A Gaussian correlation inequality for plurisubharmonic functions

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Abstract

A positive correlation inequality is established for circular-invariant plurisubharmonic functions, with respect to complex Gaussian measures. The main ingredients of the proofs are the Ornstein-Uhlenbeck semigroup, and another natural semigroup associated to the Gaussian $\overline{\partial}$-Laplacian.

1 Introduction

The motivation of the present work comes from a Gaussian moment inequality in $\mathbb{C}^n$ due to Arias de Reyna [2]. We will show that his result is in fact a very particular case of a new correlation inequality, which can be seen as the complex analogue of the following correlation inequality for convex functions in $\mathbb{R}^n$ due to Hu [8]: if μ is a centered Gaussian measure on $\mathbb{R}^n$ and if $f, g : \mathbb{R}^n \to \mathbb{R}$ are *convex* functions in $L^2(\mu)$ and f is *even*, then

$$\int f\, g\, d\mu \geq \int f\, d\mu \int g\, d\mu.$$

We will say that a function on $\mathbb{C}^n$ is *circular-symmetric* if it is invariant under the action of S^1 (i.e. multiplication by complex numbers of modulus one) ; in other words a function f defined on $\mathbb{C}^n$ is circular-symmetric if

$$f(e^{i\theta} w) = f(w) \qquad \forall \theta \in \mathbb{R}, \quad \forall w \in \mathbb{C}^n.$$

A function $u : \mathbb{C}^n \to [-\infty, +\infty)$ is plurisubharmonic (psh) if it is upper semi-continuous and for all $a, b \in \mathbb{C}^n$ the function $z \in \mathbb{C} \mapsto u(a + zb)$ is subharmonic. Classically, a twice continuously differentiable function $u : \mathbb{C}^n \to \mathbb{R}$ is psh if and only if for all $w, z \in \mathbb{C}^n$

$$\sum_{j,k} \partial^2_{z_j \overline{z_k}} u(z)\, w_j \overline{w_k} \geq 0,$$

where

$$\partial_{z_j} = \frac{1}{2}\big(\partial_{x_j} - i\partial_{y_j}\big) \quad \text{ and } \quad \partial_{\overline{z_j}} = \frac{1}{2}\big(\partial_{x_j} + i\partial_{y_j}\big),$$

$z = x + iy$ with $x, y \in \mathbb{R}^n$. The later condition means that the complex Hessian $D^2_{\mathbb{C}} u$ is pointwize Hermitian semi-definite positive. Functions of the form $\log|F|$ with F is holomorphic on $\mathbb{C}^n$ are known to be psh. We refer e.g. to the textbook [7, Chapter 4] for more details. Let us note that combining Jensen's inequality and the definition of subharmonicity, we see that a convex increasing function of a psh function is again psh; in particular if $\log(g)$ is psh then g is psh.

We consider the standard complex Gaussian measure γ on $\mathbb{C}^n$,

$$d\gamma(w) = d\gamma_n(w) = \pi^{-n} e^{-w\cdot\overline{w}} d\ell(w) = \pi^{-n} e^{-|w|^2} d\ell(w),$$

where ℓ denotes the Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and for $w, w' \in \mathbb{C}^n$

$$w \cdot w' = \sum w_j w'_j.$$

For convenience, let us introduce the following class of $L^2(\gamma)$ functions with controlled growth at infinity:

$$\mathcal{G} := \Big\{ f : \mathbb{C}^n \to \mathbb{C}\,;\; f \in L^2_{loc}(\lambda) \text{ and } \exists \epsilon, c, C > 0 \text{ such that } |f(w)| \leq e^{c|w|^{2-\epsilon}},\ \forall |w| \geq C \Big\}.$$

In particular any function (locally L^2) dominated by a polynomial function, or even an exponential $e^{c|w|}$, on $\mathbb{R}^{2n}$ belongs to $\mathcal{G}$. Our main result reads as follows:

Theorem 1 (Correlation for psh functions)**.** *Let $f, g : \mathbb{C}^n \to [-\infty, \infty)$ be two plurisubharmonic functions belonging to $\mathcal{G}$. If f is* circular-symmetric*, then*

$$\int f\, g\, d\gamma \geq \int f\, d\gamma \int g\, d\gamma.$$

One can extend the result by approximation to more general psh functions in $L^2(\gamma)$. The inequality also extends to arbitrary centered complex Gaussian measure, which are images of γ by $\mathbb{C}$-linear maps. Indeed composing a psh function with a $\mathbb{C}$-linear map gives another psh function.

Let us give some direct consequences of this theorem. First, we see that when F, G are holomorphic functions, or simply complex polynomial functions, belonging to $L^2(\gamma)$ and F is homogeneous, then for any $\alpha, \beta \geq 0$ we have

$$\int |F|^\alpha \, |G|^\beta \, d\gamma \geq \int |F|^\alpha \, d\gamma \int |G|^\beta \, d\gamma.$$

Indeed, if F is holomorphic $f = |F|^\beta$ is psh, and if F is homogeneous, then f is also circular-symmetric. This argument can also be used for products of the form

$$f := |F_1|^{\alpha_1} \dots |F_k|^{\alpha_k}$$

where the F_j are holomorphic and the α_j's are nonnegative real numbers, so that f is log-psh, in the sense that

$$\log f(w) = \sum_{\ell=1}^{k} \alpha_\ell \, \log |F_\ell(w)|$$

is psh. This implies that f is also psh, and if the holomorphic functions F_j are homogeneous then f is also circular-symmetric.

Theorem 2. *Let $F_1, \dots, F_N$ be a family of homogeneous polymomial functions on $\mathbb{C}^n$. Then for any $\alpha, \dots, \alpha_N \geq 0$ and $k \leq N-1$ we have*

$$\int \prod_{j=1}^{N} |F_j|^{\alpha_j} \, d\gamma \geq \Big(\int \prod_{j=1}^{k} |F_j|^{\alpha_j} \, d\gamma \Big) \Big(\int \prod_{j=k+1}^{N} |F_j|^{\alpha_j} \, d\gamma \Big) \geq \prod_{j=1}^{N} \int |F_j|^{\alpha_j} \, d\gamma.$$

A standard complex Gaussian vector in $\mathbb{C}^n$ is a random vector taking values in $\mathbb{C}^n$ according to the distribution $\gamma = \gamma_n$. A random vector $X = (X_1, \dots, X_N) \in \mathbb{C}^N$ is a centered complex Gaussian vector if there is an n, a standard complex Gaussian vector G in $\mathbb{C}^n$ and a $\mathbb{C}$-linear map $A : \mathbb{C}^n \to \mathbb{C}^N$ such that $X = AG$. It turns out that the law for X is then characterized by its complex covariance matrix $\big(\mathbb{E}(X_k \overline{X_\ell})\big)_{1 \leq k, \ell \leq N}$. Denoting by $a_1, \dots a_N \in \mathbb{C}^n$ the rows the matrix of A in the canonical basis, $X_j = G \cdot a_j$. Applying the latter theorem to the complex linear forms $F_j(w) = w \cdot a_j$ yields the following result.

Theorem 3. *Let $(X_1,\dots,X_N)\in\mathbb{C}^N$ be a centered complex Gaussian vector, and let $\alpha_1,\dots,\alpha_N\in\mathbb{R}^+$. Then, for any $k\le N-1$*

$$\mathbb{E}\prod_{j=1}^{N}|X_j|^{\alpha_j}\ge\Big(\mathbb{E}\prod_{j=1}^{k}|X_j|^{\alpha_j}\Big)\Big(\mathbb{E}\prod_{j=k+1}^{N}|X_j|^{\alpha_j}\Big)\tag{1}$$

and in particular

$$\mathbb{E}\prod_{j=1}^{N}|X_j|^{\alpha_j}\ge\prod_{j=1}^{N}\mathbb{E}\,|X_j|^{\alpha_j}.\tag{2}$$

In other words, among centered complex Gaussian vectors $(X_1,\dots,X_N)\in\mathbb{C}^N$ with fixed diagonal covariance (i.e. $(\mathbb{E}|X_j|^2)_{j\le N}$ fixed) the expectation of $\prod_{j=1}^N|X_j|^{\alpha_j}$ is minimal when the variables are independent.

Inequality (2) is an extension of an inequality of Arias de Reyna [2], who established the particular case where all the $\alpha_j=2p_j$ are even integers by rewriting the left hand side in terms of a permanent of a $2m$ matrix ($m=\sum p_j$) and using an inequality for permanents due to Lieb. Actually, Inequality (1) in the case where the α_j are even integers is equivalent to Lieb's permanent inequality, so in particular we are giving a new proof of this inequality.

In the next section we will introduce the tools that will be used in the proof, that is two semi-groups: the usual Ornstein-Uhlenbeck semi-group and another natural semi-group associated to the $\overline{\partial}$ operator (the generator of which could be called, depending from the context, Landau or magnetic Laplacian). In the last section we give the proof of our correlation inequality.

2 Semi-groups

To get the result, we will let the circular-symmetric psh function evolve along the Ornstein-Uhlenbeck semi-group on $\mathbb{C}^n=\mathbb{R}^n+i\mathbb{R}^n\simeq\mathbb{R}^{2n}$ associated with the measure γ and the scalar product : $\langle w,w'\rangle=\Re(w\cdot\overline{w'})$. We recall that its generator is given, for smooth functions f, writing $w=x+iy$, $x,y\in\mathbb{R}^n$, by

$$\begin{aligned}L^{\rm ou}f(w)&:=\frac14\Delta f(w)-\frac12\langle w,\nabla\rangle f(w)\\&=\sum_{j=1}^n\frac14\Big(\partial^2_{x_jx_j}f(w)+\partial^2_{y_jy_j}f(w)\Big)-\frac12\left(x_j\partial_{x_j}f(w)+y_j\partial_{y_j}f(w)\right).\end{aligned}$$

Note that the normalization differs slightly from the usual one on $\mathbb{R}^{2n}$ because our Gaussian measure has complex covariance Id_n but real covariance equal to $\frac12\mathrm{Id}_{2n}$. Accordingly, the spectrum of $-L^{\rm ou}$ on $L^2(\gamma)$ is here $\frac12\mathbb{N}=\{0,\frac12,1,\dots\}$. The Ornstein-Uhlenbeck semi-group $P_t^{\rm ou}=e^{tL^{\rm ou}}$admits the representation, for suitable functions $f:\mathbb{C}^n\to\mathbb{C}$,

$$\begin{aligned}P_t^{\rm ou}f(z)&=\int f(e^{-t/2}z+\sqrt{1-e^{-t}}\,w)\,d\gamma(w)\\&=\pi^{-n}(1-e^{-t})^{-n}\int f(w)\,e^{-\frac{1}{1-e^{-t}}|w-e^{-t/2}z|^2}\,d\ell(w)\end{aligned}\tag{3}$$

As usual in semi-group methods, it is convenient to work with a nice stable space of functions. Here, we can for instance consider

$$\mathcal{G}^\infty:=\big\{f\in C^\infty(\mathbb{C}^n)\,;\,f\text{ and all its derivatives belong to }\mathcal{G}\,\big\}.$$

As is common we will define $P_t^{\rm ou}f$ for any $f\in L^2(\gamma)$ and $t>0$ by (3). Then it is classical and elementary to check that for $f\in\mathcal{G}$, say, the map $(t,z)\mapsto P_t^{\rm ou}f(z)$ is smooth on $(0,\infty)\times\mathbb{C}^n$, with $P_t^{\rm ou}f\in\mathcal{G}^\infty$ and $\partial_tP_t^{\rm ou}f=L^{\rm ou}P_t^{\rm ou}f$ for every $t>0$, and $P_t^{\rm ou}f\to f$ in $L^2(\gamma)$ as $t\to0$. We refer to

[1, Section 2.3] and [4, Section 2.7.1] for details. Let us also mention here that with the definition (3) it is readily checked that properties like convexity, subharmonicity, pluri-subharmonicity are preserved along $P_t^{\rm ou}$.

Another natural operator will be used. Indeed, since pluri-subharmonicity is characterized through a "$\partial^2_{z\bar z}$ operation", we shall also use the following differential operator on smooth functions f on $\mathbb{C}^n$:

$$Lf = \sum_{j=1}^n \left(\partial^2_{z_j\overline{z_j}} f - \overline{z_j}\,\partial_{\overline{z_j}} f\right) = \sum_{j=1}^n e^{|z|^2}\partial_{z_j}\left(e^{-|z|^2}\partial_{\overline{z_j}} f\right)$$

Note that $Lf = 0$ if f is holomorphic. Formally $L = -\overline{\partial}^*\overline{\partial}$ on $L^2(\gamma)$ equipped with the Hermitian structure $(f,g) = \int f\overline{g}\,d\gamma$. More precisely, denoting for a differentiable function

$$\partial_{\bar z} f = (\partial_{\overline{z_1}} f, \ldots, \partial_{\overline{z_n}} f) \quad \text{and} \quad \partial_z f := (\partial_{z_1} f, \ldots, \partial_{z_n} f)$$

we have the following standard fact.

Fact 4 (Integration by parts)**.** *For regular enough functions $f, g : \mathbb{C}^n \to \mathbb{C}$, for instance for functions in $\mathcal{G}^\infty$, we have*

$$\int (Lf)\,\overline{g}\,d\gamma = -\int \partial_{\bar z} f \cdot \overline{\partial_{\bar z} g}\,d\gamma \tag{4}$$

and in particular $\int (Lf)\,\overline{f}\,d\gamma = -\int |\partial_{\bar z} f|^2\,d\gamma \le 0$. We can also write

$$\int (Lf)\,g\,d\gamma = -\int \partial_{\bar z} f \cdot \partial_z g\,d\gamma = \int f(\overline{L}g)\,d\gamma \tag{5}$$

where $\overline{L}f := \sum_{j=1}^n \left(\partial^2_{z_j\overline{z_j}} f - z_j\,\partial_{z_j} f\right)$.

Indeed, it suffices to sum over j the equations

$$\begin{aligned}\int \left[e^{|z|^2}\partial_{z_j}\left(e^{-|z|^2}\partial_{\overline{z_j}} f\right)\right]\overline{g}\,d\gamma &= \pi^{-n}\int \partial_{z_j}\left(e^{-|z|^2}\partial_{\overline{z_j}} f\right)\overline{g}\,d\lambda(z)\\ &= -\int \partial_{\overline{z_j}} f\,\partial_{z_j}\overline{g}\,d\gamma = -\int \partial_{\overline{z_j}} f\,\overline{\partial_{\overline{z_j}} g}\,d\gamma.\end{aligned}$$

The assumption that $f, g \in \mathcal{G}^\infty$ guarantees that the boundary terms (at infinity) in the integration by parts vanish.

As a consequence of (4), we see that the Gaussian measure γ is invariant for L, and actually that L is a symmetric nonpositive operator on $L^2(\gamma)$ with the above-mentioned Hermitian structure. The kernel of L is the Bargmann space $\mathcal{H}_0$ formed by the holomorphic functions on $\mathbb{C}^n$ that belong to $L^2(\gamma)$.

We want to work with the semi-group $P_t = e^{tL}$ which is also Hermitian (formally):

$$\int (P_t f)\,\overline{g}\,d\gamma = \int f\,\overline{P_t g}\,d\gamma \tag{6}$$

Remark 5 (Spectral theory for L)**.** *Although we will not explicitly use it, let us discuss a bit, in a non rigorous way, the (well known) spectral analysis of L on the complex Hilbert space $L^2(\gamma)$. This analysis is indeed fairly standard using the ideas introduced by Landau (see the example of the harmonic oscillator in [6] or [10, Chapter 4]). Following for instance the presentation given in [5, Section 4], consider the "annihilation" operators $a_j = \partial_{\overline{z_j}}$ and their adjoints, the "creation" operators $b_j := a_j^* = \overline{z_j} - \partial_{z_j}$. Then $L = -\sum_{j\le n} b_j \circ a_j$, with $[a_j, b_j] = 1$, and all these operators commute for distinct indices j. Plainly, if a function f and a scalar $\lambda \in \mathbb{C}$ satisfy $-Lf = \lambda f$, then $-L(a_j f) = (\lambda - 1)a_j f$ and $-L(b_j f) = (\lambda + 1)b_j f$. This implies that the spectrum of $-L$ is $\mathbb{N}$ and that the eigenspace associated to the eigenvalue $k \in \mathbb{N}$ is given by the sum of the spaces $b^m\mathcal{H}_0$ with $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, $|m| := \sum_{j\le n} m_j = k$ and the convention $b^m := b_1^{m_1} \circ \ldots \circ b_n^{m_n}$.*

Moreover, if we introduce the classical projection $\Pi_0 : L^2(\gamma) \to \mathcal{H}_0$ onto holomorphic functions, which is known (see e.g. [10, Chapter 4]) to be given by

$$\Pi_0 f(z) := \int f(w)\, e^{z\cdot\overline{w}} d\gamma(w) = \int f(z+w)\, e^{-\overline{z}\cdot w} d\gamma(w), \tag{7}$$

then the projector Π_k on the k-eigenspace can be expressed in terms of Π_0 and the creation and annihilation operators. This allows to compute the reproducing kernel of Π_k, in terms of classical families of orthogonal polynomials. Next, one can sum over k and obtain the kernel $K_t(z,w)$ for $e^{tL} = \sum_k e^{-kt}\Pi_k$. Only the formula for K_t will be useful in the sequel and we shall actually check *below that this suggested formula is indeed the kernel of e^{tL}.*

An explicit formula for the kernel K_t of e^{tL} can be found in [3]: setting

$$K_t(z,w) := \frac{1}{\pi^n(1-e^{-t})^n} \exp\Big(z\cdot\overline{w} - \frac{e^{-t}|z-w|^2}{1-e^{-t}} - |w|^2 \Big), \tag{8}$$

then

$$\begin{aligned} P_t f(z) &= \int f(w)\, K_t(z,w) d\ell(w) \\ &= (1-e^{-t})^{-n} \int f(w)\, e^{z\cdot\overline{w} - \frac{e^{-t}|z-w|^2}{1-e^{-t}}}\, d\gamma(w) \end{aligned} \tag{9}$$

Next, let us note that by performing the change of variable $w = z + \sqrt{1-e^{-t}}\,\xi$ for fixed z we find

$$P_t f(z) = \int f(z+\sqrt{1-e^{-t}}\,\xi)\, e^{-\sqrt{1-e^{-t}}\,\overline{z}\cdot\xi}\, d\gamma(\xi). \tag{10}$$

On this formula, we see immediately that $P_0 = \mathrm{Id}$ and $P_\infty = \Pi_0$ given by (7).

To avoid discussions regarding unbounded operators and existence of semi-groups, we will proceed in the opposite direction and use the previous formula to *define* $P_t f$. Actually, to be fair, we should mention that later, in the proof of our result, we only need to work with smooth functions $f \in \mathcal{G}^\infty$; these functions provide nice initial data, ensuring existence and uniqueness of strong solutions for the semi-group equation. Nevertheless, we feel it is of independent interest to start from the integral formula (9) or (10). The drawback is that some properties that are obvious (formally) for $e^{tL} f$ need to be checked thoroughly when using this kernel representation, in particular because the kernel (8) is not Markovian.

Definition-Proposition 6. *Given $f \in L^2(\gamma)$, $t > 0$ and $z \in \mathbb{C}^n$, we define $P_t f(z)$ by formula (9). We then have for $f \in L^2(\gamma)$ that*

(i) for any $t > 0$, $P_t f \in L^2(\gamma)$ and for any $s > 0$

$$P_s(P_t f) = P_{s+t} f.$$

Moreover, for a given $f \in \mathcal{G}$:

(ii) the function $(t,z) \to P_t f(z)$ is smooth on $(0,\infty)\times\mathbb{C}^n$ and

$$\partial_t P_t f = L P_t f. \tag{11}$$

(iii) $\|P_t f\|_{L^2(\gamma)} \le \|f\|_{L^2(\gamma)}$

(iv) $P_t f \to f$ in $L^2(\gamma)$ as $t \to 0$.

Proof. For $f \in L^2(\gamma)$, applying the Cauchy-Schwarz inequality in (9) we get the pointwise estimate $|P_t f(z)| \leq C(n,t,z)\|f\|_{L^2(\gamma)}$ for some constant $C(n,t,z) > 0$ such that $C(n,t,\cdot) \in L^2(\gamma)$. The semi-group property can be checked in a pedestrian way, by noticing that

$$\int K_t(z,w)K_s(\xi,z)\, d\ell(z) = K_{t+s}(\xi,w).$$

This follows for instance from the observation that given $z, \xi \in \mathbb{C}^n$ and $c \in \mathbb{C}$ with $\Re(c) > 0$,

$$\int e^{z\cdot w + \xi\cdot \overline{w}} e^{-c|w|^2}\, d\ell(w) = \pi^n c^{-n}\, e^{z\cdot\xi\,/c}. \tag{12}$$

In order to derive some properties of $P_t f$, the stronger integrability condition $f \in \mathcal{G}$ will be assumed.

It is readily checked that, for any fixed $w \in \mathbb{C}^n$ one has

$$\partial_t K_t(\cdot, w) = L\, K_t(\cdot, w).$$

Moreover, for $T, R, k > 0$ fixed, there exists constants $c = c(T,R,k), C = C(T,R,k)$ such that for $F(t,z,w) = K_t(z,w)$, or any derivative of $K_t(z,w)$ with respect to t or z of order at most k, it holds that $|F(t,z,w)| \leq Ce^{-c|w|^2}$ for all $w \in \mathbb{C}^n$, all $t \in (\frac{1}{T}, T)$ and all $|z| \leq R$. From this and the definition of $\mathcal{G}$, we can call upon dominated convergence to conclude to the smoothness of $(t,z) \to P_t f(z)$ and to the fact that $P_t f \in \mathcal{G}^\infty$ with $\partial_t P_t f = L P_t f$.

Regarding the contraction property (iii), we want to avoid direct computations or spectral arguments, and so we make a detour and use some obvious but important properties of P_t. Besides the semi-group property that we proved above, note that P_t is indeed Hermitian, in the sense (6), on $\mathcal{G} \subset L^2(\gamma)$; this can be seen directly from the integral formula (9) since $e^{-|z|^2}K_t(z,w) = e^{-|w|^2}\overline{K_t(w,z)}$. Note also that $\|P_t f\|_{L^2(\gamma)}$ decreases for $t \in (0,\infty)$:

$$\frac{d}{dt}\int |P_t f|^2 d\gamma = 2\Re \int (LP_t f)\overline{P_t f}\, d\gamma = -2\int |\partial_{\overline{z}} P_t f|^2\, d\gamma \leq 0.$$

Indeed, $P_t f \in \mathcal{G}^\infty$ for $t > 0$ and the differentiation argument above (for $t \in [\frac{1}{T}, T]$) ensures that we can move the derivative inside the integral; we then use Fact 4.

So for our $f \in \mathcal{G}$ and $t > 0$ we have

$$\|P_t f\|^2_{L^2(\gamma)} = \int (P_{2t} f)\, \overline{f}\, d\gamma \leq \|P_{2t} f\|_{L^2(\gamma)}\, \|f\|_{L^2(\gamma)} \leq \|P_t f\|_{L^2(\gamma)}\, \|f\|_{L^2(\gamma)}$$

which implies the contraction property in $L^2(\gamma)$.

Finally, to prove the continuity at $t = 0$ in $L^2(\gamma)$ we first assume that f is smooth and compactly supported. Using (10) we see that $P_t f$ converge point-wise to f and that for $t \in (0,1)$ we have $|P_t f(z)| \leq C\, e^{c|z|}$ for some constant $c, C > 0$; so we can conclude by dominated convergence. For $f \in \mathcal{G}$ and $\epsilon > 0$, introduce g smooth compactly supported such that $\|f - g\|_{L^2(\gamma)} \leq \epsilon$ and let $\delta > 0$ be such that $t \leq \delta$ ensures that $\|P_t g - g\|_{L^2(\gamma)} \leq \epsilon$ holds. For $t \leq \delta$,

$$\|P_t f - f\|_{L^2(\gamma)} \leq \|P_t f - P_t g\|_{L^2(\gamma)} + \|P_t g - g\|_{L^2(\gamma)} + \|g - f\|_{L^2(\gamma)} \leq 2\|f - g\|_{L^2(\gamma)} + \epsilon \leq 3\epsilon.$$

This establishes the desired continuity. □

Remark 7 (Contraction property)**.**

1. *It could be interesting to approach some results using soft semi-group techniques in place of spectral Hilbertian tools. Sticking to our definition of P_t above, we have proved that $\|P_t f\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)}$ on the dense subspace $\mathcal{G}$, which together with the pointwise estimate given at the beginning of the proof above implies by density that*

$$\|P_t\|_{L^2(\gamma)\to L^2(\gamma)} \leq 1.$$

Since $P_\infty = \Pi_0$ in view of the formula (7), this inequality formally extends the estimate

$$\|\Pi_0\|_{L^2(\gamma)\to L^2(\gamma)} \le 1.$$

Actually, since Π_0 is the orthogonal projection onto holomorphic functions in $L^2(\gamma)$, the convergence of P_t towards Π_0 can be quantified rigorously, through Hörmander's L^2 estimate [7, Chapter 4],

$$\|\varphi - \Pi_0\varphi\|^2_{L^2(\gamma)} \le \|\partial_{\overline{z}}\varphi\|^2_{L^2(\gamma)} = \int (-L\varphi)\varphi \, d\gamma$$

valid for any suitable φ, for instance for $\varphi \in \mathcal{G}^\infty$. Note that from formula (10), P_t acts as the identity on holomorphic functions, so $P_t\Pi_0 = \Pi_0 P_t = \Pi_0$. Reproducing the connection between Poincaré's inequality and the convergence of Markov semi-groups (see [4]), a classical Grönwall type argument (using the previous lemma to justify the computation of $\frac{d}{dt}\int |P_t(f - \Pi_0 f)|^2 \, d\gamma$) indeed ensures that for $f \in \mathcal{G}$ and $t \ge 0$

$$\|P_t f - \Pi_0 f\|^2_{L^2(\gamma)} \le e^{-2t} \, \|f - \Pi_0 f\|^2_{L^2(\gamma)}.$$

2. *In analogy with the Markovian case P_t^{ou} we may wonder if P_t is also a contraction on some $L^p(\gamma)$. However, for any $p \ne 2$ we have*

$$\|P_t\|_{L^p(\gamma)\to L^p(\gamma)} = +\infty,$$

as it can be seen by taking, in dimension $n = 1$, for $a \in \mathbb{R}$,

$$f_a(w) := e^{aw + \overline{w}}, \qquad w \in \mathbb{C}.$$

Indeed, repeated applications of (12) with $c = 1$ show, setting $s_t := \sqrt{1 - e^{-t}}$ and using (10), that $P_t f_a(z) = e^{s_t^2 a} e^{az + (1-s_t^2)\overline{z}}$ and that

$$\frac{\|P_t f_a\|^p_{L^p(\gamma)}}{\|f_a\|^p_{L^p(\gamma)}} = C(t,p) \, e^{a\, s_t^2 (p - p^2/2)}.$$

The next result describes how derivatives and P_t commute, an important issue in semi-group methods.

Lemma 8 (Commutation relations)**.** *For any suitable f, say $f \in \mathcal{G}^\infty$, and $t > 0$ we have for every $1 \le j \le n$ and $z \in \mathbb{C}^n$,*

$$\partial_{z_j}(P_t f(z)) = P_t(\partial_{z_j} f)(z) \qquad \text{and} \qquad \partial_{\overline{z_j}}(P_t f(z)) = e^{-t} P_t(\partial_{\overline{z_j}} f)(z).$$

Proof. We use (10). The first equality is obvious. For the second one, setting $s_t = \sqrt{1 - e^{-t}}$, we have

$$\partial_{\overline{z_j}}(P_t f)(z) = P_t(\partial_{\overline{z_j}} f)(z) - s_t \int f(z + s_t\xi) \, e^{-s_t \, \overline{z}\xi} \, \xi_j \, d\gamma(\xi),$$

and

$$\pi^{-n} \int f(z + s_t\xi) \, e^{-s_t \, \overline{z}\xi} \, \xi_j e^{-\xi\cdot\overline{\xi}} \, d\ell(\xi) = -\pi^{-n} \int f(z + s_t\xi) \, e^{-s_t \, \overline{z}\xi} \, \partial_{\overline{\xi_j}}(e^{-\xi\cdot\overline{\xi}}) \, d\ell(\xi)$$
$$= s_t \int (\partial_{\overline{z_j}} f)(z + s_t\xi) \, e^{-s_t \, \overline{z}\xi} \, d\gamma(\xi) = s_t \, P_t(\partial_{\overline{z_j}} f)(z).$$

□

Now, and for the rest of this section, we focus on the case of circular-symmetric functions. Given $\theta \in \mathbb{R}$ and a function f we denote f_θ the function $f_\theta(w) = f(e^{i\theta} w)$. Note that

$$P_t(f_\theta) = (P_t f)_\theta. \tag{13}$$

Recall that a function f is said to be circular-symmetric if $f_\theta = f$ for every θ. It is worth noting that a holomorphic function on $\mathbb{C}^n$ is necessarily constant when circular-symmetric. Indeed if $h : \mathbb{C} \to \mathbb{C}$ has both properties then invariance and the Cauchy formula give $h(1) = \int_0^{2\pi} h(e^{i\theta}) d\theta/(2\pi) = h(0)$; next if $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic and circular symmetric, then for any $(z_1, \ldots, z_n) \in \mathbb{C}^n$, the function $h : z \in \mathbb{C} \mapsto f(zz_1, \ldots, zz_n)$ is also holomorphic and circular-symmetric, hence $f(z_1, \ldots, z_n) = h(1) = h(0) = f(0)$. Accordingly if $f \in L^2(\gamma)$ is circular-symmetric then $\Pi_0 f \equiv \int f \, d\gamma$ since the gaussian density is also circular-symmetric. Actually, much more can be said, as we shall see below.

Let us first investigate the relation between L and L^{ou}. Note that one can write

$$Lf = L^{\mathrm{ou}} f + \frac{i}{2} \sum_{j=1}^{n} \left(y_j \partial_{x_j} f - x_j \partial_{y_j} f \right). \tag{14}$$

So we have $L^{\mathrm{ou}} = \Re(L) = \frac{L + \overline{L}}{2}$ where $\overline{L} f = \sum_{j=1}^n \left(\partial^2_{z_j \overline{z_j}} f - z_j \, \partial_{z_j} f \right)$ has a kernel formed by the anti-holomorphic functions. The operators L and $\overline{L}$ are Hermitian symmetric, whereas L^{ou} is symmetric for the real and the Hermitian product, and preserves the subspace of real valued functions. As we said, its spectrum is $-\frac{1}{2}\mathbb{N}$, as can be seen also from the formula $L^{\mathrm{ou}} = \frac{L+\overline{L}}{2}$. Let us illustrate this on two examples, obtained by applying the creation operator $b_1 = \overline{z_1} - \partial_{z_1}$ to the holomorphic functions $z \mapsto 1$, and $z \mapsto z_1$. The function $z \mapsto \overline{z_1}$ is an eigenfunction for L with eigenvalue -1, for $\overline{L}$ with eigenvalue 0, and for L^{ou} with eigenvalue $-1/2$. The function $z \mapsto |z_1|^2 - 1$ is an eigenfunction for L with eigenvalue -1, for $\overline{L}$ with eigenvalue -1, and for L^{ou} with eigenvalue -1.

The special role played by circular-symmetric functions is due to the fact that these operators, and the associated semi-groups, coincide for them.

Lemma 9 (Action of L and P_t on circular-symmetric functions). *If f is a smooth circular-symmetric function, then we have*

$$Lf = \overline{L} f = L^{\mathrm{ou}} f.$$

In particular we have, when $f, g \in \mathcal{G}^\infty$ and f is circular-symmetric,

$$\int (Lf) g \, d\gamma = \int f L g \, d\gamma = - \int \partial_z f \cdot \partial_{\overline{z}} g \, d\gamma.$$

Also, if $f \in L^2(\gamma)$ is circular-symmetric then we have

$$P_t f = P_t^{\mathrm{ou}} f$$

for every $t \geq 0$.

Proof. Writing $w = x + iy$, $x, y \in \mathbb{R}^n$, the symmetry rewrites as $f((\cos(\theta)x - \sin(\theta)y) + i(\cos(\theta)y + \sin(\theta)x)) = f(x + iy)$. Taking the derivative at $\theta = 0$ we find

$$\sum_{j=1}^{n} \Big(- y_j \partial_{x_j} f(x+iy) + x_j \partial_{y_j} f(x+iy) \Big) = 0,$$

and this for every $x, y \in \mathbb{R}^n$. This implies in view of (14) that $Lf = L^{\mathrm{ou}} f = \overline{L} f$. Next, for any smooth function g we have, using that $Lf = \overline{L} f$ and (5),

$$\int (Lf) g \, d\gamma = \int (\overline{L} f) g \, d\gamma = \int f L g \, d\gamma = - \int \partial_z f \cdot \partial_{\overline{z}} g \, d\gamma.$$

Although it is formally trivial that equality of L and $L^{\rm ou}$ on circular-symmetric functions implies equality of the semi-groups P_t and $P_t^{\rm ou}$, a bit more should be said since we defined the semi-group using the explicit formula (9). And it is anyway instructive to compute the kernels on circular-symmetric functions. Denote by $K_t^{\rm ou}$ the kernel of the Ornstein-Uhlenbeck semi-group that we recalled above: $K_t^{\rm ou}(z,w) = \pi^{-n}(1-e^{-t})^{-n}\, e^{-\frac{1}{1-e^{-t}}|w-e^{-t/2}z|^2}$. So we have, setting $c_t := e^{-t/2}$ and $s_t := \sqrt{1-e^{-t}}$,

$$K_t^{\rm ou}(z,w) = \pi^{-n} s_t^{-2n}\, e^{-s_t^{-2}|w|^2 - s_t^{-2}c_t^2|z|^2}\, e^{s_t^{-2}c_t(w\cdot\overline{z}+\overline{w}\cdot z)}$$

and

$$K_t(z,w) = \pi^{-n} s_t^{-2n} e^{-s_t^{-2}|w|^2 - s_t^{-2}c_t^2|z|^2}\, e^{s_t^{-2}(c_t^2 w\cdot\overline{z}+\overline{w}\cdot z)}.$$

Note that only the last exponentials differ in these two formulas. When f is circular-symmetric, in order to check that $P_t f = P_t^{\rm ou} f$ it suffices to check that for fixed w, z, t one has

$$\frac{1}{2\pi}\int_0^{2\pi} K_t(z, e^{i\theta}w)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} K_t^{\rm ou}(z, e^{i\theta}w)\, d\theta.$$

Observe that for $a, b \in \mathbb{C}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ae^{i\theta}+be^{-i\theta}}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sum_{p,q\in\mathbb{N}} \frac{a^p b^q}{p!q!} e^{i(p-q)\theta} d\theta = \sum_{n\geq 0}\frac{(ab)^n}{(n!)^2} = B(ab)$$

with $B(x) := \sum_{n\geq 0}\frac{x^n}{(n!)^2}$. Therefore, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} K_t(z, e^{i\theta}w)\, d\theta &= \pi^{-n} s_t^{-2n}\, e^{-s_t^{-2}|w|^2 - s_t^{-2}c_t^2|z|^2} B(s_t^{-4}c_t^2|w\cdot\overline{z}|^2)\\ &= \frac{1}{2\pi}\int_0^{2\pi} K_t^{\rm ou}(z, e^{i\theta}w)\, d\theta,\end{aligned}$$

as wanted. □

3 Proof of Theorem 1

First, let us note that we can assume that g is smooth, and actually that $g \in \mathcal{G}^\infty$. Indeed, if $g \in \mathcal{G}$ then $P_t^{\rm ou} g \in \mathcal{G}^\infty$ and we mentionned that $P_t^{\rm ou} g$ converges to g in $L^2(\gamma)$ and therefore also in $L^1(\gamma)$, as $t \to 0$. Consequently, if we know the conclusion for a function in $\mathcal{G}^\infty$, then

$$\int f P_t^{\rm ou} g\, d\gamma \geq \int f\, d\gamma \int P_t^{\rm ou} g\, d\gamma$$

and by passing to the limit when $t \to 0$ we know it also for $g \in \mathcal{G}$. For the same reason, we can assume that $f \in \mathcal{G}^\infty$, recalling that $P_t^{\rm ou} f$ is circular-symmetric when f is.

So in the sequel, we are given two psh functions $f, g \in \mathcal{G}^\infty$, with f circular-symmetric.

As in the proof of the correlation for convex functions [8], we will compute some kind of second derivative in t for integrals involving $P_t^{\rm ou} f$; recall that $\partial_t P_t^{\rm ou} f = L^{\rm ou} P_t^{\rm ou} f$. The novelty is that, along the way, we will also use $P_t f$ which satisfies $\partial_t P_t f = L P_t f$ (Definition-Proposition 6).

Consider

$$\alpha(t) := \int (P_t^{\rm ou} f)\, g\, d\gamma = \int (P_t f)\, g\, d\gamma \ \in \mathbb{R}.$$

The function α is, by construction, smooth on $(0,\infty)$ and continuous on $[0,\infty)$ (see Lemma 6 for the continuity at zero). Since $P_t^{\rm ou} f$ tends to the constant $\int f\, d\gamma$ when $t \to \infty$, we have

$$\alpha(t) \to \int f\, d\gamma \int g\, d\gamma$$

In order to conclude, it suffices to show that α decreases. Actually we will prove that α is convex; which is enough, since a convex function with a bounded limit at $+\infty$ cannot increase. It holds

$$\alpha'(t) = \int (L^{\mathrm{ou}} P_t^{\mathrm{ou}} f)\, g\, d\gamma = \int (LP_t f)\, g\, d\gamma. \tag{15}$$

Since $P_t f$ is also circular-symmetric, we can invoke Lemma 9 and write

$$\alpha'(t) = -\int \partial_z P_t f \cdot \partial_{\overline{z}} g\, d\gamma.$$

Next, using the first commutation relation from Lemma 8 we get

$$\alpha'(t) = -\sum_{j=1}^{n} \int P_t(\partial_{z_j} f) \partial_{\overline{z}_j} g\, d\gamma$$

We stress that $\partial_z f$ is no longer circular-symmetric, so we cannot exchange P_t and P_t^{ou}. The second derivative of α is, using Fact 4,

$$\alpha''(t) = -\sum_{j=1}^{n} \int (LP_t(\partial_{z_j} f)) \partial_{\overline{z}_j} g\, d\gamma = \sum_{j=1}^{n} \int \partial_{\overline{z}}(P_t(\partial_{z_j} f)) \cdot \partial_z(\partial_{\overline{z}_j} g)\, d\gamma = \sum_{j,k=1}^{n} \int \partial_{\overline{z_k}} P_t(\partial_{z_j} f)\, \partial^2_{z_k \overline{z_j}} g\, d\gamma.$$

Using the commutation relation from Lemma 8 we can write

$$\alpha''(t) = \sum_{j,k=1}^{n} \int \partial^2_{\overline{z_k} z_j} (P_t f)\, \partial^2_{z_k \overline{z_j}} g\, d\gamma = \int \mathrm{Tr}\Big((D^2_{\mathbb{C}} P_t f)(z)\ (D^2_{\mathbb{C}} g)(z) \Big)\, d\gamma$$

where for a C^2 function h on $\mathbb{C}^n$ the notation $D^2_{\mathbb{C}} h(z)$ refers to the Hermitian matrix $\Big(\partial^2_{z_j \overline{z_k}} h(z) \Big)_{j,k \le n}$. Since $P_t f = P_t^{\mathrm{ou}} f$ and g are psh, the corresponding matrices are nonnegative Hermitian matrices, which means that the trace of their product is still nonnegative. This shows that $\alpha'' \geq 0$ and finishes the proof of the theorem.

□

We would like to conclude with a discussion of the differences between the real case and the complex case. After all, we are computing second derivatives of the same object

$$\alpha(t) = \int (P_t^{\mathrm{ou}} f)\, g\, d\gamma$$

along the Ornstein-Uhlenbeck semi-group exactly as in the case of convex functions, so what is going on?

In both cases we prove that α decreases by showing that $\alpha' \leq 0$ using the next derivative somehow, but we compute these derivatives differently. The argument for convex function goes as follows. A direct computation and usual commutation properties show that, if $\int \nabla f\, d\gamma = 0$, which is the case when f is even, then

$$\alpha'(t) = -e^{-t/2} \int_t^{\infty} \Big(\int \mathrm{Tr}\Big((D^2 P_s^{\mathrm{ou}} f)(z)\ (D^2 g)(z) \Big)\, d\gamma(z) \Big) e^{s/2} ds$$

where D^2 refers to the usual (real) Hessian on $\mathbb{R}^{2n}$; from this we conclude to the correlation for convex functions. On the other hand, we have proved, when f is circular-symmetric, that

$$\alpha'(t) = -\int_t^{\infty} \Big(\int \mathrm{Tr}\Big((D^2_{\mathbb{C}} P_s^{\mathrm{ou}} f)(z)\ (D^2_{\mathbb{C}} g)(z) \Big)\, d\gamma(z) \Big)\, ds.$$

Note that we have used here that $\alpha'(t)$ tends to 0 when $t \to +\infty$: this follows from the fact that α is convex and has a finite limit at $+\infty$, and can also be seen from (15) since $P_t^{\mathrm{ou}} f$ tends to a constant when $t \to +\infty$. It is because we wanted to work with complex derivatives that we aimed at inserting L in place of $L^{\rm ou}$; recall that $\partial_{z_j} f$ need not be circular-symmetric even when f is, although the second derivatives are again circular-symmetric.

Finally, let us observe that if we consider in dimension 1 the circular-symmetric psh functions $f(w) = |w|^{1/3}$ and $g(w) = |w|^4$ on $\mathbb{C} \simeq \mathbb{R}^2$, then

$$\mathrm{Tr}\Big((D^2_{\mathbb{C}} f)\ (D^2_{\mathbb{C}} g) \Big) = \frac{1}{16} \Delta f\, \Delta g \geq 0,$$

but a direct computation shows that

$$\mathrm{Tr}\Big((D^2 f)(z)\ (D^2 g)(z) \Big) = -\frac{4}{3} |z|^{1/3} \leq 0 \quad \forall z \in \mathbb{C}.$$

Of course, this discrepancy cannot hold at all times for $P_t^{\rm ou} f$ in place of f (and moreover f is not smooth at zero, although this is not really an issue). But it suggests that the two formulas above for $\alpha'(t)$ are indeed quite different.

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