Prismatic F-gauges and a result of T. Liu

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Abstract

We give a new proof of a recent result of Tong Liu, which gives a general control on the torsion in the graded pieces of the so-called integral Hodge filtration associated to a crystalline Galois lattice. Our approach is stack-theoretic, and is inspired on the one hand by a result of Gee–Kisin on the shape of mod p crystalline Breuil–Kisin modules, and on the other hand by the structures seen on the diffracted Hodge complex studied by Bhatt–Lurie. Along the way, we also obtain an explicit description of the Hodge–Tate locus in the Nygaard stack $\mathcal{O}_K^{\mathcal{N}}$ for a general extension $K/\mathbf{Q}_p$.

Contents

1. Introduction

Let $k/\mathbf{F}_p$ be a perfect field and let $K := W(k)[1/p]$. Let $T \in \mathrm{Rep}^{\mathrm{crys}}_{\mathbf{Z}_p}(G_K)$ be a crystalline G_K-lattice. Fix a choice of uniformizer $\pi \in \mathcal{O}_K$, with Eisenstein polynomial $E(x)$[1]. Let $\mathfrak{M}$ be the Breuil–Kisin module associated to T for this choice of π. Pulling the (full $\mathbf{Z}$-indexed) filtration $E(x)^{\mathbf{Z}}\mathfrak{M}$ along the Frobenius $\varphi^*\mathfrak{M} \to \mathfrak{M}[1/E(x)]$, and then pushing along the natural map $\varphi^*\mathfrak{M} \twoheadrightarrow \varphi^*\mathfrak{M}/E(x)\varphi^*\mathfrak{M}$ gives a filtration $\mathrm{Fil}^\bullet_H\mathfrak{M}_{dR}$ on $\mathfrak{M}_{dR} := \varphi^*\mathfrak{M}/E(x)\varphi^*\mathfrak{M}$, which we will refer to as the (integral) Hodge filtration. (The terminology is justified by the fact that, after inverting p this recovers the usual Hodge filtration on $D_{dR}(T[1/p]) \simeq \mathfrak{M}_{dR}[1/p]$.) It was observed in [11] that the torsion in $\mathrm{gr}^\bullet_H\mathfrak{M}_{dR}$ bears some relation to the shape of the Frobenius acting on $\mathfrak{M}$. For instance, if there is no torsion, then for any choice of $\mathfrak{S}$-basis $e = (e_1, \dots, e_n)$ of $\mathfrak{M}$ we have $\varphi(e) = eA\Lambda B$ where $A, B \in \mathrm{GL}_n(\mathfrak{S})$ and Λ is a diagonal matrix $\mathrm{diag}(E(u)^{r_i})$ (in general one only has $A, B \in \mathrm{GL}_n(\mathfrak{S}[1/p])$). An immediate consequence is that in this case the mod p Breuil–Kisin module $\mathfrak{M}/p\mathfrak{M}$ remembers the Hodge–Tate weights of the rational representation $T[1/p]$.

[1] The variable is typically denoted by u but here we reserve the notation u for other use.

The goal of this note is to give a new proof of the following recent result of Tong Liu, which gives a general control on the torsion appearing in $\mathrm{gr}^{\bullet}_H\mathfrak{M}_{dR}$. (In the statement below, the notation M_{tor} denotes the torsion submodule of an $\mathcal{O}_K$-module M.)

Theorem 1.1 ([12, Thm. 1.1])**.** *We have*

$$(\mathrm{gr}^i_H\mathfrak{M}_{dR})_{\mathrm{tor}} \neq 0 \quad \Longrightarrow \quad i = r + mp \quad \textit{for some } r \in \mathrm{HT} \textit{ and } m \in \mathbf{Z}_{>0},$$

where HT *denotes the set of Hodge–Tate weights of* $T[1/p]$.

The key new structure to our proof of Theorem 1.1 is a so-called Sen operator Θ on $\mathfrak{M}_{HT} := \mathfrak{M}/E(x)\mathfrak{M}$. Our use of Θ is guided by the structures seen on the diffracted Hodge complex introduced by Bhatt–Lurie in [4]. In turn, we construct Θ from a certain differential operator D. While it is possible to extract D from the *rational* monodromy operator in the classical theory of Breuil–Kisin modules by some delicate approximations (see Section 4), its existence — together with the additional symmetries that it satisfies — seems best explained by the theory of prismatic F-gauges by Bhatt–Lurie [3]. See Subsection 2.1 below. Our considerations here are inspired by a recent result of Gee–Kisin on the shape of mod p crystalline Breuil–Kisin modules; in particular, we follow their strategy and realize the objects of interest as quasi-coherent sheaves on certain stacks.

Remark 1.2. In fact, we will construct the operator D for a general (possibly ramified) extension $K/\mathbf{Q}_p$. As a key ingredient for this, we extend the explicit description of the so-called Hodge–Tate locus $(\mathcal{O}_K^{\mathcal{N}})_{t=0}$ given in [3, Prop. 5.3.7] to the case of a general extension $K/\mathbf{Q}_p$; see Proposition 3.20 below. More generally, there is a similar description for $(R^{\mathcal{N}})_{t=0}$ where R is an arbitrary complete regular local Noetherian ring with perfect residue field of characteristic $p > 0$; see Remark 3.25.

Remark 1.3. Our proof is partially inspired by Drinfeld and Bhatt–Lurie's approach to the Deligne–Illusie theorem via the Sen operator (cf. [4, Rem. 4.7.18]). As an illustration of this analogy, note that Theorem 1.1 implies in particular that in the case where $\mathfrak{M}$ is effective (which is the case for representations coming from geometry), $\mathrm{gr}^i M$ is torsion free for all $i < p$. We can thus roughly think of this special case as an incarnation of (a weaker form of) the Deligne–Illusie theorem (with the bound $i < p$ corresponding to the bound $\dim(\mathfrak{X}) < p$ in Deligne–Illusie result).

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Notation. We follow the conventions of [3]. In particular, our p-adic formal schemes are assumed to be bounded, and all stacks appearing are defined on p-nilpotent test rings with the flat topology.

2. Proof of Main Theorem

To avoid distractions, we will first prove a more general result (Proposition 2.1) by isolating the key hypothesis. In Subsection 2.1 below we will indicate how it specializes to the situation of Theorem 1.1.

Set up. Consider an *increasing* (honest) filtration of finite free $\mathcal{O}_K$-modules

$$\mathrm{Fil}^{\mathrm{conj}}_{\bullet}: \quad \ldots \subseteq \mathrm{Fil}^{\mathrm{conj}}_0 \subseteq \ldots \subseteq \mathrm{Fil}^{\mathrm{conj}}_i \subseteq \ldots.$$

We assume that this is a finite filtration, i.e., $\mathrm{Fil}^{\mathrm{conj}}_i$ stabilizes for $i \gg 0$, and is 0 for $i \ll 0$.

Hypothesis. Assume there is a filtered endomorphism $\Theta : \mathrm{Fil}^{\mathrm{conj}}_\bullet \to \mathrm{Fil}^{\mathrm{conj}}_\bullet$ with the property that Θ acts on the ith graded piece $\mathrm{gr}^{\mathrm{conj}}_i$ via multiplication by $-i$. (The superscript "conj" stands for "conjugate".)

Inverting p gives a filtration of K-vector spaces, and we define as usual its Hodge–Tate weights as the set of filtration jumps, i.e.,

$$\mathrm{HT} := \{i \in \mathbf{Z} \mid \mathrm{gr}^{\mathrm{conj}}_i[1/p] \neq 0\}.$$

Proposition 2.1. *We have*

$$(\mathrm{gr}^{\mathrm{conj}}_i)_{\mathrm{tor}} \neq 0 \quad \Longrightarrow \quad i = r + mp \quad \textit{for some } r \in \mathrm{HT} \textit{ and } m > 0.$$

Proof. Set

$$I := \{r + mp \mid r \in \mathrm{HT}, m > 0\}.$$

We need to show that if $i \notin I$, then $\mathrm{gr}^{\mathrm{conj}}_i$ is $\mathcal{O}_K$-free. We will do this by induction on i. If $i \ll 0$, then $\mathrm{gr}^{\mathrm{conj}}_i = 0$ and there is nothing to prove. Assume the result for $i' < i$ (and $i' \notin I$), we now deduce it for i. In fact we will show the stronger assertion that the sequence

$$0 \to \mathrm{Fil}^{\mathrm{conj}}_{i-1} \to \mathrm{Fil}^{\mathrm{conj}}_i \to \mathrm{gr}^{\mathrm{conj}}_i \to 0$$

of $\mathcal{O}_K[\Theta]$-modules splits (as $\mathrm{Fil}^{\mathrm{conj}}_i$ is $\mathcal{O}_K$-free, this implies in particular that $\mathrm{gr}^{\mathrm{conj}}_i$ is free, as wanted), which in turn will follow from

$$\mathrm{Ext}^1_{\mathcal{O}_K[\Theta]}(\mathrm{gr}^{\mathrm{conj}}_i, \mathrm{Fil}^{\mathrm{conj}}_{i-1}) = 0.$$

By dévissage, it suffices to show

$$\mathrm{Ext}^1_{\mathcal{O}_K[\Theta]}(\mathrm{gr}^{\mathrm{conj}}_i, \mathrm{gr}^{\mathrm{conj}}_j) = 0 \quad \text{for each } j < i.$$

We consider two cases. If $p \nmid i - j$, then we are done as the LHS is killed by $(\Theta + i) - (\Theta + j) = i - j$, a unit. If $p | i - j$, then by definition of the set I, $j \notin \mathrm{HT}$ and we still have $j \notin I$. Thus, $\mathrm{gr}^{\mathrm{conj}}_j[1/p] = 0$ but also $\mathrm{gr}^{\mathrm{conj}}_j$ is p-flat by the inductive hypothesis for j. This forces $\mathrm{gr}^{\mathrm{conj}}_j = 0$, and the result trivially holds. (Note that the same argument also shows that $\mathrm{Hom}_{\mathcal{O}_K[\Theta]}(\mathrm{gr}^{\mathrm{conj}}_i, \mathrm{Fil}^{\mathrm{conj}}_{i-1}) = 0$, i.e. the splitting is unique.) $\square$

2.1 A stacky perspective

After Proposition 2.1, to finish the proof of Theorem 1.1 we need to construct an increasing filtration $\mathrm{Fil}^{\mathrm{conj}}_\bullet$ together with an endomorphism Θ as above with the additional property that $\mathrm{gr}^{\mathrm{conj}}_\bullet \simeq \mathrm{gr}^\bullet_H \mathfrak{M}_{dR}$.

Our construction of $\mathrm{Fil}^{\mathrm{conj}}_\bullet$ and Θ is guided by the structure seen on the diffracted Hodge complex studied by Bhatt–Lurie in [4, §4.7], and is explained in [3, Remark 6.5.11] in a geometric context. The material in this subsection is therefore presumably well-known to the experts, although we do not know of a treatment in the literature in the level of generality that we require.

Construction of $\mathrm{Fil}^{\mathrm{conj}}_\bullet$ and Θ

Consider the so-called conjugate filtration

$$\mathrm{Fil}^{\mathrm{conj}}_\bullet \mathfrak{M}_{HT} : \quad \ldots \hookrightarrow \underbrace{\mathrm{Fil}^{i-1}\varphi^*\mathfrak{M}/\mathrm{Fil}^i\varphi^*\mathfrak{M}}_{\mathrm{Fil}^{\mathrm{conj}}_{i-1}\mathfrak{M}_{HT}} \overset{u}{\hookrightarrow} \underbrace{\mathrm{Fil}^i\varphi^*\mathfrak{M}/\mathrm{Fil}^{i+1}\varphi^*\mathfrak{M}}_{\mathrm{Fil}^{\mathrm{conj}}_i\mathfrak{M}_{HT}} \hookrightarrow \ldots, \tag{2.1.1}$$

where the transition map u is induced by the multiplication by $E(x)$. One checks easily that this is a finite increasing filtration of finite free $\mathcal{O}_K$-modules, whose underlying non-filtered module is $\mathfrak{M}_{HT} := \mathfrak{M}/E(x)\mathfrak{M}$ (justifying the notation). Moreover, there is a natural graded isomorphism

$$\mathrm{gr}^{\mathrm{conj}}_{\bullet}\mathfrak{M}_{HT} \simeq \mathrm{gr}^{\bullet}_{H}\mathfrak{M}_{dR}.$$

(This identification also admits a geometric explanation; see Corollary 3.32 below.) In what follows, we will often omit $\mathfrak{M}_{HT}$ from the notation and simply write $\mathrm{Fil}^{\mathrm{conj}}_{\bullet}$, etc. We now explain the construction of Θ.

From now on we drop the assumption that $K/\mathbf{Q}_p$ is unramified, i.e. we allow $K/\mathbf{Q}_p$ to be any complete discretely valued extension with perfect residue field.

In this generality, we will construct a filtered operator Θ on $\mathrm{Fil}^{\mathrm{conj}}_{\bullet}$ with the property that Θ acts as multiplication by $-iE'(\pi)$ on $\mathrm{gr}^{\mathrm{conj}}_{i}$. Note that if $K/\mathbf{Q}_p$ is unramified, then clearly $E'(\pi) = 1$; hence in this case Θ acts as $-i$ on $\mathrm{gr}^{\mathrm{conj}}_{i}$, as desired.

It is easy to see that the datum of such a Θ is equivalent to the datum of an operator $D : \mathrm{Fil}^{\mathrm{conj}}_{\bullet} \to \mathrm{Fil}^{\mathrm{conj}}_{\bullet}[-1]$ with the property that

$$Du - uD = E'(\pi)^{2}. \tag{2.1.2}$$

For instance, given D, the corresponding Θ is given by

$$\Theta := uD - iE'(\pi) \quad \text{on } \mathrm{Fil}^{\mathrm{conj}}_{i}. \tag{2.1.3}$$

(We refer the reader to Lemma 3.28 below for a geometric explanation of this relation.)

Thus, it suffices to construct D satisfying (2.1.2). As mentioned in the Introduction, while one can extract D from the *rational* monodromy operator in the classical theory of Breuil–Kisin modules, its existence seems best explained by the theory of prismatic F-gauges by Bhatt–Lurie, as we now explain.

More precisely, the existence of D and Θ as well as the relation (2.1.3) is encoded in the following commutative diagram of stacks over $\mathrm{Spf}(\mathbf{Z}_p)$:

$$\begin{tikzcd}
(B\mathbf{G}_m)_{\mathcal{O}_K} \arrow[r, hook, "t=0"] \arrow[d, hook, "u=0"'] & (\mathbf{A}^1_+/\mathbf{G}_m)_{\mathcal{O}_K} \arrow[r, "i_{dR}"] & \mathcal{O}_K^{\mathcal{N}} \\
(\mathbf{A}^1_-/\mathbf{G}_m)_{\mathcal{O}_K} \arrow[r, "can"] & (\mathbf{A}^1_-/\mathbf{G}_a^{\sharp} \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \arrow[r, "\pi_{\mathcal{O}_K}", "\simeq"'] & (\mathcal{O}_K^{\mathcal{N}})_{t=0} \arrow[u, hook, "t=0"] \\
\mathrm{Spf}(\mathcal{O}_K) = (\mathbf{G}_m/\mathbf{G}_m)_{\mathcal{O}_K} \arrow[u, hook, "u\neq 0"] \arrow[r, "can"] \arrow[rr, bend right, "\overline{\rho}_{(\mathfrak{S},I)}"'] & (\mathbf{G}_m/\mathbf{G}_a^{\sharp} \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \arrow[u, hook, "u\neq 0"] \arrow[r, "\simeq"] & \mathcal{O}_K^{HT}. \arrow[u, hook, "j_{HT}"]
\end{tikzcd} \tag{2.1.4}$$

Let us briefly explain the objects appearing in the diagram.

- $\mathbf{A}^1_+$ (resp. $\mathbf{A}^1_-$) denotes the affine line $\mathbf{A}^1$ where the coordinate t (resp. u) is placed in grading degree 1 (resp. -1). Furthermore, we let $\mathbf{G}_a^{\sharp}$ act on $\mathbf{A}^1_-$ by $a \cdot_{\mathcal{O}_K} x := E'(\pi)a + x$; one checks that this then extends to an action of $\mathbf{G}_a^{\sharp} \rtimes \mathbf{G}_m$, where the semidirect product is formed by letting $\mathbf{G}_m$ act on $\mathbf{G}_a^{\sharp}$ by $(\lambda, a) \mapsto \lambda^{-1}a$.

- $\mathcal{O}_K^{\mathcal{N}}$ is the filtered prismatization of $\mathrm{Spf}(\mathcal{O}_K)$. This is a filtered stack, i.e., comes with a map $t : \mathcal{O}_K^{\mathcal{N}} \to (\mathbf{A}^1_+/\mathbf{G}_m)_{\mathbf{Z}_p}$. Moreover, there is an open embedding $j_{HT} : \mathcal{O}_K^{\Delta} \hookrightarrow \mathcal{O}_K^{\mathcal{N}}$ from the prismatization $\mathcal{O}_K^{\Delta}$. See [3, §5.3] for more details. The map i_{dR} in the top row is the de Rham map defined in [3, Construction 5.3.13]. The map $\overline{\rho}_{(\mathfrak{S},I)}$ is the usual map associated to the chosen Breuil–Kisin prism $(\mathfrak{S}, I)$, viewed as an object in the absolute prismatic site of $\mathrm{Spf}(\mathcal{O}_K)$. See [5, Construction 3.10].

[2]We abusively also write $E'(\pi)$ for the multiplication by $E'(\pi)$ on an $\mathcal{O}_K$-module, etc.

- The isomorphism $(\mathbf{A}^1_-/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \simeq (\mathcal{O}_K^{\mathcal{N}})_{t=0}$ in the middle row will be defined and proved in Subsection 3.3 below.

Finally, we explain the relevance of diagram (2.1.4) to the construction of the operators D and Θ. More details will be given in Section 3 below.

- The graded isomorphism $\mathrm{gr}^{\mathrm{conj}}_\bullet \mathfrak{M}_{HT} \simeq \mathrm{gr}^\bullet_H \mathfrak{M}_{dR}$ results from commutativity of the top rectangle. See Subsection 3.6 below.

- It follows from the isomorphism $(\mathcal{O}_K^{\mathcal{N}})_{t=0} \simeq (\mathbf{A}^1_-/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K}$ that quasi-coherent sheaves on $(\mathcal{O}_K^{\mathcal{N}})_{t=0}$ are equivalent to p-complete graded modules over the (p-completed) Weyl algebra $\mathcal{O}_K\{u, D\}/(Du - uD - E'(\pi))$, i.e. p-complete graded $\mathcal{O}_K[u]$-modules M together with a graded endomorphism $D : M \to M[-1]$ satisfying the commutation relation $Du - uD = E'(\pi)$ (and a certain nilpotent condition), cf. Lemma 3.27. Pulling back along the natural map $[\mathbf{A}^1_-/\mathbf{G}_m] \to (\mathcal{O}_K^{\mathcal{N}})_{t=0}$ then simply amounts to forgetting the derivation. Thus if $E \in \mathrm{Coh}(\mathcal{O}_K^{\mathrm{Syn}})$ denotes the F-gauge associated to the given crystalline lattice T, then $E|_{(\mathcal{O}_K^{\mathcal{N}})_{t=0}}$ gives rise to an operator $D : \mathrm{Fil}_\bullet \to \mathrm{Fil}_\bullet[-1]$ satisfying $Du - uD = E'(\pi)$, where $\mathrm{Fil}_\bullet$ denotes the increasing filtration corresponding (via the Rees construction) to $E|_{[\mathbf{A}^1_-/\mathbf{G}_m]}$. We then shows in Lemma 3.30 below that $\mathrm{Fil}_\bullet$ is nothing but the conjugate filtration (2.1.1). This finishes the construction of D.

- It is known that pulling back along the map $\overline{\rho}_{(\mathfrak{S},I)}$ lifts to an equivalence between quasi-coherent sheaves on $\mathcal{O}_K^{HT}$ and p-complete modules over $\mathcal{O}_K$ equipped with a so-called Sen operator Θ (see Theorem 3.26 below). In Lemma 3.28 below, we will show that under this equivalence, the restriction $E|_{\mathcal{O}_K^{HT}}$ corresponds to the module $\varinjlim \mathrm{Fil}_i^{\mathrm{conj}}$ underlying the conjugate filtration equipped with the Sen operator given by $\Theta = uD - iE'(\pi)$ on $\mathrm{Fil}_i^{\mathrm{conj}}$. This explains the above construction of Θ in terms of D, as given in (2.1.3).

□

3. Identifications with the stacky picture

In this section, we justify the various identifications with the stacky picture, as alluded to in Subsection 2.1. In particular, we extend the explicit description of the Hodge–Tate locus given in [3, Prop. 5.3.7] from the case $K = \mathbf{Q}_p$ to the case of a general extension $K/\mathbf{Q}_p$; see Proposition 3.20 below. As mentioned earlier, the materials here will not surprise an expert; however since the proofs are not available in the literature, we work them out here for completeness.

3.1 Preliminaries

We begin by briefly recalling a few basic objects and facts from [3] that will be important in what follows; for a detailed account, we refer the reader to *loc. cit.*

3.1.1 The prismatization

Definition 3.1. A Cartier–Witt divisor on a p-nilpotent ring S is a generalized Cartier–Witt divisor $I \xrightarrow{\alpha} W(S)$ such that

(i) the ideal generated by the image of $I \xrightarrow{\alpha} W(S) \xrightarrow{\gamma_0} S$ is nilpotent, and

(ii) the image of the map $I \xrightarrow{\alpha} W(S) \xrightarrow{\delta} W(S)$ generates $W(S)$.

For a Cartier–Witt divisor $I \to W(S)$, we write $W(S)/I := \mathrm{Cone}(I \to W(S))$; this is naturally a 1-truncated animated $W(S)$-algebra which is moreover p-nilpotent by [5, Lem. 3.3].

Definition 3.2. For a bounded p-adic formal scheme X, the *prismatization of* X, denoted X^{Δ}, is the stack over $\mathrm{Spf}(\mathbf{Z}_p)$ taking a p-nilpotent ring S to the groupoid consisting of Cartier–Witt divisors $I \to W(S)$ together with a map $\mathrm{Spec}(W(S)/I) \to X$ of derived formal schemes (with obvious morphisms). If $X = \mathrm{Spf}(R)$ is affine, we simply write R^{Δ} for X^{Δ}.

Remark 3.3. By construction, $\mathbf{G}_a^{\Delta}$ is the ring stack over $\mathbf{Z}_p^{\Delta}$ taking a Cartier–Witt divisor $I \to W(S)$ to the quotient $W(S)/I$. For a general X, the natural map $X^{\Delta} \to \mathbf{Z}_p^{\Delta}$ of stacks then realizes X^{Δ} as the so-called *transmutation* of $\mathbf{Z}_p^{\Delta}$ from the ring stack $\mathbf{G}_a^{\Delta}$. See [3, Remark 2.3.8].

Remark 3.4. If $I \xrightarrow{\alpha} W(S)$ is a Cartier–Witt divisor on a p-nilpotent ring S, then $F^*I \xrightarrow{F^*(\alpha)} W(S)$ is also a Cartier–Witt divisor (where F is the Witt vector Frobenius on $W(S)$). Moreover, $F : W(S) \to W(S)$ induces a map $W(S)/I \to W(S)/F^*I$ of animated rings. From this, one obtains for any bounded p-adic formal scheme X a natural map $F : X^{\Delta} \to X^{\Delta}$, called the Frobenius on X^{Δ}.

Construction 3.5. Let $(\mathrm{Spf}(A) \hookleftarrow \mathrm{Spf}(A/I) \to X)$ be an object in the absolute prismatic site X_{Δ}. Let S be a (p, I)-nilpotent A-algebra. As usual, the structure map $A \to S$ lifts uniquely to a δ-map $A \to W(S)$. The generalized Cartier divisor $I \otimes_A W(S) \xrightarrow{can} W(S)$ is then easily checked to be a Cartier–Witt divisor. Letting S vary, this construction yields a map

$$\rho_{(A,I)} : \mathrm{Spf}(A) \to X^{\Delta}$$

of stacks. One checks easily that $\rho_{(A,I)}$ is also Frobenius equivariant.

3.1.2 The filtered prismatization

Construction 3.6. Let M be a W-module scheme over a p-nilpotent ring S.

(1) The functor $I \mapsto I \otimes_{W(S)} W$ gives an equivalence from the category of invertible $W(S)$-modules onto that of invertible W-modules.

(2) M is called an invertible F_*W-module if it is locally isomorphic to the W-module F_*W. The functor $M' \mapsto F_*M'$ then gives an equivalence from the category of invertible W-modules onto that of invertible F_*W-modules.

(3) M is called $\sharp$-invertible if it is locally isomorphic to $\mathbf{G}_a^{\sharp}$. The functor $L \mapsto \mathbf{V}(L)^{\sharp}$ then gives an equivalence from the category of invertible S-modules onto that of $\sharp$-invertible W-modules.

Remark 3.7. Let S be a p-nilpotent ring. Via the functor $I \mapsto I \otimes_{W(S)} W$ from Construction 3.6 (1), we can and will identify the category of Cartier–Witt divisors $I \to W(S)$ on S as a full subcategory of the category of pairs $(M, M \xrightarrow{d} W)$, where M is an invertible W-module scheme and d is a W-linear map of W-module schemes. Note that if $(I \to W(S)) \mapsto (M := I \otimes_{W(S)} W \to W)$ under this identification, then $W(S)/I$ identifies naturally with $(W/M)(S) := R\Gamma(\mathrm{Spec}(S), W/M)$ as animated $W(S)$-algebras.

Definition 3.8. Let S be a p-nilpotent ring. An *admissible* W-module over S is an affine W-module scheme M which can be written as an extension

$$0 \to \mathbf{V}(L_M)^{\sharp} \to M \to F_*M' \to 0 \tag{3.8.1}$$

of an invertible F_*W-module by a $\sharp$-invertible module.

It turns out that the exact sequence (3.8.1) is uniquely determined by the underlying W-module M, which we can therefore refer to as the admissible sequence of M. More precisely, any map $M \to N$ of admissible W-modules lifts (necessarily uniquely) to a map (in the obvious sense) of admissible sequences.

Definition 3.9 ([3, Defn. 5.3.1])**.** A *filtered Cartier–Witt divisor* over a p-nilpotent ring S is a map $M \xrightarrow{d} W$ of admissible W-modules such that the induced map $F_*M' \to F_*W$ of associated invertible F_*W-modules comes from (after undoing F_*) a Cartier–Witt divisor over S (cf. Remark 3.7).

By [3, Cor. 5.3.9], given a filtered Cartier–Witt divisor $M \xrightarrow{d} W$ on a p-nilpotent ring S, $(W/M)(S) := R\Gamma(\mathrm{Spec}(S), W/M)$ is naturally a 1-truncated animated $W(S)$-algebra.

Definition 3.10 ([3, Defn. 5.3.10])**.** Let X be a bounded p-adic formal scheme. The *filtered prismatization of* X is the stack $X^{\mathcal{N}}$ over $\mathrm{Spf}(\mathbf{Z}_p)$ taking a p-nilpotent ring S to the groupoid consisting of filtered Cartier–Witt divisors $M \xrightarrow{d} W$ on S together with a map $\mathrm{Spf}((W/M)(S)) \to X$ of derived p-adic formal schemes. (Again, if $X = \mathrm{Spf}(R)$ is affine, we simply write $R^{\mathcal{N}}$ for $X^{\mathcal{N}}$, etc.)

As explained above, given a filtered Cartier–Witt divisor $M \xrightarrow{d} W$ on a p-nilpotent ring S, we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{V}(L_M)^\sharp \arrow[r] \arrow[d, "t(d)^\sharp"] & M \arrow[r] \arrow[d, "d"] & F_*M' \arrow[r] \arrow[d, "F_*(d')"] & 0 \\ 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0 \end{tikzcd} \tag{3.10.1}$$

for unique maps $t(d) : L_M \to S$ and $d' : M' \to W$.

Construction 3.11. Sending $d \mapsto t(d)$ defines a map of stacks

$$t : X^{\mathcal{N}} \to \mathbf{A}^1_+/\mathbf{G}_m$$

which we call the *Rees map*[3].

Construction 3.12. By definition, the map $d' : M' \to W$ is a Cartier–Witt divisor. Moreover, the right square of (3.10.1) gives a map $(W/M)(S) \to (W/M')(S)$ of animated rings[4]. Thus we obtain a map of stacks

$$\pi : X^{\mathcal{N}} \to X^{\Delta}$$

defined by sending a point $(M \xrightarrow{d} W, \mathrm{Spf}((W/M)(S)) \to X)$ of $X^{\mathcal{N}}(S)$ to the Cartier–Witt divisor d' together with the composition $\mathrm{Spec}((W/M')(S)) \to \mathrm{Spf}((W/M)(S)) \to X$. We will refer to π as the *structure map.*

Construction 3.13. Given a p-nilpotent ring S and a Cartier–Witt divisor $I \to W(S)$, the induced map $I \otimes_{W(S)} W \to W$ is a filtered Cartier–Witt divisor whose source $I \otimes_{W(S)} W$ is invertible. Conversely, if $M \xrightarrow{d} W$ is a filtered Cartier–Witt divisor with M invertible, then it arises from this construction. In this way, we obtain an embedding

$$j_{HT} : X^{\Delta} \hookrightarrow X^{\mathcal{N}},$$

which turns out to be an open immersion. It is also easy to check that $\pi \circ j_{HT} = F$ is the Frobenius on X^{Δ}.

[3]The notation is consistent with our convention that t denotes the coordinate on $\mathbf{A}^1_+$.

[4]More precisely, the source and target have natural $W(S)$-algebra structures, and the map is linear over $F : W(S) \to W(S)$.

Construction 3.14. Given a p-nilpotent ring S and a Cartier–Witt divisor $I \xrightarrow{\alpha} W(S)$, one can produce a filtered Cartier–Witt divisor over S by forming the pullback

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & F_*(I \otimes_{W(S)} W) \arrow[r] \arrow[d, "F_*(\alpha)"] & 0 \\ 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0 \end{tikzcd}$$

Moreover, since the left vertical map is an isomorphism, the map $(W/M)(S) \to W(S)/I$ of animated rings (induced by the right square, as saw earlier) is an isomorphism. In this way, we obtain a map of stacks

$$j_{dR} : X^{\Delta} \hookrightarrow X^{\mathcal{N}}$$

which in fact identifies the source with the open substack $(X^{\mathcal{N}})_{t\neq 0} \subseteq X^{\mathcal{N}}$, the preimage of $\mathbf{G}_m/\mathbf{G}_m \subseteq \mathbf{A}^1_+/\mathbf{G}_m$ under the Rees map $t : X^{\mathcal{N}} \to \mathbf{A}^1_+/\mathbf{G}_m$. It is also easy to check that $\pi \circ j_{dR} = \mathrm{id}$ is the identity on X^{Δ}.

Construction 3.15 (The Hodge–Tate locus in $X^{\mathcal{N}}$ and the Hodge–Tate structure map)**.** Recall the Rees map $t : X^{\mathcal{N}} \to \mathbf{A}^1_+/\mathbf{G}_m$ from Construction 3.11. We denote by $(X^{\mathcal{N}})_{t=0} \subseteq X^{\mathcal{N}}$ the preimage of $B\mathbf{G}_m \subseteq \mathbf{A}^1_+/\mathbf{G}_m$, and call it the Hodge–Tate locus in $X^{\mathcal{N}}$. We also denote by $X^{HT} \subseteq X^{\Delta}$ the preimage of $(X^N)_{t=0}$ under the map $j_{HT} : X^{\Delta} \hookrightarrow X^{\mathcal{N}}$.

Let S be a p-nilpotent ring. Assume that $M \xrightarrow{d} W$ is a filtered Cartier–Witt divisor contained in the Hodge–Tate locus $(\mathbf{Z}_p^{\mathcal{N}})_{t=0}(S)$. By the proof of [3, Prop. 5.3.7], d factors as $M \twoheadrightarrow F_*W \xhookrightarrow{V} W$ where the first map is part of the admissible sequence of M. In particular, we obtain a natural map $(W/M)(S) \to W(S)/VW(S) \simeq S$ of animated rings. Consequently, given a point $(M \xrightarrow{d} W, \mathrm{Spf}((W/M)(S)) \to X)$ in $(X^{\mathcal{N}})_{t=0}(S)$, one obtains a map $\mathrm{Spec}(S) \to \mathrm{Spf}((W/M)(S)) \to X$. This construction gives a map

$$(X^{\mathcal{N}})_{t=0} \to X$$

which we refer to as the *Hodge–Tate structure map*. One can check that after restricting to the open $X^{HT} \xhookrightarrow{j_{HT}} (X^{\mathcal{N}})_{t=0}$, this recovers the map with the same name from [5, Construction 3.7].

3.1.3 The syntomification

Definition 3.16 ([3, Defn. 6.1.1])**.** Let X be a bounded p-adic formal scheme. We define X^{Syn}, the *syntomification of* X, to be the co-equalizer

$$X^{\mathrm{Syn}} := \mathrm{coeq}\left(X^{\Delta} \underset{j_{HT}}{\overset{j_{dR}}{\rightrightarrows}} X^{\mathcal{N}} \right).$$

of the open embeddings j_{dR} and j_{HT}. The category of *prismatic F-gauges on* X is then defined to be $D_{qc}(X^{\mathrm{Syn}})$. (Again for $X = \mathrm{Spf}(R)$ affine, we simply write R^{Syn} for X^{Syn}, etc.)

3.2 Relation with crystalline Galois representations

We now recall the relation, as discussed in [3, §6], between crystalline Galois lattices and coherent sheaves on $\mathcal{O}_K^{\mathrm{Syn}}$.

Let X be a quasi-syntomic p-adic formal scheme. Recall from Construction 3.5 that for each object $(A, I) \in X_{\Delta}$ in the absolute prismatic site of X, there is an associated morphism $\rho_{(A,I)} : \mathrm{Spf}(A) \to X^{\Delta}$. By [5, Prop. 8.15], pulling back along these maps gives an equivalence

$$\mathrm{Perf}(X^{\Delta}) \simeq \varprojlim_{(A,I)\in X_{\Delta}} \mathrm{Perf}(A) =: \mathrm{Perf}(X_{\Delta}, \mathcal{O}_{\Delta})$$

onto the category of prismatic crystals in perfect complexes on X.

By definition of X^{Syn} as a coequalizer, there is a natural (étale) map $X^{\mathbb{\Delta}} \to X^{\mathrm{Syn}}$. Restricting along this and using the equivalence above, we obtain a functor

$$\mathrm{Perf}(X^{\mathrm{Syn}}) \to \mathrm{Perf}(X^{\mathbb{\Delta}}) \simeq \mathrm{Perf}(X_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}),$$

which in fact naturally lifts to a functor

$$\mathrm{Perf}(X^{\mathrm{Syn}}) \to \mathrm{Perf}^{\varphi}(X_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$$

into the category of prismatic F-crystals in perfect complexes on X. As it will be useful in the proof of Lemma 3.17 below, let us recall the construction of this lift. To this end, note that for any $E \in \mathrm{Perf}(X^{\mathcal{N}})$, there is a natural correspondence

$$j_{HT}^* E \xleftarrow{a} \varphi^* \pi_* E \xrightarrow{b} \varphi^* j_{dR}^* E.$$

Namely, a (resp. b) comes from adjunction and the identity $\pi \circ j_{HT} = \varphi$ (resp. $\pi \circ j_{dR} = \mathrm{id}$). As M is perfect, the maps a and b are in fact I-isogenies (where $I \subseteq \mathcal{O}_{X^{\mathbb{\Delta}}}$ is the Hodge–Tate ideal sheaf), and so we obtain a natural (in E) isomorphism

$$\iota_E : \varphi^*(j_{dR}^* E)[1/I] \simeq j_{HT}^* E[1/I]. \tag{3.16.1}$$

Now lifting E to an object in $\mathrm{Perf}(X^{\mathrm{Syn}})$ amounts to specifying an isomorphism $j_{HT}^* E \simeq j_{dR}^* E$, and so we obtain the desired functor

$$\mathrm{Perf}(X^{\mathrm{Syn}}) \to \mathrm{Perf}^{\varphi}(X_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}).$$

See [3, §6.3] for more details.

Assume now that $X = \mathrm{Spf}(\mathcal{O}_C)$. Let $(A_{\mathrm{inf}}, (\xi))$ be the perfect prism associated to $\mathcal{O}_C$ (where ξ is a generator of the kernel of the (non-twisted) Fontaine's theta map). By the preceding discussion, there is a natural functor

$$\begin{aligned} \mathrm{Perf}(\mathcal{O}_C^{\mathcal{N}}) &\to \{(N, M, \iota) \text{ where } N, M \in \mathrm{Perf}(A_{\mathrm{inf}}) \text{ and } \iota : N[1/\xi] \simeq M[1/\xi]\} \\ E &\mapsto (\varphi^*(j_{dR}^* E), j_{HT}^* E, \iota_E). \end{aligned} \tag{3.16.2}$$

In [3, §6.6.1], Bhatt isolates a subcategory $\mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_C^{\mathcal{N}})$ of $\mathrm{Perf}(\mathcal{O}_C^{\mathcal{N}})$ with the property that the functor (3.16.2) restricts to an equivalence

$$\mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_C^{\mathcal{N}}) \simeq \{(N, M, \iota) \text{ where } N, M \in \mathrm{Vect}(A_{\mathrm{inf}}) \text{ and } \iota : N[1/\xi] \simeq M[1/\xi]\}. \tag{3.16.3}$$

Now by [3, Prop. 5.5.8] (see also [3, Example 5.5.6] for its construction), there is an isomorphism $\mathcal{R}(\varphi^{-1}(\xi)^{\bullet} A_{\mathrm{inf}}) \simeq \mathcal{O}_C^{\mathcal{N}}$, where the LHS denotes the (p, ξ)-completed Rees stack for the Nygaard filtration $\varphi^{-1}(\xi)^{\bullet} A_{\mathrm{inf}}$. Composing further with the isomorphism $\varphi : \varphi^{-1}(\xi)^{\bullet} A_{\mathrm{inf}} \simeq \xi^{\bullet} A_{\mathrm{inf}}$, we obtain an isomorphism

$$\pi_{\mathcal{O}_C} : \mathrm{Spf}(A_{\mathrm{inf}}[u, t]/(ut - \xi))/\mathbf{G}_m \simeq \mathcal{R}(\xi^{\bullet} A_{\mathrm{inf}}) \simeq \mathcal{O}_C^{\mathcal{N}}, \tag{3.16.4}$$

where as usual t is the degree 1 Rees parameter and u has degree -1.

The following observation will be used in the proof of Lemma 3.29 below.

Lemma 3.17. *Let $E \in \mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_C^{\mathcal{N}})$. Then the filtration over $\xi^{\bullet} A_{\mathrm{inf}}$ corresponding to $\pi_{\mathcal{O}_C}^* E$ (via the Rees dictionary) can be recovered from the tuple (N, M, ι) associated to E as $N \cap \xi^{\mathbf{Z}} M = \varphi^*(j_{dR}^* E) \cap \xi^{\mathbf{Z}}(j_{HT}^* E)$ (where the intersection is taken via ι_E). (In particular, this is an honest filtration.)*

Proof. This follows from the construction of the functor G in the proof of [3, Prop. 6.6.3], as well as from comparing the above construction of the isomorphism (3.16.1) and that of the functor F in *loc. cit.* $\square$

Definition 3.18 (cf. [3, Defn. 6.6.4])**.** Define the category $\mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_C^{\mathrm{Syn}})$ of reflexive F-gauges on $\mathcal{O}_C$ to be the full subcategory of $\mathrm{Perf}(\mathcal{O}_C^{\mathrm{Syn}})$ consisting of E's such that $E|_{\mathcal{O}_C^{\mathcal{N}}}$ belongs to $\mathrm{Coh}^{\mathrm{ref}}(\mathcal{O}_C^{\mathcal{N}})$.

By (3.16.3), restricting along $\mathcal{O}_C^{\mathbb{\Delta}} \to \mathcal{O}_C^{\mathrm{Syn}}$ yields an equivalence

$$\mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_C^{\mathrm{Syn}}) \simeq \mathrm{Vect}^{\varphi}((\mathcal{O}_C)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$$

onto the category of prismatic F-crystals in vector bundles on $\mathcal{O}_C$ (cf. [3, Cor. 6.6.5]).

Definition 3.19 (cf. [3, Defn. 6.6.11])**.** An F-gauge $E \in \mathrm{Perf}(\mathcal{O}_K^{\mathrm{Syn}})$ is called reflexive if $E|_{\mathcal{O}_C^{\mathrm{Syn}}}$ is reflexive in the sense of Definition 3.18. Write $\mathrm{Coh}^{\mathrm{refl}}(\mathcal{O}_K^{\mathrm{Syn}})$ for the full subcategory spanned by such F-gauges.

By design, restriction again defines a functor

$$\mathrm{Coh}^{\mathrm{ref}}(\mathcal{O}_K^{\mathrm{Syn}}) \to \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$$

which turns out to be an equivalence by [3, Thm. 6.6.13]. Combining with the main result of [7] then gives an equivalence

$$\mathrm{Coh}^{\mathrm{ref}}(\mathcal{O}_K^{\mathrm{Syn}}) \simeq \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \simeq \mathrm{Rep}^{\mathrm{crys}}_{\mathbf{Z}_p}(G_K).$$

In particular it makes sense to talk about the F-gauge $E \in \mathrm{Coh}^{\mathrm{ref}}(\mathcal{O}_K^{\mathrm{Syn}})$ associated to the given crystalline lattice T.

3.3 Explicit description of the Hodge–Tate locus

The main result in this subsection is Proposition 3.20, which gives an explicit presentation of the Hodge–Tate locus $(\mathcal{O}_K^{\mathcal{N}})_{t=0}$ as a quotient stack. This extends [3, Prop. 5.3.7], which treats the case $\mathcal{O}_K = \mathbf{Z}_p$.

We first recall a general construction from [3, Rem. 5.5.19], which will be used repeatedly in what follows. Namely, given any prism (A, I) and any map $\mathrm{Spf}(A/I) \to X$ of bounded p-adic formal schemes, there is a natural map of filtered stacks

$$\pi_X : \mathcal{R}(I^{\bullet}A) \to X^{\mathcal{N}},$$

where $\mathcal{R}(I^{\bullet}A)$ denotes the (p, I)-completed Rees stack of the I-adic filtration on A. Note that *loc. cit.* seems to assume more than just this data, but this is all that is needed to *construct* the map, as we now recall[5]. Recall that, given a p-nilpotent test ring S, a point $x \in \mathcal{R}(I^{\bullet}A)(S)$ is given by a map $A \to S$ that kills some power of I, a line bundle $L \in \mathrm{Pic}(S)$, and a factorization $I \otimes_A S \xrightarrow{u} L \xrightarrow{t} S$ of the canonical map. As usual, the map $A \to S$ lifts uniquely to give a δ-A-algebra structure on the Witt ring scheme W over S, and one can consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & I \otimes_A \mathbf{G}_a^{\sharp} \arrow[r] \arrow[d, "u^{\sharp}"] \arrow[dd, bend right, "can"'] & I \otimes_A W \arrow[r] \arrow[d] \arrow[dd, bend right, "can"'] & I \otimes_A F_*W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{G}_a^{\sharp} \arrow[r] \arrow[d, "t^{\sharp}"] & M_u \arrow[r] \arrow[d, "d_{u,t}"] & I \otimes F_*W \arrow[r] \arrow[d, "can"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^{\sharp} \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0.
\end{tikzcd} \tag{3.19.1}$$

Note that the middle arrow defines a map $A/I \to (W/M_u)(S)$ of animated rings, so the filtered Cartier–Witt divisor $M_u \xrightarrow{d_{u,t}} W$ naturally lifts to a point $\pi_X(x) \in X^{\mathcal{N}}(S)$, as wanted.

[5]The map π_X depends on the prism (A, I) but we omit it for ease of notation. It will also be clear that the map $\pi_{\mathcal{O}_C}$ from (3.16.4) above is a special case of this construction, justifying our notation.

We now apply this construction to the case $X = \mathrm{Spf}(\mathcal{O}_K)$ and $(A, I) = (W(k)[[x]], E(x))$, our fixed Breuil–Kisin prism. In this case, as explained in *loc. cit.*, the map $\mathcal{R}(I^\bullet A) \xrightarrow{\pi_{\mathcal{O}_K}} \mathcal{O}_K^{\mathcal{N}}$ is in fact a flat cover. The choice of the generator $E(x)$ of I identifies $\mathcal{R}(I^\bullet A)_{t=0} \simeq (\mathbf{A}^1_-/\mathbf{G}_m)_{\mathcal{O}_K}$. Explicitly, a point $(S \xrightarrow{u} L) \in (\mathbf{A}^1_-/\mathbf{G}_m)(S)$ corresponds to the point $I \otimes_A S \xrightarrow{u} L \xrightarrow{t=0} S$ of $\mathcal{R}(I^\bullet A)_{t=0}$; here we view u as a map $I \otimes_A S \to S$ via the trivialization $I = E(x)A \simeq A$.

Thus $\pi_{\mathcal{O}_K}$ restricts to a map

$$\pi_{\mathcal{O}_K} : (\mathbf{A}^1_-)_{\mathcal{O}_K} \to (\mathbf{A}^1_-/\mathbf{G}_m)_{\mathcal{O}_K} \to (\mathcal{O}_K^{N})_{t=0} \tag{3.19.2}$$

of stacks over $\mathrm{Spf}(\mathcal{O}_K)$. Of course this is still a flat cover.

Proposition 3.20. *The map (3.19.2) factors through an isomorphism*

$$(\mathbf{A}^1_-/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \simeq (\mathcal{O}_K^{\mathcal{N}})_{t=0}.$$

of stacks over $\mathrm{Spf}(\mathcal{O}_K)$*. Here the action of* $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ *on* $\mathbf{A}^1_-$ *is given by* $(a, \lambda) \cdot_{\mathcal{O}_K} u := E'(\pi)a + \lambda^{-1}u$[6].

We begin with some preparations. The following is simply an elaboration of [3, Prop. 5.2.1 (2)].

Lemma 3.21 ([3, Prop. 5.2.1])**.** *Applying* $\mathrm{Hom}_W(-, \mathbf{G}_a^\sharp)$ *to the standard sequence* $0 \to I \otimes_A \mathbf{G}_a^\sharp \to I \otimes_A W \to I \otimes_A F_*W \to 0$ *gives an exact sequence*

$$\mathrm{Hom}_W(I \otimes_A W, \mathbf{G}_a^\sharp) \to \mathrm{Hom}_W(I \otimes_A \mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \to \mathrm{Ext}^1_W(I \otimes_A F_*W, \mathbf{G}_a^\sharp) \to 0. \tag{3.21.1}$$

Using the trivialization $I = E(x)A \simeq A$*, we identify*

$$\begin{gathered} \mathbf{G}_a^\sharp(S) \simeq \mathrm{Hom}_W(I \otimes_A W, \mathbf{G}_a^\sharp) \\ a \mapsto (E(x) \otimes w \mapsto wa), \end{gathered}$$

and

$$\begin{gathered} S = \mathbf{A}^1_-(S) \simeq \mathrm{Hom}_W(I \otimes_A \mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \\ u \mapsto (E(x) \otimes a \xrightarrow{u^\sharp} ua); \end{gathered}$$

Under these identifications, the first map in (3.21.1) identifies with the natural map $\mathbf{G}_a^\sharp(S) \to \mathbf{A}^1_-(S) = S$*, and the second map takes a point* $u \in S$ *to the pushout*

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_A \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] & I \otimes_A W \arrow[r] \arrow[d] & I \otimes_A F_*W \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & M_u \simeq \frac{\mathbf{G}_a^\sharp \oplus (I \otimes_A W)}{\{(u^\sharp(x), -x) \mid x \in I \otimes \mathbf{G}_a^\sharp\}} \arrow[r] & I \otimes F_*W \arrow[r] & 0. \end{tikzcd}$$

Furthermore, given $a \in \mathbf{G}_a^\sharp(S)$*, the associated isomorphism* $\iota(a) : M_u \simeq M_{a+u}$ *of extensions is given by*

$$\begin{gathered} \iota(a) : M_u \simeq \frac{\mathbf{G}_a^\sharp \oplus (I \otimes_A W)}{\{(u^\sharp(x), -x) \mid x \in I \otimes \mathbf{G}_a^\sharp\}} \to \frac{\mathbf{G}_a^\sharp \oplus (I \otimes_A W)}{\{(a+u)^\sharp(x), -x) \mid x \in I \otimes \mathbf{G}_a^\sharp\}} \simeq M_{a+u} \\ (x, y) \mapsto (x - a(y), y). \end{gathered} \tag{3.21.2}$$

Lemma 3.22 (A twisted version of [3, Prop. 5.3.7])**.** *The composition*

$$(\mathbf{A}^1_-)_{\mathcal{O}_K} \xrightarrow{\pi_{\mathcal{O}_K}} (\mathcal{O}_K^{\mathcal{N}})_{t=0} \to (\mathbf{Z}_p^{\mathcal{N}})_{t=0} \times \mathrm{Spf}(\mathcal{O}_K)$$

factors through an isomorphism

$$(\mathbf{A}^1_-/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \simeq (\mathbf{Z}_p^{\mathcal{N}})_{t=0} \times \mathrm{Spf}(\mathcal{O}_K). \tag{3.22.1}$$

Here the action of $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ *on* $\mathbf{A}^1_-$ *is given by* $(a, \lambda) \cdot_{\mathbf{Z}_p} u := a + \lambda^{-1}u$*.*

[6] The appearance of λ^{-1} (rather than λ) is simply due to our convention that the coordinate u of $\mathbf{A}^1_-$ has degree -1.

Proof. The proof is similar to that of [3, Prop. 5.3.7], except that one needs to "twist by (A, I)". Let S be a p-nilpotent test $\mathcal{O}_K$-algebra. By construction, the above composition takes a point $u \in \mathbf{A}^1_-(S)$ (viewed as a linear map $I \otimes_A S \xrightarrow{u} S$, as above) to the filtered Cartier–Witt divisor $M_u \xrightarrow{d_u} W$ determined by commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \otimes_A \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] \arrow[dd, bend right, "can"'] & I \otimes_A W \arrow[r] \arrow[d] \arrow[dd, bend right, "can"'] & I \otimes_A F_*W \arrow[r] \arrow[d, equal] \arrow[dd, bend left, "can"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "0"] & M_u \arrow[r] \arrow[d, "d_u"] & I \otimes F_*W \arrow[r] \arrow[d, "can"] \arrow[dll, hook, "V \circ \beta"] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0.
\end{tikzcd}$$

By the proof of [4, Prop. 3.6.6] we have a factorization

$$\begin{tikzcd}
I \otimes_A W \arrow[r, "F"] \arrow[rrr, bend right, "can"'] & I \otimes_A F_*W \arrow[r, "\beta", "\simeq"'] & F_*W \arrow[r, hook, "V"] & W
\end{tikzcd}$$

for some (necessarily unique) isomorphism β[7]. By diagram chasing it then follows that the map $M_u \xrightarrow{d_u} W$ factors as $M_u \twoheadrightarrow I \otimes F_*W \xhookrightarrow{V \circ \beta} W$.

By definition and by the preceding paragraph, an S-point of $\mathbf{A}^1_- \times_{(\mathbf{Z}_p^{\mathcal{N}})_{t=0} \times \mathrm{Spf}(\mathcal{O}_K)} \mathbf{A}^1_-$ is a triple (u, u', ι) where $u, u' \in \mathbf{A}^1_-(S)$ and ι is a W-linear isomorphism $M_u \simeq M_{u'}$ commuting with the maps onto $I \otimes F_*W$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "\lambda^{-1}"', "\simeq"] & M_u \arrow[r] \arrow[d, "\iota"', "\simeq"] & I \otimes F_*W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & M_{u'} \arrow[r] & I \otimes F_*W \arrow[r] & 0.
\end{tikzcd}$$

The induced isomorphism on the left is then of the form $(\lambda^{-1})^\sharp$ for a unique $\lambda \in S^\times$. One can then view ι as an isomorphism *of extensions*

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, equal] & M_{\lambda^{-1}u} \arrow[r] \arrow[d, "\simeq"', "\iota"] & I \otimes F_*W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & M_{u'} \arrow[r] & I \otimes F_*W \arrow[r] & 0.
\end{tikzcd}$$

By Lemma 3.21, we must have $u' = a + \lambda^{-1}u$ for some (unique) $a \in \mathbf{G}_a^\sharp(S)$ and $\iota = \iota(\alpha)$, the isomorphism (3.21.2).

Thus there is an identification

$$\mathbf{A}^1_- \times_{(\mathbf{Z}_p^{\mathcal{N}})_{t=0} \times \mathrm{Spf}(\mathcal{O}_K)} \mathbf{A}^1_- \simeq (\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m) \times \mathbf{A}^1_- \tag{3.22.2}$$

of groupoids over $\mathbf{A}^1_-$, where the action of $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ on $\mathbf{A}^1_-$ is given by $(a, \lambda) \cdot_{\mathbf{Z}_p} u := a + \lambda^{-1}u$. The composition $(\mathbf{A}^1_-)_{\mathcal{O}_K} \xrightarrow{\pi_{\mathcal{O}_K}} (\mathcal{O}_K^{\mathcal{N}})_{t=0} \to (\mathbf{Z}_p^{\mathcal{N}})_{t=0} \times \mathrm{Spf}(\mathcal{O}_K)$ therefore factors through a monomorphism

$$(\mathbf{A}^1_- / \mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \hookrightarrow (\mathbf{Z}_p^{\mathcal{N}})_{t=0} \times \mathrm{Spf}(\mathcal{O}_K).$$

[7] Note that β is not induced by the trivialization $I = E(x)A \simeq A$!

It remains to show that the map is surjective flat locally. Given a p-nilpotent $\mathcal{O}_K$-algebra S, by the proof of [3, Prop. 5.3.7] any S-point $(M \to W) \in (\mathbf{Z}_p^{\mathcal{N}})_{t=0}(S)$ arises as a composition $M \twoheadrightarrow I \otimes F_*W \xhookrightarrow{V\circ\beta} W$ where the first map is part of the extension $0 \to \mathbf{V}(L)^\sharp \to M \to I \otimes F_*W \to 0$ defining the admissible module M. Working locally one can trivialize the line bundle L, and then by Lemma 3.21 this extension arises as the pushout of the standard sequence $0 \to I \otimes_A \mathbf{G}_a^\sharp \to I \otimes_A W \to I \otimes_A F_*W \to 0$ along some map $I \otimes_A \mathbf{G}_a^\sharp \xrightarrow{u} \mathbf{G}_a^\sharp$. This finishes the proof. $\square$

Remark 3.23. We warn the reader that the composition

$$(\mathbf{A}^1_-)_{\mathcal{O}_K} \to (\mathbf{A}^1_-/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \xrightarrow[\simeq]{\pi_{\mathcal{O}_K}} (\mathbf{Z}_p^{\mathcal{N}})_{t=0} \times \mathrm{Spf}(\mathcal{O}_K) \tag{3.23.1}$$

however does not agree with the analogous composition

$$(\mathbf{A}^1_-)_{\mathcal{O}_K} \to (\mathbf{A}^1_-/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \xrightarrow[\simeq]{[3,\ \text{Prop. } 5.3.7]} (\mathbf{Z}_p^{\mathcal{N}})_{t=0} \times \mathrm{Spf}(\mathcal{O}_K) \tag{3.23.2}$$

defined using the isomorphism from [3, Prop. 5.3.7]. In fact, this phenomenon already occurs for $\mathcal{O}_K = \mathbf{Z}_p$ when restricting to the open $j_{HT} : \mathbf{Z}_p^{HT} \hookrightarrow (\mathbf{Z}_p^{\mathcal{N}})_{t=0}$. More precisely, by construction it is easy to see that for $\mathcal{O}_K = \mathbf{Z}_p$ the restriction of (3.23.2) to $\mathbf{Z}_p^{HT} \xhookrightarrow{j_{HT}} (\mathbf{Z}_p^{N})_{t=0}$ is the map $\eta : \mathrm{Spf}(\mathbf{Z}_p) \to \mathbf{Z}_p^{HT}$ given by the Cartier–Witt divisor $W(\mathbf{Z}_p) \xrightarrow{V(1)} W(\mathbf{Z}_p)$. Recall from [3, Prop. 5.1.4] that η induces an isomorphism $B\mathbf{G}_m^\sharp \simeq \mathbf{Z}_p^{HT}$. Similarly, the restriction of (3.23.1) is the map $\overline{\rho}_{(\mathfrak{S},I)} : \mathrm{Spf}(\mathbf{Z}_p) \to \mathbf{Z}_p^{HT}$ associated to our chosen Breuil–Kisin prism $(\mathfrak{S}, I)$ of $\mathbf{Z}_p$. This map turns out to also induce an isomorphism $B\mathbf{G}_m^\sharp \simeq \mathbf{Z}_p^{HT}$ (see e.g. the discussion in Subsection 3.28 below). However, the maps η and $\overline{\rho}_{(\mathfrak{S},I)}$ do not coincide in general. For instance, for $(\mathfrak{S}, I) := (\mathbf{Z}_p[[x]], (x-p))$ one can check that they agree if and only if $p > 2$.

We also note here that the description of $(\mathbf{Z}_p^{\mathcal{N}})_{t=0}$ from [3, Prop. 5.3.7] is completely canonical. This seems to be specific to the case $K = \mathbf{Q}_p$: for a general K our similar description in Proposition 3.20 requires the choice of a uniformizer.

Remark 3.24. For future reference, we record here the isomorphism (of filtered Cartier–Witt divisors)

$$\iota(a,\lambda) : (M_u \xrightarrow{d_u} W) \simeq (M_{(a,\lambda)\cdot_{\mathbf{Z}_p} u} \xrightarrow{d_{(a,\lambda)\cdot_{\mathbf{Z}_p} u}} W),$$

associated to an element $(a,\lambda) \in (\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)(S)$ via the identification (3.22.2) above. It suffices to consider the factors $\mathbf{G}_a^\sharp$ and $\mathbf{G}_m$ separately. First, the isomorphism $\iota(\lambda) : M_u \simeq M_{\lambda^{-1}u}$ for $\lambda \in \mathbf{G}_m(S)$ is simply given by the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "\lambda^{-1}"', "\simeq"] & M_u \arrow[r] \arrow[d, "\simeq"] & I \otimes F_*W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & M_{\lambda^{-1}u} \arrow[r] & I \otimes F_*W \arrow[r] & 0.
\end{tikzcd}$$

Now let $a \in \mathbf{G}_a^\sharp(S)$. Recall that by its construction as a pushout, $M_u \simeq \frac{\mathbf{G}_a^\sharp \oplus (I \otimes_A W)}{\{(u^\sharp(x), -x) | x \in I \otimes \mathbf{G}_a^\sharp\}}$ and similarly for M_{a+u}. The isomorphism $\iota(a) : M_u \simeq M_{a+u}$ is then given by

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, equal] & M_u \simeq \frac{\mathbf{G}_a^\sharp \oplus (I \otimes_A W)}{\{(u^\sharp(x), -x) | x \in I \otimes \mathbf{G}_a^\sharp\}} \arrow[r] \arrow[d, "\simeq"', "(x,y)\mapsto(x-a(y),y)"] & I \otimes F_*W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & M_{a+u} \simeq \frac{\mathbf{G}_a^\sharp \oplus (I \otimes_A W)}{\{((a+u)^\sharp(x), -x) | x \in I \otimes \mathbf{G}_a^\sharp\}} \arrow[r] & I \otimes F_*W \arrow[r] & 0.
\end{tikzcd} \tag{3.24.1}$$

We are now ready to prove Proposition 3.20.

Proof of Proposition 3.20. Inspired by [5, Construction 9.4], we introduce the following auxiliary functor. Let $\mathcal{F}$ be the functor taking a p-nilpotent $\mathcal{O}_K$-algebra S to the set of pairs $(u \in S, \tau)$, where τ is an A-linear map $I \to M_u$ making the diagram

$$\begin{tikzcd} I \arrow[r, "\tau"] \arrow[d] & M_u \arrow[d, "d_u"] \\ A \arrow[r] & W \end{tikzcd} \tag{3.24.2}$$

commute. Here as before $M_u \xrightarrow{d_u} W$ is the filtered Cartier–Witt divisor underlying the image of $u \in S = \mathbf{A}^1_-(S)$ under the map $\pi_{\mathcal{O}_K}$, which (by construction) is determined by the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_A \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] & I \otimes_A W \arrow[r] \arrow[d, "\tau_{u,0}"] \arrow[dd, bend right, "can"'] & I \otimes_A F_*W \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "0"] & M_u \arrow[r] \arrow[d, "d_u"] & I \otimes F_*W \arrow[r] \arrow[d, "can"] \arrow[dl, "V \circ \beta"'] & 0 \\ 0 \arrow[r] & \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0. \end{tikzcd} \tag{3.24.3}$$

Note that the middle column gives a map $I \xrightarrow{\tau_{u,0}} M_u$ which is an instance of the maps τ appearing in the definition of $\mathcal{F}$. This gives a map $(\mathbf{A}^1_-)_{\mathcal{O}_K} \to \mathcal{F}, u \mapsto (u, \tau_{u,0})$. Furthermore the commutative square (3.24.2) can be viewed as a map of quasi-ideals, and thus induces a map $\bar{\tau} : A/I \to (W/M_u)(S)$ of animated rings. The assignment $(u \in S, \tau) \mapsto (M_u \xrightarrow{d_u} W, \bar{\tau})$ then defines a map $\mathcal{F} \to (\mathcal{O}_K^{\mathcal{N}})_{t=0}$ which clearly fits into

$$\begin{tikzcd} (\mathbf{A}^1_-)_{\mathcal{O}_K} \arrow[r] \arrow[rr, bend left, "\pi_{\mathcal{O}_K}"] & \mathcal{F} \arrow[r] & (\mathcal{O}_K^{\mathcal{N}})_{t=0}. \end{tikzcd}$$

In particular the map $\mathcal{F} \to (\mathcal{O}_K^{\mathcal{N}})_{t=0}$ is also a surjection in the flat topology.

We next explain that the preferred element $\tau_{u,0}$ gives an identification $\mathcal{F} \simeq \mathbf{A}^1_- \times \mathbf{G}_a^\sharp$. Indeed, as explained in the proof of Lemma 3.22, the map d_u factors as $M_u \twoheadrightarrow I \otimes F_*W \xhookrightarrow{V \circ \beta} W$; in particular $\ker(d_u)$ identifies with $\mathbf{G}_a^\sharp \hookrightarrow M_u$. Thus the assignment $\tau \mapsto \tau - \tau_{u,0}$ induces (for each fixed $u \in S$) a bijection between the set of τ making diagram (3.24.2) commute and $\mathrm{Hom}_A(I, \mathbf{G}_a^\sharp) \simeq \mathbf{G}_a^\sharp\{-1\} \simeq \mathbf{G}_a^\sharp$ (for the last identification we use again the chosen generator $E(x)$ of I). Under this identification, the map $(\mathbf{A}^1_-)_{\mathcal{O}_K} \to \mathcal{F}$ is simply $\mathbf{A}^1_- \to \mathbf{A}^1_- \times \mathbf{G}_a^\sharp, u \mapsto (u, 0)$.

Now we compute $\mathcal{F} \times_{\mathcal{O}_K^{\mathcal{N}}} \mathcal{F}$. By definition, for a p-nilpotent $\mathcal{O}_K$-algebra S, $(\mathcal{F} \times_{\mathcal{O}_K^{\mathcal{N}}} \mathcal{F})(S)$ is the groupoid of tuples $((u, \tau), (u', \tau'); \iota, \kappa)$ where $(u, \tau), (u', \tau') \in \mathcal{F}(S)$; ι is an isomorphism

$$\begin{tikzcd} M_u \arrow[rr, "\iota", "\simeq"'] \arrow[dr, "d_u"'] & & M_{u'} \arrow[dl, "d_{u'}"] \\ & W & \end{tikzcd}$$

of filtered Cartier–Witt divisors, *and* κ is a homotopy

$$\begin{tikzcd} A/I \arrow[r, bend left=50, "\overline{\iota \circ \tau}", ""{name=U, below}] \arrow[r, bend right=50, "\overline{\tau}'"', ""{name=D, above}] & (W/M_{u'})(S) \arrow[from=U, to=D, Rightarrow, "\kappa"'] \end{tikzcd}$$

from $\overline{\iota \circ \tau}$ to $\overline{\tau}'$ (recall that these are maps of animated rings).

By Lemma 3.22, such isomorphisms ι are precisely of the forms $\iota = \iota(a, \lambda)$ for elements $(a, \lambda) \in (\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)(S)$ such that $u' = (a, \lambda) \cdot_{\mathbf{Z}_p} u$. Moreover, using the explicit description of $\iota(a, \lambda)$ given in Remark 3.23, one checks that, under the identification $\mathcal{F} \simeq \mathbf{A}^1_- \times \mathbf{G}_a^\sharp$ above, the action $((a, \lambda), (u, \tau)) \mapsto ((a, \lambda) \cdot_{\mathbf{Z_p}} u, \iota(a, \lambda) \circ \tau)$ corresponds precisely to the action of $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ on $\mathbf{A}^1_- \times \mathbf{G}_a^\sharp$ given by the above action $\cdot_{\mathbf{Z}_p}$ on $\mathbf{A}^1_-$, and the following action

$$(a, \lambda) * x := -a + \lambda^{-1} x$$

on $\mathbf{G}_a^\sharp$. (Note the minus sign in $-a$(!); this comes precisely from the minus sign appearing in the formula of the isomorphism $\iota(a)$ in (3.24.1).)

It remains to consider the homotopy κ. The collection of such κ's naturally identifies with the set of derivations $D \in \mathrm{Der}(A, \mathbf{G}_a^\sharp)$ such that $D|_I = \tau' - \iota \circ \tau$. This follows from the equivalence between quasi-ideals and DG algebras concentrated in degree $[-1, 0]$ from [9, §3.3]. Note that under the identifications $\{\tau\} \simeq \mathbf{G}_a^\sharp$ above and $\mathrm{Der}(A, \mathbf{G}_a^\sharp) \simeq \mathrm{Hom}_{\mathcal{O}_K}(\Omega^1_A \otimes_A \mathcal{O}_K, \mathbf{G}_a^\sharp) = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K dx, \mathbf{G}_a^\sharp) \simeq \mathbf{G}_a^\sharp$, the action $(D, \tau) \mapsto D|_I + \tau$ corresponds to the action of $\mathbf{G}_a^\sharp$ on $\mathbf{G}_a^\sharp$ given by $a \cdot_{\mathcal{O}_K} x := E'(\pi) a + x$.

In summary, we have shown that the map $\mathcal{F} \to (\mathcal{O}_K^{\mathcal{N}})_{t=0}$ factors through an isomorphism

$$\left(\frac{\mathbf{A}^1_- \times \mathbf{G}_a^\sharp}{\mathbf{G}_a^\sharp \rtimes (\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)} \right)_{\mathcal{O}_K} \simeq (\mathcal{O}_K^{\mathcal{N}})_{t=0}.$$

Here,

- in the formation of the semidirect product $\mathbf{G}_a^\sharp \rtimes (\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$, $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ acts on $\mathbf{G}_a^\sharp$ via the factor $\mathbf{G}_m$ and the multiplication action $(\lambda, a) \mapsto \lambda^{-1} a$ of $\mathbf{G}_m$ on $\mathbf{G}_a^\sharp$;
- the factor $\mathbf{G}_a^\sharp$ acts trivially on $\mathbf{A}^1_-$, and acts on $\mathbf{G}_a^\sharp$ via $a \cdot_{\mathcal{O}_K} x := E'(\pi) a + x$;
- the factor $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ acts on $\mathbf{A}^1_-$ via $(a, \lambda) \cdot_{\mathbf{Z}_p} x := a + \lambda^{-1} x$, and acts on $\mathbf{G}_a^\sharp$ via $(a, \lambda) * x := -a + \lambda^{-1} x$.

Note that the above action $*$ of $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ on $\mathbf{G}_a^\sharp$ is transitive. In fact, under the embedding $\mathbf{G}_m^\sharp \hookrightarrow \mathbf{G}_a^\sharp \rtimes \mathbf{G}_m, \lambda \mapsto (1 - \lambda^{-1}, \lambda)$, the induced action of $\mathbf{G}_m^\sharp$ on $\mathbf{G}_a^\sharp$ is $(\lambda, a) \mapsto \lambda^{-1}(a + 1) - 1$, which is simply transitive (if we identify $\mathbf{G}_m^\sharp = \mathbf{G}_a^\sharp + 1$ inside W, this is nothing but the (inverted) multiplication action of the group scheme $\mathbf{G}_m^\sharp$ on itself).

Thus the map $\mathbf{A}^1_- \to \mathbf{A}^1_- \times \mathbf{G}_a^\sharp, u \mapsto (u, 0)$ induces a surjection

$$\mathbf{A}^1_- \twoheadrightarrow \frac{\mathbf{A}^1_- \times \mathbf{G}_a^\sharp}{\mathbf{G}_a^\sharp \rtimes (\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)}.$$

By unraveling the various actions, one then checks that this factors through an isomorphism

$$(\mathbf{A}^1_- / \mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \simeq \left(\frac{\mathbf{A}^1_- \times \mathbf{G}_a^\sharp}{\mathbf{G}_a^\sharp \rtimes (\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)} \right)_{\mathcal{O}_K} \simeq (\mathcal{O}_K^{\mathcal{N}})_{t=0},$$

where in the LHS the action of $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ on $\mathbf{A}^1_-$ is given by $(a, \lambda) \cdot_{\mathcal{O}_K} x := E'(\pi) a + \lambda^{-1} x$. This finishes the proof. □

Remark 3.25. In fact, the above argument applies to give a similar description of $(R^{\mathcal{N}})_{t=0}$ for a general complete Noetherian regular local ring R with perfect residue field k of characteristic $p > 0$. Namely, let R be such a ring. *Choose* a prism (A, I) and an isomorphism $A/I \simeq R$. Such a choice always exists by the Cohen structure theorem; furthermore A is necessarily formally smooth over $W(k)$. See [5, Notation 9.3][8].

Then the map $\pi_R : \mathcal{R}(I^\bullet A) \to R^{\mathcal{N}}$ induces a natural isomorphism

$$\frac{\mathbf{A}^1_-\{-1\}}{(T_A \otimes_A R)^\sharp \rtimes \mathbf{G}_m} \simeq (R^{\mathcal{N}})_{t=0}$$

of stacks over $\mathrm{Spf}(R)$. Here,

[8] More concretely, as explained in [6, Remark 3.11], (A, I) can be chosen to be of the form $(W(k)[[x_1, \ldots, x_d]], (f))$ with the δ-structure given by the standard one on $W(k)$ and $\delta(x_i) = 0$ for all i, and f a power series whose constant term has p-adic valuation 1.

- in the formation of the semiproduct, $\mathbf{G}_m$ acts on $(T_A \otimes_A R)^\sharp \simeq \mathrm{Hom}_R(\Omega^1_A \otimes_A R, \mathbf{G}_a^\sharp) \simeq \mathrm{Der}(A, \mathbf{G}_a^\sharp)$ by the (inverted) scalar multiplication on $\mathbf{G}_a^\sharp$;
- $\mathbf{G}_m$ acts on $\mathbf{A}^1_-\{-1\} \simeq \mathrm{Hom}_A(I, \mathbf{A}^1_-)$ by the (inverted) scalar multiplication on $\mathbf{A}^1_-$;
- $(T_A \otimes_A R)^\sharp \simeq \mathrm{Der}(A, \mathbf{G}_a^\sharp)$ acts on $\mathbf{A}^1_-\{-1\} \simeq \mathrm{Hom}_A(I, \mathbf{A}^1_-)$ via $(D, \tau) \mapsto D|_I + \tau$.

In the case $R = \mathcal{O}_K$ and $(A, I) = (\mathfrak{S}, (E(x)))$ a Breuil–Kisin prism for R, after trivializing $\Omega^1_A \simeq A \cdot dx$ and $I/I^2 \simeq R \cdot E(x)$ this yields the description from Proposition 3.20. Moreover, this also recovers the description in [5, Prop. 9.5] after restricting to the open $R^{HT} \xhookrightarrow{j_{HT}} (R^{\mathcal{N}})_{t=0}$.

3.4 Identifying the Sen operator

We now check that the lower diagram[9]

$$\begin{tikzcd}
 & (\mathbf{A}^1_-/\mathbf{G}_m)_{\mathcal{O}_K} \arrow[r, "can"] & (\mathbf{A}^1_-/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \arrow[r, "\simeq"] & (\mathcal{O}_K^{\mathcal{N}})_{t=0} \\
\mathrm{Spf}(\mathcal{O}_K) = & (\mathbf{G}_m/\mathbf{G}_m)_{\mathcal{O}_K} \arrow[u, hook, "u\neq 0"] \arrow[r, "can"] \arrow[rr, bend right, "\overline{\rho}_{(\mathfrak{S},I)}"'] & (\mathbf{G}_m/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \arrow[u, hook, "u\neq 0"] \arrow[r, "\simeq"] & \mathcal{O}_K^{HT} \arrow[u, hook, "j_{HT}"]
\end{tikzcd}$$

from (2.1.4) is cartesian. By tracing through definitions, it is easy to see that it commutes. For the cartesian property, note that the middle column of diagram (3.24.3) defines a morphism $(I \otimes_A W \xrightarrow{can} W) \to (M_u \xrightarrow{d_u} W)$ of filtered Cartier–Witt divisors. Since $\mathcal{O}_K^{\Delta} \xhookrightarrow{j_{HT}} \mathcal{O}_K^{\mathcal{N}}$ identifies with the subgroupoid of invertible filtered Cartier–Witt divisors, we have by rigidity [3, Lem. 5.1.5] that, $(M_u \xrightarrow{d_u} W) \in j_{HT}(\mathcal{O}_K^{\Delta})$ if and only if $(I \otimes_A W \xrightarrow{can} W) \to (M_u \xrightarrow{d_u} W)$ is an isomorphism if and only if $I \otimes_A S \xrightarrow{u} S$ is an isomorphism, as wanted. $\square$

Our next goal is to identify the so-called Sen operator corresponding to the restriction $E|_{\mathcal{O}_K^{HT}}$; see Lemma 3.28 below. This will explain our construction of Θ from D, as given in (2.1.3).

To this end, note that since the action $\cdot_{\mathcal{O}_K}$ of $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ (or even just the subgroup $1 \rtimes \mathbf{G}_m$) on $\mathbf{G}_m$ is transitive, the map $\overline{\rho}_{(\mathfrak{S},I)} : \mathrm{Spf}(\mathcal{O}_K) \to \mathcal{O}_K^{HT}$ factors through an isomorphism

$$B(\mathrm{Stab}_{\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m}(1 \in \mathbf{G}_a)) \simeq (\mathbf{G}_m/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \simeq \mathcal{O}_K^{HT}.$$

Let $G_\pi := \mathrm{Stab}_{\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m}(1 \in \mathbf{G}_a)$. By definition

$$G_\pi = \{(a, \lambda) \in \mathbf{G}_a^\sharp \rtimes \mathbf{G}_m \mid E'(\pi)a + \lambda^{-1} = 1\}.$$

Via the projection $(a, \lambda) \mapsto a$, G_π identifies with $\mathbf{G}_a^\sharp$ as a formal scheme. The group structure on G_π then transfers to the operation

$$a \bullet b := a + (1 - E'(\pi)a)b \tag{3.25.1}$$

on $\mathbf{G}_a^\sharp$.

Thus pulling back along $\overline{\rho}_{(\mathfrak{S},I)}$ identifies quasi-coherent sheaves on $\mathcal{O}_K^{HT}$ with p-complete $\mathcal{O}_K$-modules M equipped with a continuous coaction $M \to M \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}(\mathbf{G}_a^\sharp)$, where $\mathbf{G}_a^\sharp \simeq G_\pi$ is equipped with the

[9] Over $\mathrm{Spf}(\mathcal{O}_K)$, $\mathbf{G}_m \subseteq \mathbf{A}^1_-$ is stable under the action $\cdot_{\mathcal{O}_K}$ of $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$, so it really makes sense to consider the quotient $\mathbf{G}_m/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$. Indeed, if R is p-nilpotent and $a \in R$ admits divided powers, then $a^n \in n!R = 0$ for $n \gg 0$, and so $a \cdot_{\mathcal{O}_K} x = E'(\pi)a + x \in R^\times$ for any $x \in R^\times$.

group structure given by (3.25.1) above. For such M, the Sen operator $\Theta_M : M \to M$ is defined as the infinitesimal action of the element $\epsilon \in \mathrm{Lie}(\mathbf{G}_a^\sharp) \subseteq \mathbf{G}_a^\sharp(\mathcal{O}_K[\epsilon])$, i.e. $1+\epsilon\Theta_M$ is given by the composition

$$M \to M \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}(\mathbf{G}_a^\sharp) \xrightarrow{a \mapsto \epsilon} M \otimes_{\mathcal{O}_K} \mathcal{O}_K[\epsilon] = M \oplus \epsilon M$$

(where as before a denotes the coordinate on $\mathbf{G}_a^\sharp$). See [1, §2.2] for more details. Concretely, Θ_M is given by

$$\Theta_M : M \to M \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}(\mathbf{G}_a^\sharp) \xrightarrow{(d/da)|_{a=0}} M \otimes \mathcal{O}_K \simeq M.$$

Theorem 3.26 (cf. [1, Thm. 2.5])**.** *The functor*

$$\begin{aligned} \mathrm{QCoh}(\mathcal{O}_K^{HT}) &\to \mathrm{Mod}_{\mathcal{O}_K[\Theta]} \\ E &\mapsto (\eta^* E, \Theta_{\eta^* E}) \end{aligned}$$

is fully faithful. Its essential image consists of those M which are p-complete and for which the action of $\Theta^p - E'(\pi)^{p-1}\Theta$ on the cohomology $H^\bullet(k \otimes^L_{\mathcal{O}_K} M)$ is locally nilpotent[10]*.*

Lemma 3.27. *We have*

$$D_{\mathrm{qc}}((\mathbf{A}^1_-/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m) \times \mathrm{Spec}(\mathcal{O}_K)) \simeq D_{\mathrm{gr},D\text{-nilp}}(\mathcal{O}_K\{u, D\}/(Du - uD - 1)).$$

*Here "*gr*" means "graded" (where as before* $\deg(u) = -1$ *and* $\deg(D) = +1$*), and "D-*nilp*" is the condition that D is locally nilpotent.*

Note that this implies a similar equivalence over $\mathrm{Spf}(\mathcal{O}_K)$ by restricting to p-complete objects on the RHS and requiring that D is locally nilpotent modulo p.

Proof of the result at the abelian level. We first show that given a $\mathcal{O}_K[u] = \mathcal{O}(\mathbf{A}^1_-)$-module M, the datum of an equivariant action of $\mathbf{G}_a^\sharp$ on M is equivalent to the datum of a locally nilpotent endomorphism $D : M \to M$ satisfying $Du - uD = E'(\pi)$. As in [3, Prop. 2.4.4], giving a coaction $\mu : M \to M \otimes_{\mathcal{O}_K} \mathcal{O}(\mathbf{G}_a^\sharp)$ amounts to giving a locally nilpotent operator $D : M \to M$: given D, the corresponding coaction is $m \mapsto \sum_{i \geq 0} D^i(m) a^i/i!$ (where the infinite sum makes sense by local nilpotence of D). We check that the coaction is equivariant, or equivalently, it is linear over the ring map $\mu : \mathcal{O}_K[u] \to \mathcal{O}_K[u] \otimes_{\mathcal{O}_K} \mathcal{O}(\mathbf{G}_a^\sharp), u \mapsto u + E'(\pi)a$ if and only if d satisfies $Du - uD = E'(\pi)$. For this, we compute

$$\mu(um) = \sum_{i \geq 0} D^i(um) a^i/i!,$$

and

$$\mu(u)\mu(m) = \sum_{i \geq 0} (u + E'(\pi)a) D^i(m) a^i/i!.$$

By comparing the coefficients of $a^i/i!$, one deduces that $\mu(um) = \mu(u)\mu(m)$ if and only if $D^i u - uD^i = E'(\pi) i D^{i-1}$ for all $i \geq 1$ if and only if $Du - uD = E'(\pi)$, as claimed. It remains to incorporate a $\mathbf{G}_m$-action. Recall that giving a coaction $\mu_\lambda : M \to M \otimes_{\mathcal{O}_K} \mathcal{O}(\mathbf{G}_m) = M[\lambda^{\pm 1}]$ is the same as giving a grading $M = \bigoplus_n M^n$: $\mu_\lambda(m) = m\lambda^n$ for $m \in M^n$. Moreover, it is compatible with the action of $\mathbf{G}_m$ on $\mathbf{A}^1_-$ if and only if $u : M \to M$ is homogeneous of degree -1. Thus, we need to show that the two actions of $\mathbf{G}_a^\sharp$ and $\mathbf{G}_m$ extend to an action of $\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m$ if and only if D is homogeneous of degree $+1$.

The compatibility of the two actions is precisely the condition that $h(n(h^{-1}m)) = (h \cdot n)(m)$ for all $h \in \mathbf{G}_m, n \in \mathbf{G}_a^\sharp, m \in M$ and $h \cdot n$ denotes the action of h on $n \in \mathbf{G}_a^\sharp$. We can express the map $(h, n, m) \mapsto h(n(h^{-1}m))$ as

$$\begin{aligned} \mathbf{G}_m \times \mathbf{G}_a^\sharp \times M &\to \mathbf{G}_m \times \mathbf{G}_m \times \mathbf{G}_a^\sharp \times M \to \mathbf{G}_m \times \mathbf{G}_a^\sharp \times M \to \mathbf{G}_m \times M \to M \\ (h, n, m) &\mapsto ((h, h^{-1}), n, m) \mapsto (h, n, h^{-1}(m)) \mapsto (h, n(h^{-1}m)) \mapsto h(n(h^{-1}m)). \end{aligned}$$

[10] There is a similar equivalence for quasi-coherent complexes, but we will only need the result at the abelian level.

Translating in terms of coactions, this is given by the composition

$$\begin{aligned} M &\to M\otimes\mathcal{O}(\mathbf{G}_m)\to M\otimes\mathcal{O}(\mathbf{G}_a^\sharp)\otimes\mathcal{O}(\mathbf{G}_m)\to M\otimes\mathcal{O}(\mathbf{G}_a^\sharp)\otimes\mathcal{O}(\mathbf{G}_m)\otimes\mathcal{O}(\mathbf{G}_m)\to M\otimes\mathcal{O}(\mathbf{G}_a^\sharp)\otimes\mathcal{O}(\mathbf{G}_m) \\ m\in M^n &\mapsto m\otimes\lambda^n\mapsto\sum_i D^i(m)a^i/i!\otimes\lambda^n\mapsto\sum_i\mu_{\lambda'}(D^i(m))a^i/i!\otimes\lambda^n\mapsto\sum_i(\mu_{\lambda'}(D^i(m))|_{\lambda':=\lambda^{-1}})a^i/i!\otimes\lambda^n,\end{aligned}$$

where as above $\mu_t : M\to M[\lambda^{\pm1}]$ records the action of $\mathbf{G}_m$ on M and $\lambda' := \lambda^{-1}$. On the other hand, in terms of coactions, the map $(h,n,m)\mapsto(h\cdot n)(m)$ is given by the composition

$$\begin{aligned} M &\to M\otimes\mathcal{O}(\mathbf{G}_a^\sharp)\to M\otimes\mathcal{O}(\mathbf{G}_a^\sharp)\otimes\mathcal{O}(\mathbf{G}_m) \\ m &\mapsto \sum_i D^i(m)a^i/i!\mapsto\sum_i D^i(m)(\lambda^{-1}a)^i/i!.\end{aligned}$$

Again by comparing coefficients of $a^i/i!$, we see that the two composition maps agree if and only if

$$\lambda^n\mu_{\lambda'}(D^i(m))|_{\lambda':=\lambda}=\lambda^{-i}D^i(m)\Longleftrightarrow\mu_\lambda(D^i(m))=\lambda^{n+i}D^i(m)\in M[\lambda^{\pm1}]$$

for all $m\in M^n$. This happens if and only if $D^i(m)\in M^{n+i}$(recall that $M^n=\mu_\lambda^{-1}(M\lambda^n)$), i.e. D is of homogeneous degree $+1$, as claimed. $\square$

Lemma 3.28 (Identifying the Sen operator)**.** *Consider the open immersion*

$$j_{HT}:\mathcal{O}_K^{HT}\simeq B\mathrm{Stab}_{\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m}(1)\simeq(\mathbf{G}_m/\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m)\hookrightarrow(\mathbf{A}^1_-/\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m)\simeq(\mathcal{O}_K^{\mathcal{N}})_{t=0}.$$

Let E be a quasi-coherent sheaf on $(\mathcal{O}_K^{\mathcal{N}})_{t=0}$. Let M be the graded $\mathcal{O}_K\{u,D\}/(Du-uD-E'(\pi))$-module corresponding to E under the identification in Lemma 3.27. By the Rees construction, M corresponds to an increasing filtration $\mathrm{Fil}_\bullet$ (with transition maps $u:\mathrm{Fil}_i\to\mathrm{Fil}_{i+1}$) of p-complete $\mathcal{O}_K$-modules together with a map $D:\mathrm{Fil}_\bullet\to\mathrm{Fil}_\bullet[-1]$ satisfying $Du-uD=E'(\pi)$. (Explicitly, $\mathrm{Fil}_i=M^{\deg=-i}$[11].)

Then, under the identification in Theorem 3.26, the restriction $E|_{\mathcal{O}_K^{HT}}$ corresponds to the $\mathcal{O}_K$-module given by the underlying non-filtered module $\varinjlim_i\mathrm{Fil}_i$ together with the Sen operator given by $\Theta:=uD-iE'(\pi)$ on Fil_i.

Proof. We will unwind the various identifications. First, the restriction of E to $\mathbf{G}_m/\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m$ corresponds to the graded $\mathcal{O}_K[u,1/u]$-module $M[1/u]$ equipped with the obvious extension of D, i.e., $D(m/u^i):=D(m)/u^i-E'(\pi)im/u^{i+1}$[12]). Now the restriction of E to $B(\mathrm{Stab}_{\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m}(1))$ corresponds to the quotient module $N:=M[1/u]/(u-1)$ together with the induced action of the subgroup $\mathrm{Stab}_{\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m}(1)$.

We need to compute the Sen operator on N in terms of the usual identification (from the Rees dictionary)

$$\begin{aligned} \varinjlim_i\mathrm{Fil}_i &\simeq(M[1/u])^{\deg=0}\simeq M[1/u]/(u-1)=:N \\ m\in\mathrm{Fil}_i &\mapsto m/u^i.\end{aligned}\tag{3.28.1}$$

Recall that $\mathbf{G}_a^\sharp\simeq\mathrm{Stab}_{\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m}(1)$ via $a\mapsto(a,(1-E'(\pi)a)^{-1})$. The induced action of $\mathbf{G}_a^\sharp\simeq\mathrm{Stab}_{\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m}(1)\subseteq\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m$ on $M[1/u]$ is thus given by the composition

$$\begin{aligned} M[1/u]\to M[1/u]\otimes\mathcal{O}(\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m) &\xrightarrow{a\mapsto a,\lambda\mapsto(1-E'(\pi)a)^{-1}} M[1/u]\otimes\mathcal{O}(\mathbf{G}_a^\sharp) \\ m\mapsto\sum_{i\geq0}D^i(m)\tfrac{a^i}{i!}(1-E'(\pi)a)^{-(\deg(m)+i)} &\end{aligned}$$

[11]Recall again our convention that $\deg(u)=-1$.

[12]One checks that D indeed acts locally nilpotently mod p; this is related to the fact we have seen above that $\mathbf{G}_a^\sharp\rtimes\mathbf{G}_m$ preserves $\mathbf{G}_m\subseteq\mathbf{G}_a$ and follows again from the fact that we are working in a p-complete setting.

(where recall that a and λ respectively denote the coordinates on $\mathbf{G}_a^\sharp$ and $\mathbf{G}_m$). After applying $(d/da)|_{a=0}$ we see that the Sen operator on $M[1/u]$ is given by $\Theta_{M[1/u]}(m) = D(m) + E'(\pi)(\deg(m))m$. In particular for $m/u^i \in (M[1/u])^{\deg=0}$ (so that $m \in \mathrm{Fil}_i$), we have

(3.28.2)
$$\Theta_{M[1/u]}(m/u^i) = D(m/u^i) = (uD - E'(\pi)i)(m)/u^{i+1} \equiv (uD - E'(\pi)i)(m)/u^i \bmod (u-1)M[1/u].$$

As the formation of the Sen operator is functorial, we have a commutative square

$$\begin{tikzcd} M[1/u] \arrow[r, "\Theta_{M[1/u]}"] \arrow[d, two heads] & M[1/u] \arrow[d, two heads, "\bmod (u-1)"] \\ N \arrow[r, "\Theta_N"] & N. \end{tikzcd}$$

As $(uD - E'(\pi)i)(m) \in \mathrm{Fil}_i$, it follows from (3.28.2) that, via the identification (3.28.1), the Sen operator on N is given by $\Theta_N = uD - E'(\pi)i$ on Fil_i, as desired. (Equivalently, $\Theta_N = uD$ under the identification $N \simeq (M[1/u])^{\deg=0}$; this is the description used in [3, §6.5.4, second bullet point].) $\square$

3.5 Identifying the Nygaard filtration

Consider again the flat cover

$$\pi_{\mathcal{O}_K} : \mathcal{R}(E(x)^\bullet \mathfrak{S}) \simeq \mathrm{Spf}(W(k)[[x]][u,t]/(ut - E(x)))/\mathbf{G}_m \to \mathcal{O}_K^{\mathcal{N}}.$$

Lemma 3.29. *The filtration over $E(x)^\bullet \mathfrak{S}$ associated (via the Rees dictionary) to the pullback $\pi_{\mathcal{O}_K}^* E$ is precisely the Nygaard filtration $\mathrm{Fil}^\bullet \varphi^* \mathfrak{M} := \varphi^* \mathfrak{M} \cap E(u)^{\mathbf{Z}} \mathfrak{M}$ on $\varphi^* \mathfrak{M}$.*

Proof. We first check that the non-filtered module underlying the filtration $\pi_{\mathcal{O}_K}^* E$ is indeed $\varphi^* \mathfrak{M}$. To see this, note that the restriction of $\pi_{\mathcal{O}_K} : \mathcal{R}(E(x)^\bullet \mathfrak{S}) \to \mathcal{O}_K^{\mathcal{N}}$ to the open locus $j_{dR}(\mathcal{O}_K^{\mathbb{\Delta}}) = (\mathcal{O}_K^{\mathcal{N}})_{t \neq 0}$ identifies the composition

$$\mathrm{Spf}(\mathfrak{S}) \xrightarrow{F \circ \rho_{(\mathfrak{S}, I)}} \mathcal{O}_K^{\mathbb{\Delta}} \xhookrightarrow{j_{dR}} \mathcal{O}_K^{\mathcal{N}}.$$

(Note the Frobenius twist!) This follows easily by unraveling the various constructions. This implies the claim since restricting to the open $\{t \neq 0\}$ amounts to passing to the underlying non-filtered module.

Consider now the commutative square (arising from the map of prisms $(\mathfrak{S}, E(x)) \to (A_{\mathrm{inf}}, \xi), u \mapsto [\pi^\flat]$)

$$\begin{tikzcd} \mathcal{R}(\xi^\bullet A_{\mathrm{inf}}) \arrow[r, "\pi_{\mathcal{O}_C}"] \arrow[d] & \mathcal{O}_C^{\mathcal{N}} \arrow[d] \\ \mathcal{R}(E(x)^\bullet \mathfrak{S}) \arrow[r, "\pi_{\mathcal{O}_K}"] & \mathcal{O}_K^{\mathcal{N}}. \end{tikzcd}$$

By Lemma 3.17 above, the pullback $\pi_{\mathcal{O}_C}^*(E|_{\mathcal{O}_C^{\mathcal{N}}})$ corresponds to the (honest) filtration $\mathrm{Fil}^\bullet(\varphi^* M_{\mathrm{inf}}) := \varphi^* M_{\mathrm{inf}} \cap E(u)^{\mathbf{Z}} M_{\mathrm{inf}}$ on $\varphi^* M_{\mathrm{inf}}$, where $M_{\mathrm{inf}} := \mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$. As the map $\mathfrak{S} \to A_{\mathrm{inf}}$ is (classically) faithfully flat, we have a natural identification $\mathrm{Fil}^\bullet(\varphi^* M_{\mathrm{inf}}) \simeq \mathrm{Fil}^\bullet(\varphi^* \mathfrak{M}) \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$. In summary, we have shown that the filtration given by $\pi^* E$ is an honest filtration on $\varphi^* \mathfrak{M}$, which, after base change along the faithfully flat map $\mathfrak{S} \to A_{\mathrm{inf}}$, agrees with the filtration $\mathrm{Fil}^\bullet(\varphi^* \mathfrak{M}) \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ on $\varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$. Hence it must agree with the filtration $\mathrm{Fil}^\bullet \varphi^* \mathfrak{M}$, as wanted. $\square$

Corollary 3.30. *The increasing filtration corresponding to the pullback of E under the map*

$$(\mathbf{A}^1_-/\mathbf{G}_m) \to (\mathbf{A}^1_-/\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)_{\mathcal{O}_K} \xrightarrow[\simeq]{\pi_{\mathcal{O}_K}} (\mathcal{O}_K^{\mathcal{N}})_{t=0}$$

is precisely the conjugate filtration

$$\mathrm{Fil}^{\mathrm{conj}}_\bullet \mathfrak{M}_{HT}: \quad \ldots \hookrightarrow \underbrace{\mathrm{Fil}^{i-1}\varphi^*\mathfrak{M}/\mathrm{Fil}^i\varphi^*\mathfrak{M}}_{\mathrm{Fil}^{\mathrm{conj}}_{i-1}} \xhookrightarrow{\times E(x)} \underbrace{\mathrm{Fil}^{i}\varphi^*\mathfrak{M}/\mathrm{Fil}^{i+1}\varphi^*\mathfrak{M}}_{\mathrm{Fil}^{\mathrm{conj}}_{i}} \hookrightarrow \ldots.$$

Proof. We have a commutative (even cartesian) diagram

$$\begin{tikzcd}
\mathcal{R}(E(x)^\bullet\mathfrak{S}) \simeq \mathrm{Spf}(W(k)[[x]][u,t]/(ut-E(x)))/\mathbf{G}_m \arrow[r, "\pi_{\mathcal{O}_K}"] & \mathcal{O}_K^{\mathcal{N}} \\
(\mathbf{A}^1_-/\mathbf{G}_m)_{\mathcal{O}_K} \arrow[u, hook, "t=0"] \arrow[r] & (\mathcal{O}_K^{\mathcal{N}})_{t=0}. \arrow[u, hook]
\end{tikzcd}$$

Since the conjugate filtration is by definition the associated graded of the $E(x)^\bullet\mathfrak{S}$-filtration $\mathrm{Fil}^\bullet\varphi^*\mathfrak{M}$, the result follows from Lemma 3.29 because restricting to the closed $\{t=0\}$ amounts to passing to the associated graded. □

3.6 Identifying the Hodge filtration

Lemma 3.31. *Let X be a bounded p-adic formal scheme. Then for any object $(A,I) \in X_{\Delta}$, the diagram*

$$\begin{tikzcd}
\mathrm{Spf}(A/I) \times B\mathbf{G}_m \arrow[r, hook, "t=0"] \arrow[d, hook, "u=0"] & \mathrm{Spf}(A/I) \times \mathbf{A}^1_+/\mathbf{G}_m \arrow[r] \arrow[d, hook, "u=0"] & X \times \mathbf{A}^1_+/\mathbf{G}_m \arrow[d, "i_{dR}"] \\
\mathrm{Spf}(A/I) \times \mathbf{A}^1_-/\mathbf{G}_m \simeq \mathcal{R}(I^\bullet A)_{t=0} \arrow[r, hook] & \mathcal{R}(I^\bullet A) \arrow[r, "\pi_X"] & X^{\mathcal{N}}
\end{tikzcd}$$

commutes. Here the left vertical map is induced by the map of filtered rings $I^\bullet A \to A/I$ where the target has the trivial filtration; and the right vertical map is the de Rham map from [3, Construction 5.3.13].

Proof. Commutativity of the left square is clear. We now show commutativity of the right square after further composing with the map $X^{\mathcal{N}} \to \mathbf{Z}_p^{\mathcal{N}}$; the rest of the proof is left to the reader. Let S be a p-nilpotent test A/I-algebra and let $t: L \to S$ be an S-point of $\mathrm{Spf}(A/I) \times \mathbf{A}^1_+/\mathbf{G}_m$. In terms of the usual moduli description of $\mathcal{R}(I^\bullet A)$, the image of t under the left vertical map corresponds to the factorization $(I \otimes_A S \xrightarrow{u=0} L \xrightarrow{t} S)$ of the natural map. Then by construction of π_X (see diagram (3.19.1)), the image of t under $\mathrm{Spf}(A/I) \times \mathbf{A}^1_+/\mathbf{G}_m \to \mathcal{R}(I^\bullet A) \xrightarrow{\pi_X} X^{\mathcal{N}} \to \mathbf{Z}_p^{\mathcal{N}}$ is the filtered Cartier–Witt divisor

$$V(L)^\sharp \oplus (I \otimes_A F_*W) \xrightarrow{(t^\sharp, V\circ\beta)} W,$$

where as before β is the isomorphism fitting into

$$\begin{tikzcd}
I \otimes_A W \arrow[r, "F"] \arrow[rrr, bend right, "can"'] & I \otimes_A F_*W \arrow[r, "\beta", "\simeq"'] & F_*W \arrow[r, hook, "V"] & W.
\end{tikzcd}$$

Thus $V(L)^\sharp \oplus (I \otimes_A F_*W) \xrightarrow{(t^\sharp, V\circ\beta)} W$ identifies with $V(L)^\sharp \oplus F_*W \xrightarrow{(t^\sharp, V)} W$ as filtered Cartier–Witt divisors. We are done since by definition of the de Rham map, the latter is precisely the image of t under the composition $\mathrm{Spf}(A/I) \times \mathbf{A}^1_+/\mathbf{G}_m \to X \times \mathbf{A}^1_+/\mathbf{G}_m \xrightarrow{i_{dR}} X^{\mathcal{N}} \to \mathbf{Z}_p^{\mathcal{N}}$. □

We apply this for $X = \mathrm{Spf}(\mathcal{O}_K)$ and $(A,I) = (\mathfrak{S}, (E(x)))$, our fixed Breuil–Kisin prism:

Corollary 3.32. *The pullback of E under the de Rham map*

$$(\mathbf{A}^1_+/\mathbf{G}_m)_{\mathcal{O}_K} \xrightarrow{i_{dR}} \mathcal{O}_K^{\mathcal{N}}$$

corresponds (via the Rees dictionary) to the Hodge filtration $\mathrm{Fil}^\bullet_H \mathfrak{M}_{dR}$*. Moreover, there is a natural graded isomorphism*

$$\mathrm{gr}^{\mathrm{conj}}_\bullet \mathfrak{M}_{HT} \simeq \mathrm{gr}^\bullet_H \mathfrak{M}_{dR}.$$

Proof. This follows from Lemma 3.31 and Lemma 3.29 since the Hodge filtration is by definition the image of the Nygaard filtration on $\varphi^*\mathfrak{M}$ under the natural map $\varphi^*\mathfrak{M} \to \varphi^*\mathfrak{M}/E(x)\varphi^*\mathfrak{M} = \mathfrak{M}_{dR}$. □

4. Relation with the classical theory

We finish by briefly indicating a more explicit construction of the operators D and Θ, following the work [10] by Gao–Liu. We refer the reader to *loc. cit.* for additional details.

By the work of Kisin, given a Breuil–Kisin module $\mathfrak{M}$ coming from a crystalline Galois lattice, there is a canonical monodromy operator $N : \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$ over the derivation $N_\nabla := \lambda \frac{d}{dx}$[13] on $\mathcal{O}$. Here as usual $\mathcal{O}$ denotes the ring of functions on the rigid open unit disk over K_0 (in the coordinate x), and $\lambda \in \mathcal{O}$ denotes the element $\prod_{n\geq 0} \varphi^n(E(x)/E(0))$. The construction of N however only uses $\mathfrak{M}[1/p]$, so one may ask if it can be actually defined over an integral variant of $\mathcal{O}$. By the Dwork's trick, φ-modules over $\mathcal{O}$ extends uniquely to $\mathfrak{S}\langle E(x)/p\rangle[1/p]$ (the ring of functions on the closed disc $\{|x| \leq |\pi|\}$) so we can also equivalently consider N as being defined on $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}\langle E(x)/p\rangle[1/p]$ and linear over the derivation $N_\nabla := E(x)\frac{d}{dx}$ on the coefficient ring. Note that there is now an obvious integral candidate, namely $S_{\max} := \mathfrak{S}\langle E(x)/p\rangle$, and one can ask if N in fact extends to $\mathfrak{M} \otimes_{\mathfrak{S}} S_{\max}$. By exploiting integral properties of the G_K-action on $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\inf}$, it is shown in [10] that this is indeed the case. (In [2], Bartlett also proves this result using similar arguments.)

We claim that after extending scalars along the evaluation map $\mathrm{ev}_\pi : S_{\max} \twoheadrightarrow \mathcal{O}_K$, N recovers our Sen operator Θ on $M := \mathfrak{M}_{HT} := \mathfrak{M}/E(x)\mathfrak{M}$. To see this, recall that the map $\mathrm{Spf}(\mathcal{O}_K) \xrightarrow{\overline{\rho}_{(\mathfrak{S},I)}} \mathcal{O}_K^{HT}$ induces an isomorphism $B\mathbf{G}_a^\sharp \simeq \mathcal{O}_K^{HT}$, and our Θ is then defined as the composition

$$\Theta_M : M \to M \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}(\mathbf{G}_a^\sharp) \xrightarrow{(d/da)|_{a=0}} M \otimes \mathcal{O}_K \simeq M,$$

where the first map is the associated coaction map. On the other hand, similar to [4, Prop. 3.2.8], one can show that the natural diagram of stacks

$$\begin{tikzcd} \mathrm{Spf}(\mathfrak{S}^{(1)}/I) \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathrm{Spf}(\mathcal{O}_K) \arrow[d, "\overline{\rho}_{(\mathfrak{S},I)}"] \\ \mathrm{Spf}(\mathcal{O}_K) \arrow[r, "\overline{\rho}_{(\mathfrak{S},I)}"] & \mathcal{O}_K^{HT} \end{tikzcd}$$

is cartesian, where $\mathfrak{S}^{(1)}$ denotes the self-coproduct of $(\mathfrak{S}, I)$ as an object in the absolute prismatic site $(\mathcal{O}_K)_{\Delta}$. Thus, there is an identification $\mathfrak{S}^{(1)}/I \simeq \mathcal{O}(\mathbf{G}_a^\sharp) = \widehat{\bigoplus}_{n\geq 0} \mathcal{O}_K \frac{a^n}{n!}$, and our Θ is also given by the composition

$$M \xrightarrow{can} M \otimes_{\mathcal{O}_K, i_1} (\mathcal{O}_K^{(1)}/I) \simeq M \otimes_{\mathcal{O}_K, i_2} (\mathcal{O}_K^{(1)}/I) = \widehat{\bigoplus_{n\geq 0}} M \frac{a^n}{n!} \xrightarrow{proj} Ma \simeq M,$$

with the middle isomorphism being induced by the descent datum $\mathfrak{M} \otimes_{\mathfrak{S}, i_1} \mathfrak{S}^{(1)} \simeq \mathfrak{M} \otimes_{\mathfrak{S}, i_2} \mathfrak{S}^{(1)}$. This is what is called the "prismatic Sen operator" in [10], and it is shown in Proposition 9.8 of *loc. cit.* that this indeed agrees with the base change of Kisin's operator $N : \mathfrak{M} \otimes_{\mathfrak{S}} S_{\max} \to \mathfrak{M} \otimes_{\mathfrak{S}} S_{\max}$ along the map $ev_\pi : S_{\max} \twoheadrightarrow \mathcal{O}_K$, as claimed.

[13] Kisin considers instead the derivation $\lambda x \frac{d}{dx}$ (with an additional factor x) to accommodate the case of semistable representations, but this will not concern us.

Remark 4.1. An advantage of the stacky approach is that the construction of D and Θ works uniformly in the uniformizer π. In contrast, the argument in [10] (which makes crucial use of the Galois action on $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$) requires some additional care in the case $p = 2$ (due to the usual issue that $\pi^{1/p}$ may belong to $K(\zeta_{p^\infty})$ in this case).

Above we have used Kisin's operator N_∇, but one can also use the monodromy operator in Breuil's theory to construct Θ (or equivalently D). We refer the reader to [12, Lem. 2.3] for more details. An important difference here is that unlike Kisin's N_∇, Breuil's monodromy operator reduces to D (rather than to Θ).

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