The Annals of Probability
2023, Vol. 51, No. 3, 987–1003
https://doi.org/10.1214/22-AOP1604

IMPROVED LOG-CONCAVITY FOR ROTATIONALLY INVARIANT MEASURES OF SYMMETRIC CONVEX SETS

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We prove that the (B) conjecture and the Gardner–Zvavitch conjecture are true for all log-concave measures that are rotationally invariant, extending previous results known for Gaussian measures. Actually, our result apply beyond the case of log-concave measures, for instance, to Cauchy measures as well. For the proof, new sharp weighted Poincaré inequalities are obtained for even probability measures that are log-concave with respect to a rotationally invariant measure.

1. Introduction and main results. Improved log-concavity inequalities under the assumption of symmetry have become a central topic in the Brunn–Minkowski theory of convex bodies with several fascinating consequences and conjectures. Maybe one of the first appearances of this phenomenon was the (B) inequality established in [10] for a centered Gaussian measure γ on $\mathbb{R}^n$. It states that for a symmetric convex set $K \subset \mathbb{R}^n$ (here symmetry means origin-symmetry, i.e., $K = -K$), the function

$$t \to \gamma(e^t K) \quad \text{is log-concave on } \mathbb{R}. \tag{1}$$

A nonnegative function m is said to be log-concave if $(-\log m)$ is a convex function with values in $\mathbb{R} \cup \{+\infty\}$. The indicator of a convex set C is a log-concave function, and it is common to denote the corresponding convex function by $\mathbf{1}_C^\infty := -\log \mathbf{1}_C$ which is equal to 0 on K and to $+\infty$ outside.

Property (1) was first conjectured by Banaszczyk and popularized by Latała [19]. It found applications outside Brunn–Minkowski theory, for instance, in the setting of small ball estimates in high dimensions; see [15, 20].

A Borel measure μ on $\mathbb{R}^n$ is said to satisfy the (B) property if (1) holds for every symmetric convex set $K \subset \mathbb{R}^n$ with μ in place of γ. It is conjectured that every even log-concave measure μ, and by this we mean $\mathrm{d}\mu(x) = e^{-V(x)}\,\mathrm{d}x$ with V convex and even, satisfies the (B) property. Prékopa's theorem [23] implies that every log-concave measure μ satisfies a Brunn–Minkowski inequality: For all convex sets K, L the function

$$[0,1] \ni t \to \mu\big((1-t)K + tL\big) \quad \text{is log-concave.} \tag{2}$$

This immediately implies that the function $s \to \mu(sK)$ is log-concave on $\mathbb{R}^+$. The conjecture is a strengthening of this property under the extra assumption of symmetry. Saroglou [25] showed that this (B) conjecture follows from another celebrated conjecture for symmetric convex sets, namely, the log-Brunn–Minkowski conjecture [6]. Combining the results of [25] and [6], it follows that the conjecture holds in $\mathbb{R}^2$. Conversely, a certain strong form of the (B) conjecture will also imply the log-Brunn–Minkowski conjecture [24].

In dimension $n \geq 3$, very few examples of measures verifying the (B) inequality were known, and until now they all somehow relied on the result for the Gaussian measure. These

Received November 2021; revised September 2022.
MSC2020 subject classifications. Primary 52A40; secondary 60E15, 26D10.
Key words and phrases. (B) conjecture, Gardner–Zvavitch conjecture, log-concavity, Brunn–Minkowski, Brascamp–Lieb inequality, Poincaré inequality.

few non-Gaussian examples were obtained by Eskenazis, Nayar and Tkocz in [13] and will be discussed in Section 6.

In a similar vein, a striking recent result of Eskenazis and Moschidis [12] gives the following improvement of the log-concavity (2) to $1/n$-concavity for a centered Gaussian measure γ: if K and L are symmetric convex sets in $\mathbb{R}^n$ and $\lambda \in [0,1]$, then

$$\gamma\big((1-\lambda)K + tL\big)^{1/n} \geq (1-\lambda)\gamma(K)^{1/n} + \lambda\gamma(L)^{1/n}. \tag{3}$$

This implies that the function $[0,1] \ni t \to \gamma((1-t)K + tL)^{1/n}$ is concave (because we work with convex sets). This property was conjectured by Gardner and Zvavitch [14], and again it is connected to several natural problems in the geometry of convex bodies. Note that here the prototype of a measure satisfying this property is the Lebesgue measure restricted to a convex set by the Brunn–Minkowski inequality. It is remarkable that the Gaussian measure also behaves this way. One can ask whether every even log-concave measure satisfies this Gardner–Zvavitch property. This conjecture will again be a corollary of the log-Brunn–Minkowski conjecture [22], so in particular, it is known to hold in $\mathbb{R}^2$. Building on earlier ideas of Kolesnikov and Milman [17, 18], Kolesnikov and Livshyts [16] proposed a convenient spectral inequality that allowed them to show (3) with exponent $\frac{1}{2n}$ (see Theorem 10 below). This was improved to the optimal exponent $\frac{1}{n}$ in [12]. In [9] it was shown that (3) holds for arbitrary rotation invariant log-concave measures, instead of γ but only when K and L are small perturbations of a ball. In [21] Livshyts proved (3) for all even log-concave measures but with the optimal exponent $\frac{1}{n}$ replaced with a worse exponent $c_n = \frac{1}{n^{4+o(1)}}$.

In the present paper, we show that there is nothing special about the Gaussian measure and that properties (1) and (3) hold for every rotationally invariant log-concave measure on $\mathbb{R}^n$, providing the first large class of measures on $\mathbb{R}^n$ beyond Gaussian measures satisfying the (B) conjecture and the Gardner–Zvavitch conjecture. Actually, we will go beyond log-concave measures; for instance, we will show that the Cauchy measures also satisfy these properties.

Let us fix some notation in order to state our results. We consider a finite dimensional Euclidean space $(\mathbb{R}^n, |\cdot|, \langle\cdot,\cdot\rangle)$. For notational convenience we assume we work with the standard structure—note that the problems we study are affine invariant. A Borel measure μ is rotationally invariant if $\mu(A) = \mu(RA)$ for every Borel set A and every linear map $R \in O(n)$. Since we are only considering measures that are absolutely continuous with respect to the Lebesgue measure $\mathrm{d}x$, this means that we are considering measures of the form

$$\mathrm{d}\mu(x) = e^{-w(|x|)}\,\mathrm{d}x$$

for some function $w : \mathbb{R}^+ \to \mathbb{R} \cup \{+\infty\}$. In this setting we have

$$\begin{aligned} \mu \text{ is log-concave} \quad &\Leftrightarrow \quad x \to w\big(|x|\big) \text{ is convex on } \mathbb{R}^n \\ &\Leftrightarrow \quad w \text{ is increasing and convex on } \mathbb{R}^+ \\ &\Rightarrow \quad w \text{ increasing and } t \to w\big(e^t\big) \text{ is convex on } \mathbb{R}. \end{aligned}$$

This last condition will prove to be sufficient for establishing the results (here and everywhere in the paper, "increasing" is understood in the weak sense, i.e., as nondecreasing). Note that for a smooth w, the log-concavity of μ amounts to the conditions $w' \geq 0$, $w'' \geq 0$, whereas our weaker assumption is equivalent to $w' \geq 0$, $sw''(s) + w'(s) \geq 0$. Also, unless w is constant (which is not a situation of interest) we will always have that $w(t) \to +\infty$ as $t \to +\infty$.

Throughout the paper the notion of "being log-concave with respect to a measure" will play an important role, and so let us introduce a specific notation for that.

DEFINITION (Log-concavity preorder $\lhd$ on measures). Given two Borel measures μ, ν on $\mathbb{R}^n$, we say that ν is log-concave with respect to μ and write $\nu \lhd \mu$ if ν has a log-concave density with respect to μ, that is,

$$\mathrm{d}\nu(x) = e^{-v(x)}\,\mathrm{d}\mu(x)$$

with $v : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex.

Of course, for any measure μ and any constant $c > 0$ we have $c\,\mu \lhd \mu$; the central example is $\mu_K \lhd \mu$, where μ_K is the restriction of a measure μ to a convex set $K \subset \mathbb{R}^n$, defined by $\mu_K(A) = \mu(A \cap K)$, possibly renormalized into a probability measure when $\mu(K) < \infty$. In Bakry–Emery comparison geometry terms, one could say that $\nu \lhd \mu$ means that ν is *more curved* than μ, although one must pay attention that no curvature is a priori imposed on μ. It is, nonetheless, natural to ask whether some (well-chosen) functional inequality is valid for the whole class of such ν's. Our main contribution in this direction for rotationally invariant measures will be Theorem 4 below.

We begin with the (B) conjecture. Actually, we are able to extend the same *strong* form that was established for the Gaussian measure to every log-concave (and beyond) rotationally invariant measure.

THEOREM 1. *Let $w : (0,\infty) \to (-\infty,\infty]$ be an increasing function such that $t \to w(e^t)$ is convex, and let μ be the measure on $\mathbb{R}^n$ with density $e^{-w(|x|)}$. Then, for every symmetric convex body $K \subseteq \mathbb{R}^n$ and every* symmetric *matrix A, the function*

$$t \mapsto \mu(e^{tA}K)$$

is log-concave.

Let us mention that we will actually prove the following more general statement: under the same assumptions on w, if $v : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is an even convex function, then the function

$$t \to \int_{\mathbb{R}^n} e^{-v(e^{tA}x) - w(|x|)}\,\mathrm{d}x \quad \text{is log-concave on } \mathbb{R}. \tag{4}$$

The theorem corresponds to the choice $v = 1_K^\infty$, replacing t by $-t$. This "functional" version of the (B) property was previously studied in [11].

Note that the functions $w(t) = w_p(t) := t^p/p$ satisfy the assumptions of the theorem for all $p > 0$. Hence, the corresponding measures

$$\mathrm{d}\mu_p = e^{-|x|^p/p}\,\mathrm{d}x$$

all have the strong (B) property. Taking $w = \mathbf{1}_{[0,1]}^\infty$ (i.e., $p \to +\infty$) we see that the uniform measure on B_2^n also has the strong (B) property. Recall we are free to pick any Euclidean structure, or equivalently, we can work with measures on $\mathbb{R}^n$ of the form

$$\mathrm{d}\mu(x) = e^{-w(\langle Cx, x\rangle)}\,\mathrm{d}x, \tag{5}$$

where C is a symmetric positive matrix; one just has to be careful when stating the strong (B) property, as in this case the symmetry condition on the matrix A is that $CA = A^*C$. For the classical (B) property ($A = \mathrm{Id}$), there is no issue here, and, in particular, the uniform measure on an ellipsoid satisfies the (B) inequality. Note also that if $\mathcal{E}$ is an ellipsoid, we may use (4) with $A = \mathrm{Id}$, $w = \mathbf{1}_{[0,1]}^\infty$ and the norm associated to $\mathcal{E}$. Performing the change of variables $y = e^{tA}x$ (whose Jacobian is log-linear and hence immaterial), we derive the following corollary.

COROLLARY 2. *Let ν be an arbitrary even log-concave measure on $\mathbb{R}^n$ and $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid. Then we have that*

$$t \to \nu(e^t \mathcal{E}) \quad \text{is log-concave on } \mathbb{R}.$$

It is also worth mentioning that by approximation, our results apply to degenerate nonnegative quadratic forms as well, that is, to the case where the matrix C in (5) is degenerate. For instance, we can consider measures of the form $e^{-w(\sqrt{x_1^2+\cdots x_k^2})}\,\mathrm{d}x$ with $k \le n$.

Let us give some further examples of nonlog-concave measures that satisfy our assumptions and for which our results hold. For instance, by taking $w(t) = a \log t + \tilde{w}(t)$ for any $a \ge 0$ and $\tilde{w}$ satisfying our assumptions (possibly $\tilde{w} \equiv 0$), we can consider measures of the form

$$\mathrm{d}\mu(x) = |x|^{-a} e^{-\tilde{w}(|x|)}\,\mathrm{d}x.$$

It is reasonable to impose local integrability (around zero) of the density, that is, $0 \le a < n$, for if not the measure of every symmetric convex body is $+\infty$, and the result is, therefore, less interesting. We can also take, for instance, $w(t) = a \log(1 + t^b)$ for any $a, b \ge 0$ and work with measures of the form

$$\mathrm{d}\mu(x) = \big(1 + |x|^b\big)^{-a}\,\mathrm{d}x$$

which include Cauchy-type measures on $\mathbb{R}^n$.

Note that our condition on the density $e^{-w(|x|)}$ of the rotationally measure μ is stable under products, because the condition on the corresponding function w is stable under additions. In particular, we can replace $\mathrm{d}x$ in the previous examples by a suitable rotationally invariant measure. For example, we can restrict these measures to a centered Euclidean ball.

Let us now comment on the proof of the (B) inequality. It is well known that taking second derivatives reduces Brunn–Minkowski type inequalities to spectral inequalities for some differential operator. The proof of Theorem 1 follows a scheme similar to [10] which handled the Gaussian case μ_2 by establishing a connection with a "second eigenvalue problem" associated to measures that are log-concave with respect to μ_2. We will reduce the problem to a spectral inequality of Brascamp–Lieb type in an improved form for even functions. By examining the Gaussian case, one could seek an improvement in the constant of a "classical" spectral inequality. However, we believe this would not be the right way to go (see the remark at the end of Section 4). Moreover, already for our rotationally invariant measure μ, we do not know the exact whole spectrum (unlike the Gaussian case), and we need anyway to work with measures that are log-concave with respect to μ. We will instead establish the following sharp spectral inequality from which the result follows.

THEOREM 3. *Let $w : [0, \infty) \to \mathbb{R}$ be a C^2-smooth increasing function such that $t \mapsto w(e^t)$ is convex. Define $W : \mathbb{R}^n \to \mathbb{R}$ by $W(x) = w(|x|)$, and let ν be an even probability measure with $\nu \lhd e^{-W(x)}\,\mathrm{d}x$.*

Then, for every even *C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int |f|^2\,\mathrm{d}\nu < \infty$, we have*

$$\mathrm{Var}_\nu f \le \int \left\langle \left(\nabla^2 W + \frac{w'(|x|)}{|x|}\,\mathrm{Id} \right)^{-1} \nabla f, \nabla f \right\rangle \mathrm{d}\nu.$$

Here and for the rest of the paper, we omit the dependence on the variable, except where necessary. Also, as we will see in Section 3, the matrix $\nabla^2 W + \frac{w'(|x|)}{|x|}\,\mathrm{Id}$ is always positive *semi*definite, but it can be singular (formulation should then be adapted, as explained in Remark 9 below), although in applications we can assume by approximation that this situation does not arise.

One can check that equality holds in Theorem 3 when $\mathrm{d}\nu = \frac{e^{-W(x)}}{\int e^{-W}}\,\mathrm{d}x$, with $\int e^{-W} < \infty$ and $f(x) = \langle \nabla W, x\rangle = w'(|x|)|x|$. This can be seen without explicit computations by inspecting the use of Theorem 3 in the proof of Theorem 1 in Section 4 and using the fact that for $K = \mathbb{R}^n$ the function $t \mapsto \mu(e^t K)$ is constant and thus log-linear.

In the case of the Gaussian measure μ_2 (i.e., $w(t) = t^2/2$), this inequality reduces to the fact that for an even probability measure $\nu \lhd \mu_2$, one has $\mathrm{Var}_\nu f \le \frac{1}{2}\int |\nabla f|^2\,\mathrm{d}\nu$ for every even smooth f.

The matrix $\nabla^2 W + \frac{w'(|x|)}{|x|}\,\mathrm{Id}$ is a rank one perturbation of a scalar matrix (see (7) below), so we can compute its inverse explicitly. The result is that under the assumptions of Theorem 3 we have the inequality

$$\mathrm{Var}_\nu f \le \int \left(\frac{|x|}{2w'(|x|)}|\nabla f|^2 - \frac{|x|w''(|x|) - w'(|x|)}{2|x|w'(|x|)(|x|w''(|x|) + w'(|x|))}\langle \nabla f, x\rangle^2 \right)\mathrm{d}\nu.$$

For example, taking $w(t) = w_p(t) = t^p/p$, we see that for an even $\nu \lhd \mu_p$ we have

$$\mathrm{Var}_\nu f \le \int \left(\frac{1}{2}|x|^{2-p}|\nabla f|^2 - \frac{p-2}{2p}\cdot\frac{\langle \nabla f, x\rangle^2}{|x|^p} \right)\mathrm{d}\nu.$$

Using the trivial bounds $0 \le \langle \nabla f, x\rangle^2 \le |\nabla f|^2|x|^2$, we can deduce the less precise but more elegant inequality

$$\mathrm{Var}_\nu f \le \max\left\{\frac{1}{p}, \frac{1}{2}\right\}\cdot \int |x|^{2-p}|\nabla f|^2\,\mathrm{d}\nu.$$

This inequality is still sharp when $\nu = c\cdot\mu_p$ for $0 < p \le 2$ and c a normalization constant with equality when $f(x) = |x|^p$. Similarly, taking $w_C(t) = a\cdot\log(1+t^2)$, we see that when ν is log-concave with respect to the Cauchy-type measure $\mathrm{d}\mu_C = \frac{1}{(1+|x|^2)^a}\,\mathrm{d}x$, we obtain the inequalities

$$\mathrm{Var}_\nu f \le \frac{1}{4a}\int (1+|x|^2)\big(|\nabla f|^2 + \langle \nabla f, x\rangle^2\big)\,\mathrm{d}\nu \le \frac{1}{4a}\int (1+|x|^2)^2|\nabla f|^2\,\mathrm{d}\nu.$$

Again, both of these inequalities are sharp when ν is the (normalized) reference measure μ_C. This last inequality is similar in spirit to a result of Bobkov and Ledoux [2] for Cauchy measures which is only sharp up to a universal constant but holds for noneven functions; this was recently sharpened in [3] in the case $a = n$.

In the Gaussian case, the above-mentioned inequality $\mathrm{Var}_\nu f \le \frac{1}{2}\int |\nabla f|^2\,\mathrm{d}\nu$ for f even was at the heart of the argument in [10]. It was established using an L^2 argument with a Bochner integration by parts (a second argument using Cafferelli's contraction property was also given). The argument used the following classical Poincaré inequality which follows from the variance Brascamp–Lieb inequality [7] or the Bakry–Emery criterion [1]: For a probability measure ν with $\nu \lhd \mu_2$, one has for every smooth h that $\mathrm{Var}_\nu h \le \int |\nabla h|^2\,\mathrm{d}\nu$. For our general μ this inequality needs to be replaced by a *weighted* Poincaré inequality that appears to be new, even in the simple case of of the measure $e^{-w(|x|)}\,\mathrm{d}x$ for which it is sharp. In fact, we will only prove such an inequality when the function is *odd* which is good enough for our purposes.

THEOREM 4. *Let $w : (0,\infty) \to \mathbb{R}$ be C^1-smooth and increasing, and let ν be an even finite measure on $\mathbb{R}^n$ with $\nu \lhd e^{-w(|x|)}\,\mathrm{d}x$.*

Then, for every C^1-smooth and ***odd*** *function $h : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\int \frac{w'(|x|)}{|x|}h^2\,\mathrm{d}\nu \le \int |\nabla h|^2\,\mathrm{d}\nu.$$

As we will see after the proof, equality holds in Theorem 4 when $\mathrm{d}\nu = e^{-w(|x|)}\,\mathrm{d}x$ and $h(x) = \langle x, \theta\rangle$ for a fixed $\theta \in \mathbb{R}^n$.

In the two cases of interest from before $w = w_p$ and $w = w_C$, Theorem 4 reduces to the inequalities

$$\int |x|^{p-2} h^2 \,\mathrm{d}\nu \le \int |\nabla h|^2 \,\mathrm{d}\nu \quad \text{and} \quad \int \frac{h^2}{1+|x|^2}\,\mathrm{d}\nu \le \frac{1}{2a}\int |\nabla h|^2 \,\mathrm{d}\nu,$$

respectively. Both of these inequalities are sharp when $\mathrm{d}\nu = e^{-|x|^p/p}\,\mathrm{d}x$ and $\mathrm{d}\nu = \frac{1}{(1+|x|^2)^a}\,\mathrm{d}x$ with $a > n/2$ (condition for finiteness), respectively, with equality for linear functions.

It turns out that our weighted Poincaré inequality above (Theorem 4) allows us to solve the problem of the Gardner–Zvavitch conjecture for rotationally invariant measures with the same condition as for the (B) inequality.

THEOREM 5. *Let $w : [0, \infty) \to (-\infty, \infty]$ be an increasing function such that $t \mapsto w(e^t)$ is convex, and let μ be the measure on $\mathbb{R}^n$ with density $e^{-w(|x|)}$. Then, for every symmetric convex bodies $K, L \subset \mathbb{R}^n$ and $\lambda \in [0, 1]$,*

$$\mu\big((1-\lambda)K + \lambda L\big)^{1/n} \ge (1-\lambda)\mu(K)^{1/n} + \lambda\mu(L)^{1/n}.$$

As before, our result includes all rotationally invariant log-concave measures but applies also beyond this class; see the examples given above, such as Cauchy type measures.

There is some surprising phenomenon here that we would like to outline. It was observed by Borell [4, 5] that if a measure μ with density f on $\mathbb{R}^n$ satisfies any kind of Brunn–Minkowski inequality, even in the weakest form $\mu((1-\lambda)K + \lambda L) \ge \min\{\mu(K), \mu(L)\}$ for every convex sets K, L and $\lambda \in (0, 1)$, then f must satisfy some concavity property. It follows from Borell's observation that the family of measures $\mathrm{d}\mu_C = \frac{1}{(1+|x|^2)^a}\,\mathrm{d}x$ with $a > 0$ satisfy *no* Brunn–Minkowski inequality when $2a < n$. However, when restricted to *symmetric* convex sets, all these measures satisfy the strong Brunn–Minkowski inequality given by the previous theorem.

The rest of the paper is devoted to the proofs of the main results and some further comments and extensions. In the next section, we prove the weighted Poincaré inequality (Theorem 4). Then in Section 3 we will use it to establish our spectral estimate of Brascamp–Lieb type for even functions (Theorem 3). We show in Section 4 that this spectral estimate in turn implies the strong (B) inequality (Theorem 1). In Section 5 we give the proof of the dimensional Brunn–Minkowski inequality (Theorem 5). In the final Section 6, following an argument of [13], we explain how to extend the (B) inequality to mixtures of rotationally invariant measures, thus providing new examples of measures satisfying this property.

2. Weighted Poincaré inequalities. In this section we give the proof of Theorem 4. We will proceed by integration in polar coordinates: for an integrable or nonnegative function F on $\mathbb{R}^n$,

$$\int F(x)\,\mathrm{d}x = c_n \int_{\mathbb{S}^{n-1}} \int_0^\infty F(r\theta)\, r^{n-1}\,\mathrm{d}r\,\mathrm{d}\theta,$$

where $\mathrm{d}\theta$ refers to the usual normalized measure on the sphere $\mathbb{S}^{n-1} = \{x : |x| = 1\}$. Therefore, we will need two Poincaré-type inequalities, one for the spherical part and one for the radial part.

In order to treat the spherical part, we will need the following weighted Poincaré inequality on the sphere, that is, a particular case of a general result of Kolesnikov and Milman [17], as we shall see.

PROPOSITION 6. *Let $v:\mathbb{R}^n\to\mathbb{R}$ be a convex C^1 function, and let μ be the measure on $\mathbb{S}^{n-1}$ with density e^{-v}. Then, for every C^1 function $g:\mathbb{S}^{n-1}\to\mathbb{R}$ with $\int_{\mathbb{S}^{n-1}} g\,\mathrm{d}\mu=0$, one has*

$$\int_{\mathbb{S}^{n-1}}(n-1-\mathrm{R}v)g^2\,\mathrm{d}\mu\le\int_{\mathbb{S}^{n-1}}|\nabla_{\mathbb{S}}g|^2\,\mathrm{d}\mu.$$

Here and after, we use the notation $\mathrm{R}v(x)=\langle\nabla v(x),x\rangle$ for the radial derivative and $\nabla_{\mathbb{S}}g$ for the spherical gradient of g.

Indeed, generalizing a result of Colesanti [8], Kolesnikov and Milman proved the following very general inequality.

THEOREM 7 ([17]). *Let (M,g) be a compact, smooth, complete, connected and oriented n-dimensional Riemannian manifold with boundary ∂M. Let $\mathrm{d}\mu=e^{-v}\,\mathrm{d}Vol_M$ be a measure on M, where $v:M\to\mathbb{R}$ is C^2-smooth.*

Assume (M,g,μ) satisfies the $CD(0,N)$ condition for some N such that $\frac{1}{N}\in[-\infty,\frac{1}{n}]$, and that $\mathrm{II}_{\partial M}\succ 0$. Then, for every $f\in C^1(\partial M)$, we have

$$\int_{\partial M}H_\mu f^2\,\mathrm{d}\mu_{\partial M}-\frac{N-1}{N}\cdot\frac{(\int_{\partial M}f\,\mathrm{d}\mu_{\partial M})^2}{\mu(M)}\le\int_{\partial M}\langle\mathrm{II}_{\partial M}^{-1}\nabla_{\partial M}f,\nabla_{\partial M}f\rangle\,\mathrm{d}\mu_{\partial M}.$$

To explain the notation of the theorem, we say that (M,g,μ) satisfies the $CD(0,N)$ condition if

$$\mathrm{Ric}_{g,\mu}:=\mathrm{Ric}_g+\nabla^2v-\frac{1}{N-n}\,\mathrm{d}v\otimes\mathrm{d}v\succeq 0$$

as a 2-tensor, where Ric_g denotes the classical Ricci curvature. Furthermore, $\mathrm{II}_{\partial M}$ denotes the second fundamental form, and $H_\mu(x)=\mathrm{tr}(\mathrm{II}_{\partial M}(x))-\langle\nabla v(x),\nu(x)\rangle$ denotes the weighted mean curvature of ∂M at $x\in\partial M$, where $\nu(x)$ is the outer unit normal to ∂M at x.

To see why Proposition 6 follows from Theorem 7, we simply choose $M=B_2^n\subset\mathbb{R}^n$, the unit Euclidean ball with the standard Euclidean metric. By approximation we may assume v is C^2. Then, for $\mathrm{d}\mu=e^{-v}\,\mathrm{d}x$, the weighted manifold (M,g,μ) satisfies the $CD(0,\infty)$ condition since

$$\mathrm{Ric}_{g,\mu}=0+\nabla^2v+0=\nabla^2v\succeq 0.$$

Moreover, in this case $\mathrm{II}_{\mathbb{S}^{n-1}}(x)$ is given by the standard inner product on $\mathbb{R}^n$ for all $x\in\mathbb{S}^{n-1}=\partial B_2^n$, so

$$H_\mu(x)=n-1-\langle\nabla v(x),x\rangle=n-1-\mathrm{R}v(x).$$

Plugging this into Theorem 7, one obtains Proposition 6.

Let us comment a bit more on this result. Kolesnikov and Milman obtained their inequality using a general Reilly-type integration by parts formula for the solution u of the problem $\Delta_g u-\langle\nabla u,\nabla v\rangle\equiv\frac{1}{\mu(M)}\int_{\partial M}f\,\mathrm{d}\mu_{\partial M}$ in the interior of M and the normal derivative of u on ∂M equal to f. However, when M is a convex body in $\mathbb{R}^n$, this inequality can be derived in a more elementary way by differentiating the Brunn–Minkowski inequality (2) for the log-concave measure $\mathrm{d}\mu=e^{-v}\,\mathrm{d}x$; see [17] (in particular, Theorem 6.6) and [16] (in particular, Proposition 3.2). When $v=0$, this is exactly what was done in [8], but it is absolutely crucial for us to have the correct inequality for the weight e^{-v}.

The second ingredient we will need for the proof is the following one dimensional lemma.

LEMMA 8. *Let $w, v : [0,\infty) \to \mathbb{R}$ be continuous functions and C^1 on $(0,\infty)$. Let f be a C^1 function on $[0,\infty)$ which is compactly supported (for simplicity) and satisfies $f(0) = 0$. Then, for every $\alpha \geq 0$, we have*

$$\int_0^\infty \frac{w'}{t} f^2 t^\alpha e^{-w-v}\,\mathrm{d}t \leq \int_0^\infty \Big((f')^2 + \alpha \cdot \Big(\frac{f}{t}\Big)^2 - v'\frac{f^2}{t}\Big)t^\alpha e^{-w-v}\,\mathrm{d}t.$$

PROOF. Since f is C^1-smooth and $f(0) = 0$, we may write $f(t) = t \cdot g(t)$ for a function g continuous on $[0,\infty)$, C^1 on $(0,\infty)$ and compactly supported. It follows using integration by parts, since boundary terms vanish, that

$$\begin{aligned}\int_0^\infty \frac{w'}{t} f^2 t^\alpha e^{-w-v}\,\mathrm{d}t &= \int_0^\infty w' g^2 t^{\alpha+1} e^{-w-v}\,\mathrm{d}t = -\int_0^\infty (g^2 t^{\alpha+1} e^{-v})(e^{-w})'\,\mathrm{d}t.\\ &= \int_0^\infty (g^2 t^{\alpha+1} e^{-v})' e^{-w}\,\mathrm{d}t \\ &= \int_0^\infty (2tgg' + (\alpha+1)g^2 - v'g^2 t)t^\alpha e^{-w-v}\,\mathrm{d}t.\end{aligned}$$

On the other hand, we have for the right-hand side,

$$\begin{aligned}&\int_0^\infty \Big((f')^2 + \alpha \cdot \Big(\frac{f}{t}\Big)^2 - v'\frac{f^2}{t}\Big)t^\alpha e^{-w-v}\,\mathrm{d}t \\ &\quad= \int_0^\infty ((g+tg')^2 + \alpha \cdot g^2 - v'tg^2)t^\alpha e^{-w-v}\,\mathrm{d}t \\ &\quad= \int_0^\infty (g^2 + 2tgg' + t^2(g')^2 + \alpha g^2 - v'tg^2)t^\alpha e^{-w-v}\,\mathrm{d}t.\end{aligned}$$

Comparing the two expressions, we see that the difference between the right-hand side and the left-hand side is exactly $\int_0^\infty (g')^2 t^{\alpha+2} e^{-w-v}\,\mathrm{d}t$ which is clearly nonnegative. $\square$

We are now ready to prove Theorem 4.

PROOF OF THEOREM 4. Our finite measure ν is of the form $\mathrm{d}\nu(x) = e^{-v(x)-w(|x|)}\,\mathrm{d}x$ with $v : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex. We can assume that our h satisfies $\nabla h \in L^2(\nu)$ since otherwise there is nothing to prove.

We begin with some standard approximation arguments. First, let us note that we can assume that $h \in L^2(\nu)$. Actually, we can assume that h is bounded. Indeed, let us introduce for every $k \in \mathbb{N}^*$ a C^1 smooth nondecreasing odd function $R_k : \mathbb{R} \to \mathbb{R}$ such that $R_k(t) = t$ for $t \in [0,k]$, $R_k(t) \equiv k+1$ for $t \geq k+2$, $R_k(t) \leq t$ and $R_k'(t) \leq 1$ for every $t \in \mathbb{R}^+$. Then the functions $h_k := R_k(h)$ satisfy $h_k = h$ on the open set $\{|h| < k\}$, $|h_k| \uparrow |h|$ and $|\nabla h_k| \xrightarrow{\leq} |\nabla h|$. Hence, by monotone and dominated convergence, respectively, we can pass from the bounded functions h_k to h in our inequality.

Next, we reduce to the case that h is compactly supported. The classical argument is to introduce a smooth and radially decreasing function χ on $\mathbb{R}^n$ with values in $[0,1]$ that is compactly supported and equals to 1 in a neighborhood of 0 and to set $\chi_k(x) := \chi(x/k)$. Then $\chi_k \uparrow 1$ and $|\nabla \chi_k| \leq C/k$ for some constant $C > 0$. On one hand, we have $\frac{w'(|x|)}{|x|}(h\chi_k)^2 \uparrow \frac{w'(|x|)}{|x|}h^2$, and on the other hand,

$$\begin{aligned}\int |\nabla(h\chi_k)|^2\,\mathrm{d}\nu &= \int |\nabla h|^2 \chi_k^2\,\mathrm{d}\nu + 2\int h\chi_k \langle \nabla h, \nabla \chi_k\rangle\,\mathrm{d}\nu + \int h^2 |\nabla \chi_k|^2\,\mathrm{d}\nu \\ &\leq \int |\nabla h|^2\,\mathrm{d}\nu + \frac{2C}{k}\sqrt{\int h^2\,\mathrm{d}\nu \int |\nabla h|^2\,\mathrm{d}\nu} + \frac{C^2}{k^2}\int h^2\,\mathrm{d}\nu,\end{aligned}$$

and this upper bound converges to $\int |\nabla h|^2\,\mathrm{d}\nu$, as wanted.

Finally, we approximate w and v. By replacing $w(t)$ with $\max(w(t), -k)$ and invoking monotone convergence as $k \to \infty$, we can assume that w is continuous on the closed ray $[0, \infty)$ and C^1 on $(0, \infty)$, except maybe at one point which is irrelevant. By standard approximation we may also assume without loss of generality that v is smooth.

These remarks being made, we compute the integrals for our compactly supported function h using polar coordinates. We obtain

$$\int \frac{w'(|x|)}{|x|} h^2 \,\mathrm{d}\nu = c_n \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{w'(r)}{r} h^2(r\theta) r^{n-1} e^{-w(r)-v(r\theta)} \,\mathrm{d}r \,\mathrm{d}\theta.$$

For a fixed $\theta \in \mathbb{S}^{n-1}$, we will now apply Lemma 8 with $f_\theta(r) = h(r\theta)$, $v_\theta(r) = v(r\theta)$ and $\alpha = n-1$. Note that $v'_\theta(r) = \langle \nabla v(r\theta), \theta \rangle = \frac{1}{r} \mathrm{R}v(r\theta)$. Therefore, we can bound our integral by

$$c_n \int_{S^{n-1}} \int_0^\infty \Big(\underbrace{\langle \nabla h(r\theta), \theta \rangle^2}_{\mathrm{I}} + \underbrace{(n-1)\Big(\frac{h(r\theta)}{r}\Big)^2 - \frac{1}{r} \mathrm{R}v(r\theta) \cdot \frac{h(r\theta)^2}{r}}_{\mathrm{II}} \Big)$$
$$\times\, r^{n-1} e^{-w(r)-v(r\theta)} \,\mathrm{d}r \,\mathrm{d}\theta.$$

We will leave term I as is for now. In order to bound term II, we change the order of integration,

$$\begin{aligned} \mathrm{II} &= c_n \int_0^\infty \int_{S^{n-1}} (n-1-\mathrm{R}v(r\theta)) \Big(\frac{h(r\theta)}{r}\Big)^2 r^{n-1} e^{-w(r)-v(r\theta)} \,\mathrm{d}\theta \,\mathrm{d}r \\ &= c_n \int_0^\infty r^{n-1} e^{-w(r)} \Big(\int_{\mathbb{S}^{n-1}} (n-1-\mathrm{R}v(r\theta)) \Big(\frac{h(r\theta)}{r}\Big)^2 e^{-v(r\theta)} \,\mathrm{d}\theta \Big) \,\mathrm{d}r. \end{aligned}$$

We now apply Proposition 6 to the inner integral, with $v_r(\theta) = v(r\theta)$, $\mu_r = e^{-v_r}\,\mathrm{d}\theta$ and $g_r(\theta) = \frac{h(r\theta)}{r}$. Note that since v_r is even and g_r is odd, we indeed have $\int_{\mathbb{S}^{n-1}} g_r \,\mathrm{d}\mu_r = 0$ (this is, in fact, the only place where we use the fact that h is odd). Also, note that $\mathrm{R}v_r(\theta) = \mathrm{R}v(r\theta)$ and $\nabla_{\mathbb{S}} g_r(\theta) = \nabla_{\mathbb{S}} h(r\theta)$, where for a function $h : \mathbb{R}^n \to \mathbb{R}$ the notation

$$\nabla_{\mathbb{S}} h(x) = \nabla h(x) - \Big\langle \nabla h(x), \frac{x}{|x|} \Big\rangle \cdot \frac{x}{|x|}$$

denotes the tangential part of the gradient of h. We may, therefore, apply the proposition and conclude that

$$\mathrm{II} \le c_n \int_0^\infty r^{n-1} e^{-w(r)} \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}} h(r\theta)|^2 e^{-v(r\theta)} \,\mathrm{d}\theta \,\mathrm{d}r.$$

Using this estimate for II, we conclude that

$$\begin{aligned} \int \frac{w'(|x|)}{|x|} h^2 \,\mathrm{d}\nu &\le c_n \int_{\mathbb{S}^{n-1}} \int_0^\infty (\langle \nabla h(r\theta), \theta \rangle^2 + |\nabla_{\mathbb{S}} h(r\theta)|^2) r^{n-1} e^{-w(r)-v(r\theta)} \,\mathrm{d}r \,\mathrm{d}\theta \\ &= c_n \int_{\mathbb{S}^{n-1}} \int_0^\infty |\nabla h(r\theta)|^2 r^{n-1} e^{-w(r)-v(r\theta)} \,\mathrm{d}r \,\mathrm{d}\theta \\ &= \int |\nabla h(x)|^2 e^{-w(|x|)-v(x)} \,\mathrm{d}x = \int |\nabla h|^2 \,\mathrm{d}\nu, \end{aligned}$$

completing the proof of Theorem 4. $\square$

We remark that the proof above strongly uses the fact that h is odd to deduce that the functions $g_r e^{-v_r}$ are all centered. It is possible that Theorem 4 is true under a weaker assumption on h, but at the moment we do not know how to address this interesting question.

To conclude this section, let us prove that when $v=0$, that is, $\mathrm{d}\nu = e^{-w(|x|)}\,\mathrm{d}x$, we have equality in Theorem 4 for every linear function $h(x)=\langle x,\theta\rangle$. By homogeneity and rotation invariance, it is enough to consider the function $h(x)=x_1$. Formally, the result follows by integration by parts:

$$\int \frac{w'(|x|)}{|x|}h^2\,\mathrm{d}\nu = -\int \partial_1\big(e^{-w(|x|)}\big)x_1\,\mathrm{d}x = \int 1\cdot e^{-w(|x|)}\,\mathrm{d}x = \int |\nabla h|^2\,\mathrm{d}\nu.$$

To check this rigorously, introduce $A_{\epsilon,R}=\{x\in\mathbb{R}^n : \epsilon<|x|<R\}$ for $0<\epsilon<R<\infty$. Then using polar coordinates and integration by parts, we have

$$\begin{aligned}
\int_{A_{\epsilon,R}} \frac{w'(|x|)}{|x|}h^2\,\mathrm{d}\nu &= \frac{1}{n}\int_{A_{\epsilon,R}} \frac{w'(|x|)}{|x|}|x|^2 e^{-w(|x|)}\,\mathrm{d}x \\
&= \frac{c_n}{n}\int_\epsilon^R w'(r)r^n e^{-w(r)}\,\mathrm{d}r = -\frac{c_n}{n}\int_\epsilon^R r^n\big(e^{-w(r)}\big)'\,\mathrm{d}r \\
&= \frac{c_n}{n}\cdot\big(\epsilon^n e^{-w(\epsilon)} - R^n e^{-w(R)}\big) + c_n\int_\epsilon^R r^{n-1}e^{-w(r)}\,\mathrm{d}r \\
&= \frac{c_n}{n}\cdot\big(\epsilon^n e^{-w(\epsilon)} - R^n e^{-w(R)}\big) + \int_{A_{\epsilon,R}} |\nabla h|^2\,\mathrm{d}\nu.
\end{aligned} \tag{6}$$

Since the integrands are nonnegative, the integrals $\int_{A_{\epsilon,R}} \frac{w'(|x|)}{|x|}h^2\,\mathrm{d}\nu$ and $\int_{A_{\epsilon,R}}|\nabla h|^2\,\mathrm{d}\nu$ have a limit when $\epsilon\to 0^+$ and $R\to\infty$, and the limits are finite since $\int \frac{w'(|x|)}{|x|}h^2\,\mathrm{d}\nu \le \int|\nabla h|^2\,\mathrm{d}\nu<\infty$. Therefore, the limits $\lim_{\epsilon\to 0^+}\epsilon^n e^{-w(\epsilon)}$ and $\lim_{R\to\infty}R^n e^{-w(R)}$ also exist. Since $\int_0^\infty r^{n-1}e^{-w(r)}\,\mathrm{d}r = \frac{1}{c_n}\int e^{-w(|x|)}\,\mathrm{d}x<\infty$, both of these limits have to be 0. We may, therefore, let $\epsilon\to 0^+$, $R\to\infty$ in (6) and deduce that

$$\int \frac{w'(|x|)}{|x|}h^2\,\mathrm{d}\nu = \int |\nabla h|^2\,\mathrm{d}\nu,$$

as claimed.

3. Improved Brascamp–Lieb inequality. In this section, we give the proof of Theorem 3. So we are working with a probability measure ν whose density is of the form $e^{-W(x)-v(x)}$ with $W(x)=w(|x|)$ where w is smooth and satisfies the assumptions of the theorem, and v is an arbitrary even convex function on $\mathbb{R}^n$ with values on $\mathbb{R}\cup\{+\infty\}$. In the applications, e^{-v} will be the indicator of a symmetric convex set. But by approximation, we can easily assume that v is finite and smooth on $\mathbb{R}^n$.

The classical Hörmander–Brascamp–Lieb inequality states that for a smooth integrable f one has

$$\begin{aligned}
\operatorname{Var}_\nu f &\le \int\big\langle\big(\nabla^2 W+\nabla^2 v\big)^{-1}\nabla f,\nabla f\big\rangle\,\mathrm{d}\nu \\
&\le \int\big\langle\big(\nabla^2 W\big)^{-1}\nabla f,\nabla f\big\rangle\,\mathrm{d}\nu.
\end{aligned}$$

Since $\frac{w'(|x|)}{|x|}\cdot\mathrm{Id}\succeq 0$, the conclusion of Theorem 3 is clearly stronger than this last inequality, but of course, we are assuming that f is even. Recall that $f(x)=\langle(\nabla W+\nabla v)(x),\theta\rangle$ is an equality case in the first inequality, but this function is odd in our case.

A direct computation of $\nabla^2 W$ shows that, for every $x\neq 0$,

$$\nabla^2 W(x) = w''\big(|x|\big)\frac{x}{|x|}\otimes\frac{x}{|x|} + \frac{w'(|x|)}{|x|}\cdot\bigg(\mathrm{Id}-\frac{x}{|x|}\otimes\frac{x}{|x|}\bigg),$$

so one can write

$$
\begin{aligned}
&\nabla^2 W(x) + \frac{w'(|x|)}{|x|} \cdot \mathrm{Id} \\
&\quad = \left(w''(|x|) + \frac{w'(|x|)}{|x|}\right)\frac{x}{|x|} \otimes \frac{x}{|x|} + 2\frac{w'(|x|)}{|x|} \cdot \left(\mathrm{Id} - \frac{x}{|x|} \otimes \frac{x}{|x|}\right).
\end{aligned} \tag{7}
$$

The condition that $t \mapsto w(e^t)$ is convex implies that $w''(s) + \frac{w'(s)}{s} \geq 0$ for all $s > 0$. Hence, the expression above shows that $\nabla^2 W(x) + \frac{w'(|x|)}{|x|} \cdot \mathrm{Id} \succeq 0$, which we will use in the proof.

REMARK 9. In the statement of Theorem 3 and in its proof below, we encounter expression like $\langle(\nabla^2 W + \frac{w'(|x|)}{|x|} \cdot \mathrm{Id})^{-1} a, a\rangle$ for some $a \in \mathbb{R}^n$. When the matrix is singular, one should rather use the polar form $Q_x^\circ(a) = \sup\{\langle a, b\rangle^2 \ : \ Q_x(b) \leq 1\} \in [0, +\infty]$ of the quadratic form $b \mapsto Q_x(b) := \langle(\nabla^2 W + \frac{w'(|x|)}{|x|} \cdot \mathrm{Id})b, b\rangle$. Indeed, the only property we need is that $\frac{1}{2}Q_x^\circ(a) + \frac{1}{2}Q_x(b) \geq \langle a, b\rangle$ for all $a, b \in \mathbb{R}^n$.

PROOF OF THEOREM 3. With the notation above, consider the even function $V := W + v$ so that $\frac{\mathrm{d}\nu}{\mathrm{d}x} = e^{-V}$; we already mentioned that by approximation v can be assumed to be C^2-smooth, so V is C^2 as well. Since ν is log-concave with respect to e^{-W}, it follows that $\nabla^2 V \succeq \nabla^2 W$ as positive definite matrices. Also, write $A(x) = \nabla^2 W(x) + \frac{w'(|x|)}{|x|} \cdot \mathrm{Id}$.

Consider the operator $Lu = \Delta u - \langle \nabla V, \nabla u\rangle$, that is, the Laplace operator $\nabla^*\nabla$ on $L^2(\nu)$. We are given an even function f. We can add a constant to f and assume without loss of generality that $\int f \,\mathrm{d}\nu = 0$. It is well known then that f can be approximated in $L^2(\nu)$ by functions of the form Lu for smooth compactly supported u (see, for instance, [10]). Moreover, since V is even and f is even, we can also assume that u is even. Therefore it is enough to prove

$$\int \big((Lu - f)^2 - f^2 + \langle A^{-1}\nabla f, \nabla f\rangle\big)\,\mathrm{d}\nu \geq 0,$$

that is,

$$\int \big((Lu)^2 - 2Lu \cdot f + \langle A^{-1}\nabla f, \nabla f\rangle\big)\,\mathrm{d}\nu \geq 0.$$

Integrating by parts, we see that

$$
\begin{aligned}
\int Lu \cdot f \,\mathrm{d}\nu &= -\int \langle \nabla u, \nabla f\rangle \,\mathrm{d}\nu, \\
\int (Lu)^2 \,\mathrm{d}\nu &= \int \big(\|\nabla^2 u\|_2^2 + \langle(\nabla^2 V) \cdot \nabla u, \nabla u\rangle\big)\,\mathrm{d}\nu \\
&\geq \int \big(\|\nabla^2 u\|_2^2 + \langle(\nabla^2 W) \cdot \nabla u, \nabla u\rangle\big)\,\mathrm{d}\nu,
\end{aligned}
$$

where $\|A\|_2 = \sqrt{\mathrm{tr}(AA^*)}$ is the Hilbert–Schmidt norm. Therefore, it is enough to prove the inequality

$$\int \big(\|\nabla^2 u\|_2^2 + \langle(\nabla^2 W)\nabla u, \nabla u\rangle + 2\langle \nabla u, \nabla f\rangle + \langle A^{-1}\nabla f, \nabla f\rangle\big)\,\mathrm{d}\nu \geq 0.$$

We have the pointwise identity

$$\big|A^{-\frac{1}{2}}\nabla f + A^{\frac{1}{2}}\nabla u\big|^2 = \langle A^{-1}\nabla f, \nabla f\rangle + 2\langle \nabla f, \nabla u\rangle + \langle A \cdot \nabla u, \nabla u\rangle,$$

so our goal can be written as

$$\int \left(\|\nabla^2 u\|_2^2 + \langle (\nabla^2 W)\nabla u, \nabla u\rangle + \left|A^{-\frac{1}{2}}\nabla f + A^{\frac{1}{2}}\nabla u\right|^2 - \langle A\cdot\nabla u, \nabla u\rangle\right) \mathrm{d}\nu \geq 0.$$

As $|A^{-\frac{1}{2}}\nabla f + A^{\frac{1}{2}}\nabla u|^2 \geq 0$ (this corresponds to the duality relation recalled in Remark 9 above), it is, therefore, enough to prove that

$$\int \frac{w'(|x|)}{|x|}|\nabla u|^2 \,\mathrm{d}\nu = \int \langle (A - \nabla^2 W)\cdot\nabla u, \nabla u\rangle \,\mathrm{d}\nu \leq \int \|\nabla^2 u\|_2^2 \,\mathrm{d}\nu. \tag{8}$$

But this follows form Theorem 4: Every derivative $h_i = \frac{\partial u}{\partial x_i}$ is odd, so by Theorem 4 we have

$$\int \frac{w'(|x|)}{|x|} h_i^2 \,\mathrm{d}\nu \leq \int |\nabla h_i|^2 \,\mathrm{d}\nu.$$

Summing over $1 \leq i \leq n$, we obtain the desired inequality (8). $\square$

4. The (B) property. In this section we prove Theorem 1 in the functional form (4).

PROOF OF THEOREM 1. By approximation we may assume w is well defined on $[0,\infty)$ and C^2-smooth there with $w'' > 0$. Write $W(x) := w(|x|)$.

As we said, we will prove the more general form (4). So we fix an arbitrary *even* convex function $v : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, and our goal is to prove that

$$t \mapsto \int_{\mathbb{R}^n} e^{-v(e^{tA}y)-W(y)} \,\mathrm{d}y = e^{-t\cdot \mathrm{tr}\, A} \cdot \int_{\mathbb{R}^n} e^{-v(x)-W(e^{-tA}x)} \,\mathrm{d}x \tag{9}$$

is log-concave. Since the function $t \mapsto e^{-t\cdot\mathrm{tr}\, A}$ is clearly log-linear and since the change of variables $t \mapsto -t$ preserves log-concavity, we are led to prove that the function

$$\rho_v(t) = \int_{\mathbb{R}^n} e^{-v(x)-W(e^{tA}x)} \,\mathrm{d}x$$

is log-concave.

To do so, we need to show that $\rho_v(t)\rho_v''(t) \leq \rho_v'(t)^2$ for every $t \in \mathbb{R}$. However, from the change of variables (9) one can check that $\rho_v(t+h) = \rho_{\tilde{v}}(h)$, where $\tilde{v}(x) = v(e^{-tA}x) + t\cdot \mathrm{tr}\, A$ is again an even convex function. So we see it is enough to show that $\rho(0)\rho''(0) \leq \rho'(0)^2$ for $\rho := \rho_v$ and v is an arbitrary even convex function.

Computing these derivatives, we obtain

$$\rho'(t) = -\int e^{-W(e^{tA}x)} \langle \nabla W(e^{tA}x), Ae^{tA}x\rangle e^{-v(x)} \,\mathrm{d}x$$

$$\rho''(t) = \int e^{-W(e^{tA}x)} \langle \nabla W(e^{tA}x), Ae^{tA}x\rangle^2 e^{-v(x)} \,\mathrm{d}x$$
$$- \int e^{-W(e^{tA}x)} \left(\langle \nabla^2 W(e^{tA}x)\cdot Ae^{tA}x, Ae^{tA}x\rangle + \langle \nabla W(e^{tA}x), A^2 e^{tA}x\rangle\right) e^{-v(x)} \,\mathrm{d}x,$$

so the condition $\rho(0)\rho''(0) \leq \rho'(0)^2$ becomes

$$\int e^{-v(x)} \,\mathrm{d}\mu(x) \cdot \int \left(\langle \nabla W, Ax\rangle^2 - \langle \nabla^2 W\cdot Ax, Ax\rangle - \langle \nabla W, A^2 x\rangle\right) e^{-v(x)} \,\mathrm{d}\mu(x)$$
$$\leq \left(\int \langle \nabla W, Ax\rangle e^{-v(x)} \,\mathrm{d}\mu(x)\right)^2.$$

Introduce the probability measure

$$\mathrm{d}\nu(x) = \frac{e^{-v(x)}}{\int e^{-v}\,\mathrm{d}\mu}\,\mathrm{d}\mu(x).$$

Our aim is to prove that

$$\int \big(\langle \nabla W, Ax\rangle^2 - \langle \nabla^2 W \cdot Ax, Ax\rangle - \langle \nabla W, A^2 x\rangle\big)\,\mathrm{d}\nu \le \Big(\int \langle \nabla W, Ax\rangle\,\mathrm{d}\nu\Big)^2,$$

that is

$$\int \langle \nabla W, Ax\rangle^2\,\mathrm{d}\nu - \Big(\int \langle \nabla W, Ax\rangle\,\mathrm{d}\nu\Big)^2 \le \int \big(\langle \nabla^2 W \cdot Ax, Ax\rangle + \langle \nabla W, A^2 x\rangle\big)\,\mathrm{d}\nu. \tag{10}$$

Actually, this aimed inequality is really equivalent to the strong (B) inequality for μ. We claim that this inequality follows from Theorem 3 for the function

$$f_0(x) := \langle \nabla W(x), Ax\rangle.$$

Indeed, note first that Theorem 3 is applicable since ν is even and log-concave with respect to μ and f_0 is even. Next, we have to interpret correctly the right-hand side of (10). Note that

$$\nabla f_0 = \nabla^2 W \cdot Ax + A \cdot \nabla W,$$

and so

$$\langle \nabla^2 W \cdot Ax, Ax\rangle + \langle \nabla W, A^2 x\rangle = \langle \nabla f_0, Ax\rangle,$$

where we have used the symmetry of A. This means that (10) can be written as

$$\mathrm{Var}_\nu(f_0) \le \int \langle \nabla f_0, Ax\rangle\,\mathrm{d}\nu(x).$$

But since $\nabla W(x) = \frac{w'(|x|)}{|x|}x$, we can write

$$\nabla f_0 = \Big(\nabla^2 W + \frac{w'(|x|)}{|x|}\mathrm{Id}\Big)Ax \tag{11}$$

which implies that

$$\langle \nabla f_0, Ax\rangle = \Big\langle \Big(\nabla^2 W + \frac{w'(|x|)}{|x|}\mathrm{Id}\Big)^{-1} \nabla f_0, \nabla f_0\Big\rangle.$$

This shows that (10) follows from Theorem 3, as claimed. □

There is a hidden but crucial choice behind the apparently trivial relation (11). Indeed, let us consider the simple case where $w(t) = t^p/p$ (so $W(x) = |x|^p/p$) and $A = \mathrm{Id}$. The equivalent formulation (10) of the (B) inequality is then

$$\mathrm{Var}_\nu(f_0) = \mathrm{Var}_\nu\big(|\cdot|^p\big) \le p \int |x|^p\,\mathrm{d}\nu(x) = \int \langle \nabla f_0, x\rangle\,\mathrm{d}\nu$$

for $f_0(x) = |x|^p$ and ν an even probability measure which is log-concave with respect to $e^{-W(x)}\,\mathrm{d}x = e^{-|x|^p/p}\,\mathrm{d}x$. We see that there are several possible interpretations of the last term, since we have both

$$x = \frac{p-1}{p}(\nabla^2 W)^{-1} \cdot \nabla f_0 \quad \text{and} \quad x = \frac{1}{p|x|^{p-2}}\nabla f_0.$$

By the way, here we can invoke homogeneity to shorten computations since $f_0 = pW$. Every choice of a matrix-valued function $B(x)$ such that $B \cdot x = \nabla f_0$ leads to the natural question whether the corresponding Brascamp–Lieb-type inequality $\mathrm{Var}_\nu(f) \le \int \langle B^{-1}\nabla f, \nabla f\rangle \, d\nu$ holds for every smooth even f; This would imply the (B) inequality. The two formulas above coincide in the case of the Gaussian measure ($p = 2$) but not in general. Our choice (11) is some combination of the two:

$$x = \left(\nabla^2 W + |x|^{p-2}\mathrm{Id}\right)^{-1} \nabla f_0(x).$$

5. Dimensional Brunn–Minkowski inequality. In this section we prove Theorem 5, that is, the Gardner–Zvavitch conjecture for rotationally invariant measures. We heavily rely on the computations done by Kolesnikov and Livshyts and on the ideas introduced by Eskenazis and Moschidis in their solution of the Gaussian case.

As for the Brunn–Minkowski inequality and the (B) inequality, we can prove this $\frac{1}{n}$-concavity by computing the second derivative in the parameter λ. This was done by Kolesnikov and Livshyts [16] (see Lemma 2.3), who found the following neat *sufficient* condition; this result is somehow a substitute to the duality argument used in the proof of Theorem 3. Below, the notation μ_K refers to the normalized restriction of a measure μ to a set K with $\mu(K) < \infty$.

THEOREM 10 ([16]). *Let μ be a locally finite measure on $\mathbb{R}^n$ with density e^{-W}. Assume that for every symmetric convex body $K \subset \mathbb{R}^n$ and every smooth even function $u : K \to \mathbb{R}$ with $Lu := \Delta u - \langle \nabla W, \nabla u\rangle \equiv 1$ in K, we have*

$$\int \left(\|\nabla^2 u\|_2^2 + \langle \nabla^2 W \cdot \nabla u, \nabla u\rangle\right) d\mu_K \ge \frac{1}{n}.$$

Then, for every symmetric convex bodies $K, L \subseteq \mathbb{R}^n$ and every $0 \le \lambda \le 1$, we have

$$\mu\left((1-\lambda)K + \lambda L\right)^{\frac{1}{n}} \ge (1-\lambda)\mu(K)^{\frac{1}{n}} + \lambda\mu(L)^{\frac{1}{n}}.$$

Let us mention that this formulation builds upon previous ideas introduced by Kolesnikov and Milman in [17] and [18], in particular, the idea to obtain Poincaré-type inequalities on the *boundary* of the given domain K by expressing a function on ∂K as a Neumann data of a function in the interior of K.

We now establish the condition in Theorem 10 using our weighted Poincaré inequality (Theorem 4). Loosely speaking, we have "Theorem 4 $\Rightarrow$ Theorem 10 $\Rightarrow$ Brunn–Minkowski," whereas previously we had "Theorem 4 $\Rightarrow$ Theorem 3 $\Rightarrow$ (B) inequality," with the difference that the formulation of Theorem 3 was specific to the rotationally invariant case and possibly of independent interest.

PROOF OF THEOREM 5. Write $W(x) = w(|x|)$, and assume by approximation that w is smooth. We begin by following the argument of Eskenazis and Moschidis [12].

Define $r = \frac{|x|^2}{2n}$, and note that

$$\|\nabla^2 u\|_2^2 = \|\nabla^2(u-r)\|_2^2 + \frac{2}{n}\Delta u - \frac{1}{n}.$$

Since $Lu = 1$, we have $\Delta u = \langle \nabla W, \nabla u\rangle + 1$, so

$$\|\nabla^2 u\|_2^2 = \|\nabla^2(u-r)\|_2^2 + \frac{2}{n}\langle \nabla W, \nabla u\rangle + \frac{1}{n}. \tag{12}$$

Next, we apply our weighted Poincaré inequality (Theorem 4). Since K is symmetric, the measure μ_K is even, and of course, it is log-concave with respect to μ. Moreover, for every i the derivative $\partial_i(u-r)$ is odd, since $u - r$ is even. Hence,

$$\int |\nabla \partial_i(u-r)|^2 \, d\mu_K \ge \int \frac{w'(|x|)}{|x|}\left(\partial_i(u-r)\right)^2 d\mu_K.$$

Summing over $1 \le i \le n$, we get

$$\begin{aligned}\int \|\nabla^2(u-r)\|_2^2 \,\mathrm{d}\mu_K &\ge \int \frac{w'(|x|)}{|x|}|\nabla(u-r)|^2 \,\mathrm{d}\mu_K \\ &= \int \frac{w'(|x|)}{|x|}\big(|\nabla u|^2 - 2\langle \nabla u, \nabla r\rangle + |\nabla r|^2\big)\,\mathrm{d}\mu_K \\ &\ge \int \frac{w'(|x|)}{|x|}\bigg(|\nabla u|^2 - \frac{2}{n}\langle \nabla u, x\rangle\bigg)\,\mathrm{d}\mu_K \\ &= \int \bigg(\frac{w'(|x|)}{|x|}|\nabla u|^2 - \frac{2}{n}\langle \nabla W, \nabla u\rangle\bigg)\,\mathrm{d}\mu_K,\end{aligned}$$

where in the last equality we used the fact that $\nabla W(x) = \frac{w'(|x|)}{|x|}x$. Therefore, using (12), we obtain

$$\begin{aligned}\int \big(\|\nabla^2 u\|_2^2 + \langle \nabla^2 W \cdot \nabla u, \nabla u\rangle\big)\,\mathrm{d}\mu_K &\ge \int \bigg\langle \bigg(\nabla^2 W + \frac{w'(|x|)}{|x|}\mathrm{Id}\bigg)\nabla u, \nabla u\bigg\rangle \mathrm{d}\mu_K + \frac{1}{n} \\ &\ge 0 + \frac{1}{n} = \frac{1}{n}.\end{aligned}$$

The last inequality is true since our assumption on w implies that the matrix $(\nabla^2 W + \frac{w'(|x|)}{|x|}\mathrm{Id})$ is nonnegative (see the computation in the beginning of Section 3).

By Theorem 10 we conclude that

$$\mu\big((1-\lambda)K + \lambda L\big)^{\frac{1}{n}} \ge (1-\lambda)\mu(K)^{\frac{1}{n}} + \lambda\mu(L)^{\frac{1}{n}}$$

for all symmetric K, L and all $0 \le \lambda \le 1$. □

6. Mixtures. As was mentioned in the Introduction, before the results of this paper there were very few examples of measures known to have the (B) property, except the Gaussian measure. The only such examples we are aware of in dimension $n \ge 3$ come from a result of Eskenazis, Nayar and Tkocz [13] about Gaussian mixtures. We will now briefly explain and slightly extend their result.

PROPOSITION 11. *Let $X = (X_1, X_2, \dots, X_n)$ be a random vector with a probability density on $\mathbb{R}^n$ which is rotationally invariant and log-concave. Let $Y = (Y_1, Y_2, \dots, Y_n)$ be a random random vector on $(\mathbb{R}^+)^n$ independent of X with probability density $h : (0,\infty)^n \to \mathbb{R}$ such that $(s_1, s_2, \dots, s_n) \mapsto h(e^{s_1}, e^{s_2}, \dots, e^{s_n})$ is log-concave. Let ν denote the distribution of $(X_1Y_1, X_2Y_2, \dots, X_nY_n)$. Then, for every symmetric convex body $K \subseteq \mathbb{R}^n$, the function*

$$(t_1, t_2, \dots, t_n) \mapsto \nu\big(e^{\Delta(t_1,\dots,t_n)}K\big)$$

is log-concave on $\mathbb{R}^n$; In particular, $t \mapsto \nu(e^t K)$ is log-concave on $\mathbb{R}$.

Here we use the notation $\Delta(t_1, \dots, t_n)$ for the diagonal matrix with entries $t_1, t_2, \dots, t_n$ on its diagonal.

PROOF. For every Borel set $K \subseteq \mathbb{R}^n$, we have

$$\begin{aligned}\nu(K) &= \mathbb{P}\big((X_1Y_1, X_2Y_2, \dots, X_nY_n) \in K\big) \\ &= \int_{(0,\infty)^n} \mathbb{P}\big((y_1X_1, y_2X_2, \dots, y_nX_n) \in K\big)h(y)\,\mathrm{d}y.\end{aligned}$$

We perform a change of variables $e^{-s} = y$ (i.e., $e^{-s_i} = y_i$ for $1 \le i \le n$). Then $\mathrm{d}y = e^{-\sum_{i=1}^n s_i}\,\mathrm{d}s$, so

$$\begin{aligned}\nu(K) &= \int_{\mathbb{R}^n} \mathbb{P}\big(e^{-\Delta(s_1,\dots,s_n)} \cdot X \in K\big) h\big(e^{-s}\big) e^{-\sum s_i}\,\mathrm{d}s \\ &= \int_{\mathbb{R}^n} \mathbb{P}\big(X \in e^{\Delta(s_1,\dots,s_n)} K\big) h\big(e^{-s}\big) e^{-\sum s_i}\,\mathrm{d}s \\ &= \int_{\mathbb{R}^n} \mu\big(e^{\Delta(s_1,\dots,s_n)} K\big) \cdot h\big(e^{-s}\big) e^{-\sum s_i}\,\mathrm{d}s,\end{aligned}$$

where μ denotes the distribution of X.

Therefore, if we now assume that K is a symmetric convex body, then

$$\nu\big(e^{\Delta(t_1,\dots,t_n)} K\big) = \int_{\mathbb{R}^n} \mu\big(e^{\Delta(s_1+t_1,\dots,s_n+t_n)} K\big) \cdot h\big(e^{-s}\big) e^{-\sum s_i}\,\mathrm{d}s.$$

By Theorem 1 the function $(t, s) \mapsto \mu(e^{\Delta(s_1+t_1,\dots,s_n+t_n)} K)$ is log-concave on $\mathbb{R}^{2n}$, so by our assumption on h

$$(t, s) \mapsto \mu\big(e^{\Delta(s_1+t_1,\dots,s_n+t_n)} K\big) h\big(e^{-s}\big) e^{-\sum s_i}$$

is also log-concave. It is a well-known corollary of the Prékopa inequality (2) that marginals of log-concave functions are also log-concave. Hence, the function

$$t \mapsto \int_{\mathbb{R}^n} \mu\big(e^{\Delta(s_1+t_1,\dots,s_n+t_n)} K\big) \cdot h\big(e^{-s}\big) e^{-\sum s_i}\,\mathrm{d}s = \nu\big(e^{\Delta(t_1,\dots,t_n)} K\big)$$

is log-concave. $\square$

The result of [13] is identical to the proposition above, with an identical proof, except the fact that they have to assume X is Gaussian in order to use the original result of [10], while we can use instead Theorem 1. Of course, the assumption in the proposition that the distribution μ of X is log-concave may be replaced with the weaker assumption on μ of Theorem 1.

Proposition 11 is only useful if one can identify measures ν which satisfy its assumptions. It is shown in [13] that if $0 < p \le 1$ and if ν has density proportional to $e^{-\|x\|_p^p} = e^{-\sum |x_i|^p}$, then ν satisfies the assumptions of the proposition (with a Gaussian random vector X) and, therefore, has the (B) property. The same is shown for the product measure $\nu = \nu_1^{\otimes n}$, where ν_1 is the distribution of a p-stable random variable for $0 < p \le 1$. No other examples are constructed.

Since we now have more freedom in the choice of X, our proposition applies to more measures ν than the theorem of [13]. However, at the moment we don't have any natural measure to propose that could be handled using this extra freedom.

Funding. The second author is partially supported by ISF Grant 1468/19 and BSF Grant 2016050.

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