THE 2D EULER EQUATIONS ON SINGULAR DOMAINS

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ABSTRACT. We establish the existence of global weak solutions of the 2D incompressible Euler equations, for a large class of non-smooth open sets. Losely, these open sets are the complements (in a simply connected domain) of a finite number of obstacles with positive Sobolev capacity. Existence of weak solutions with L^p vorticity is deduced from a property of domain continuity for the Euler equations, that relates to the so-called γ-convergence of open sets. Our results complete those obtained for convex domains in [18], or for domains with asymptotically small holes [6, 12].

CONTENTS

1. INTRODUCTION

Our concern in this paper is the existence theory for the 2D incompressible Euler flow: for Ω an open subset of $\mathbb{R}^2$, we consider the equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & t > 0, x \in \Omega \\ \text{div } u = 0, & t > 0, x \in \Omega \end{cases} \tag{1.1}$$

endowed with an initial condition and an impermeability condition at the boundary $\partial\Omega$:

$$u|_{t=0} = u^0, \quad u \cdot n|_{\partial\Omega} = 0. \tag{1.2}$$

As usual, $u(t,x) = (u_1(t,x_1,x_2), u_2(t,x_1,x_2))$ and $p = p(t,x_1,x_2)$ denote the velocity and pressure fields, and the vorticity

$$\operatorname{curl} u := \partial_1 u_2 - \partial_2 u_1$$

plays a crucial role in their dynamics.

The well-posedness of system (1.1)-(1.2) has been of course the matter of many works, starting from the seminal paper of Wolibner for smooth data in bounded domains [20]. For the case of smooth data in the whole plane, respectively in exterior domains, see [14], resp. [7]. In the case where the vorticity is only assumed to be bounded, existence and uniqueness of a weak solution was established by Yudovich in [21]. We quote that the well-posedness result of Yudovich applies to smooth bounded domains, and to unbounded ones under further decay assumptions. Since the work of Yudovich, the theory of weak solutions has been considerably improved, accounting for vorticities that are only in $L^1 \cap L^p$ (see the work of Di Perna and Majda [4]), or that are positive Radon measures in H^{-1} (*cf* the paper of Delort [3]). We refer to the textbook [13] for extensive discussion and bibliography.

A common point in all above studies is that $\partial\Omega$ is at least $C^{1,1}$. Roughly, the reason is the following: due to the non-local character of the Euler equations, these works rely on global in space estimates of u in terms of $\operatorname{curl} u$. These estimates *up to the boundary* involve Biot and Savart type kernels, corresponding to operators such as $\nabla\Delta^{-1}$. Unfortunately, such operators are known to behave badly in general non-smooth domains. This explains why well-posedness results are dedicated to regular domains, with a few exceptions.

Among those exceptions, one can mention the work [18] of M. Taylor related to convex domains Ω. Indeed, it is well known that if Ω is convex, the solution ψ of the Dirichlet problem

$$\Delta\psi = f \text{ in } \Omega, \quad \psi|_{\partial\Omega} = 0$$

belongs to $H^2(\Omega)$ when the source term f belongs to $L^2(\Omega)$, no matter the regularity of the domain. Pondering on this remark, Taylor was able to prove in [18] the *existence of global weak solutions in bounded convex domains*. Nevertheless, this interesting result still leaves aside many situations of practical interest, notably flows around irregular obstacles.

The special case of a flow outside a curve has been partly studied in a recent paper by the second author: [8]. This paper yields the existence of Yudovich like solution of the Euler equations in the exterior of a smooth Jordan arc. However, this work relies heavily on the Joukowski transform, and can not be extended easily to more general domains.

Our ambition in the present paper is to recover the existence of weak solutions with $L^1 \cap L^p$ vorticity, for a large class of non-smooth domains. To do so, we will establish a general property of domain continuity for the Euler equations.

The first part of the paper is devoted to bounded sets. These sets Ω are obtained by retrieving to a simply connected domain $\widetilde{\Omega}$ a finite number of obstacles $\mathcal{C}^1, \dots, \mathcal{C}^k$. More precisely, they can be written as

$$\Omega := \widetilde{\Omega} \setminus \left(\cup_{i=1}^k \mathcal{C}^i\right), \quad k \in \mathbb{N} \tag{1.3}$$

with the following assumptions

- **(H1) (connectedness):** $\widetilde{\Omega}$ is a bounded simply connected domain, $\mathcal{C}^1, \dots, \mathcal{C}^k$ are disjoint connected compact subsets of $\widetilde{\Omega}$.
- **(H2) (capacity):** For all $i = 1..k$, $\operatorname{cap}(\mathcal{C}^i) > 0$, where cap denotes the Sobolev H^1 capacity.

Reminders on the notion of capacity are provided in Appendix A. In particular, our assumptions allow to handle flows around obstacles of positive Lebesgue measure, as well as flows around Jordan arcs or curves. They do not cover the case of point obstacles, which have zero capacity. Let us insist that no regularity is assumed on Ω: exotic geometries, such as the Koch snowflake, can be considered.

Within this setting, it is possible to establish the existence of global weak solutions of the Euler equations with L^p vorticity. More precisely, we consider initial data satisfying

$$u^0 \in L^2(\Omega), \quad \operatorname{curl} u^0 \in L^p(\Omega), \quad \operatorname{div} u^0 = 0, \quad u^0 \cdot n|_{\partial\Omega} = 0, \tag{1.4}$$

for some $p \in]1, \infty]$. Note that, due to the irregularity of Ω, the condition $u^0 \cdot n|_{\partial\Omega} = 0$ has to be understood in a weak sense: for any $\varphi \in C^1_c(\mathbb{R}^2)$,

$$\int_\Omega u^0 \cdot \nabla\varphi = -\int_\Omega \operatorname{div} u^0 \, \varphi = 0. \tag{1.5}$$

Let us stress that this set of initial data is large: we will show later that for any function $\omega^0 \in L^p(\Omega)$, there exists u^0 verifying (1.4) and $\operatorname{curl} u^0 = \omega^0$.

Similarly to (1.5), the weak form of the divergence free and tangency conditions on the Euler solution u will read:

$$\forall \varphi \in \mathcal{D}\left([0, +\infty); C^1_c(\mathbb{R}^2)\right), \quad \int_{\mathbb{R}^+} \int_\Omega u \cdot \nabla\varphi = 0. \tag{1.6}$$

Finally, the weak form of the momentum equation on u will read:

$$\text{for all } \varphi \in \mathcal{D}([0, +\infty[\times\Omega) \text{ with } \operatorname{div} \varphi = 0, \quad \int_0^\infty \int_\Omega \left(u \cdot \partial_t\varphi + (u \otimes u) : \nabla\varphi\right) = -\int_\Omega u^0 \cdot \varphi(0, \cdot). \tag{1.7}$$

Our first main theorem is

Theorem 1. *Assume that Ω is of type* (1.3)*, with (H1)-(H2). Let $p \in (1, \infty]$ and u^0 as in* (1.4)*. Then there exists*

$$u \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \quad \textit{with } \operatorname{curl} u \in L^\infty(\mathbb{R}^+; L^p(\Omega))$$

which is a global weak solution of (1.1)-(1.2) *in the sense of* (1.6) *and* (1.7).

In a few words, our existence theorem will follow from a property of domain continuity for the Euler equations. Namely, we will show that smooth solutions u_n of the Euler equations in smooth approximate domains Ω_n converge to a solution u in Ω. By *approximate domains*, we mean converging to Ω in the Hausdorff topology. These approximate domains, to be built in Section 2, read

$$\Omega_n := \widetilde{\Omega}_n \setminus \left(\cup_{i=1}^k \overline{O^i_n}\right)$$

for some smooth Jordan domains $\widetilde{\Omega}_n$ and O^i_n. A keypoint is the so-called γ-convergence of Ω_n to Ω. All necessary prerequisites on Hausdorff, resp. γ-convergence will be given in Appendix B, resp. Appendix C. The compactness argument will be given in Section 2 ($p = \infty$) and Section 4 (finite p). Further discussion of domain continuity for the Euler equations is provided in Section 5. Possible extension of Theorem 1 to weaker settings (Delort's solutions) is also discussed there.

In the second part of the paper, we consider general exterior domains Ω. We assume that Ω is the exterior of a bounded obstacle with positive capacity. It reads

$$\Omega := \mathbb{R}^2 \setminus \mathcal{C} \tag{1.8}$$

with

- **(H1') (connectedness):** $\mathcal{C}$ is a connected compact set.
- **(H2') (capacity):** $\operatorname{cap}(\mathcal{C}) > 0$.

Let us point out that to work with square integrable velocities in exterior domains is too restrictive. Therefore, we relax the condition (1.4) on the initial data into

$$u^0 \in L^2_{\text{loc}}(\overline{\Omega}), \quad u^0 \to 0 \text{ as } |x| \to +\infty, \quad \operatorname{curl} u^0 \in L^p(\Omega), \quad \operatorname{div} u^0 = 0, \quad u^0 \cdot n|_{\partial\Omega} = 0. \tag{1.9}$$

We make the additional assumption that

$$\operatorname{curl} u^0 \text{ is supported in a compact subset of } \mathbb{R}^2 \tag{1.10}$$

which is classical in this context. We prove in Sections 3 and 4 the following result:

Theorem 2. *Assume that* Ω *is of type* (1.8), *with (H1')-(H2'). Let* $p \in (2,\infty]$ *and* u^0 *satisfying* (1.9)-(1.10). *Then, there exists*

$$u \in L^\infty_{\rm loc}(\mathbb{R}^+; L^2_{\rm loc}(\overline{\Omega})), \quad \textit{with} \ \operatorname{curl} u \in L^\infty(\mathbb{R}^+; L^1 \cap L^p(\Omega))$$

which is a global weak solution of (1.1)-(1.2) *in the sense of* (1.6) *and* (1.7).

Again, the weak solution u is obtained from the compactness of a sequence of smooth solutions u_n in the approximate domains $\Omega_n := \mathbb{R}^2 \setminus \overline{O_n}$. The special case $p=\infty$ will be treated with full details in Section 3. The extension to finite p will be sketched in Section 4.

Note that our analysis improves the recent results obtained by the second author for the exterior of a Jordan arc. We improve both the result in [8] (existence of solution outside a Jordan arc) and [9] (continuity property for a special class of approximating obstacles $\overline{O_n}$): we treat more shapes than just C^2 Jordan arcs, and our convergence of Ω_n to Ω is expressed through the Hausdorff distance, which is more general and simple than the conditions in [9]. Therein, one needs stringent convergence properties of the biholomorphisms that map Ω_n to the set $\{|z|>1\}$. In particular, to obtain the uniform convergence of the first derivatives requires the convergence of the tangent angles of ∂O_n. We refer to [9] for detailed statements.

We point out that the limit dynamics in Theorem 2 is expressed differently than in [8]. Indeed, in this article, extensions $\tilde{u}_n$ of u_n to the whole plane are considered, resulting in a modified Euler system in the whole plane at the limit. This system is expressed in vorticity form, and reads

$$\partial_t \omega + u\cdot\nabla\omega = 0, \quad \omega := \operatorname{curl} u - g_\omega \delta_{\mathcal{C}}, \quad t>0,\ x\in\mathbb{R}^2, \tag{1.11}$$

with an additional Dirac mass along the curve. The equivalence between (1.11) and the standard formulation (1.7) of our theorem will be discussed in Section 5. In particular, it is proved in [8] that the velocity blows up near the end-points like the inverse of the square root of the distance, which belongs to $L^p_{\rm loc}$ for $p<4$. Here, we will obtain some uniform estimates of the velocity in $L^2_{\rm loc}$ (see (3.14)-(3.16)) which are in agreement with the former estimates.

The general idea here is to get uniform L^2 estimates for the velocity (i.e. H^1 for the Laplace problem). Combined with L^p bounds on the vorticity, they will allow us to establish the existence of weak solutions. As we will show, such uniform local estimates do not require any assumption on the regularity of the boundary, but they require that the obstacles have a non zero capacity, which means that we cannot treat the material points. Indeed, in the case of an obstacle which shrinks to a point P, Iftimie, Lopes-Filho and Nussenzveig-Lopes in [6] (one obstacle in $\mathbb{R}^2$) and Lopes-Filho in [12] (several obstacles in a bounded domain) proved that one has at the limit a term like $x^\perp/(2\pi|x|^2)$ centered at the point P, which is not square integrable. Therefore, the assumption of positive capacity appears to be necessary for a L^2 approach, and the goal of this article is to prove that it is sufficient to establish the existence.

Let us finally insist that even for Yudovich type solutions ($p=\infty$), we only deal with global existence, not uniqueness. The classical proof of uniqueness requires accurate velocity estimates in $W^{1,p}(\Omega)$ for any $p<\infty$. Unfortunately, the velocity only belongs to $W^{1,p}$ for $p<4/3$ if $\partial\Omega$ admits a bad corner (e.g. the exterior of a curve) and the uniqueness in singular domains appears to be a hard issue, that has been so far only resolved in special cases: see [10].

2. Theorem 1 for $p=\infty$

This section is devoted to the proof of Theorem 1, in the case $p=\infty$. Our starting point is the approximation of Ω by smooth domains Ω_n.

2.1. Approximation procedure.

We state

Proposition 1. *Let* Ω *of type* (1.3), *satisfying (H1). Then,* Ω *is the Hausdorff limit of a sequence*

$$\Omega_n := \widetilde{\Omega}_n \setminus \left(\cup_{i=1}^k \overline{O^i_n}\right).$$

where $\widetilde{\Omega}_n$ *and the* O_n^i*'s are smooth Jordan domains, and such that* $\widetilde{\Omega}_n$*, resp.* $\overline{O_n^i}$*, converges in the Hausdorff sense to* $\widetilde{\Omega}$*, resp.* $\mathcal{C}^i$*.*

Sketch of proof: Let U be a bounded simply connected domain. By the Riemann mapping theorem, there exists a unique biholomorphism $\mathcal{T} : \{|z| < 1\} \mapsto \widetilde{\Omega}$ satisfying $\mathcal{T}'(0) > 0$. The sets $U_n := \mathcal{T}(\{|z| < 1 - 1/n\})$ are smooth Jordan domains, with (U_n), resp. $\overline{U_n}$ converging respectively to U and $\overline{U}$. In particular, applying this argument with $U = \widetilde{\Omega}$ yields the sequence $(\widetilde{\Omega}_n)$ from the lemma. To conclude the proof, it remains to show that any connected compact set $\mathcal{C}$ can be approximated in the Hausdorff topology by the closure of a bounded simply connected domain. Clearly, $\mathcal{C}$ can be approximated by the closure of a non-disjoint and finite union of open disks $O = \cup D_i$ (the fact that the union is not disjoint comes from the connectedness of $\mathcal{C}$). Then, to approximate O by a simply connected domain, one makes slits in the disks, connecting the gaps left by the disks either to each other or to the outside. Details are left to the reader. □

We then need to approximate the initial data u^0, to generate some strong Euler solution in Ω_n. We proceed as follows. Let $\omega^0 := \operatorname{curl} u^0$. By truncation and convolution, there exists some sequence $\omega_k^0 \in C_c^\infty(\Omega)$ such that

$$\omega_k^0 \to \omega^0 \quad \text{weakly in } L^p(\Omega), \quad \|\omega_k^0\|_{L^p} \le \|\omega^0\|_{L^p}, \quad \forall p \in [1, \infty].$$

As $(\Omega_n)_{n\in\mathbb{N}}$ converges to Ω in the Hausdorff sense, it follows from Proposition 10 that $\omega_k^0 \in C_c^\infty(\Omega_{n_k})$ for n_k large enough. Hence, up to extract a subsequence from $(\Omega_n)_{n\in\mathbb{N}}$ one can assume

$$\omega_n^0 \in C_c^\infty(\Omega_n), \quad \forall n \in \mathbb{N}.$$

To uniquely determine a velocity field u_n^0 from ω_n^0, we still need to specify the circulation around each obstacle.

First, we introduce some cutoff functions. For all $i = 1 \dots k$ and $\varepsilon > 0$, we denote $\mathcal{C}^{i,\varepsilon} := \{x, d(x, \mathcal{C}^i) \le \varepsilon\}$ the ε-neighborhood of $\mathcal{C}^i$. Let $\chi^{i,\varepsilon} \in C_c^\infty(\mathbb{R}^2)$ smooth functions satisfying

$$\chi^{i,\varepsilon} = 1 \text{ on } \mathcal{C}^{i,\varepsilon}, \quad \chi^{i,\varepsilon} = 0 \text{ on } \mathbb{R}^2 \setminus \mathcal{C}^{i,2\varepsilon}. \tag{2.1}$$

By Assumptions (H1)-(H2), there exists $\varepsilon > 0$ and $n_0 = n_0(\varepsilon)$ such that

$$\chi^{i,\varepsilon} = 1 \text{ on } \overline{O_n^i}, \quad \chi^{i,\varepsilon} = 0 \text{ on } \overline{O_n^j}, \quad j \neq i, \quad \chi^{i,\varepsilon} = 0 \text{ on } \partial\widetilde{\Omega}_n, \quad \text{for all } n \ge n_0.$$

For brevity, we drop the upperscript ε.

Then, we define the weak circulation of u^0 around $\mathcal{C}^j$

$$\gamma^j = \gamma^j(u^0) := -\int_\Omega \omega^0 \chi^j - \int_\Omega u^0 \cdot \nabla^\perp \chi^j. \tag{2.2}$$

By standard results related to the Hodge-De Rham theorem, there exists a unique field $u_n^0 \in C_c^\infty(\overline{\Omega_n})$ satisfying

$$\operatorname{curl} u_n^0 = \omega_n^0, \quad \operatorname{div} u_n^0 = 0, \quad u_n^0 \cdot n|_{\partial\Omega_n} = 0, \quad \int_{\partial O_n^i} u_n^0 \cdot \tau ds = \gamma^i,$$

where τ denotes the unit tangent vector rotating counterclockwise.

We take (u_n^0) as our sequence of initial data. We shall postpone the convergence of u_n^0 to u^0 to the end of the section. We consider for all n the unique smooth solution u_n of the Euler equations in Ω_n, such that

$$u_n \cdot n|_{\partial\Omega_n} = 0, \quad u_n|_{t=0} = u_n^0.$$

Again, from classical results related to the Hodge-De Rham theorem, the divergence-free smooth fields u_n satisfy in Ω_n

$$u_n(t,x) = \nabla^\perp \psi_n^0(t,x) + \sum_{i=1}^k \alpha_n^i(t) \nabla^\perp \psi_n^i(x) \tag{2.3}$$

where ψ_n^0 satisfies the Dirichlet problem

$$\Delta \psi_n^0 = \omega_n := \operatorname{curl} u_n \text{ in } \Omega_n, \quad \psi_n^0|_{\partial\Omega_n} = 0 \tag{2.4}$$

whereas ψ_n^i, $i = 1 \dots k$ are harmonic functions satisfying

$$\Delta\psi_n^i \;=\; 0 \;\text{ in }\; \Omega_n, \quad \frac{\partial \psi_n^i}{\partial \tau}|_{\partial\Omega_n} \;=\; 0, \quad \int_{\partial O_n^j} \frac{\partial \psi_n^i}{\partial n} = -\delta_{ij}, \quad \psi_n^i|_{\partial\widetilde{\Omega}_n} = 0, \tag{2.5}$$

where δ_{ij} is the Kronecker symbol and n denotes the unit vector pointing outside Ω_n. Note that α_n^i, $i = 1 \dots k$ only depends on time (the formula will be given in Proposition 2).

We refer to [7] and [12] for all details. The key point in proving Theorem 1 is to obtain some compactness on u_n through the study of the ψ_n^i's.

2.2. Study of the harmonic part.

We first focus on the harmonic part of u_n, that is the sum at the r.h.s. of (2.3). Note that the harmonic functions ψ_n^i, $i = 1 \dots k$ are defined up to an additive constant. We fix this constant by imposing

$$\psi_n^i = 0 \;\text{ on }\; \partial\widetilde{\Omega}_n.$$

We then introduce the auxiliary harmonic functions ϕ_n^i, $i = 1 \dots k$, that satisfy

$$\Delta\phi_n^i = 0, \quad \phi_n^i|_{\partial\widetilde{\Omega}_n} = 0, \quad \phi_n^i|_{\partial O_n^j} = \delta_{ij}, \quad j = 1 \dots k. \tag{2.6}$$

Clearly, one can decompose each ψ_n^i on the ϕ_n^j's:

$$\psi_n^i \;=\; \sum_{j=1}^k c_n^{i,j}\, \phi_n^j. \tag{2.7}$$

Our ambition in this section is to prove the convergence of the ϕ_n^j's, the $c_n^{i,j}$'s and the α_n^i's as n goes to infinity.

Using the cutoff function introduced in (2.1), we notice that the function $\Phi_n^i \;:=\; \phi_n^i - \chi^i$ satisfies

$$\Delta\Phi_n^i = -\Delta\chi^i \quad \text{in }\; \Omega_n, \quad \Phi_n^i|_{\partial\Omega_n} = 0.$$

Let D some big open ball containing all the Ω_n's. We can use Proposition 12: as $(\Omega_n)_{n\in\mathbb{N}}$ converges to Ω in the Hausdorff sense and the complement in D of Ω_n has at most $k+1$ connected components for all n, $(\Omega_n)_{n\in\mathbb{N}}$ γ-converges to Ω. We deduce that Φ_n^i converges in $H_0^1(D)$ to the solution $\Phi^i \in H_0^1(\Omega)$ of

$$\Delta\Phi^i = -\Delta\chi^i \quad \text{in }\; \Omega, \quad \Phi^i|_{\partial\Omega} = 0.$$

Setting $\phi^i \;:=\; \Phi^i + \chi^i$, we have for $i = 1 \dots k$ the convergence of ϕ_n^i to ϕ^i strongly in $H_0^1(D)$.

Let us now turn to the convergence of the constants $c_n^{i,j}$. We take the normal derivative at both sides of (2.7) and integrate along ∂O_n^m, for $m \in \{1, \dots k\}$. We obtain thanks to (2.5) and (2.6):

$$-\delta_{im} \;=\; \sum_{j=1}^k c_n^{i,j} \int_{\partial O_n^m} \frac{\partial \phi_n^j}{\partial n} \;=\; \sum_{j=1}^k c_n^{i,j} \int_{\Omega_n} \nabla\phi_n^j \cdot \nabla\phi_n^m.$$

Introducing the $k \times k$ identity matrix Id, this last line reads:

$$-Id \;=\; C_n\, P_n, \qquad \text{with }\; C_n = \left(c_n^{i,j}\right)_{1\le i,j\le k}, \quad P_n = \left(\int_{\Omega_n} \nabla\phi_n^i \cdot \nabla\phi_n^j \right)_{1\le i,j\le k}.$$

Our goal is to show the convergence of C_n: it is therefore enough to prove the convergence of P_n to an invertible matrix P. But from the previous step, that is the convergence of ϕ_n^i to ϕ^i in $H_0^1(D)$, we know that P_n converges to

$$P := \left(\int_D \nabla\phi^i \cdot \nabla\phi^j \right)_{1\le i,j\le k}.$$

The matrix P is selfadjoint and nonnegative: namely, for any vector $\lambda \in \mathbb{R}^k$,

$$P\lambda\cdot\lambda \;=\; \int_D |\nabla \sum_{i=1}^k \lambda_i\, \phi^i|^2.$$

Thus, to prove the invertibility of P, it is enough to show that the ϕ^i's are linearly independant. Assume *a contrario* that

$$\sum \lambda_i \, \phi^i = 0 \text{ almost everywhere, for some non-zero vector } \lambda.$$

Up to reindex the functions, one can assume that $\lambda_1 \neq 0$. We remind that the functions $\Phi^i := \phi^i - \chi^i$ belong to $H^1_0(\Omega)$ (see above). Thus, there exists a sequence of functions $\tilde{\Phi}^i_n$ in $C^\infty_c(\Omega)$ converging to Φ^i in $H^1_0(\Omega)$, $i = 1...k$. We set $\tilde{\phi}^i_n := \tilde{\Phi}^i_n + \chi^i$, and introduce

$$v_n := \frac{1}{\lambda_1} \left(\sum_{i=1}^k \lambda_i \, \tilde{\phi}^i_n \right) \to 0 \quad \text{in } H^1_0(D).$$

Clearly, $v_n = 1$ on a neighborhood of $\mathcal{C}^1$. It follows that

$$\int_D |\nabla v_n|^2 \geq \mathrm{cap}_D(\mathcal{C}^1),$$

and letting n go to infinity leads to $\mathrm{cap}_D(\mathcal{C}^1) = 0$. This contradicts Assumption (H3).

Eventually, we obtain that P is invertible, which yields a uniform bound on the $c^{i,j}_n$'s, and their convergence (up to subsequences) to some limit constants $c^{i,j}$. From the above lines and from relation (2.7), we deduce that $(\psi^i_n)_{n\in\mathbb{N}}$ converges (up to a subsequence) to $\psi^i := \sum_{j=1}^{j} c^{i,j} \, \phi^i$ in $H^1_0(D)$, for all $i = 1, \ldots, k$.

Moreover, from the definition of ψ^i_n (2.5), we know that the weak circulations verify

$$\gamma^j(\nabla^\perp \psi^i) = - \int_\Omega \nabla^\perp \psi^i \cdot \nabla^\perp \chi^j = \lim_{n\to\infty} - \int_{\Omega_n} \nabla^\perp \psi^i_n \cdot \nabla^\perp \chi^j = \delta_{ij}. \tag{2.8}$$

To completely control the harmonic part of the velocity u_n, it remains to show convergence of the time dependent functions α^i_n, $i = 1 \ldots k$ in (2.3). We shall use the following proposition, to be found in [12]:

Proposition 2. *For all* $i = 1 \ldots k$, $\alpha^i_n = \int_{\Omega_n} \phi^i_n \, \omega_n \, dx + \int_{\partial O^i_n} u_n \cdot \tau \, ds$.

By Kelvin's theorem, the circulation of u_n on each ∂O^i_n is constant in time so that

$$\alpha^i_n = \int_{\Omega_n} \phi^i_n \, \omega_n \, dx + \int_{\partial O^i_n} u^0_n \cdot \tau \, ds.$$

Moreover, we remind that the vorticity ω_n obeys the transport equation

$$\partial_t \omega_n + u_n \cdot \nabla \omega_n = 0$$

so that the L^p norms are conserved:

$$\|\omega_n(t,\cdot)\|_{L^p(\Omega_n)} = \|\omega^0_n\|_{L^p(\Omega_n)} \leq \|\omega^0\|_{L^p(\Omega)}, \quad 1 \leq p \leq \infty. \tag{2.9}$$

We now extend ω_n by 0 outside Ω_n for all n, so that the sequence $(\omega_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^+ \times D)$. Up to an extraction, we deduce that

$$\omega_n \to \omega \text{ weakly * in } L^\infty(\mathbb{R}^+ \times D). \tag{2.10}$$

One has easily that

$$\omega = 0 \quad \text{outside } \overline{\Omega}. \tag{2.11}$$

From this convergence, we infer that α^i_n converges weakly* in $L^\infty(\mathbb{R}^+)$ to

$$\alpha^i := \int_\Omega \phi^i \, \omega \, dx + \gamma^i$$

Unfortunately, we cannot establish rightaway strong convergence of (α^i_n). We need some uniform $L^\infty L^2$ bounds on u_n, to be obtained in the section below.

2.3. **Study of the rotational part.**

A simple energy estimate on (2.4) yields

$$\|\nabla\psi_n^0(t,\cdot)\|^2_{L^2(\Omega_n)} \;\le\; \|\omega_n(t,\cdot)\|_{L^2(\Omega_n)}\|\psi_n^0(t,\cdot)\|_{L^2(\Omega_n)}, \quad \forall t, n.$$

Extending ψ_n^0 by zero outside Ω_n, we can see it as an element of $H^1_0(D)$. By applying the Poincaré inequality on D, we end up with

$$\|\psi_n^0(t,\cdot)\|_{H^1_0(D)} \;=\; \|\psi_n^0(t,\cdot)\|_{H^1_0(\Omega_n)} \;\le\; C\,\|\omega_n(t,\cdot)\|_{L^2(\Omega_n)} \;\le\; C'$$

uniformly in t, n, in particular for $t=0$. Combining this bound on $\psi_n^0(0,\cdot)$ with the estimates on ψ_n^i and $\alpha_n^i(0)$, we obtain that u_n^0 is uniformly bounded in L^2. Then, the conservation of energy implies that

$$\|u_n(t)\|_{L^2(\Omega_n)} \;=\; \|u_n^0\|_{L^2(\Omega_n)} \;\le\; C, \quad \forall t,$$

that is a uniform $L^\infty L^2$ estimate on u_n.

On one hand, this estimate implies the strong convergence of α_n^i (and completes the analysis of the harmonic part). Indeed, we compute

$$(\alpha_n^i)' \;=\; \int_{\Omega_n} \phi_n^i\,\partial_t\omega_n\,dx \;=\; -\int_{\Omega_n} \phi_n^i\,\mathrm{div}\,(u_n\omega_n)\,dx \;=\; \int_{\Omega_n} \nabla\phi_n^i\cdot u_n\omega_n\,dx.$$

Using the uniform L^2 bounds on $\nabla\phi_n^i$ and u_n, we infer that α_n^i is uniformly bounded in $W^{1,\infty}(\mathbb{R}^+)$ which means that the converge holds strongly in $C^0_{\mathrm{loc}}(\mathbb{R}^+)$.

On the other hand, this estimate allows a control of the time derivatives of ψ_n^0. Indeed, we observe that $\partial_t\psi_n^0$ satisfies

$$\Delta\left(\partial_t\psi_n^0\right) \;=\; \partial_t\omega_n \;=\; -\mathrm{div}\;(u_n\omega_n) \;\text{ in }\; \Omega_n, \quad \partial_t\psi_n^0|_{\partial\Omega_n} = 0.$$

Using the uniform $L^\infty L^2$ and L^∞ bounds on u_n and ω_n respectively, we get similarly

$$\|\partial_t\psi_n^0(t,\cdot)\|_{H^1_0(D)} \;\le\; C, \quad \forall t, n.$$

From these bounds and standard compactness lemma [16], there exists $\psi^0 \in W^{1,\infty}(\mathbb{R}^+; H^1_0(D))$ such that up to a subsequence:

$$\psi_n^0 \to \psi^0 \;\text{ weakly* in }\; W^{1,\infty}(\mathbb{R}^+; H^1_0(D)) \text{ and strongly in } C^0(0,T; L^2(D)), \quad \forall T>0.$$

From the weak convergence of ψ_n^0 and ω_n, we infer that

$$\Delta\psi^0(t,\cdot) = \omega(t,\cdot) \text{ in } \mathcal{D}'(\Omega), \text{ for almost every } t \tag{2.12}$$

using again that any compact subset of Ω is included in Ω_n for n large enough.

As Ω_n γ-converges to Ω, we can use Proposition 13: $\psi_n^0(t,\cdot)$ has for every t a subsequence that converges weakly in $H^1_0(D)$ to a limit in $H^1_0(\Omega)$. Thus, for every t, $\psi^0(t,\cdot)$ belongs to $H^1_0(\Omega)$.

Finally, let us prove the strong convergence of ψ_n^0 to ψ^0 in $L^2(0,T; H^1_0(D))$ for all $T>0$. Therefore, we go back to the equation (2.4). We compute:

$$\int_0^T\int_D |\nabla\psi_n^0|^2 = \int_0^T\int_{\Omega_n} |\nabla\psi_n^0|^2 = -\int_0^T\int_{\Omega_n} \omega_n\,\psi_n^0 = -\int_0^T\int_D \omega_n\,\psi_n^0 \;\to -\int_0^T\int_D \omega\,\psi^0$$

As we know from the previous paragraph that $\psi^0(t,\cdot)$ belongs to $H^1_0(\Omega)$ for every t, we can perform an energy estimate on (2.12) as well. We get

$$\int_0^T\int_D |\nabla\psi^0|^2 = \int_0^T\int_\Omega |\nabla\psi^0|^2 = -\int_0^T\int_\Omega \omega\,\psi^0 = -\int_0^T\int_D \omega\,\psi^0$$

Hence,

$$\int_0^T\int_D |\nabla\psi_n^0|^2 \;\to\; \int_0^T\int_D |\nabla\psi^0|^2$$

which together with the weak convergence in $W^{1,\infty}(0,T; H^1_0(D))$ yields the strong convergence of ψ_n^0 to ψ^0 in $L^2(0,T; H^1_0(D))$ for all $T>0$.

2.4. **Conclusion of the proof.**

We can now conclude the proof of Theorem 1. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of Euler solutions in Ω_n, associated to the initial data u^0_n. Each field u_n has the Hodge decomposition (2.3). Through obvious extension of ψ^m_n, $m = 0\dots k$, it can be seen as an element of $L^\infty(\mathbb{R}^+; L^2(D))$. By the results of the previous subsections, it *converges strongly in* $L^2((0,T)\times D)$ *and weakly* in* $L^\infty(\mathbb{R}^+; L^2(D))$, $T > 0$, to the field

$$u(t,x) \;=\; \nabla^\perp \psi^0(t,x) \;+\; \sum_{i=1}^k \alpha^i(t) \nabla^\perp \psi^i(x).$$

Note that ψ^0 belongs to $L^\infty(\mathbb{R}^+; H^1_0(\Omega))$ whereas for $i = 1,\dots,k$, α^i belongs to $C^0(\mathbb{R}^+)$ and ψ^i belongs to $H^1_0(\widetilde{\Omega})$. Moreover, by construction, one has $\operatorname{curl} u \;=\; \Delta\psi^0 \;=\; \omega \in L^\infty(\mathbb{R}^+\times\Omega)$ as well as the divergence-free and tangency conditions, *cf* (1.6).

An important remark is that all the reasoning we have made so far also applies to the initial data (without the difficulties linked to time dependence). In particular, it can be seen that the sequence (u^0_n) converges strongly in $L^2(\Omega)$ (up to a subsequence). Moreover, its limit $\tilde{u}^0$ has a Hodge decomposition,

$$\tilde{u}^0(x) \;=\; \nabla^\perp \psi^{0,0}(x) \;+\; \sum_{i=1}^k \alpha^{0,i} \nabla^\perp \psi^i(x).$$

with $\psi^{0,0} \in H^1_0(\Omega)$, $\Delta\psi^{0,0} = \omega^0$ and $\alpha^{0,i} := \int_\Omega \phi^i\, \omega^0\, dx \;+ \gamma^i$. In particular, it satisfies

$$\operatorname{curl} \tilde{u}^0 \;=\; \omega^0,$$

as well as the divergence-free and tangency conditions (1.5).

Noting that $\chi^j = \phi^j - \Phi^j$ with $\Phi^j \in H^1_0(\Omega)$, we compute the weak circulation of $\nabla^\perp\psi^{0,0}$:

$$\begin{aligned}\gamma^j(\nabla^\perp\psi^{0,0}) \;&=\; -\int_\Omega \omega^0\phi^j - \int_\Omega \nabla^\perp\psi^{0,0}\cdot\nabla^\perp\phi^j + \int_\Omega \omega^0\Phi^j + \int_\Omega \nabla^\perp\psi^{0,0}\cdot\nabla^\perp\Phi^j \\ &=\; -\int_\Omega \omega^0\phi^j\end{aligned}$$

where we have integrated by parts and used that $\psi^{0,0}, \Phi^j \in H^1_0(\Omega)$ and $\Delta\phi^j = 0$. Moreover, we remind (see (2.8)) that

$$\gamma^j(\nabla^\perp\psi^i) = \delta_{ij}.$$

Combining everything, it follows that the difference $\tilde{u}^0 - u^0$ is curl-free, divergence free, with zero weak circulation around each $\mathcal{C}^i$ and a tangency condition. By a slight modification of the argument used in [6, Proposition 2.1], it follows that $\tilde{u}^0 - u^0 = 0$. In particular, u^0_n converges to u^0 strongly in L^2. As a byproduct, we obtain the existence of a Hodge decomposition for data u^0 satisfying (1.4) in the irregular domain Ω.

Finally, let $\varphi \in \mathcal{D}\left([0,+\infty[\times\Omega\right)$, with $\operatorname{div}\varphi = 0$. For n large enough, the support of φ is included in Ω_n so that:

$$\int_0^\infty \int_\Omega (u_n\cdot\partial_t\varphi + (u_n\otimes u_n) : \nabla\varphi) = -\int_\Omega u^0_n\cdot\varphi(0,\cdot)$$

By the strong L^2 convergence of u_n to u, and of u^0_n to u^0, it follows that u satisfies the weak form of the Euler equations (1.7).

3. Theorem 2 for $p = \infty$

Let Ω of type (1.8), satisfying (H1'). Following the proof of Proposition 1 , it can be shown that Ω is the Hausdorff limit of a sequence

$$\Omega_n \;:=\; \mathbb{R}^2 \setminus \overline{O_n}$$

where O_n is a smooth Jordan domain, whose closure converges in the Hausdorff sense to $\mathcal{C}$.

We then introduce a sequence (ω^0_n) such that $\omega^0_n \in C^\infty_c(\Omega_n)\cap C^\infty_c(\Omega)$ for all n,

$$\omega^0_n \to \omega^0 := \operatorname{curl} u^0 \quad \text{weakly in } L^p(\Omega) \quad \text{with } \|\omega^0_n\|_{L^p} \le \|\omega^0\|_{L^p} \text{ for all } p\in[1,\infty],$$

and such that for $\rho_0 > 0$ large enough, $\omega_n^0 = 0$ outside $B(0, \rho_0)$ for all n (see(1.9)-(1.10)).

To build up an appropriate initial velocity u_n^0 in $\Omega_n := \mathbb{R}^2 \setminus \overline{O_n}$ from the vorticity, we need to specify the value of the circulation somewhere. Let J be a smooth closed Jordan curve in Ω such that $\mathcal{C}$ is included in the interior of J. For any n, we consider as an initial velocity u_n^0 the unique vector field in Ω_n which verifies

$$\operatorname{div} u_n^0 = 0, \quad \operatorname{curl} u_n^0 = \omega_n^0, \quad u_n^0|_{\partial O_n} \cdot n = 0, \quad \int_J u_n^0 \cdot \tau \, ds = \int_J u^0 \cdot \tau \, ds \text{ and } \lim_{|x|\to\infty} u_n^0(x) = 0.$$

Note that the quantity $\int_J u^0 \cdot \tau \, ds$ is well-defined because u^0 belongs to $W^{1,q}_{\rm loc}(\Omega)$ for all finite q. As Ω_n is smooth, u_n^0 generates a unique global strong solution of the Euler equations (see e.g. [7]). The transport equation governing the vorticity implies that the L^p norms are conserved:

$$\|\omega_n(t,\cdot)\|_{L^p(\Omega_n)} = \|\omega_n^0\|_{L^p} \leq \|\omega^0\|_{L^p}, \; 1 \leq p \leq +\infty. \tag{3.1}$$

As in the previous part, the Hodge-decomposition will be useful to obtain estimates on the velocity:

$$u_n(t,x) \; = \; \nabla^\perp \psi_n^0(t,x) \; + \; \alpha_n(t) \nabla^\perp \psi_n(x) \tag{3.2}$$

where ψ_n^0 satisfies for any t the Dirichlet problem

$$\Delta \psi_n^0 \; = \; \omega_n \text{ in } \Omega_n, \quad \psi_n^0|_{\partial\Omega_n} = 0, \quad \psi_n^0(x) = \mathcal{O}\big(\frac{1}{|x|}\big) \text{ as } x \to \infty, \tag{3.3}$$

whereas ψ_n is the harmonic function satisfying

$$\Delta \psi_n \; = \; 0 \text{ in } \Omega_n, \quad \frac{\partial \psi_n}{\partial \tau}|_{\partial\Omega_n} \; = \; 0, \quad \int_{\partial O_n} \frac{\partial \psi_n}{\partial n} = -1, \quad \psi_n(x) = \mathcal{O}\big(\ln|x|\big) \text{ as } x \to \infty. \tag{3.4}$$

The function $\alpha_n(t)$ is the sum of the circulation of the velocity around O_n and the mass of the vorticity $\int_{\Omega_n} \omega_n(t,\cdot)$. By the Kelvin's circulation theorem and the transport nature of the vorticity equation, we infer that these two quantities are conserved, hence

$$\alpha_n(t) \equiv \alpha_n = \int_{\partial O_n} u_n^0 \cdot \tau \, ds + \int_{\Omega_n} \omega_n^0 = \int_{\partial J} u_n^0 \cdot \tau \, ds - \int_{A_n} \omega_n^0 + \int_{\Omega_n} \omega_n^0 \tag{3.5}$$

where $A_n := \Omega_n \setminus Ext(J)$ (with $Ext(J)$ the exterior of J), hence

$$\alpha_n \; \to \; \alpha \; := \; \int_J u^0 \cdot \tau \, ds + \int_{Ext(J)} \omega^0. \tag{3.6}$$

3.1. **Poincaré inequality in exterior domains.**

Thanks to the properties in (3.3), we integrate by parts to obtain:

$$\|\nabla \psi_n^0\|^2_{L^2(\Omega_n)} \leq \|\omega_n\|_{L^2(\Omega_n)} \|\psi_n^0\|_{L^2(\Omega_n \cap \operatorname{supp} \omega_n)}. \tag{3.7}$$

In the case of a bounded domain, we used the Poincaré inequality on a domain D containing all the Ω_n's. The idea here is to establish a similar inequality, thanks to the γ-convergence of $\overline{O_n}$ to $\mathcal{C}$ with cap $\mathcal{C} > 0$.

Lemma 1. *Let ρ be a positive such that $\mathcal{C} \subset B(0,\rho)$. As Ω is of type* (1.8)*, with (H1')-(H2'), then there exists $C_\rho > 0$ and N_ρ, depending only on ρ, such that*

$$\|\varphi\|_{L^2(\Omega_n \cap B(0,\rho))} \leq C_\rho \|\nabla \varphi\|_{L^2(\Omega_n \cap B(0,\rho))}, \;\; \forall \varphi \in C_c^\infty(\Omega_n), \;\; \forall n \geq N_\rho.$$

Proof. Let us assume that the conclusion is false, which means that for any $k \in \mathbb{N}$, if we choose $C_\rho = k$ and $N_\rho = max(k, n_{k-1})$, then there exist $n_k \geq N_\rho$ and $\varphi_k \in C_c^\infty(\Omega_{n_k})$ such that

$$\|\varphi_k\|_{L^2(\Omega_{n_k} \cap B(0,\rho))} > k \|\nabla \varphi_k\|_{L^2(\Omega_{n_k} \cap B(0,\rho))}.$$

Dividing φ_k by $\|\varphi_k\|_{L^2(\Omega_{n_k} \cap B(0,\rho))}$, we can consider that $\|\varphi_k\|_{L^2(\Omega_{n_k} \cap B(0,\rho))} = 1$, which implies that $\|\nabla \varphi_k\|_{L^2(\Omega_{n_k} \cap B(0,\rho))}$ tends to zero as k tends to infinity. Therefore, extracting a subsequence if necessary, we have that

$$\varphi_k \to \varphi \text{ weakly in } H^1(B(0,\rho)) \text{ and strongly in } L^2(B(0,\rho)).$$

It follows that φ is a non zero contant, because $\|\varphi\|_{L^2(B(0,\rho))} = 1$ and $\nabla\varphi_k \rightharpoonup 0$ weakly in $L^2(B(0,\rho))$.

We introduce a cutoff function χ which is equal to zero in $B(0,\rho)^c$, and equal to 1 in some neighborhoods of O_n's and $\mathcal{C}$. Then,

$$\chi\varphi_k \text{ belongs to } H^1_0(B(0,\rho)\setminus\overline{O_n}), \tag{3.8}$$

and

$$\chi\varphi_k \rightharpoonup \chi\varphi \text{ weakly in } H^1_0(B(0,\rho)). \tag{3.9}$$

However, the sequence $(B(0,\rho)\setminus\overline{O_n})$ converges to $B(0,\rho)\setminus\mathcal{C}$ in the Hausdorff sense and as $\overline{O_n}$ is connected for all n, Proposition 12 implies:

$$B(0,\rho)\setminus\overline{O_n}\ \gamma\text{-converges to } B(0,\rho)\setminus\mathcal{C}.$$

Combining (3.8), (3.9) and Proposition 13, we obtain that $\chi\varphi$ belongs to $H^1_0(B(0,\rho)\setminus\mathcal{C})$.

Next, Proposition 6 implies that $\chi\varphi = 0$ quasi everywhere in $\mathcal{C}$. This is in contradiction with $\text{cap}(\mathcal{C}) > 0$ and the fact that $\chi\varphi$ is equal to a non zero constant in $\mathcal{C}$. The conclusion of the proof follows. ☐

We want to apply the previous lemma to (3.7), but we remark that an important issue is to control the size of the support of ω_n independently of n. As ω_n is transported by u_n, we will prove that the velocity is uniformly bounded far from the domains O_n.

3.2. **Uniform estimates of the velocity far from the boundaries.**

The advantage of working outside one simply connected domain is the explicit formula of ψ^0_n and ψ_n in terms of biholomorphisms. We denote $D := \{|z| < 1\}$ the open unit disk, $\Omega_n := \mathbb{R}^2\setminus\overline{O_n}$ the approximate exterior domain given by Proposition 1, and $\Delta := \{|z| > 1\}$ the exterior of the closed unit disk. From the Riemann mapping theorem, it is easily seen that there is a unique biholomorphism

$$\mathcal{T}_n : \Omega_n \mapsto \Delta, \quad \text{with } \mathcal{T}_n(\infty) = \infty, \quad \mathcal{T}_n'(\infty) > 0.$$

We remind that the last two conditions mean

$$\mathcal{T}_n(z) \sim \lambda_n z, \quad |z| \sim +\infty, \quad \text{for some } \lambda_n > 0.$$

With such notations, we have

$$\nabla^\perp\psi^0_n(t,x) = \frac{1}{2\pi}D\mathcal{T}_n^T(x)\int_{\Omega_n}\Big(\frac{\mathcal{T}_n(x)-\mathcal{T}_n(y)}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)|^2} - \frac{\mathcal{T}_n(x)-\mathcal{T}_n(y)^*}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)^*|^2}\Big)^\perp\omega_n(t,y)\,dy \tag{3.10}$$

and

$$\nabla^\perp\psi_n(t,x) = \frac{1}{2\pi}D\mathcal{T}_n^T(x)\frac{\mathcal{T}_n(x)^\perp}{|\mathcal{T}_n|^2} \tag{3.11}$$

with the notation $z^* = \frac{z}{|z|^2}$ (see e.g. [6, 9] for an introduction to the Biot-Savart law in exterior domains).

Like in [6, 8, 9], a key point is to control $\mathcal{T}_n$ when $\overline{O_n}$ tends to $\mathcal{C}$. We state

Proposition 3. *Let Π be the unbounded connected component of Ω. There is a unique biholomorphism $\mathcal{T}$ from Π to Δ, satisfying $\mathcal{T}(\infty) = \infty$, $\mathcal{T}'(\infty) > 0$. Moreover, one has the following convergence properties:*

i) *$\mathcal{T}_n^{-1}$ converges uniformly locally to $\mathcal{T}^{-1}$ in Δ.*
ii) *$\mathcal{T}_n$ (resp. $\mathcal{T}_n'$) converges uniformly locally to $\mathcal{T}$ (resp. to $\mathcal{T}'$) in Π.*
iii) *$|\mathcal{T}_n|$ converges uniformly locally to 1 in $\Omega\setminus\Pi$.*

This proposition is a consequence of the celebrated theorem of Caratheodory on the convergence of biholomorphisms; we refer to Appendix D for all details and proof. From there, we deduce:

Lemma 2. *Let R_0 large enough so that $\mathcal{C}\subset B(0,R_0)$. Then, there exists $C_0=C(\|\omega^0\|_{L^1},\|\omega^0\|_{L^\infty},R_0)$ such that*

$$f_n(t,x) := \frac{1}{2\pi} D\mathcal{T}^T(x)\int_\Pi \Big(\frac{\mathcal{T}(x)-\mathcal{T}(y)}{|\mathcal{T}(x)-\mathcal{T}(y)|^2}-\frac{\mathcal{T}(x)-\mathcal{T}(y)^*}{|\mathcal{T}(x)-\mathcal{T}(y)^*|^2}\Big)^\perp \omega_n(t,y)\,dy \tag{3.12}$$

verifies

$$\|f_n(t,x)\|_{L^\infty(\mathbb{R}^+\times B(0,R_0)^c)}\leq C_0,\ \forall n.$$

Moreover, for any compact K outside $\overline{B(0,R_0)}$, there exists N_K such that

$$\Big\|\nabla^\perp\psi_n^0(t,x)\Big\|_{L^\infty(\mathbb{R}^+\times K)}\leq 2C_0+1,\ \forall n\geq N_K.$$

Proof. Let $\tilde R_0<R_0$ such that $\mathcal{C}\subset B(0,\tilde R_0)$. We decompose the integral (3.12) into three parts:

$$\begin{aligned} f_n(t,x) &= \frac{1}{2\pi}D\mathcal{T}^T(x)\Big[\int_{\Pi\cap B(0,\tilde R_0)}\frac{(\mathcal{T}(x)-\mathcal{T}(y))^\perp}{|\mathcal{T}(x)-\mathcal{T}(y)|^2}\omega_n(t,y)\,dy \\ &\quad+\int_{B(0,\tilde R_0)^c}\frac{(\mathcal{T}(x)-\mathcal{T}(y))^\perp}{|\mathcal{T}(x)-\mathcal{T}(y)|^2}\omega_n(t,y)\,dy-\int_\Pi\frac{(\mathcal{T}(x)-\mathcal{T}(y)^*)^\perp}{|\mathcal{T}(x)-\mathcal{T}(y)^*|^2}\omega_n(t,y)\,dy\Big] \\ &= \frac{1}{2\pi}D\mathcal{T}^T(x)(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3). \end{aligned}$$

By the definition of $\mathcal{T}$ (see Proposition 3), there exists some $(\beta,\tilde\beta)\in\mathbb{R}^+_*\times\mathbb{C}$ such that:

$$\mathcal{T}(z)=\beta z+\tilde\beta+\mathcal{O}(\frac{1}{z}) \text{ as } z\to\infty.$$

Then there exists C_1 such that $\|D\mathcal{T}\|_{L^\infty(B(0,R_0)^c)}\leq C_1$. If the boundary $\partial\Pi$ is rough, we recall that such an inequality does not hold in $L^\infty(\Pi)$ (see for instance [11, 10]). This remark underlines the importance of R_0.

As $\mathcal{T}$ is continuous and one-to-one, there exists $\delta>0$ such that

$$\operatorname{dist}\Big(\mathcal{T}(\partial B(0,\tilde R_0));\mathcal{T}(\partial B(0,R_0))\Big)\geq\delta.$$

Then $|\mathcal{T}(x)-\mathcal{T}(y)|\geq\delta$ for any $x\in B(0,R_0)^c$ and $y\in\Pi\cap B(0,\tilde R_0)$. Hence, for any $x\in B(0,R_0)^c$, we have

$$|\mathcal{I}_1|\leq\frac{1}{\delta}\int_{\Pi\cap B(0,\tilde R_0)}|\omega_n(t,y)|\,dy\leq\frac{\|\omega^0\|_{L^1}}{\delta},$$

where we have used (3.1).

As $|\mathcal{T}(y)^*|\leq 1\leq|\mathcal{T}(y)|$, we also have $|\mathcal{T}(x)-\mathcal{T}(y)^*|\geq\delta$ for any $x\in B(0,R_0)^c$. Therefore, we obtain

$$|\mathcal{I}_3|\leq\frac{1}{\delta}\int_\Pi|\omega_n(t,y)|\,dy\ \leq\ \frac{\|\omega^0\|_{L^1}}{\delta},$$

Concerning the last part $\mathcal{I}_2$, we introduce $z=\mathcal{T}(x)$ and

$$g(t,\eta):=\omega_n(t,\mathcal{T}^{-1}(\eta))|\det D\mathcal{T}^{-1}(\eta)|\mathbf{1}_{\mathcal{T}(B(0,\tilde R_0)^c)}(\eta).$$

Changing variables $\eta=\mathcal{T}(y)$, we compute

$$\mathcal{I}_2=\int_{\mathbb{R}^2}\frac{(z-\eta)^\perp}{|z-\eta|^2}g(t,\eta)\,d\eta.$$

Changing variables back, we obtain that

$$\|g(t,\cdot)\|_{L^1(\mathbb{R}^2)}=\|\omega_n(t,\cdot)\|_{L^1(B(0,\tilde R_0)^c)}\leq\|\omega^0\|_{L^1}.$$

Using the behavior at infinity of $\mathcal{T}^{-1}$, we infer that there exists C_2 such that

$$|\det D\mathcal{T}^{-1}(\eta)|\leq C_2,\ \forall\eta\in\mathcal{T}(B(0,\tilde R_0)^c),$$

hence

$$\|g(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\leq C_2\|\omega_n(t,\cdot)\|_{L^\infty}\leq C_2\|\omega^0\|_{L^\infty}.$$

This last argument explains why we split the integral into several parts: we cannot prove that $\det D\mathcal{T}^{-1}$ is bounded up to the boundary, in particular when its boundary is rough. Using a classical estimate for the Biot-Savart kernel in $\mathbb{R}^2$, we write

$$|\mathcal{I}_2| \leq C_3 \|g(t,\cdot)\|^{1/2}_{L^1(\mathbb{R}^2)} \|g(t,\cdot)\|^{1/2}_{L^\infty(\mathbb{R}^2)} \leq C_3 \sqrt{C_2} \|\omega^0\|^{1/2}_{L^1} \|\omega^0\|^{1/2}_{L^\infty},$$

where C_3 is a universal constant.

Putting $C_0 := \frac{C_1}{2\pi}\Big(\frac{2\|\omega^0\|_{L^1}}{\delta} + C_3\sqrt{C_2}\|\omega^0\|^{1/2}_{L^1}\|\omega^0\|^{1/2}_{L^\infty}\Big)$, we have established the first inequality:

$$\|f_n(t,x)\|_{L^\infty(\mathbb{R}^+\times B(0,R_0)^c)} \leq C_0, \ \forall n.$$

We treat now $\nabla^\perp \psi^0_n$. Let K be a compact set in $B(0,R_0)^c$. Let $\tilde K$ a compact set satisfying

$$K \subset \tilde K \subset B(0,\tilde R_0)^c, \quad \text{ and } \operatorname{dist}\Big(\mathcal{T}(\partial K);\mathcal{T}(\partial \tilde K)\Big) \geq \delta.$$

One can take for instance $\tilde K = \{\tilde R_0 \leq |z| \leq R_1\}$ for R_1 large enough. Again, we decompose the integral (3.10) into three parts:

$$\begin{aligned}
\nabla^\perp \psi^0_n(t,x) &= \frac{1}{2\pi} D\mathcal{T}_n^T(x)\Big[\int_{\Omega_n\setminus\tilde K} \frac{(\mathcal{T}_n(x)-\mathcal{T}_n(y))^\perp}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)|^2}\omega_n(t,y)\,dy \\
&\quad + \int_{\tilde K} \frac{(\mathcal{T}_n(x)-\mathcal{T}_n(y))^\perp}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)|^2}\omega_n(t,y)\,dy - \int_{\Omega_n} \frac{(\mathcal{T}_n(x)-\mathcal{T}_n(y)^*)^\perp}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)^*|^2}\omega_n(t,y)\,dy\Big] \\
&= \frac{1}{2\pi} D\mathcal{T}_n^T(x)(\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3).
\end{aligned}$$

By the uniform convergence of $D\mathcal{T}_n$ to $D\mathcal{T}$ in K (see Proposition 3), for any $\varepsilon_1 > 0$ there exists N_1 such that

$$\|D\mathcal{T}_n\|_{L^\infty(K)} \leq C_1 + \varepsilon_1, \ \forall n \geq N_1.$$

By the uniform convergence of $\mathcal{T}_n$ to $\mathcal{T}$ in $\tilde K$, there exists $N_2 > 0$ such that

$$\operatorname{dist}\Big(\mathcal{T}_n(\partial\tilde K);\mathcal{T}_n(\partial K)\Big) \geq \delta/2, \ \forall n \geq N_2.$$

Then $|\mathcal{T}_n(x) - \mathcal{T}_n(y)| \geq \delta/2$ for any $x \in K$ and $y \in \Omega_n \setminus \tilde K$. Hence, for any $x \in K$, we have

$$|\mathcal{J}_1| \leq \frac{2}{\delta}\int_{\Omega_n\setminus\tilde K} |\omega_n(t,y)|\,dy \leq \frac{2\|\omega^0\|_{L^1}}{\delta}, \ \forall n \geq N_2.$$

As $|\mathcal{T}_n(y)^*| \leq 1 \leq |\mathcal{T}_n(y)|$, we also have $|\mathcal{T}_n(x) - \mathcal{T}_n(y)^*| \geq \delta/2$ for any $x \in K$. Therefore, we obtain

$$|\mathcal{J}_3| \leq \frac{2}{\delta}\int_{\Omega_n} |\omega_n(t,y)|\,dy \ \leq \ \frac{2\|\omega^0\|_{L^1}}{\delta},$$

Concerning the last part $\mathcal{J}_2$, we introduce $z = \mathcal{T}_n(x)$ and

$$g_n(t,\eta) := \omega_n(t,\mathcal{T}_n^{-1}(\eta))|\det D\mathcal{T}_n^{-1}(\eta)|\mathbf{1}_{\mathcal{T}_n(\tilde K)}(\eta).$$

Changing variables $\eta = \mathcal{T}_n(y)$, we compute

$$\mathcal{J}_2 = \int_{\mathbb{R}^2} \frac{(z-\eta)^\perp}{|z-\eta|^2} g_n(t,\eta)\,d\eta.$$

Changing variables back, we obtain that

$$\|g_n(t,\cdot)\|_{L^1(\mathbb{R}^2)} = \|\omega_n(t,\cdot)\|_{L^1(\tilde K)} \leq \|\omega^0\|_{L^1}.$$

Using the uniform convergence of $D\mathcal{T}_n^{-1}$ to $D\mathcal{T}^{-1}$ in a compact big enough (such that $\mathcal{T}_n(\tilde K) \subset D$), for any $\varepsilon_3 > 0$ there exists N_3 such that

$$|\det D\mathcal{T}_n^{-1}(\eta)| \leq C_2 + \varepsilon_3, \ \forall \eta \in \mathcal{T}_n(\tilde K), \ \forall n \geq N_3$$

hence

$$\|g_n(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq (C_2+\varepsilon_3)\|\omega_n(t,\cdot)\|_{L^\infty} \ \leq \ (C_2+\varepsilon_3)\|\omega^0\|_{L^\infty}.$$

Finnaly, we use the classical estimate for the Biot-Savart kernel in $\mathbb{R}^2$:

$$|\mathcal{J}_2| \leq C_3 \|g_n(t,\cdot)\|^{1/2}_{L^1(\mathbb{R}^2)} \|g_n(t,\cdot)\|^{1/2}_{L^\infty(\mathbb{R}^2)} \leq C_3 \sqrt{C_2+\varepsilon_3} \|\omega^0\|^{1/2}_{L^1} \|\omega^0\|^{1/2}_{L^\infty}.$$

Choosing well ε_1 and ε_3, we find $N_K = \max(N_1, N_2, N_3)$ such that

$$\left\|\nabla^\perp \psi^0_n(t,x)\right\|_{L^\infty(\mathbb{R}^+\times K)} \leq 2C_0 + 1, \ \forall n \geq N_K,$$

which ends the proof. □

The reason why we divide the proof in two parts is to obtain a constant C_0 independent of the compact set K. Although N_K depends on K, the independence of C_0 with respect to K will be crucial for the uniform estimate of the vorticity support. The harmonic part is easier to estimate.

Lemma 3. *Let R_0 a positive number such that $\mathcal{C} \subset B(0,R_0)$. Then, there exists $C_0 = C(R_0)$ such that*

$$\psi(x) := \frac{1}{2\pi} \ln |\mathcal{T}(x)| \tag{3.13}$$

verifies

$$\left\|\nabla^\perp \psi(x)\right\|_{L^\infty(B(0,R_0)^c)} \leq C_0.$$

Moreover, for any compact K outside $\overline{B(0,R_0)}$, there exists N_K such that

$$\left\|\nabla^\perp \psi_n(x)\right\|_{L^\infty(K)} \leq 2C_0 + 1, \ \forall n \geq N_K.$$

Proof. The first part comes from the behavior of $\mathcal{T}$ at infinity:

$$\mathcal{T}(z) = \beta z + \tilde{\beta} + \mathcal{O}(\frac{1}{z}) \text{ as } z \to \infty.$$

The second point is a direct consequence of the uniform convergence of $\mathcal{T}_n$ in K (see Proposition 3). □

3.3. **Support of the vorticity and H^1 estimates.**

Let ρ_0 such that $\cup_n \operatorname{supp} \omega^0_n \cup \operatorname{supp} \omega^0 \cup \Pi^c \subset B(0,\rho_0)$.

$$C_0 = C_0(\|\omega^0\|_{L^1}, \|\omega^0\|_{L^\infty}, \rho_0)$$

the constant of Lemmata 2 and 3. Let $C := (2C_0+1)(2+|\alpha|)$, where α was defined in (3.6). We fix a time $T>0$ and we introduce

$$K_T := \overline{B(0,\rho_0+CT)} \setminus B(0,\rho_0).$$

Together with (3.2)-(3.6), Lemmata 2 and 3 provide some N_T such that

$$\|u_n\|_{L^\infty(\mathbb{R}^+\times K_T)} \leq C, \ \forall n \geq N_T.$$

As ω_n is transported by u_n, we can conclude that

$$\operatorname{supp} \omega_n(t,\cdot) \subset B(0,\rho_0+Ct), \ \forall t \in [0,T], \ \forall n \geq N_T.$$

Finally we can complete the estimate of $\|\nabla \psi^0_n\|_{L^2(\Omega_n)}$. Let $\rho_T := \rho_0 + CT$, then Lemma 1 implies that there exist C_{ρ_T} and N_{ρ_T} such that

$$\|\psi^0_n\|_{L^2(\Omega_n \cap \operatorname{supp} \omega_n)} \leq C_{\rho_T} \|\nabla \psi^0_n\|_{L^2(\Omega_n \cap B(0,\rho_T))}, \ \forall t \in [0,T], \ \forall n \geq \max(N_{\rho_T}, N_T).$$

Combining with (3.7) and (3.1), we obtain: $\forall t \in [0,T], \ \forall n \geq \max(N_{\rho_T}, N_T)$:

$$\|\nabla \psi^0_n\|_{L^2(\Omega_n)} \leq C_{\rho_T} \|\omega_n\|_{L^2(\Omega_n)} = C_{\rho_T} \|\omega^0\|_{L^2}. \tag{3.14}$$

Using again Lemma 1 on any compact K of $\mathbb{R}^2$, we conclude that there exist C_K and N_K such that

$$\|\psi^0_n\|_{L^2(\Omega_n \cap K)} \leq C_K \|\nabla \psi^0_n\|_{L^2(\Omega_n)} \leq C_K C_{\rho_T} \|\omega^0\|_{L^2}, \ \forall t \in [0,T], \ \forall n \geq \max(N_{\rho_T}, N_T, N_K), \tag{3.15}$$

where C_K depends on the diameter of K. We recall that ψ^0_n is not square integrable at infinity (see (3.3)), but (3.15) will be sufficient to obtain local convergence.

We end this subsection with a $L^2_{\rm loc}$ estimate of $\nabla^\perp\psi_n$ up to the boundary. Let $R_0>0$ and χ be a cutoff function equal to 1 in $B(0,R_0)$ and to 0 outside $B(0,R_0+1)$. Then, $\chi\psi_n$ verifies

$$\Delta(\chi\psi_n) \;=\; \tilde\omega_n := 2\nabla\chi\cdot\nabla\psi_n+\psi_n\Delta\chi \;\text{ in }\; \Omega_n\cap B(0,R_0+1),\quad \chi\psi_n \;=\; 0 \;\text{ on }\; \partial\Omega_n\cup\partial B(0,R_0+1).$$

Note that the connectedness of $\partial\Omega_n$ allows to impose a Dirichlet condition on ψ_n. This Dirichlet condition can also be read on the formula $\psi_n=\frac{1}{2\pi}\ln|\mathcal{T}_n(x)|$, as $\mathcal{T}_n$ maps $\partial\Omega_n$ to $\partial B(0,1)$. Therefore, by a classical energy estimate and Poincaré inequality applied in $B(0,R_0+1)$, we obtain that:

$$\|\nabla(\chi\psi_n)\|^2_{L^2(\Omega_n)}\le\|\tilde\omega_n\|_{L^2(\Omega_n)}\|\chi\psi_n\|_{L^2(\Omega_n\cap B(0,R_0+1))}\le C_{R_0}\|\tilde\omega_n\|_{L^2(\Omega_n)}\|\nabla(\chi\psi_n)\|_{L^2(\Omega_n)}.$$

Using that ψ_n and $\nabla\psi_n$ converge uniformly to ψ and $\nabla\psi$ in $B(0,R_0+1)\setminus B(0,R_0)$ (see Proposition 3), we get that $\|\tilde\omega_n\|_{L^2(\Omega_n)}$ is uniformly bounded. This yields the existence of a constant C, depending only on R_0, such that

$$\|\nabla\psi_n\|_{L^2(\Omega_n\cap B(0,R_0))}\le C,\ \forall n. \tag{3.16}$$

3.4. **Conclusion of the proof.**

The proof follows the one for bounded domains, taking care of integrability at infinity. We fix $T>0$ and a compact K in Ω. We denote

$$D:=K\cup B(0,\rho_T),$$

with ρ_T defined in the previous subsection.

a) Compactness of the rotational part.

We deduce from (3.14) and (3.15) that

$$\|\psi^0_n(t,\cdot)\|_{H^1(D)}\;\le\;C_{T,K}\quad\forall t\in[0,T],\ \forall n\ge N_K.$$

As regards time derivatives, we observe that $\partial_t\psi^0_n$ satisfies

$$\Delta\left(\partial_t\psi^0_n\right)\;=\;\partial_t\omega_n\;=\;-\text{div }(u_n\omega_n)\;\text{ in }\;\Omega_n,\quad \partial_t\psi^0_n|_{\partial\Omega_n}=0.$$

Using the uniform $L^\infty([0,T],L^2(B(0,\rho_T)))$ bound of u_n (see (3.14) and (3.16)) and the L^∞ bounds on ω_n, we get

$$\|\partial_t\psi^0_n(t,\cdot)\|_{H^1(D)}\;\le\;C,\quad\forall t\in[0,T],\ \forall n\ge N_K.$$

From these bounds and standard compactness lemma, there exists $\psi^0\in W^{1,\infty}(0,T;H^1(D))$ such that up to a subsequence:

$$\psi^0_n\to\psi^0\;\text{ weakly* in }\;W^{1,\infty}(0,T;H^1(D))\text{ and strongly in }C^0(0,T;L^2(D)).$$

We now extend ω_n by 0 outside Ω_n for all n, so that the sequence $(\omega_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^+;L^1\cap L^\infty(\mathbb{R}^2))$. Up to another extraction, we deduce that

$$\omega_n\to\omega\;\text{ weakly * in }\;L^\infty(\mathbb{R}^+;L^1\cap L^\infty(\mathbb{R}^2)). \tag{3.17}$$

From the weak convergence of ψ^0_n and ω_n, we infer that

$$\Delta\psi^0(t,\cdot)=\omega(t,\cdot)\text{ in }\mathcal{D}'(\Omega\cap D),\text{ for almost every }\;t \tag{3.18}$$

using again that any compact subset of Ω is included in Ω_n for n large enough.

Now, we use Proposition 12: as $(\Omega_n\cap B(0,\rho_T))_{n\in\mathbb{N}}$ converges to $\Omega\cap B(0,\rho_T)$ in the Hausdorff sense and as the complement in a large closed ball B of $\Omega_n\cap B(0,\rho_T)$ has at most 2 connected components for all n, $(\Omega_n\cap B(0,\rho_T))_{n\in\mathbb{N}}$ γ-converges to $\Omega\cap B(0,\rho_T)$. Let χ be a cutoff function equal to 1 in a neighborhood of the O_n's, and to 0 outside $B(0,\rho_T)$. By Proposition 13, $\chi\psi^0_n(t,\cdot)$ has for every t a subsequence that converges weakly in $H^1_0(B(0,\rho_T))$ to a limit in $H^1_0(\Omega\cap B(0,\rho_T))$. Thus, for every $t\in[0,T]$, $\chi\psi^0(t,\cdot)$ belongs to $H^1_0(\Omega\cap B(0,\rho_T))$.

Finally, let us prove the strong convergence of ψ^0_n to ψ^0 in $L^2(0,T;H^1(D))$ for all $T>0$. Therefore, we go back to the equation (3.3). As $|\nabla\psi^0_n||\psi^0_n|=\mathcal{O}(1/|x|^3)$, we can integrate by part:

$$\int_0^T\int_{\mathbb{R}^2}|\nabla\psi^0_n|^2=\int_0^T\int_{\Omega_n}|\nabla\psi^0_n|^2=-\int_0^T\int_{\Omega_n}\omega_n\,\psi^0_n=\,-\int_0^T\int_D\omega_n\,\psi^0_n\;\to-\int_0^T\int_D\omega\,\psi^0$$

As we know from the previous paragraph that $\chi\psi^0(t,\cdot)$ belongs to $H^1_0(\Omega\cap B(0,\rho_T))$ for every t, we can perform an energy estimate on (3.18) as well. We get

$$\int_0^T\int_{\mathbb{R}^2}|\nabla\psi^0|^2=\int_0^T\int_{\Omega}|\nabla\psi^0|^2=-\int_0^T\int_\Omega\omega\,\psi^0=-\int_0^T\int_D\omega\,\psi^0$$

Hence,

$$\int_0^T\int_{\mathbb{R}^2}|\nabla\psi^0_n|^2\;\to\;\int_0^T\int_{\mathbb{R}^2}|\nabla\psi^0|^2$$

which together with the weak convergence in $W^{1,\infty}(0,T;H^1(D))$ yields the strong convergence of ψ^0_n to ψ^0 in $L^2(0,T;H^1(K))$.

b) Compactness of the harmonic part.

By the convergence results on $(\mathcal{T}_n)$ (see Proposition 3), we obtain directly that $\psi_n=\frac{1}{2\pi}\ln|\mathcal{T}_n(x)|$ converges uniformly to ψ, resp. to 0, in any compact subset K of Π (the unbounded connected component of Ω), resp. of $\Omega\setminus\Pi$ (the bounded connected components of Ω). As $\psi_n-\psi$ is harmonic, local uniform convergence implies $H^1_{\rm loc}$ convergence by the mean-value theorem.

c) Limit equation.

We can now conclude the proof of Theorem 2. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of Euler solutions in Ω_n, associated to the initial data u^n_0. Each field u_n has the Hodge decomposition (3.2). By diagonal extraction, it *converges strongly in* $L^2_{\rm loc}(\mathbb{R}^+\times\overline{\Omega})$ *and weakly* in* $L^\infty_{\rm loc}(\mathbb{R}^+;L^2_{\rm loc}(\overline{\Omega}))$ to the field

$$u(t,x)\;=\;\begin{cases}\nabla^\perp\psi^0(t,x)\;+\;\alpha\nabla^\perp\psi(x), & \text{if } x\in\Pi,\\ \nabla^\perp\psi^0(t,x), & \text{if } x\in\Omega\setminus\Pi.\end{cases}$$

Note that $\nabla^\perp\psi^0$ belongs to $L^\infty_{\rm loc}(\mathbb{R}^+;L^2(\Omega))$ (see (3.14)) whereas $\nabla^\perp\psi$ is only locally square integrable. It follows that $u\in L^\infty_{\rm loc}(\mathbb{R}^+;L^2_{\rm loc}(\overline{\Omega}))$. From this explicit form, we deduce that u is divergence free, tangent to the boundary, with a conserved circulation along the closed curve J. Moreover, inside Ω, one has

$$\operatorname{curl}u\;=\;\Delta\psi^0\;=\;\omega\in L^\infty(\mathbb{R}^+;L^1\cap L^\infty(\Omega)).$$

The uniform estimate of the support of ω_n means that ω is also compactly supported.

Finally, let $\varphi\in\mathcal{D}\left([0,+\infty[,\mathcal{V}(\Omega)\right)$. For n large enough, the support of φ is included in Ω_n so that:

$$\int_0^\infty\int_\Omega\left(u_n\cdot\partial_t\varphi+(u_n\otimes u_n):\nabla\varphi\right)=-\int_\Omega u^0_n\cdot\varphi(0,\cdot)$$

By the strong $L^2_{\rm loc}$ convergence of u_n to u, and also the strong $L^2_{\rm loc}$ convergence of u^0_n to u^0 (see the previous section for details), it follows that u satisfies the weak form of the Euler equations (1.7).

Let us emphasize that this convergence in $L^2_{\rm loc}$ does not hold for the situation studied in [6] (one small obstacle shrinking to a point), or in [12] (bounded domain with several holes, one of them shrinking to a point). In such situations, the limit velocity is the sum of a smooth part and a harmonic part $x^\perp/|x|^2$, so it does not even belong to $L^2_{\rm loc}$.

4. Initial vorticity in L^p

We complete in this section the proof of Theorems 1 and 2, by passing from L^∞ vorticity to L^p vorticity.

4.1. Theorem 1 for $p > 1$.

Let $p > 1$, u^0 satisfying (1.4). Let $\omega^0 := \operatorname{curl} u^0$. We introduce a sequence of smooth functions $(\omega_n^0)_{n\in\mathbb{N}}$ such that $\omega_n^0 \to \omega^0$ strongly in $L^p(\Omega)$. We remind that we have established in Section 3 that there is for each n and each real k-uplet $c^1,\dots,c^k$ a unique $u_n^0 \in L^2(\Omega)$ satisfying

$$\operatorname{curl} u_n^0 = \omega_n^0, \quad \gamma^i(u_n^0) = c^i, \quad \forall i = 1,\dots,k$$

together with the divergence-free and tangency conditions. We choose here $c^i := \gamma^i(u^0)$. We then denote by u_n a weak solution constructed in Section 3. We denote $\omega_n := \operatorname{curl} u_n$.

Assuming as in Section 3 that u_n is the limit of a sequence of smooth solutions $u_{n,N}$ of Euler in smooth domains Ω_N, we notice by (2.9) that:

$$\|\omega_n\|_{L^\infty(L^p(\Omega))} \le \liminf_{N\to\infty} \|\omega_{n,N}\|_{L^\infty(L^p(\Omega_N))} \le \|\omega_n^0\|_{L^p(\Omega)} \le C_p. \tag{4.1}$$

Then we have, up to a subsequence, the weak $*$ convergence of ω_n to some ω in $L^\infty(\mathbb{R}^+; L^p(\Omega))$.

Moreover, we proved that the velocity can be written as

$$u_n(t,x) \;=\; \nabla^\perp\psi_n^0(t,x) \;+\; \sum_{i=1}^k \alpha_n^i(t)\nabla^\perp\psi^i(x)$$

where

$$\psi_n^0 \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)) \text{ and } \Delta\psi_n^0(t,\cdot) = \omega_n(t,\cdot) \text{ in } \mathcal{D}'(\Omega), \text{ for almost every } t;$$

$$\psi^i \;=\; \sum_{j=1}^k c^{i,j}\,\phi^i \text{ for all } i = 1,\dots,k;$$

$$\alpha_n^i \;=\; \int_\Omega \phi^i\,\omega_n\,dx \;+\gamma^i, \tag{4.2}$$

with

$$\phi^i \in H_0^1(\tilde\Omega),\ \Delta\phi^i = 0 \quad \text{in } \Omega, \quad \phi^i|_{\partial\mathcal{C}^j} = \delta_{ij} \text{ in a weak sense, see Section 3,}$$

and

$$C = \left(c^{i,j}\right)_{1\le i,j\le k} = -\left(\int_\Omega \nabla\phi^i\cdot\nabla\phi^j\right)^{-1}_{1\le i,j\le k}.$$

Note that ϕ^i, ψ^i and C do not depend on n.

By the energy estimate, we obtain that $\|\nabla\psi_n^0(t,\cdot)\|^2_{L^2(\Omega)} \le \|\omega_n(t,\cdot)\|_{H^{-1}(\Omega)}\|\psi_n^0(t,\cdot)\|_{H^1(\Omega)}$, which implies by (4.1) and the Poincaré inequality on a big ball D that ψ_n^0 is uniformly bounded in $L^\infty(\mathbb{R}^+; H_0^1(\Omega))$. Also by (4.1), the sequences $(\alpha_n^i)_{n\in\mathbb{N}}$ are bounded in $L^\infty(\mathbb{R}^+)$.

Therefore, we can write $u_n = \nabla^\perp\psi_n$ with ψ_n bounded in $L^\infty(\mathbb{R}^+; H^1(\Omega))$. By this estimate, we extract a subsequence such that $u_n \to u$ weakly* in $L^\infty(\mathbb{R}^+; L^2(\Omega))$, which implies that u verifies the divergence-free and tangency conditions (1.6).

The last step consists in obtaining strong compactness of (u_n) in $C^0\left([0,T]; L^2_{\rm loc}(\Omega)\right)$, as it allows to pass to the limit in the momentum equation (1.7). A natural idea is to rely on a compactness lemma of Aubin-Lions type, in any set $\Omega' \Subset \Omega$. Indeed, it follows from the uniform L^p bounds on (ω_n) that (u_n) is bounded in $L^\infty(W^{1,p}(\Omega'))$, whereas $(\partial_t u_n)$ is easily shown to be bounded in $L^\infty(H^{-2}_\sigma(\Omega'))$. However, as u_n is not tangent at $\partial\Omega'$, there is no imbedding of $L^2(\Omega')$ into $H^{-2}_\sigma(\Omega')$. Moreover, multiplying by a cutoff function does not solve this problem, as it breaks the divergence-free condition. Hence, we rather follow the ideas developed in [1, Sect. 6]. Let Ω' be a smooth bounded subdomain of Ω:

a) *Decomposition of the pressure.* We remind that $W^{1,p}(\Omega')$ is embedded in L^{p_0} for some $p_0 > 2$. We can apply [19, Theo. 2.6] or [1, Lem. 6], with the couple

$$(u_n, \mathbb{Q}_n := u_n \otimes u_n) \in C_{weak}(0,T; L^2(\Omega')) \times L^{q_0}((0,T) \times \Omega'), \quad q_0 := \frac{p_0}{2} > 1)$$

(see (1.6)-(1.7)). One can write

$$\partial_t u_n + \text{div } Q_n \ = \ \nabla(p_n^r + \partial_t p_n^h)$$

with

$$\Delta p_n^h = 0 \text{ in } \mathcal{D}'((0,T)\times\Omega'), \quad \int_{\Omega'} p_n^h \, dx = 0,$$

and

$$\begin{aligned} \|p_n^r\|_{L^{q_0}((0,T)\times\Omega')} &\leq c\|\mathbb{Q}_n\|_{L^{q_0}((0,T)\times\Omega')}, \\ \|p_n^h\|_{L^\infty(0,T;L^{q_0}(\Omega'))} &\leq c\Big(\|u_n\|_{L^\infty(0,T;L^2(\Omega'))} + \|\mathbb{Q}_n\|_{L^{q_0}((0,T)\times\Omega')}\Big), \end{aligned}$$

where c depend only on Ω', T and q_0.

b) *Compactness :* for any $\Omega'' \Subset \Omega' \Subset \Omega$, by harmonicity,$\nabla p_n^h$ is uniformly bounded in $L^\infty(H^k(\Omega''))$ for all k. Then $u_n - \nabla p_n^h$ is uniformly bounded in $L^\infty(W^{1,p}(\Omega''))$. Moreover, $\partial_t(u_n - \nabla p_n^h) = \nabla p_n^r - \text{div } \mathbb{Q}_n$ is uniformly bounded in $L^{q_0}(W^{-1,q_0}(\Omega''))$. In the spirit of Aubin-Lions compactness lemma, see [16], $(u_n - \nabla p_n^h)$ is strongly compact in $L^2((0,T)\times\Omega'')$ (up to a subsequence).

c) *Passing to the limit in the momentum equation:* Through a diagonal extraction, it follows from the previous step that $u_n - \nabla p_n^h$ converges strongly (up to a subsequence) in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$. Then, we compute:

$$\text{div } \Big(u_n \otimes u_n\Big) = \text{div } \Big((u_n - \nabla p_n^h) \otimes u_n\Big) + \text{div } \Big(\nabla p_n^h \otimes (u_n - \nabla p_n^h)\Big) + \text{div } \Big(\nabla p_n^h \otimes \nabla p_n^h\Big).$$

The first two terms at the right hand side pass easily to the limit. We conclude by noting that

$$\int_\Omega \nabla p_n^h \otimes \nabla p_n^h : \nabla\varphi = -\int_\Omega \Big(\frac{1}{2}\nabla|\nabla p_n^h|^2 \cdot \varphi + \Delta p_n^h \nabla p_n^h \cdot \varphi\Big) = 0,$$

for all $\varphi \in \mathcal{D}\left([0,+\infty[\times\Omega\right)$ with div $\varphi = 0$.

This ends the proof.

4.2. **Theorem 2 for $p > 2$.**

To go from $p = \infty$ to $p > 2$, one follows the lines of the bounded case. Let us remark that for solutions in Theorem 2, we have that $\alpha_n(t) = \alpha_n = \int_{Ext(J)} \omega_n^0 + \int_J u^0 \cdot \tau$ which tends easily to $\alpha := \int_{Ext(J)} \omega^0 + \int_J u^0 \cdot \tau$.

Then, it is sufficient to prove the convergence of the rotational part. In order to obtain an estimate in L^2 of $\nabla\psi_n^0$ independent of n (see (3.14)), we need to control uniformly the size of the support of ω_n. Therefore, we want to prove Lemma 2 with $\|\omega_n\|_{L^p}$ instead of $\|\omega_n\|_{L^\infty}$. Such an extension is possible for $p > 2$ (see e.g. the technics used in [9, Lemma 3.5]). More precisely, we assume $p > 2$ in the unbounded domain in order to obtain an L^∞ estimate of the velocity far from the boundaries, and a uniform control of the size of the support of ω_n.

We conclude by the same compactness argument as above.

5. Final remarks

5.1. **Domain continuity for Euler.**

Theorems 1 and 2 yield existence of global weak solutions in singular domains. However, their proof yields more, namely some domain continuity for the Euler equations. It shows that solutions of Euler in

$$\Omega_n := \widetilde{\Omega}_n \setminus \left(\cup_{i=1}^k \overline{O_n^i}\right), \quad \text{resp. } \Omega_n := \mathbb{R}^2 \setminus \overline{O_n}$$

converge to solutions of Euler in

$$\Omega := \widetilde{\Omega} \setminus \left(\cup_{i=1}^k \mathcal{C}^i\right), \quad \text{resp. } \Omega := \mathbb{R}^2 \setminus \mathcal{C}.$$

We discuss here some consequences of this convergence result.

Rugosity. A typical problem in rugosity theory is the following: let Ω be a smooth domain with a rough wall $y = 0$. Let Ω_ε be obtained from Ω by a boundary perturbation of the form $y = \varepsilon^\alpha \cos(x/\varepsilon)$ ($\alpha > 0$ fixed). What is the asymptotic behaviour of the flow in Ω_ε as $\varepsilon \to 0$? In the case of viscous flows, it has been shown that there is a drastic effect of the rugosity at the limit, see [2, 1]. In the opposite direction, *one can deduce* from our analysis that such effect does not hold for ideal incompressible flows: the solution u_ε of the Euler equations on Ω_ε converges to the solution u of the Euler equations on Ω.

Trapping of a flow. The complements of the domains Ω_n and Ω that we consider have the same number of connected components. Thus, the domain continuity that we show does not extend to the fusion of two obstacles as in Figure 1. In such a case, we do not pretend that u_n solution in Ω_n (see Figure 1) converges to u solution in Ω. Actually, we guess that it does not hold because of the Kelvin's circulation theorem.

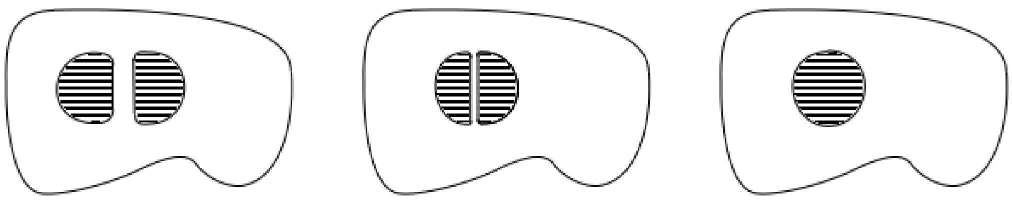

Figure 1. fusion of two obstacles.

However, an example that we can include in our analysis is an obstacle $\overline{O_n}$ which closes on itself (see Figure 2). In this picture, although Ω_n as a unique connected component, Ω has several connected components. Here, $\mathcal{C}$ is a Jordan curve, and it is an example of a compact set obtain as a Hausdorff limit, but not as a decreasing sequence of smooth simply connected domains. In such a setting, the present work still shows that u_n solution in Ω_n (see Figure 2) converges to u solution in Ω.

5.2. **The case of the Jordan arc.**

In this subsection, we pay special attention to the case of a smooth Jordan arc $\mathcal{C}$. We shall denote 0_1 and 0_2 the endpoints of the arc. As mentioned earlier, this geometry has already been investigated by the second author in [8]. In this article, the existence of Yudovitch type solutions is established through an approximation by regular domains Ω_ε. The corresponding regular solutions u_ε and their curl ω_ε are then truncated smoothly over a size ε around the obstacle. The resulting truncations $\tilde{u}_\varepsilon$

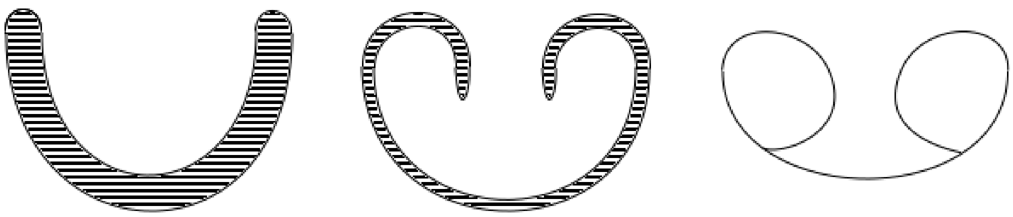

FIGURE 2. $\overline{O_n}$ tends to $\mathcal{C}$ in the Hausdorff sense.

and $\tilde{\omega}_\varepsilon$, defined over the whole of $\mathbb{R}^2$, are shown to converge in appropriate topologies to the solutions $\tilde{u}$, $\tilde{\omega}$ of the system

$$\begin{cases} \operatorname{div} \tilde{u} = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ \partial_t \tilde{\omega} + \tilde{u} \cdot \nabla \tilde{\omega} = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ \tilde{\omega} \ := \ \operatorname{curl} \tilde{u} \ - \ g_{\tilde{\omega}} \delta_{\mathcal{C}}, \quad t > 0,\ x \in \mathbb{R}^2, \end{cases} \tag{5.1}$$

(plus a circulation condition). This is an Euler like system, modified by a Dirac mass along the arc. The density function $g_{\tilde{\omega}}$ is given explicitly in terms of $\tilde{\omega}$ and $\mathcal{C}$. Moreover, it is shown that it is equal to the jump of the tangential component of the velocity across the arc. We refer to [8] for all necessary details. Our point in this section is to link this formulation in the whole space to the classical formulation of the Euler equations in Ω, see (1.7)-(1.5).

More precisely, let u be the solution of (1.7)-(1.5) built in Section 3, and $\omega := \operatorname{curl} u$. We want to show that extending u and ω by 0 yields a solution of (5.1) in $\mathbb{R}^2$. Therefore, we first notice that these extensions (still defined by u and ω) satisfy

$$u \in L^\infty_{\rm loc}(\mathbb{R}^+; L^2_{\rm loc}(\mathbb{R}^2)), \quad \omega \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)).$$

It follows easily from the estimates (3.14)-(3.16), and (2.9)-(2.10). Then, we remark that $u = \nabla^\perp \psi^0 + \alpha \nabla^\perp \psi$ is clearly divergence free over the whole of $\mathbb{R}^2$.

We now turn to the transport equation for the vorticity. Taking $\varphi = \nabla^\perp \psi$ in (1.7), with some ψ compactly supported in $]0, +\infty[\times\Omega$, we first obtain

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \text{ in } \ ]0, +\infty[\times\Omega \tag{5.2}$$

in the distributional sense. Let now

$$\varphi \in \mathcal{D}\left([0, +\infty[\times(\mathbb{R}^2 \setminus \{0_1, 0_2\})\right)$$

be a scalar test function. We want to prove that

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \partial_t \varphi \, \omega + \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \nabla \varphi \cdot (\omega u) = \int_{\mathbb{R}^2} \varphi(0, \cdot) \, \omega^0, \tag{5.3}$$

meaning that the transport equation is satisfied over $\mathbb{R}^2 \setminus \{0_1, 0_2\}$. We introduce a curvilinear coordinate $s \in]0, S[$ and a transverse coordinate $r \in [-R, R]$, so that in a neighborhood of $\mathcal{C} \setminus \{0_1, 0_2\}$, one has $x = J(s) + r\, n(s)$, n a normal vector field. In view of (5.2), we can assume with no loss of generality that φ is compactly supported in this neighborhood. We then consider a truncation function that reads

$$\varphi_\varepsilon(t, x) = \varphi(t, x)(1 - \chi(r/\varepsilon))$$

where $\chi \in C^\infty_c(\mathbb{R})$, $\chi = 1$ near 0. One can use φ_ε as a test function in (5.2). Hence, to prove (5.3), it remains to prove that

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \partial_t \varphi \, \chi_\varepsilon \, \omega + \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \nabla(\varphi \, \chi_\varepsilon) \cdot (\omega u) - \int_{\mathbb{R}^2} \varphi(0, \cdot) \, \chi_\varepsilon \, \omega^0 \to 0, \quad \text{ as } \varepsilon \to 0, \quad \chi_\varepsilon(x) := \chi(r/\varepsilon).$$

The only difficult term is

$$I_\varepsilon \;:=\; \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} (\varphi\, \nabla\chi_\varepsilon)\cdot(\omega u).$$

We remind that the streamfunction $\eta = \psi^0 + \alpha\psi$ associated to u satisfies $\Delta\eta = \omega$ in Ω, and that it is constant at $\mathcal{C}$ by the impermeability condition. As ω is bounded, it follows from elliptic regularity that η has $W^{2,p}$ regularity for all finite p on each side of $\mathcal{C}$, *away from the endpoints* $0_1, 0_2$. In particular, one has

$$\|u\|_{W^{1,p}(K_\varepsilon)} \;\le\; C_p \tag{5.4}$$

over the support K_ε of $\varphi\nabla\chi_\varepsilon$. Denoting $u_n(x) := u(x)\cdot n(s)$ the "normal" component of u, one has

$$\begin{aligned}|I_\varepsilon| \;&\le\; C\int_{K_\varepsilon} \frac{1}{\varepsilon}|\chi'(r/\varepsilon)|\,|u_n(x)|\,dx \;\le\; C\int_{K_\varepsilon} \frac{r}{\varepsilon}|\chi'(r/\varepsilon)|\,\frac{|u_n(x)|}{r}\,dx\\ &\le\; C\sup_\theta\big(\theta\,|\chi'(\theta)|\big)\int_{K_\varepsilon}\frac{|u_n(x)|}{r}\,dx \;\le\; C'\sqrt{\int_{K_\varepsilon} dx}\sqrt{\int_{K_\varepsilon}|\nabla u_n(x)|^2\,dx}\end{aligned}$$

where the last bound comes from the Hardy inequality, applied on each side of $\mathcal{C}$ to u_n (which vanishes at $\mathcal{C}$ by the impermeability condition). It follows from (5.4) that I_ε vanishes to zero with ε, as expected.

Thus, to establish the transport equation for the vorticity on the whole plane, it remains to handle the neighborhood of the endpoints $0_1, 0_2$, say 0_1. This time, we introduce the truncation

$$\chi_\varepsilon(x) := \chi\left(\frac{x-0_1}{\varepsilon}\right) \quad \text{with } \chi \in C_c^\infty(\mathbb{R}^2),\ \chi = 1 \text{ near } 0.$$

As before, one is left with showing that

$$I_\varepsilon \;:=\; \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} (\varphi\, \nabla\chi_\varepsilon)\cdot(\omega u)$$

goes to zero with ε. But this time, as $\nabla\chi_\varepsilon$ is uniformly bounded in L^2, one has the simple inequality

$$|I_\varepsilon| \;\le\; C\,\|\nabla\chi_\varepsilon\|\,\|\omega u\|_{L^2(K_\varepsilon)} \;\le\; C'\|\omega u\|_{L^2(K_\varepsilon)}$$

where K_ε is the support of χ_ε. The r.h.s. goes to zero by Lebesgue dominated convergence theorem, and yields the result.

Eventually, we have to establish the third line of (5.1), which expresses ω in terms of u and a Dirac mass along the arc. Again, we notice that the streamfunction η has $W^{2,p}$ regularity for all finite p on each side of the arc, away from its endpoints. This implies that the velocity u has a trace from each side of the arc, denoted by $u_\pm$. These traces belong to $W^{1-1/p,p}_{\text{loc}}(int(\mathcal{C}))$ for any finite p. By the impermeability condition, only the tangential component of these traces is non-zero. Let now $\varphi \in C_c^\infty(\mathbb{R}^2 \setminus \{0_1, 0_2\})$ a scalar test function. Testing this function with the relation $\omega = \operatorname{curl} u$ (that clearly holds in the strong sense in $\mathbb{R}^2 \setminus \mathcal{C}$), and integrating by parts on each side of the arc, we end up with

$$\int_{\mathbb{R}^2} \omega\,\varphi \;=\; -\int_{\mathbb{R}^2} u\cdot\nabla^\perp\phi \,+\, \int_{\mathcal{C}} [u_\tau]\varphi,$$

almost surely in t, where $[u_\tau]$ *denotes the jump of the tangential component*: if n is the normal going from side $+$ to side $-$, $[u_\tau] \;:=\; (u_+ - u_-)\cdot n^\perp$. The last equation can be written

$$\omega \;=\; \operatorname{curl} u \;-\; g_\omega\delta_{\mathcal{C}} \text{ in } \mathbb{R}^2 \setminus \{0_1, 0_2\}$$

in the sense of distributions, where $g_\omega(s) := [u_\tau](s)$ (s the curvilinear coordinate).

The last step is to go from $\mathbb{R}^2 \setminus \{0_1, 0_2\}$ to $\mathbb{R}^2$. Therefore, we introduce again truncation functions near the endpoints: say

$$\chi_\varepsilon(x) := \chi\left(\frac{x-0_1}{\varepsilon}\right) \quad \text{with } \chi \in C_c^\infty(\mathbb{R}^2),\ \chi = 1 \text{ near } 0.$$

As before, the point is to show that

$$\int_{\mathbb{R}^2} \omega\, \varphi \chi_\varepsilon, \quad \int_{\mathbb{R}}^2 u \cdot \nabla^\perp \phi \chi_\varepsilon, \quad \text{and} \int_{\mathcal{C}} [u_\tau] \varphi \chi_\varepsilon$$

all go to zero with ε. The only annoying quantity is the third one: it requires a control on the jump function $[u_\tau]$ *up to the endpoint* 0_1. Therefore, we use results related to elliptic equations in polygons, see [11]. Indeed, up to a smooth change of variable, the Laplace equation for η in $\mathbb{R}^2 \setminus \mathcal{C}$ turns into a divergence form elliptic equation in the exterior of a slit. In particular, it follows from the results in [11] that $u = \nabla^\perp \eta$ decomposes into $u_1 + u_2$, where u_1 behaves like $1/|x - 0_i|^{1/2}$ near the endpoint 0_i, and u_2 has $W^{1,p}_{\rm loc}(\mathbb{R}^2 \setminus \mathcal{C})$ regularity for all $p < 2$. This allows to conclude that $\int_{\mathcal{C}} [u_\tau] \varphi \chi_\varepsilon$ goes to zero with ε. This concludes the proof.

5.3. **Extension to Delort's solutions.**

Looking closer at the proof of Theorem 1 for general $p > 1$ (see Section 4), we see that uniform bounds on the field u_n in $L^\infty L^2$ only require uniform bounds on $\operatorname{curl} u_n^0$ in $H^{-1} \cap L^1$. From there, one can recover the appropriate initial data, tangency condition and divergence-free condition. Moreover, the obtention of the Euler equation (1.7) on the limit u relies on local properties away from the boundary. Hence, one can replace our compactness (Aubin-Lions) arguments by the analysis led by Delort in [3, section 2.3, p582]. Consequently, it is possible to obtain an analogue of Delort's theorem (solutions with initial vorticity in $H^{-1}_{comp}(\mathbb{R}^2) \cap \mathcal{M}(\mathbb{R}^2)$ of definite sign) in our singular domains.

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Appendix A. Sobolev capacity

We recall here basic notions on Sobolev capacity, taken from [5]. Let $E \subset \mathbb{R}^N$, $N \geq 1$. The capacity of E (with respect to the Sobolev space $H^1(\mathbb{R}^N)$) is defined by

$$\operatorname{cap}(E) := \inf\{\|v\|^2_{H^1(\mathbb{R}^N)},\ v \geq 1 \text{ a.e. in a neighborhood of } E\},$$

with the convention that $\operatorname{cap}(E) = +\infty$ when the set at the r.h.s. is empty. The capacity is not a measure, but has similar good properties:

Proposition 4.

(1) $A \subset B \Rightarrow \operatorname{cap}(A) \leq \operatorname{cap}(B)$.
(2) *Let* $(K_n)_{n\in\mathbb{N}}$ *a decreasing sequence of compact sets, with* $K = \cap K_n$*. Then,* $\operatorname{cap}(K) = \lim \operatorname{cap}(K_n)$.
(3) *Let* $(E_n)_{n\in\mathbb{N}}$ *an increasing sequence of sets, with* $E = \cup E_n$*. Then,* $\operatorname{cap}(E) = \lim \operatorname{cap}(E_n)$.
(4) *(Strong subadditivity) For all sets* A *and* B*, one has*

$$\operatorname{cap}(A \cup B) + \operatorname{cap}(A \cap B) \leq \operatorname{cap}(A) + \operatorname{cap}(B).$$

If D is a bounded open set of $\mathbb{R}^N$, one can also define a capacity relatively to D: for $E \subset D$,

$$\operatorname{cap}_D(E) := \inf\left\{\|\nabla v\|^2_{L^2(D)},\ v \in H^1_0(D),\ v \geq 1 \text{ a.e. in a neighborhood of } E\right\},$$

with the same convention as before. It is clear from this definition and the Poincaré inequality that $\operatorname{cap}(E) \leq C \operatorname{cap}_D(E)$.

For nice sets E in $\mathbb{R}^N$, the capacity of E can be thought very roughly as some $n-1$ dimensional Hausdorff measure of its boundary. More precisely:

Proposition 5.

(1) *For all compact set K included in a bounded open set D,* $\mathrm{cap}(K) = \mathrm{cap}(\partial K)$.
(2) *If $E \subset \mathbb{R}^N$ is contained in a manifold of dimension $N-2$, then* $\mathrm{cap}(E) = 0$.
(3) *If $E \subset \mathbb{R}^N$ contains a piece of some smooth hypersurface (manifold of dimension N-1), then* $\mathrm{cap}(E) > 0$.

The last result concerns $H^1_0(\Omega)$. When Ω is a smooth open set, $H^1_0(\Omega)$ can be defined as the set of function in $H^1(\mathbb{R}^2)$ which are equal to zero almost everywhere in $\mathbb{R}^2 \setminus \Omega$. But this result does not hold for general open sets Ω. To generalize such a characterization, the notion of capacity is appropriate.

Proposition 6. *Let D and Ω be open sets such that $\Omega \subset D$. Then*

$$\Big(v \in H^1_0(\Omega)\Big) \iff \Big(v \in H^1_0(D) \text{ and } v = 0 \text{ quasi everywhere in } D \setminus \Omega\Big),$$

which means that $v = 0$ except on a set with zero capacity.

Appendix B. Hausdorff convergence

We recall here basic notions of Hausdorff topology, taken from [5]. We first introduce the Hausdorff distance for compact sets. Let $\mathcal{K}$ the set of all non-empty compact sets of $\mathbb{R}^N$, $N \geq 1$. For $K_1, K_2 \in \mathcal{K}$, we define

$$d_H(K_1, K_2) \ := \ \max\left(\rho(K_1, K_2), \rho(K_2, K_1)\right), \quad \rho(K, K') \ := \ \sup_{x \in K} d(x, K').$$

It is an easy exercise to show that d_H defines a distance on $\mathcal{K}$. Sequences that converge with respect to this distance are said to converge in the Hausdorff sense. One has the following basic properties

Proposition 7.

(1) *A decreasing sequence of non-empty compact sets converges in the Hausdorff sense to its intersection.*
(2) *An increasing sequence of non-empty compact sets converges in the Hausdorff sense to the closure of its union.*
(3) *Inclusion is stable for convergence in the Hausdorff sense.*
(4) *The Hausdorff convergence preserves connectedness. More generally, if $(K_n)_{n \in \mathbb{N}}$ converges to K, and K_n has at most p connected components, K has at most p connected components.*

A remarkable feature of the Hausdorff topology is given by the following

Proposition 8. *Any bounded sequence of $(\mathcal{K}, d_H)$ has a convergent subsequence.*

From the Hausdorff topology on $\mathcal{K}$, one can define a Hausdorff topology on *confined* open sets, that is on all open sets included in some big given compact. Thus, let B some compact domain in $\mathbb{R}^N$, $N \geq 1$, and $\mathcal{O}_B$ the set of all open sets included in B. The Hausdorff distance on $\mathcal{O}_B$ is defined by:

$$d_H(\Omega_1, \Omega_2) \ := \ d_H(B \setminus \Omega_1, B \setminus \Omega_2)$$

the r.h.s refering to the Hausdorff distance for compact sets. Let us note that this distance does not really depend on B: that is, for $B \subset B'$ two compact sets, and Ω_1, Ω_2 in $\mathcal{O}_B$,

$$d_H(B' \setminus \Omega_1, B' \setminus \Omega_2) \ = \ d_H(B \setminus \Omega_1, B \setminus \Omega_2).$$

Proposition 9.

(1) *An increasing sequence of (confined) open sets converges in the Hausdorff sense to its union.*
(2) *A decreasing sequence of (confined) open sets converges in the Hausdorff sense to the interior of its intersection.*
(3) *Inclusion is stable for convergence in the Hausdorff sense.*
(4) *Finite intersection is stable for convergence in the Hausdorff sense*
(5) *Let (Ω_n) a sequence that converges to Ω in the Hausdorff sense. Let $x \in \partial\Omega$. There exists a sequence (x_n) with $x_n \in \partial\Omega_n$ that converges to x.*

(6) *Let $(\Omega_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{O}_B$. There exists an open set $\Omega \in \mathcal{O}_B$ and a subsequence $(\Omega_{n_k})_{k\in\mathbb{N}}$ that converges to Ω in the Hausdorff sense*

Let us note that the Hausdorff convergence of open sets, contrary to the one of compact sets, does not preserve connectedness. Let us finally point out the following result, to be used later on:

Proposition 10. *If $(\Omega_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense to Ω and K is a compact subset of Ω, then there exists n_0 such that $\Omega_n \supset K$ for $n \geq n_0$.*

Appendix C. γ-convergence of open sets

Let D be a bounded open set. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets included in D. One says that $(\Omega_n)_{n\in\mathbb{N}}$ γ-converges to $\Omega \subset D$ if for any $f \in H^{-1}(D)$, the sequence of solutions $\psi_n \in H^1_0(\Omega_n)$ of

$$-\Delta\psi_n = f \ \text{ in } \ \Omega_n, \quad \psi_n|_{\partial\Omega_n} = 0.$$

converges in $H^1_0(D)$ to the solution $\psi \in H^1_0(\Omega)$ of

$$-\Delta\psi = f \ \text{ in } \ \Omega, \quad \psi|_{\partial\Omega} = 0.$$

In this definition, $H^1_0(\Omega)$ and $H^1_0(\Omega_n)$ are seen as subsets of $H^1_0(D)$, through extension by zero. In a dual way, $H^{-1}(D)$ is seen as a subset of $H^{-1}(\Omega_n)$ and $H^{-1}(\Omega)$. As for the Hausdorff convergence of open sets, the definition of γ-convergence does not depend on the choice of the confining set D.

The notion of γ-convergence is extensively discussed in [5]. The basic example of γ-convergence is given by increasing sequences:

Proposition 11. *If $(\Omega_n)_{n\in\mathbb{N}}$ is an increasing sequence in D, it γ-converges to $\Omega = \cup\Omega_n$. More generally, if $(\Omega_n)_{n\in\mathbb{N}}$ is included in Ω and converges to Ω in the Hausdorff sense, then it γ-converges to Ω.*

In general, Hausdorff converging sequences are not γ-converging. We refer to [5] for counterexamples, with domains Ω_n that have more and more holes as n goes to infinity. This kind of counterexamples, reminiscent of homogenization problems, is the only one in dimension 2, as proved by Sverak [17]:

Proposition 12. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^2$, included in D. Assume that the number of connected components of $D \setminus \Omega_n$ is bounded uniformly in n. If $(\Omega_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense to Ω, it γ-converges to Ω.*

This result will be a crucial ingredient of the next sections.

One can characterize the γ-convergence in terms of the Mosco-convergence of $H^1_0(\Omega_n)$ to $H^1_0(\Omega)$. Namely:

Proposition 13. *$(\Omega_n)_{n\in\mathbb{N}}$ γ-converges to Ω if and only if the following two conditions are satisfied:*

(1) *For all $\psi \in H^1_0(\Omega)$, there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ with ψ_n in $H^1_0(\Omega_n)$ that converges to ψ.*
(2) *For any sequence $(\psi_n)_{n\in\mathbb{N}}$ with ψ_n in $H^1_0(\Omega_n)$, weakly converging to ψ in $H^1_0(D)$, $\psi \in H^1_0(\Omega)$.*

One can also characterize γ-convergence with capacity, see [5, Proposition 3.5.5 page 114]. Let us finally mention that the notion of γ-convergence of open sets is related to the more standard Γ-convergence of Di Giorgi. Losely, Ω_n γ-converges to Ω if the corresponding Dirichlet energy functional J_{Ω_n} Γ-converges to J_Ω: see [5, section 7.1.1] for all details.

Appendix D. Convergence of biholomorphisms in exterior domains

We remind here the notion of kernel convergence introduced by Caratheodory in 1912, see [15, p28] (the word *domain* refers to a connected open set):

Definition 1. *Let (F_n) be a sequence of domains, with $0 \in F_n$ for all n. Its kernel F (with respect to 0) is the set consisting of 0 together with all points $w \in \mathbb{C}$ that satisfy:* there exists a domain H including 0 and w such that $H \subset F_n$ for all n large enough.

If F is the kernel of any subsequence of (F_n), we say that (F_n) converges to F in the kernel sense.

This type of geometric convergence is related to the famous Caratheodory theorem, see [15, Theorem 1.8, p29]:

Proposition 14. *Let (f_n) be a sequence of biholomorphisms from $D := \{|z| < 1\}$ to $F_n := f_n(D)$, with $f_n(0) = 0$, $f_n'(0) > 0$. Then, f_n converges locally uniformly in D if and only if (F_n) converges to its kernel F and if $F \neq \mathbb{C}$. Moreover, the limit function maps D onto F.*

From the Caratheodory theorem, it is possible to deduce the following property, which is crucial in our proof of Theorem 2 (notations are taken from the beginning of paragraph 3.2):

Proposition 15. *Let Π be the unbounded connected component of Ω. There is a unique biholomorphism $\mathcal{T}$ from Π to Δ, satisfying $\mathcal{T}(\infty) = \infty$, $\mathcal{T}'(\infty) > 0$. Moreover, one has the following convergence properties:*

i) *$\mathcal{T}_n^{-1}$ converges uniformly locally to $\mathcal{T}^{-1}$ in Δ.*
ii) *$\mathcal{T}_n$ (resp. $\mathcal{T}_n'$) converges uniformly locally to $\mathcal{T}$ (resp. to $\mathcal{T}'$) in Π.*
iii) *$|\mathcal{T}_n|$ converges uniformly locally to 1 in $\Omega \setminus \Pi$.*

Proof. Let us first point out that, because of Hausdorff convergence and Proposition 10, any compact of Ω is included in Ω_n for n large enough. Thus, the local convergence properties stated in ii) and iii) make sense.

Up to a change of coordinates, we can always assume that $0 \in \partial\Omega \subset \mathcal{C}$. By Proposition 9, there exists $x_n \in \partial\Omega_n$ converging to 0. Then, if we introduce the domains

$$F_n := \frac{1}{(\Omega_n - x_n) \cup \{\infty\}} := \left\{\frac{1}{z},\ z + x_n \in \Omega_n\right\} \cup \{0\},$$

and

$$F := \frac{1}{\Pi \cup \{\infty\}} := \left\{\frac{1}{z},\ z \in \Pi\right\} \cup \{0\},$$

it follows easily from (H1'), Proposition 9 and Proposition 10 that F_n converges to F in the kernel sense. Note that by the choice of (x_n), the F_n's do not include ∞.

Hence, by the Caratheodory theorem, the sequence of biholomorphisms (f_n) defined by

$$f_n : D \mapsto f_n(D) = F_n, \quad f_n(z) := \frac{1}{\mathcal{T}_n^{-1}(1/z) - x_n}$$

converges uniformly locally in D to some function f from D onto F. By the Weierstrass convergence theorem, f is holomorphic over D. Moreover, by a standard application of the Rouché formula, as f_n is one-to-one for all n, so is f. Going on with standard arguments, f is the unique biholomorphism that maps D to F and that satisfies $f(0) = 0$, $f'(0) > 0$. Back to $\mathcal{T}_n^{-1}$, this yields i) with $\mathcal{T}^{-1}(z) := \dfrac{1}{f(1/z)}$. Actually, one has clearly uniform convergence of $\mathcal{T}_n^{-1}$ to $\mathcal{T}^{-1}$ in $\Delta_\delta := \{|z| \geq 1+\delta\}$ for all $\delta > 0$. Then, by the Weierstrass theorem, the sequence of derivatives $(\mathcal{T}_n^{-1})'$ converges locally uniformly to $(\mathcal{T}^{-1})'$.

As regards ii), let $z_0 \in \Pi$, and $J_n := \mathcal{T}_n^{-1}(\{z', |z' - \mathcal{T}(z_0)| = \delta\})$. By i), for $\delta > 0$ small enough and n large enough, J_n is a closed curve that encloses z_0 and is contained in Π. For all z in a small enough neighborhood of z_0, we can then write the Cauchy formula:

$$\mathcal{T}_n(z) = \frac{1}{2i\pi} \int_{J_n} \frac{\mathcal{T}_n(\xi)}{\xi - z} d\xi = \frac{1}{2i\pi} \int_{\{|\xi' - \mathcal{T}(z_0)| = \delta\}} \frac{\xi'}{\mathcal{T}_n^{-1}(\xi') - z} (\mathcal{T}_n^{-1})'(\xi')\, d\xi'$$

where the last equality comes from the change of variable $\xi = \mathcal{T}_n^{-1}(\xi')$. Thanks to i), we may let n go to infinity to obtain the convergence of $\mathcal{T}_n$ to $\mathcal{T}$ uniformly in a neighborhood of z_0. Again, the convergence of derivatives follows from the Weierstrass theorem. This ends the proof of ii).

To obtain iii), we argue by contradiction. We assume *a contrario* that there exists a $\delta > 0$ and a sequence z_n located in a given closed ball B of $\Omega \setminus \Pi$ such that $|\mathcal{T}_n(z_n)| \geq 1 + \delta$. Up to extract a subsequence, we can assume that $z_n \to z \in B$. By the uniform convergence of $\mathcal{T}_n^{-1}$ in Δ_δ (see above) we have $z \in \mathcal{T}^{-1}(\Delta_\delta) \subset \Pi$. Thus, we reach a contradiction, which proves iii).

□

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