DUALITY AND HEAT FLOW

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Abstract. We reveal the relation between the Legendre transform of convex functions and heat flow evolution, and how it applies to the functional Blaschke-Santaló inequality. We also describe local maximizers in this inequality.

A well known and useful property of the Legendre transform is to linearize infimal convolution. Here we describe its connection with Gaussian 'log'-convolution, shading new light on convexity and duality.

Our initial motivation was to provide a streamlined semi-group approach to the functional form of the Blaschke-Santaló inequality for symmetric convex bodies, in the wake of the approach proposed by the two last authors. This functional inequality states that for an even (convex) function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, with $0 < \int e^{-\phi} < \infty$,

$$M(e^{-\phi}) := \int_{\mathbb{R}^n} e^{-\phi} \int_{\mathbb{R}^n} e^{-\phi^*} \leq (2\pi)^n. \tag{1}$$

The left-hand side term is called the functional volume product and the inequality states it is maximized when $\phi = |x|^2/2$, the fixed point of the Legendre's transform,

$$\phi^*(z) := \sup_x x \cdot z - \phi(x), \qquad \forall z \in \mathbb{R}^n,$$

on the Euclidean space $(\mathbb{R}^n, \cdot, |\cdot|)$. Taking $\phi(x) = \|x\|_K^2/2$, where $\|\cdot\|_K$ is the gauge associated to a symmetric convex body $K \subset \mathbb{R}^n$, the geometric inequality $\mathrm{vol}(K)\mathrm{vol}(K^\circ) \leq \mathrm{vol}(B_2^n)^2$ is recovered; here $K^\circ = \{x \in \mathbb{R}^n \; ; \; x \cdot y \leq 1, \; \forall y \in K\}$ and $B_2^n = \{|\cdot| \leq 1\}$ (see e.g. [11, 9]).

Inequality (1) was put forward by Keith Ball in his phd dissertation. It has been extended to non-even functions in [1] and admits several proofs relying on geometric arguments (see for instance [1, 7, 8]). A new, analytical, approach based on heat flow was obtained in [14], where it is proven that $M(e^{-\phi})$ increases when we let $e^{-\phi}$ evolve along the Fokker-Planck flow. However the authors obtained this property at the limit, by using the Laplace transform in place of the Legendre transform. We aim at providing a direct approach, together with a characterization of strict monotonicity and local maximizers.

We shall be considering convex functions with $0 < \int e^{-\phi} < \infty$, which implies that ϕ is *coercive*, in the sense that $\phi(x) \geq a|x| - b$ for all $x \in \mathbb{R}^n$ for some constants $a > 0$, $b \in \mathbb{R}$. In order to avoid technicalities, we shall actually assume that ϕ is *super-linear* which means that $\limsup_\infty \frac{\phi(x)}{|x|} = \infty$. This ensures that ϕ^* is finite, for instance.

The heat flow evolution $P_t u$ of a suitable function u on $\mathbb{R}^n$, is given for $t > 0$ by $\partial_t(P_t u)(x) = \Delta_x P_t u$. The Fokker-Planck evolution $Q_t u$ is given by $\partial_t(Q_t u)(x) = \Delta_x Q_t u + \mathrm{div}_x(x\, Q_t u)$. One can pass from one flow to the other by rescaling in time and space, $Q_t u(x) = e^{nt} P_{\frac{e^{2t}-1}{2}} u(e^t x)$, and it follows that, for $u = e^{-\phi}$, $M(Q_t u) = M(P_{\frac{e^{2t}-1}{2}} u)$.

Our main observation below is simple but most useful, as we shall see.

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Proposition. *Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a super-linear convex function, and let $\phi_t = -\log P_t(e^{-\phi})$ where P_t is the heat semi-group. Let $\psi_t = (\phi_t)^*$ denote the Legendre transform of the convex function ϕ_t. Then for every $z \in \mathbb{R}^n$ and $t > 0$*

$$\partial_t \psi_t(z) = |z|^2 - \mathrm{Tr}(D_z^2 \psi_t)^{-1}. \tag{2}$$

One can derive a similar formula in the case of the Fokker-Planck flow, and also for the Gaussian reformulation in terms of infimal convolution [10, 3] and Ornstein-Uhlenbeck semi-group.

Proof of the Proposition. From the formula $e^{-\phi_t(x)} = \int e^{-\phi(y)}\, e^{-|x-y|^2/4t} \frac{dy}{(4\pi t)^{n/2}}$ we have, using that ϕ is convex coercive, that $(t,x) \to \phi_t(x)$ is C^∞ on $(0,\infty) \times \mathbb{R}^n$. For $t > 0$ fixed, it is classical (for instance by Prékopa's theorem, [2] or the earlier work [6]) that ϕ_t is convex and actually that

- $D_x^2 \phi_t > 0$ at every $x \in \mathbb{R}^n$,
- ϕ_t is super-linear.

We shall provide a proof of these facts at the end of the note. In particular, $\psi_t = (\phi_t)^*$ is finite and smooth on $\mathbb{R}^n$, and $\nabla \phi_t$ is a diffeomorphism of $\mathbb{R}^n$ with inverse $\nabla \psi_t$. We next use the following simple relation, valid for any first order perturbation of a convex function:

$$\partial_t \psi_t(z) = -\partial_t \phi_t(\nabla \psi_t(z)).$$

To check this, take for instance the derivative in t of $\psi_t(z) + \phi_t(\nabla\psi_t(z)) = z \cdot \nabla \psi_t(z)$ and use that $\nabla \phi_t(\nabla \psi_t(z)) = z$. In the case of heat flow we have

$$\partial_t \phi_t(y) = \Delta_y \phi_t - |\nabla_y \phi_t|^2 = \mathrm{Tr} D_y^2 \phi_t - |\nabla_y \phi_t|^2.$$

The relation (2) follows. □

As an example of application of the Proposition, let us go back to the Blaschke-Santaló inequality (1). We aim at reproducing the monotonicity of the functional volume product along Fokker-Planck flow proved in [14] under the convexity assumption. For the monotonicity without the convexity, we refer to [14]. We rather consider the heat flow evolution

$$\alpha(t) := \log M(P_t(e^{-\phi})) = \log\Big(\int e^{-\phi_t} \int e^{-\psi_t}\Big) = \log\Big(\int e^{-\phi} \int e^{-\psi_t}\Big),$$

where ϕ is an even super-linear convex function. We want to prove that α is non-decreasing. Note that, after re-scaling, the functions converge to Gaussians which gives back (1). Here, we concentrate on monotonicity, as our aim is to emphasize the method of the proof. It is easily checked that $\partial_t e^{-\psi_t}$ is locally uniformly in $t > 0$ dominated by an integrable function, so we have, using the Proposition, that

$$\alpha'(t) = \int (-\partial_t \psi_t) \frac{e^{-\psi_t}}{\int e^{-\psi_t}} = \int (\mathrm{Tr}(D_x^2 \psi_t)^{-1} - |x|^2) \frac{e^{-\psi_t(x)} dx}{\int e^{-\psi_t}}.$$

We now call upon the variance Brascamp-Lieb inequality [2] which states that for a C^2-smooth convex function $V : \mathbb{R}^n \to \mathbb{R}$ with $D^2 V > 0$ almost-everywhere, denoting by μ_V the probability on $\mathbb{R}^n$ with density $\frac{e^{-V}}{\int e^{-V}}$, the inequality

$$\mathrm{Var}_{\mu_V}(u) = \int \Big(u - \int u \, d\mu_V\Big)^2 d\mu_V \le \int (D^2 V)^{-1} \nabla u \cdot \nabla u \, d\mu_V, \tag{3}$$

holds for every smooth $u \in L^2(\mu_V)$. We apply this inequality with $V = \psi_t$ and to the linear functions $x \to x_i$, $i \le n$, which are centered, $\int x_i e^{-\psi_t(x)}\, dx = 0$, since ψ_t is even. Summing over $i \le n$, we then get exactly that $\alpha'(t) \ge 0$, for $t > 0$. At $t = 0^+$ we have (4) (see Appendix), so α is non-decreasing on $[0, \infty)$, as wanted. Using ideas from [4], one can adapt the argument to the non-even case.

A closer look to the previous arguments allows also to settle the cases of equality, and even better, to describe local maximizers, thus providing a functional version of the geometric result in [12]. We say that a function f is a *centered Gaussian function* if $f(x) = K\, e^{-Hx\cdot x}$ where H is a positive matrix and $K > 0$.

Theorem. *Let ϕ be an even convex super-linear function and $\epsilon > 0$ such that*

$$M(e^{-\phi}) = \sup\Big\{M(e^{-\psi}) \ ; \ \psi \text{ even convex with } \|e^{-\phi} - e^{-\psi}\|_{L^1(\mathbb{R}^n)} \leq \epsilon\Big\}.$$

Then $e^{-\phi}$ is a centered Gaussian function.

Proof. We keep the notation of the proof above. Since $e^{-\phi_t} = P_t(e^{-\phi})$ tends to $e^{-\phi}$ in $L^1(\mathbb{R}^n)$, and since $M(e^{-\phi_t})$ increases, there must exists $t_0 > 0$ such that $M(e^{-\phi_t}) = M(e^{-\phi})$ for all $t \in [0, t_0]$. Consequently $\alpha'(\frac{t_0}{2}) = 0$. This implies that the linear functions are cases of equality in the Brascamp-Lieb inequality. However, the only centered equality cases in (3) are the linear combinations of the $\partial_i V$, $i \leq n$, which forms a space of dimension at most n; this is a classical fact (one can call upon the discussion in the last section of [5] for instance). Since the linear functions $x \to x_j$ are linearly independent, it means that each $\partial_i \psi_{t_0/2}$ is a linear combination of the $x \to x_j$, $j \leq n$. This implies that $D^2\psi_{t_0/2}$ is constant on $\mathbb{R}^n$. Since $\psi_{t_0/2}$ is even, it means that $\psi_{t_0/2}$ and thus $\phi_{t_0} = \psi_{t_0}^*$ are quadratic, of the form $x \to Hx\cdot x + C$ with H a positive matrix. Recall that $e^{-\phi_{t_0/2}} = P_{t_0/2}(e^{-\phi})$ is obtained by a convolution with a centered Gaussian function. By injectivity of the Fourier transform on $L^1(\mathbb{R}^n)$ (or invoking the more sophisticated Cramer Theorem), this forces $e^{-\phi}$ to be a centered Gaussian function. □

If ϕ is an even convex super-linear function, the argument in the proof above also shows that when $e^{-\phi}$ is not Gaussian, the volume product $M(P_t(e^{-\phi}))$ is strictly increasing in $t \in [0, \infty)$.

Let us conclude with a word on linear invariance. The presence of this large class of invariance might have been seen as an obstacle to a semi-group approach of the Santaló inequality. However, we luckily have that the key inequality is the variance Brascamp-Lieb inequality (our formula (2) points right to it, in fact), and this inequality possess indeed a linear invariance, unlike usual Poincaré inequalities, say, which depend on the Euclidean structure. In this regard, it is natural to investigate other linear invariant inequalities, as in [9, 13].

We believe the result discussed in this note is only the beginning of a promising direction. It is part of a work in progress on quantitative versions, and on other formulations and applications of the principle we have put forward.

Appendix: technicalities

We aim at proving the two properties of $\phi_t = -\log P_t(e^{-\phi})$ used in the proof of the Proposition (for $t > 0$ is fixed). Then we examine the limit as $t \to 0^+$ in the volume product.

We start with the super-linearity of ϕ_t. Since ϕ is (bounded-below) super-linear, for an arbitrary $M > 0$, there exists $b = b(M)$ such that for every $x \in \mathbb{R}^n$

$$\phi(x) \geq M|x| - b.$$

Therefore we have, using $|x + \sqrt{2t}y| \geq |x| - \sqrt{2t}|y|$ that

$$e^{-\phi_t}(x) \leq e^b e^{-M|x|} \int e^{\sqrt{2t}M|y|}\, d\gamma(y) \leq e^b e^{-M|x| + tM^2 + \sqrt{2nt}M},$$

where γ is the standard Gaussian measure, that is $\phi_t(x) \geq M|x| - b - tM^2 - \sqrt{2nt}M$. Therefore there exists a $K = K(M)$ such that for every $|x| \geq K$,

$$\frac{\phi_t(x)}{|x|} \geq \frac{M}{2}.$$

This implies that ϕ_t is superlinear.

Next we prove the strict convexity of ϕ_t, in the form $D^2\phi_t > 0$. Since $e^{-\phi_t} = P_{t/2}(P_{t/2}e^{-\phi})$ and $P_{t/2}e^{-\phi}$ is an integrable, positive, smooth, log-concave function, we can assume that $\phi : \mathbb{R}^n \to \mathbb{R}$ is a convex coercive (actually we just proved it is also superlinear) twice continuously differentiable function. We can also assume that $t = 1/2$, for notational simplicity. For a fixed direction $|\theta| = 1$, we readily check from

$$e^{-\phi_{1/2}(x)} = \int e^{-\phi(y)} e^{-|x-y|^2/2} \frac{dy}{(2\pi)^{n/2}}$$

that for any $x \in \mathbb{R}^n$,

$$(D_x^2\phi_{1/2})\theta \cdot \theta = 1 - \mathrm{Var}_{d\mu_x(y)}(y \cdot \theta)$$

where the variance is computed with respect to the probability measure

$$d\mu_x(y) = e^{-\phi(y)} e^{-|x-y|^2/2} \frac{dy}{\int e^{-\phi(z)} e^{-|x-z|^2/2}\, dz}.$$

Assume that, for some fixed $x \in \mathbb{R}^n$, we have $(D_x^2\phi_{1/2})\theta \cdot \theta = 0$. We then have by the Brascamp-Lieb inequality that

$$1 = \mathrm{Var}_{d\mu_x(y)}(y \cdot \theta) \leq \int \left(D_y^2\phi + \mathrm{Id}_n\right)^{-1} \theta \cdot \theta \, d\mu_x(y) \leq 1$$

since $D_y^2\phi + \mathrm{Id}_n \geq \mathrm{Id}_n$. As the density of μ_x is continuous and positive on $\mathbb{R}^n$, we must have $\left(D_y^2\phi + \mathrm{Id}_n\right)^{-1} \theta \cdot \theta = 1 = |\theta|^2$, and thus $(D_y^2\phi)\theta \cdot \theta = 0$, at every $y \in \mathbb{R}^n$; for this recall that for a nonnegative matrix H we have $(H + \mathrm{Id}_n)^{-1}\theta - \theta = -(H + \mathrm{Id}_n)^{-1} H\theta$ and that $(H + \mathrm{Id}_n)^{-1} H$ is nonnegative. So for all $y \in \mathbb{R}^n$, the function $t \mapsto \phi(y + t\theta)$ is affine, which contradicts coercivity, for instance.

Finally, as $t \to 0^+$, one formally expects $\phi_t \to \phi$ and so $\psi_t \to \psi$. Let us prove rigorously that

$$\liminf_{t \to 0^+} \int e^{-\psi_t} \geq \int e^{-\psi}. \tag{4}$$

For every $x, y, z \in \mathbb{R}^n$ we have $-\phi(y) - |y-x|^2/4t \leq \psi(z) - z \cdot y - |y-x|^2/4t$ and therefore

$$\phi_t(x) \geq -\psi(z) + z \cdot x - t|z|^2.$$

This implies, by definition, that for every $z \in \mathbb{R}^n$,

$$\psi_t(z) \leq \psi(z) + t|z|^2,$$

and so $\int e^{-\psi_t} \geq \int e^{-\psi(x) - tx^2}\, dx$. Then (4) follows by Fatou's Lemma.

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