explains the previous claim that all even log-concave measures in dimension $n = 2$ have the (B)-property.

In the opposite direction, Saroglou also proved that the *strong* (B)-property can imply the log-Brunn-Minkowski conjecture. If Σ is an $n \times n$ positive definite matrix, let us denote by γ_Σ the Gaussian measure with covariance matrix Σ. More explicitly, γ_Σ has density

$$\frac{\mathrm{d}\gamma_\Sigma}{\mathrm{d}x} = \frac{1}{(2\pi)^{\frac{n}{2}}\left(\det\Sigma\right)^{\frac{1}{2}}} e^{-\frac{1}{2}\left\langle\Sigma^{-1}x,x\right\rangle}.$$

Let us also denote by $C_n = [-1,1]^n$ the n-dimensional hypercube. Saroglou's result then implies:

Theorem 5 ([?]). *The following are equivalent:*

1. *The log-Brunn-Minkowski inequality holds in every dimension n.*
2. *For every dimension n, every $n \times n$ covariance matrix Σ and every diagonal matrix A the function*

$$t \mapsto \gamma_\Sigma\left(e^{tA}\cdot C_n\right)$$

   is log-concave.

In fact, there is nothing special here about the Gaussian measure, and the same result holds if γ is replaced with any even log-concave measure together with all of its linear images. However, it is easy to see from Theorem 1 that the measures γ_Σ satisfy the (weak) (B)-property, so the Gaussian seems like the most natural choice.

In particular, Theorem 5 implies that if all mesures γ_Σ satisfy the strong (B)-property then the log-Brunn-Minkowski conjecture is proved. One may conjecture that maybe *all* even log-concave measures have the strong (B)-property. However, an example of Nayar and Tkocz ([?]) shows that this is not the case. But the example from [?]) is not Gaussian, so it does not contradict the idea above.

This paper has two mostly independent sections. In Section 2 we will show that not all measures γ_Σ satisfy the strong (B)-property. In fact, we will prove that in every dimension n there exist Gaussian measures γ_Σ arbitrarily close to the standard Gaussian γ_n, which nonetheless don't satisfy the strong (B)-property.

Theorem 6. *Fix $n \geq 2$. For every $\eta > 0$ there exists a positive-definite $n \times n$ matrix Σ and a symmetric convex body $K \subseteq \mathbb{R}^n$ such that:*

1. *$\|\Sigma - I\| < \eta$, where I denotes the $n \times n$ identity matrix.*

2. *The function $(t_1, t_2, \ldots, t_n) \mapsto \gamma_\Sigma\left(e^{\Delta(t_1,t_2,\ldots,t_n)}K\right)$ is not log-concave.*

Since all norms on a finite dimensional space are equivalent, the choice of the norm in property 1 is immaterial.

In Section 3 we turn our attention to the weak (B)-property. It is still a plausible conjecture that every even log-concave measure satisfies the (B)-property. We refer to this conjecture simply as the (B)-conjecture. The main goal of Section 3 is to prove several equivalent formulations of this conjecture. For example, it turns out that the (B)-conjecture is intimately related to correlation inequalities, and we prove the following result:

Theorem 7. *The following are equivalent:*

1. *Every even log-concave measure in any dimension satisfies the (B)-property.*

2. *For any dimension n, and any functions $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}$ which are convex, even, C^2-smooth and homogeneous, one has*

$$\int \varphi\psi \mathrm{d}\mu \geq \int \varphi \mathrm{d}\mu \cdot \int \psi \mathrm{d}\mu,$$

where μ is the probability measure with density $\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{e^{-\varphi-\psi}}{\int e^{-\varphi-\psi}}$.

"Homogeneous" in the above theorem means homogeneous of an arbitrary degree. In other words, φ is homogeneous if there exists $p \geq 1$ such that $\varphi(\lambda x) = \lambda^p \varphi(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$.

2 A Gaussian Counter-Example

In this section we prove Theorem 6. Recall that we denote by γ_Σ the Gaussian probability measure on $\mathbb{R}^n$ with covariance matrix Σ and simply write γ when $\Sigma = I$. For a convex body $K \subseteq \mathbb{R}^n$, γ_K denotes the standard Gaussian measure restricted to K:

$$\gamma_K(A) = \frac{\gamma(A \cap K)}{\gamma(K)}.$$

For $1 \leq i \leq n$, $\gamma_{(i)}$ denotes the 1-dimensional standard Gaussian measure supported on the i-th axis in $\mathbb{R}^n$, that is, for a test function $f : \mathbb{R}^n \to \mathbb{R}$ we have $\int_{\mathbb{R}^n} f(x) \mathrm{d}\gamma_{(i)}(x) = \int_{\mathbb{R}} f(te_i) \mathrm{d}\gamma_1(t)$, where e_i is the i-th standard unit vector. We will also need the following straighforward fact, which allows us to approximate $\gamma_{(1)}$ by measures of the form γ_K:

Fact 8. *Define $K(\varepsilon) = \left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right] \times [-\varepsilon, \varepsilon]^{n-1} \subseteq \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function, and assume that there exists a constant $C > 0$ such that $|f(x)| \leq Ce^{C|x|}$ and $|\nabla f(x)| \leq Ce^{C|x|}$ for all $x \in \mathbb{R}^n$. Then*

$$\lim_{\varepsilon \to 0^+} \int f \mathrm{d}\gamma_{K(\varepsilon)} = \int f \mathrm{d}\gamma_{(1)}.$$

The assumptions on f in the lemma are not the minimal possible assumptions, but will more than suffice for our needs.

Proof. Let us write a general point $x \in \mathbb{R}^n$ as $x = (t, y)$ where $t \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Define $F, G : (0, \infty) \to \mathbb{R}$ by

$$F(\varepsilon) = \int f \mathrm{d}\gamma_{K(\varepsilon)} = \frac{\int_{K(\varepsilon)} f(t,y) e^{-\frac{1}{2}\left(t^2+|y|^2\right)} \mathrm{d}t\mathrm{d}y}{\int_{K(\varepsilon)} e^{-\frac{1}{2}\left(t^2+|y|^2\right)} \mathrm{d}t\mathrm{d}y}$$

$$G(\varepsilon) = \frac{\int_{K(\varepsilon)} f(t,0) e^{-\frac{1}{2}\left(t^2+|y|^2\right)} \mathrm{d}t\mathrm{d}y}{\int_{K(\varepsilon)} e^{-\frac{1}{2}\left(t^2+|y|^2\right)} \mathrm{d}t\mathrm{d}y} = \frac{\int_{-1/\varepsilon}^{1/\varepsilon} f(t,0) e^{-\frac{1}{2}t^2} dt}{\int_{-1/\varepsilon}^{1/\varepsilon} e^{-\frac{1}{2}t^2} dt}.$$

Since $|f(x)| \le Ce^{C|x|}$ we may apply the dominated convergence theorem and conclude that $\lim_{\varepsilon\to 0^+} G(\varepsilon) = \int f d\gamma_{(1)}$.

For every point $(t,y) \in K(\varepsilon)$ we have $|y| \le \sqrt{n}\varepsilon$, and so $|f(t,y) - f(t,0)| \le Ce^{C|(t,y)|} \cdot (\sqrt{n}\varepsilon)$. It follows that

$$|F(\varepsilon) - G(\varepsilon)| \le \frac{\int_{K(\varepsilon)} |f(t,y) - f(t,0)|\, e^{-\frac{1}{2}|(t,y)|^2} \mathrm{d}t\mathrm{d}y}{\int_{K(\varepsilon)} e^{-\frac{1}{2}|(t,y)|^2} \mathrm{d}t\mathrm{d}y} \le C\sqrt{n}\varepsilon \cdot \frac{\int_{K(\varepsilon)} e^{C|(t,y)| - \frac{1}{2}|(t,y)|^2} \mathrm{d}t\mathrm{d}y}{\int_{K(\varepsilon)} e^{-\frac{1}{2}|(t,y)|^2} \mathrm{d}t\mathrm{d}y}. \tag{3}$$

If we now assume further that $0 < \varepsilon < 1$ then $|y| \le \sqrt{n}$ so

$$\begin{aligned}\int_{K(\varepsilon)} e^{C|(t,y)| - \frac{1}{2}|(t,y)|^2} \mathrm{d}t\mathrm{d}y &\le \int_{K(\varepsilon)} e^{C\sqrt{t^2+n} - \frac{1}{2}t^2} \mathrm{d}t\mathrm{d}y \\ &\le (2\varepsilon)^{n-1} \int_{-\infty}^{\infty} e^{C\sqrt{t^2+n} - \frac{1}{2}t^2} \mathrm{d}t = A_n \cdot \varepsilon^{n-1},\end{aligned}$$

where A_n is a constant which depends on C and n, but not on ε. Similarly we have

$$\int_{K(\varepsilon)} e^{-\frac{1}{2}|(t,y)|^2} \mathrm{d}t\mathrm{d}y \ge \int_{K(\varepsilon)} e^{-\frac{1}{2}(t^2+n)} \mathrm{d}t\mathrm{d}y \ge (2\varepsilon)^{n-1} \int_{-1}^{1} e^{-\frac{1}{2}(t^2+n)} \mathrm{d}t = B_n \varepsilon^{n-1}.$$

Plugging these estimates into (3) we see that $|F(\varepsilon) - G(\varepsilon)| \le C\sqrt{n}\varepsilon \cdot \frac{A_n}{B_n} \xrightarrow{\varepsilon\to 0^+} 0$, so $\lim_{\varepsilon\to 0^+} F(\varepsilon) = \lim_{\varepsilon\to 0^+} G(\varepsilon) = \int f d\gamma_{(1)}$. □

Using Fact 8 we can prove Theorem 6.

Proof of Theorem 6. Assume by contradiction that the function

$$(t_1, t_2, \ldots, t_n) \mapsto \gamma_\Sigma \left(e^{\Delta(t_1,t_2,\ldots,t_n)} K \right) \tag{4}$$

is log-concave for all symmetric convex bodies $K \subseteq \mathbb{R}^n$ and all positive-definite matrices Σ such that $\|\Sigma - I\|$ is small enough.

For a fixed $\eta > 0$, consider the $n \times n$ block matrix $P = \left(\begin{array}{cc|c} 1 & 2\eta & \multirow{2}{*}{0} \\ \eta & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array} \right)$, where I_{n-2} denotes the $(n-2)\times(n-2)$ identity matrix. Consider also the diagonal matrix $D = \Delta(2,1,1,\ldots,1)$. A direct computation shows that

$$A = P^{-1}DP = \left(\begin{array}{cc|c} \frac{2-2\eta^2}{1-2\eta^2} & \frac{2\eta}{1-2\eta^2} & \multirow{2}{*}{0} \\ -\frac{\eta}{1-2\eta^2} & \frac{1-4\eta^2}{1-2\eta^2} & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array} \right).$$

We claim the function $F(t) := \gamma\left(e^{tA} K(\varepsilon)\right)$ is log-concave for all $\varepsilon > 0$ and small enough $\eta > 0$. Indeed, we have

$$F(t) = \gamma\left(P^{-1} e^{tD} P \cdot K(\varepsilon)\right) = \gamma_\Sigma\left(e^{tD} P \cdot K(\varepsilon)\right),$$

where $\Sigma = PP^T$. In other words, if we take $K = P \cdot K(\varepsilon)$, then the function F is a restriction of the function in (4) to a line. Since $\Sigma \to I$ as $\eta \to 0$, the claim follows.

Writing F more explicitly we have

$$F(t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{e^{tA} K(\varepsilon)} e^{-\frac{1}{2}|y|^2} \mathrm{d}y = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{K(\varepsilon)} e^{-\frac{1}{2}\left|e^{tA}x\right|^2} \cdot e^{t \cdot \mathrm{Tr} A} \mathrm{d}x.$$

As the term $e^{t\cdot\mathrm{Tr}A}$ is log-linear, it follows that the function $G(t)=\int_{K(\varepsilon)}e^{-\frac{1}{2}\left|e^{tA}x\right|^2}\mathrm{d}x$ is log-concave, so in particular

$$(\log G)''(0)=\frac{G''(0)}{G(0)}-\left(\frac{G'(0)}{G(0)}\right)^2\leq 0. \tag{5}$$

An explicit calculation of the derivatives gives

$$\begin{aligned}G'(t) &= -\int_{K(\varepsilon)}\left\langle Ae^{tA}x,e^{tA}x\right\rangle\cdot e^{-\frac{1}{2}\left|e^{tA}x\right|^2}\mathrm{d}x\\ G''(t) &= \int_{K(\varepsilon)}\left(\left\langle Ae^{tA}x,e^{tA}x\right\rangle^2-\left\langle A^2e^{tA}x,e^{tA}x\right\rangle-\left\langle Ae^{tA}x,Ae^{tA}x\right\rangle\right)\cdot e^{-\frac{1}{2}\left|e^{tA}x\right|^2}\mathrm{d}x,\end{aligned}$$

So (5) reads

$$\int\left(\langle Ax,x\rangle^2-\left\langle A^2x,x\right\rangle-\langle Ax,Ax\rangle\right)d\gamma_{K(\varepsilon)}(x)-\left(\int\langle Ax,x\rangle\,d\gamma_{K(\varepsilon)}(x)\right)^2\leq 0.$$

Since this inequality is true for any $\varepsilon>0$, we may let $\varepsilon\to 0$. Using Fact 8 we deduce that

$$\int\left(\langle Ax,x\rangle^2-\left\langle A^2x,x\right\rangle-\langle Ax,Ax\rangle\right)d\gamma_{(1)}(x)-\left(\int\langle Ax,x\rangle\,d\gamma_{(1)}(x)\right)^2\leq 0, \tag{6}$$

However, an explicit computation gives

$$\int\langle Ax,x\rangle^2\,\mathrm{d}\gamma_{(1)}(x)=\int\langle Ate_1,te_1\rangle^2\,\mathrm{d}\gamma_1(t)=\langle Ae_1,e_1\rangle^2\cdot\int t^4\mathrm{d}\gamma_1(t)=3\cdot\left(\frac{2-2\eta^2}{1-2\eta^2}\right)^2.$$

Computing the other three integrals in the same way, (6) reduces to

$$3\cdot\left(\frac{2-2\eta^2}{1-2\eta^2}\right)^2-\frac{4-2\eta^2}{1-2\eta^2}-\frac{4\eta^4-7\eta^2+4}{\left(1-2\eta^2\right)^2}-\left(\frac{2-2\eta^2}{1-2\eta^2}\right)^2\leq 0,$$

or equivalently $\frac{\eta^2}{(1-2\eta^2)^2}\leq 0$. Since this is impossible for every $\eta>0$, we arrived at a contradiction and the theorem is proved. □

3 A Functional (B)-conjecture

In this section we discuss several equivalent formulations of the (B)-conjecture. The usual statement of the (B)-conjecture, mentioned in the introduction, involves one log-concave measure μ, and one convex body K. However, it is possible and rather standard to state equivalent conjectures dealing only with bodies, or only with functions:

Theorem 9. *The following are equivalent:*

1. *For every dimension n and every symmetric convex bodies $K,T\subseteq\mathbb{R}^n$ the function $t\mapsto\left|e^tK\cap T\right|$ is log-concave.*
2. *For every dimension n, every symmetric convex body $K\subseteq\mathbb{R}^n$ and every even convex function $\varphi:\mathbb{R}^n\to(-\infty,\infty]$, the function*

$$t\mapsto\int_{e^tK}e^{-\varphi(x)}\mathrm{d}x$$

is log-concave.

3. *For every dimension n and every even convex functions $\varphi,\psi:\mathbb{R}^n\to(-\infty,\infty]$, the function*

$$t\mapsto\int_{\mathbb{R}^n}e^{-\varphi(x)-\psi(e^tx)}\mathrm{d}x$$

is log-concave.

Note that formulation 2. is exactly the standard (B)-conjecture.

Proof. $(2\Rightarrow1)$ is obvious by taking

$$\varphi(x)=\mathbf{1}_T^\infty(x)=\begin{cases}0 & \text{if } x\in T\\ \infty & \text{if } x\notin T\end{cases}.$$

Similarly, to see that $(3\Rightarrow2)$ one chooses $\psi=\mathbf{1}_K^\infty$ and applies the change of variables $t\mapsto-t$, which preserves log-concavity.

Next, we prove that $(1\Rightarrow2)$. Our first observation is that it is not important in 1. for K to be compact, as long as T is compact. Indeed, let us define $F_{K,T}(t)=|e^tK\cap T|$. If we denote the unit Euclidean ball in $\mathbb{R}^n$ by B_n then $F_{K\cap rB_n,T}\to F_{K,T}$ pointwise as $r\to\infty$. Since the pointwise limit of log-concave functions is log-concave, the claim follows.

To show that 2. holds, we will use a standard approximation argument similar to that of [**?**]. Fix a convex body $K\subseteq\mathbb{R}^n$ and a convex function $\varphi:\mathbb{R}^n\to(-\infty,\infty]$. For every integer $m\geq1$ we define

$$\begin{aligned}K_m&=K\times\mathbb{R}^m\subseteq\mathbb{R}^{n+m}\\ T_m&=\left\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m:\ \varphi(x)\leq m \text{ and } |y|\leq c_m\cdot\left(1-\frac{\varphi(x)}{m}\right)\right\}\subseteq\mathbb{R}^{n+m}.\end{aligned}$$

Here c_m is a normalization constant chosen to have $|c_mB_m|=1$. Obviously K_m is a symmetric convex set, and it easy to check that T_m is a symmetric convex body. Therefore by 1. the function F_{K_m,T_m} is log-concave. By Fubini's theorem we have

$$\begin{aligned}F_{K_m,T_m}(t)=\left|e^tK_m\cap T_m\right|&=\int_{e^tK\cap[\varphi\leq m]}\left|c_m\left(1-\frac{\varphi(x)}{m}\right)B_m\right|\mathrm{d}x\\ &=\int_{e^tK\cap[\varphi\leq m]}\left(1-\frac{\varphi(x)}{m}\right)^m\mathrm{d}x=\int_{e^tK}\left(1-\frac{\varphi(x)}{m}\right)_+^m\mathrm{d}x,\end{aligned}$$

where $[\varphi\leq m]:=\{x\in\mathbb{R}^n:\ \varphi(x)\leq m\}$ and $a_+=\max\{a,0\}$. The functions $g_m(x)=\left(1-\frac{\varphi(x)}{m}\right)_+^m$ satisfy $g_m(x)\leq e^{-\varphi(x)}$ for all $x\in\mathbb{R}^n$ and $m\geq1$, and $g_m\to e^{-\varphi}$ pointwise as $m\to\infty$. Hence by the dominated convergence theorem we have

$$\lim_{m\to\infty}F_{K_m,T_m}(t)=\int_{e^tK}e^{-\varphi(x)}\mathrm{d}x$$

for every $t\in\mathbb{R}$. Since pointwise limits preserve log-concavity, 2. follows.

The proof that $(2\Rightarrow3)$ is similar. We are given that for every convex body $T\subseteq\mathbb{R}^n$ and convex function $\psi:\mathbb{R}^n\to(-\infty,\infty]$ the function

$$t\mapsto\int_{e^tT}e^{-\psi(x)}\mathrm{d}x=\int_Te^{-\psi(e^ty)}\cdot e^{nt}\mathrm{d}y$$

is log-concave. As e^{nt} is log-linear, we deduce that $G_{T,\psi}(t)=\int_Te^{-\psi(e^tx)}\mathrm{d}x$ is log-concave.

Given $\varphi, \psi : \mathbb{R}^n \to (-\infty, \infty]$, we define functions $\psi_m : \mathbb{R}^{n+m} \to (-\infty, \infty]$ by $\psi_m(x,y) = \psi(x)$. The functions ψ_m do not satisfy $\int e^{-\psi_m} < \infty$, but this does not matter for the same reason as before. Using the bodies T_m from before and Fubini's theorem we have

$$G_{T_m,\psi_m}(t) = \int_{T_m} e^{-\psi_m(e^t x, e^t y)} \mathrm{d}x\mathrm{d}y = \int_{[\varphi \le m]} \int_{|y| \le c_m\left(1 - \frac{\varphi(x)}{m}\right)} e^{-\psi(e^t x)} \mathrm{d}y\mathrm{d}x$$
$$= \int e^{-\psi(e^t x)} \cdot \left(1 - \frac{\varphi(x)}{m}\right)_+^m \mathrm{d}x \xrightarrow{m\to\infty} \int e^{-\varphi(x) - \psi(e^t x)} \mathrm{d}x.$$

This completes the proof. □

Since the rest of this section deals with property 3. of Theorem 9, let us give this property a name:

Definition 10. *We say that the functional (B)-conjecture holds in dimension n if for every even convex functions $\varphi, \psi : \mathbb{R}^n \to (-\infty, \infty]$ the function*

$$t \mapsto \int_{\mathbb{R}^n} e^{-\varphi(x) - \psi(e^t x)} \mathrm{d}x$$

is log-concave.

By Theorem 9 the functional (B)-conjecture is equivalent to the standard (B)-conjecture, but only if one considers all dimensions simultaneously. For example, we saw in the introduction that the standard (B)-conjecture holds in dimension 2, but the same is unknown for the functional conjecture.

The functional (B)-conjecture is more general than the standard one, but it has one advantage: by a standard approximation argument one may assume that φ and ψ are as smooth as needed. This allows the use of analytic tools such as integration by parts. For example, one can use such tools to show that the (B)-conjecture is equivalent to a certain correlation inequality:

Definition 11. *For a C^1-smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we define its radial derivative $\mathrm{R}f : \mathbb{R}^n \to \mathbb{R}$ by*

$$(\mathrm{R}f)(x) = \langle \nabla f(x), x \rangle .$$

Proposition 12. *The functional (B)-conjecture in dimension n is equivalent to the following: For every C^2-smooth even convex functions $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}$ one has*

$$\int \mathrm{R}\varphi \cdot \mathrm{R}\psi \mathrm{d}\mu \ge \int \mathrm{R}\varphi \mathrm{d}\mu \cdot \int \mathrm{R}\psi \mathrm{d}\mu, \tag{7}$$

where $\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{e^{-\varphi-\psi}}{\int e^{-\varphi-\psi}}$.

Proof. By a standard approximation argument one may assume that φ and ψ are C^2-smooth (or C^∞-smooth if desired). The functional (B)-conjecture states that the function

$$F_{\varphi,\psi}(t) = \int e^{-\varphi(x) - \psi(e^t x)} \mathrm{d}x$$

is log-concave, which is equivalent to

$$(\log F_{\varphi,\psi})''(t) = \frac{F''_{\varphi,\psi}(t)}{F_{\varphi,\psi}(t)} - \left(\frac{F'_{\varphi,\psi}(t)}{F_{\varphi,\psi}(t)}\right)^2 \le 0.$$

If we define $\psi_s(x) = \psi(e^s x)$, then ψ_s is also an even convex function and $F_{\varphi,\psi}(t+s) = F_{\varphi,\psi_s}(t)$. This implies that $(\log F_{\varphi,\psi})''(t) = (\log F_{\varphi,\psi_t})''(0)$. Therefore the functional (B)-conjecture is equivalent to the inequality

$$\frac{F''_{\varphi,\psi}(0)}{F_{\varphi,\psi}(0)} - \left(\frac{F'_{\varphi,\psi}(0)}{F_{\varphi,\psi}(0)}\right)^2 \leq 0 \tag{8}$$

holding for all smooth even convex functions $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}$.

Note that for every smooth $f : \mathbb{R}^n \to \mathbb{R}$ we have $\frac{\mathrm{d}}{\mathrm{d}t} f(e^t x) = \mathrm{R}f(e^t x)$. Hence we have

$$\begin{aligned} F'_{\varphi,\psi}(t) &= -\int \mathrm{R}\psi(e^t x) \cdot e^{-\varphi(x)-\psi(e^t x)} \mathrm{d}x \\ F''_{\varphi,\psi}(t) &= \int \left((\mathrm{R}\psi)^2 (e^t x) - \mathrm{R}^2\psi(e^t x) \right) e^{-\varphi(x)-\psi(e^t x)} \mathrm{d}x, \end{aligned}$$

and the inequality (8) becomes

$$\int \left((\mathrm{R}\psi)^2 - \mathrm{R}^2\psi \right) \mathrm{d}\mu - \left(\int \mathrm{R}\psi \mathrm{d}\mu \right)^2 \leq 0. \tag{9}$$

To continue, we need to integrate by parts. For any smooth function f that doesn't grow too quickly we have

$$\begin{aligned} \int \mathrm{R}f \mathrm{d}\mu &= \frac{\int \left\langle \nabla f, e^{-\varphi-\psi} \cdot x \right\rangle}{\int e^{-\varphi-\psi}} = -\frac{\int f \cdot \mathrm{div}\left(e^{-\varphi-\psi} \cdot x\right)}{\int e^{-\varphi-\psi}} \\ &= -\frac{\int f \cdot \left(\left\langle \nabla e^{-\varphi-\psi}, x \right\rangle + e^{-\varphi-\psi} \mathrm{div} x \right)}{\int e^{-\varphi-\psi}} = -\frac{\int f \cdot \left(\langle -\nabla\varphi - \nabla\psi, x \rangle + n \right) e^{-\varphi-\psi}}{\int e^{-\varphi-\psi}} \\ &= \int f \cdot (\mathrm{R}\varphi + \mathrm{R}\psi - n) \mathrm{d}\mu. \end{aligned}$$

In particular, by taking $f = R\psi$ we see that

$$\int \mathrm{R}^2\psi \mathrm{d}\mu = \int (\mathrm{R}\psi)^2 \mathrm{d}\mu + \int \mathrm{R}\varphi \cdot \mathrm{R}\psi \mathrm{d}\mu - n \int \mathrm{R}\psi d\mu,$$

so inequality (9) is equivalent to

$$-\left(\int \mathrm{R}\varphi \cdot \mathrm{R}\psi \mathrm{d}\mu - n \int \mathrm{R}\psi d\mu \right) - \left(\int \mathrm{R}\psi \mathrm{d}\mu \right)^2 \leq 0,$$

or

$$\int \mathrm{R}\varphi \cdot \mathrm{R}\psi \mathrm{d}\mu \geq \int \mathrm{R}\psi \mathrm{d}\mu \cdot \left(n - \int \mathrm{R}\psi \mathrm{d}\mu \right). \tag{10}$$

A second integration by parts shows that

$$\begin{aligned} \int \mathrm{R}\psi \mathrm{d}\mu &= -\frac{\int \left\langle \nabla\left(e^{-\psi}\right), e^{-\varphi} \cdot x \right\rangle}{\int e^{-\varphi-\psi}} = \frac{\int e^{-\psi} \cdot \mathrm{div}\left(e^{-\varphi} x\right)}{\int e^{-\varphi-\psi}} \\ &= \frac{\int e^{-\psi} \left(\langle \nabla e^{-\varphi}, x \rangle + e^{-\varphi} \mathrm{div} x \right)}{\int e^{-\varphi-\psi}} = \int (n - \mathrm{R}\varphi) \mathrm{d}\mu. \end{aligned}$$

so (10) is equivalent to the correlation inequality

$$\int \mathrm{R}\varphi \cdot \mathrm{R}\psi \mathrm{d}\mu \geq \int \mathrm{R}\varphi \mathrm{d}\mu \cdot \int \mathrm{R}\psi \mathrm{d}\mu,$$

and the proof is complete. □

As a corollary we obtain:

Corollary 13. *The functional (B)-conjecture holds in dimension $n = 1$.*

Proof. We should prove that for every smooth, even and convex functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ we have

$$\int \mathrm{R}\varphi \cdot \mathrm{R}\psi \mathrm{d}\mu \geq \int \mathrm{R}\varphi \mathrm{d}\mu \cdot \int \mathrm{R}\psi \mathrm{d}\mu,$$

where $\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{e^{-\varphi-\psi}}{\int e^{-\varphi-\psi}}$. Since φ and ψ are even so are $\mathrm{R}\varphi$ and $\mathrm{R}\psi$, so one may replace the integrals over $\mathbb{R}$ by integrals over $[0, \infty)$:

$$\int_0^\infty \mathrm{R}\varphi \cdot \mathrm{R}\psi \mathrm{d}\widetilde{\mu} \geq \int_0^\infty \mathrm{R}\varphi \mathrm{d}\widetilde{\mu} \cdot \int_0^\infty \mathrm{R}\psi \mathrm{d}\widetilde{\mu},$$

where $\frac{d\widetilde{\mu}}{\mathrm{d}x} = \frac{e^{-\varphi-\psi}}{\int_0^\infty e^{-\varphi-\psi}} \mathbf{1}_{[0,\infty)}$.

Since φ and ψ are convex and increasing on $[0, \infty)$, $\mathrm{R}\varphi$ and $\mathrm{R}\psi$ are also increasing on $[0, \infty)$. The assertion follows by Chebyshev's correlation inequality (see, e.g. [**?**]. In fact the inequality is true for *any* probability measure $\widetilde{\mu}$).

□

In dimension $n \geq 2$ it is no longer true that $\mathrm{R}\varphi$ and $\mathrm{R}\psi$ are correlated with respect to an arbitrary probability measure μ, even if we further assume that μ is log-concave with respect to $e^{-\varphi-\psi}$. It is not clear how to use the special choice of μ in the inequality.

We conclude this section by proving Theorem 7, which can be seen as a strengthening of Theorem 9 which allows one to check (7) only for homogeneous functions φ and ψ. This may be useful since if φ is homogeneous of degree d then $\mathrm{R}\varphi = d\varphi$. Therefore for homogeneous functions the inequality (7) no longer involves any derivatives.

Proof of Theorem 7. In one direction, assume the (B)-conjecture holds in any dimension. By Theorem 9 the functional (B)-conjecture also holds in any dimension. By Proposition 12 we deduce that

$$\int \mathrm{R}\varphi \cdot \mathrm{R}\psi \mathrm{d}\mu \geq \int \mathrm{R}\varphi \mathrm{d}\mu \cdot \int \mathrm{R}\psi \mathrm{d}\mu.$$

However, if φ is homogeneous of some degree d_1 then $\mathrm{R}\varphi = d_1\varphi$. Similarly if ψ is homogeneous of degree d_2 then $\mathrm{R}\psi = d_2\psi$. Hence we have

$$d_1 d_2 \int \varphi\psi \mathrm{d}\mu \geq \left(d_1 \int \varphi \mathrm{d}\mu \right) \cdot \left(d_2 \int \psi \mathrm{d}\mu \right),$$

which is what we wanted.

In the other direction, assume property 2. in the Theorem holds. We will prove Theorem 9's formulation 1. of the (B)-conjecture. Let K and T be even convex bodies. Recall the definition of the Minkowski functional

$$\|x\|_K = \inf \{\lambda > 0 : \ x \in \lambda K\}.$$

By approximating K and T, we may assume without loss of generality that $\|x\|_K$ and $\|x\|_T$ are C^2-smooth on $\mathbb{R}^n \setminus \{0\}$ (see, e.g., Section 2.5 of [**?**]). It follows that the functions $\varphi_m(x) = \|x\|_K^m$ and $\psi_m(x) = \|x\|_T^m$

are even, convex, C^2-smooth and homogeneous for all $m \geq 2$. The same is obviously true for the functions $\psi_{m,t}(x) = \psi_m(e^t x)$. By our assumption we have

$$\begin{aligned}\int \mathrm{R}\varphi_m \cdot \mathrm{R}\psi_{m,t} \mathrm{d}\mu &= m^2 \cdot \int \varphi_m \psi_{m,t} \mathrm{d}\mu \geq m^2 \int \varphi_m \mathrm{d}\mu \cdot \int \psi_{m,t} \mathrm{d}\mu \\ &= \int \mathrm{R}\varphi_m \mathrm{d}\mu \cdot \int \mathrm{R}\psi_{m,t} \mathrm{d}\mu.\end{aligned}$$

As we saw in the proof of Proposition 12, this inequality is equivalent to

$$\left(\log F_{\varphi_m,\psi_m}\right)''(t) = \left(\log F_{\varphi_m,\psi_{m,t}}\right)''(0) \leq 0,$$

so F_{φ_m,ψ_m} is log-concave.

For any fixed $t \in \mathbb{R}$ we have

$$\lim_{m\to\infty} e^{-\varphi_m(x)-\psi_m(e^t x)} = \mathbf{1}_{K\cap e^{-t}T}(x)$$

for almost every $x \in \mathbb{R}^n$. More precisely, the convergence holds for every $x \notin \partial K \cup \partial\left(e^{-t}T\right)$. Moreover,

$$e^{-\varphi_m(x)-\psi_m(e^t x)} \leq e^{-\varphi_m(x)} \leq \max\left\{\mathbf{1}_K, e^{-\|x\|_K}\right\},$$

which is an integrable function. Hence by dominated convergence we have

$$\lim_{m\to\infty} F_{\varphi_m,\psi_m}(t) = \left|K \cap e^{-t}T\right|.$$

It follows that the function $t \mapsto |K \cap e^{-t}T|$ is log-concave, which is what we wanted to prove up to an immaterial change of sign. $\square$