Homogenization in polygonal domains

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We consider the homogenization of elliptic systems with ε-periodic coefficients. Classical two-scale approximation yields a $O(\varepsilon)$ error inside the domain. We discuss here the existence of higher order corrections, in the case of general polygonal domains. The corrector depends in a non-trivial way on the boundary. Our analysis extends substantially previous results obtained for polygonal domains with sides of rational slopes.

1 Introduction

This paper is devoted to elliptic systems in divergence form, with Dirichlet boundary condition:

$$\begin{cases} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon = f, & x \in \Omega, \\ u^\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

set in a bounded domain $\Omega \subset \mathbb{R}^d$. For simplicity, we assume $d = 2$ or 3. Following standard notations, $\varepsilon > 0$ is a small parameter, and $A = A^{\alpha\beta}(y) \in M_n(\mathbb{R})$ is a family of functions of $y \in \mathbb{R}^d$, with values in the set of $n \times n$ matrices $M_n(\mathbb{R})$, indexed by $1 \le \alpha, \beta \le d$. The unknown and source term are $u^\varepsilon = u^\varepsilon(x) \in \mathbb{R}^n$ and $f = f(x) \in \mathbb{R}^n$. We remind, using Einstein convention for summation, that for each $1 \le i \le n$,

$$(\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u)_i \;:=\; \partial_{x_\alpha} \left[A^{\alpha\beta}_{ij}\left(\frac{x}{\varepsilon}\right) \partial_{x_\beta} u_j \right].$$

We assume that A and f are smooth. Finally, we make the following hypothesis:

i) Ellipticity: For some $\lambda > 0$, for all family of vectors $\xi = \xi^\alpha \in \mathbb{R}^n$ indexed by $1 \le \alpha \le d$,

$$\lambda\, \xi^\alpha \cdot \xi^\alpha \;\le\; A\xi \cdot \xi \;\le\; \lambda^{-1} \xi^\alpha \cdot \xi^\alpha$$

where $A\xi \cdot \xi$ denotes the sum

$$A\xi \cdot \xi \;:=\; \sum_{\alpha,\beta,i,j} A^{\alpha,\beta}_{ij}\, \xi^\beta_j\, \xi^\alpha_i.$$

ii) Periodicity:

$$A(y+h) \;=\; A(y), \quad \forall y \in \mathbb{R}^d, \forall h \in \mathbb{Z}^d.$$

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We are interested in the limit $\varepsilon \to 0$, *i.e.* the homogenization of system (1.1).

Periodic homogenization has a very long history, and we refer to the classical book [3]. The starting point of most studies is a formal two-scale expansion of the solution u^ε,

$$u^\varepsilon = u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon^2 u^2(x, x/\varepsilon) + \dots \tag{1.2}$$

The leading term u^0 satisfies the homogenized system:

$$\begin{cases} -\nabla \cdot A^0 \nabla u^0 = f, & x \in \Omega, \\ u^0 = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

The homogeneized matrix A^0 comes from the averaging of the microstructure. It involves the periodic solution $\chi = \chi^\gamma(y) \in M_n(\mathbb{R})$, $1 \le \gamma \le d$, of the famous *cell problem*:

$$-\partial_{y_\alpha} \left[A^{\alpha\beta}(y)\, \partial_{y_\beta} \chi^\gamma(y) \right] \; = \; \partial_{y_\alpha} A^{\alpha\gamma}(y), \qquad \int_{[0,1]^d} \chi^\gamma(y)\, dy = 0. \tag{1.4}$$

More precisely A^0 is given by:

$$A^{0,\alpha\beta} \; = \; \int_{[0,1]^d} A^{\alpha\beta} \; + \; \int_{[0,1]^d} A^{\alpha\gamma} \partial_{y_\gamma} \chi^\gamma.$$

The second term in the expansion (1.2) reads

$$u^1(x,y) \; := \; \tilde{u}^1(x,y) + \bar{u}^1(x) \; := \; -\chi^\alpha(y) \partial_{x_\alpha} u^0(x) + \bar{u}^1(x), \tag{1.5}$$

where χ is again the solution of (1.4).

All profiles $u^k = u^k(x,y)$ in (1.2) are periodic in y, and therefore do not satisfy the homogeneous Dirichlet boundary condition. However, the first terms of the expansion are relevant, and the following bound holds (see [3]):

$$\|u^\varepsilon - u^0(x) - \varepsilon\, u^1(x, x/\varepsilon)\|_{H^1(\Omega)} = O(\sqrt{\varepsilon}). \tag{1.6}$$

It is known that such estimate is optimal: as the approximation is not zero at the boundary, there is a boundary layer phenomenon, responsible for a $O(\sqrt{\varepsilon})$ loss in (1.6). However, if a relatively compact subset $\omega \Subset \Omega$ is considered, one may avoid this loss, as strong gradients near the boundary are filtered out. Precisely, Avellaneda and Lin prove in [2], under some regularity assumptions on A and Ω, that

$$\|u^\varepsilon - u^0(x) - \varepsilon u^1(x, x/\varepsilon)\|_{H^1(\omega)} \; = \; O(\varepsilon). \tag{1.7}$$

Following these results, a natural attempt is to derive the next order approximation, and an estimate like:

$$\|u^\varepsilon - u^0(x) - \varepsilon u^1(x, x/\varepsilon) - \varepsilon^2 u^2(x, x/\varepsilon)\|_{H^1(\omega)} \; = \; O(\varepsilon^2). \tag{1.8}$$

However, to obtain this refined approximation turns out to be very difficult, and very much dependent on the geometry of Ω*.* Before stating our results on this problem, let us describe its main difficulties and former studies.

To establish the estimate (1.8), one must first identify the average part $\bar{u}^1(x)$ and the oscillating term $\tilde{u}^2(x,y)$. Note that the choice of $\bar{u}^1(x)$ did not affect previous estimates (1.6), (1.7). Following Allaire and Amar in [1], one needs to introduce another family of 1-periodic matrices

$$\Upsilon^{\alpha\beta} = \Upsilon^{\alpha\beta}(y) \in M_n(\mathbb{R}),\ \alpha, \beta = 1, ..., d,$$

satisfying

$$-\nabla_y \cdot A\, \nabla_y \Upsilon^{\alpha\beta} \ = \ B^{\alpha\beta} - \int_y B^{\alpha\beta}, \quad \int_y \Upsilon^{\alpha\beta} = 0, \tag{1.9}$$

where

$$B^{\alpha\beta} \ := \ A^{\alpha\beta} - A^{\alpha\gamma}\frac{\partial \chi^\beta}{\partial y_\gamma} - \frac{\partial}{\partial y_\gamma}\left(A^{\gamma\alpha}\chi^\beta\right).$$

Formal considerations yield

$$u^2(x,y) \ := \ \Upsilon^{\alpha,\beta}\frac{\partial^2 u^0}{\partial x_\alpha \partial x_\beta} \ - \ \chi^\alpha \partial_\alpha \bar{u}^1. \tag{1.10}$$

The average term $\bar{u}^1 = \bar{u}^1(x)$ formally satisfies the equation

$$-\nabla \cdot A^0 \nabla \bar{u}^1 = c^{\alpha\beta\gamma}\frac{\partial^3 u^0}{\partial x_\alpha \partial x_\beta \partial x_\gamma}, \quad c^{\alpha\beta\gamma} \ := \ \int_y A^{\gamma\eta}\frac{\partial \Upsilon^{\alpha\beta}}{\partial y_\eta} - A^{\alpha\beta}\chi^\gamma. \tag{1.11}$$

We refer to [1] for all details. Note that u^2 depends on $\bar{u}^1$, and has zero average with respect to y. In other words, we take $\bar{u}^2 = 0$. This is enough for a $O(\varepsilon^2)$ approximation, in the same way as taking $\bar{u}^1 = 0$ was enough to obtain a $O(\varepsilon)$ approximation.

Note also that these relations are not enough: to close system (1.11), boundary conditions on $\bar{u}^1$ are required. To derive the correct boundary conditions and obtain the interior estimate (1.8), one needs to understand the behavior of u^ε near the boundary. This is emphasized in article [1, theorem 3.7], where it is shown that:

$$\|u^\varepsilon - u^0(x) - \varepsilon u^1(x, x/\varepsilon) - \varepsilon u^{1,\varepsilon}_{bl}(x) - \varepsilon^2 u^2(x, x/\varepsilon)\|_{H^1(\Omega)} \ = \ O(\varepsilon^{3/2}).$$

with $u^{1,\varepsilon}_{bl}(x)$ the solution of the Dirichlet problem

$$\left\{ \begin{aligned} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^{1,\varepsilon}_{bl} &= 0, \quad x \in \Omega \subset \mathbb{R}^d, \\ u^{1,\varepsilon}_{bl} &= -u^1(x, x/\varepsilon), \quad x \in \partial\Omega, \end{aligned} \right. \tag{1.12}$$

In other words, the construction of high order approximation relies on the homogenization of system (1.12). *The main problem is that the homogenization of this auxiliary system is much harder than the original one.* Indeed, the boundary data in (1.12) forces oscillations within a boundary layer. To understand the structure of these (not anymore periodic) oscillations and their averaged effect is essentially an open question.

Most works on that topic have been limited to convex polygons

$$\Omega \ := \ \cap_{k=1}^N \left\{x, \quad n^k \cdot x > c^k\right\},$$

bounded by N hyperplanes of $\mathbb{R}^d$ with inward unit normal vector n^k:

$$K^k := \left\{x, \quad n^k \cdot x = c^k\right\}, \quad n^k \in S^{d-1}, \quad c^k \in \mathbb{R}, \quad 1 \le k \le N,$$

More precisely, all results have been obtained *under the stringent assumption that the normal vector n^k can be taken in $\mathbb{R}\,\mathbb{Q}^d$, that is proportional to a vector with rational coordinates.* When $d = 2$, this corresponds to *polygons with sides of rational slopes*, and we will keep this terminology for general d. For instance, in [1], Allaire and Amar consider the special case

$$\Omega = [0,1]^d, \quad \varepsilon_n = 1/n.$$

They manage to build correctors, such that a bound of type (1.8) holds when $\varepsilon = \varepsilon_n$. They show that the appropriate boundary conditions on $\bar{u}^1$ read:

$$\bar{u}^1 \; = \; \Gamma^k \, \partial_n u^0, \quad x \in K^k \cap \partial\Omega, \quad 1 \le k \le N, \tag{1.13}$$

with the matrix coefficients $\Gamma^k \in M_n(\mathbb{R})$ linked to some auxiliary boundary layer systems. Numerical schemes based on these correctors are studied in [14, 13]. Let us mention the works [10], where the case of *layered media* is considered.

The existence of accurate approximations has also been studied by Vogelius and coauthors [12, 11], within the slightly different context of eigenvalue problems:

$$\begin{cases} -\nabla \cdot A\left(\dfrac{x}{\varepsilon}\right) \nabla u^\varepsilon = \lambda^\varepsilon u^\varepsilon, \quad x \in \Omega \subset \mathbb{R}^d, \\ \qquad\qquad u^\varepsilon = 0, \quad x \in \partial\Omega, \end{cases}$$

We refer to paper [9] for Neumann boundary conditions. The behavior of λ^ε is investigated, notably the accumulation points of the ratio

$$\frac{\lambda^\varepsilon - \lambda^0}{\varepsilon}, \quad \varepsilon \to 0$$

when λ^0 is a simple eigenvalue of the homogenized system (1.3). The analysis is performed in the case of convex polygons with sides of rational slopes, and relies on the same boundary layer systems as in [1]. It is shown that the ratio does not in general have one limit but rather a continuum of accumulation points. Recast in the framework of article [1], with $\Omega = [0,1]^d$, this result indicates that the constant matrices Γ^k in (1.13) depend on the subsequence ε_n, so that the corrector $\bar{u}^1$ in the approximation (1.8) also depends on the subsequence ε_n (which is $\varepsilon_n = 1/n$ in [1]). Crudely, one can then say that *for convex polygons with sides of rational slopes, estimate* (1.8) *does not hold uniformly in* ε.

The aim of this paper is to consider general convex polygonal domains Ω, that is without the assumption of rational slopes. We will show that *"generically", there exists a $O(\varepsilon^2)$ two-scale approximation of u^ε inside Ω.*

Our main assumption will be a diophantine condition on the normals $n := n^k$, $k = 1...N$:

(A) There is $c, l > 0$, such that: $\forall \xi \in \mathbb{Z}^d \setminus \{0\}, \; |n \times \xi| \; \ge \; c\,|\xi|^{-l}$, with $n \times \xi := n_2\xi_1 - n_1\xi_2$ when $d = 2$, and $n \times \xi$ is the usual cross product when $d = 3$. If $d = 2$, one can replace the cross product in assumption **(A)** by a scalar product, namely $|n \cdot \xi| \geq c\,|\xi|^{-l}$. If $d = 3$, then assumption **(A)** is equivalent to the fact that each two components of n, say (n_1, n_2), satisfy: $\forall \xi \in \mathbb{Z}^2 \setminus \{0\}$, $|n_1\xi_1 + n_2\xi_2| \geq c\,|\xi|^{-l}$. *We emphasize that this condition is generic, in the sense that it is satisfied for almost every* $n^1, ..., n^N$. This is a direct consequence of the following classical result (see [4]): For almost any vector $\nu \in \mathbb{R}^d$, for all $\delta > 0$, there exists $c > 0$ such that

$$|\nu \cdot \xi| \geq c\,|\xi|^{-d-\delta}, \quad \forall \xi \in \mathbb{Z}^d - \{0\}.$$

Besides this small divisor assumption, we will need technical assumptions on u^0, u^1, due to possible loss of regularity near the edges and vertices of Ω. Namely, we will assume that

(A0) The solution u^0 of (1.3) belongs to $H^3(\Omega) \cap C^2(\overline{\Omega})$.

(A1) The solution $\bar{u}^1$ of (1.11)-(1.13), with Γ^k defined in (3.2), belongs to $H^2(\Omega) \cap C^1(\overline{\Omega})$.

The relevance of hypothesis (A0), the well-posedness of (1.11)-(1.13), and the relevance of hypothesis (A1) will be discussed extensively in section 3.

We can state our main result:

Theorem 1 *Let* $\Omega = \cap_{k=1}^N \left\{x, \quad n^k \cdot x > c^k\right\}$ *be a convex polygonal domain. Suppose that for all* k*, the normal vector* $n = n^k$ *satisfies the diophantine condition* **(A)**, *and that the regularity conditions (A0) and (A1) hold. Then, for any open subset* $\omega \Subset \Omega$,

$$\|u^\varepsilon - u^0(x) - \varepsilon u^1(x, x/\varepsilon) - \varepsilon^2 u^2(x, x/\varepsilon)\|_{H^1(\omega)} = O(\varepsilon^2),$$

with u^0, $\bar{u}^1$ *as in (A0) and (A1), and* u^1, u^2 *as in* (1.5) *and* (1.10).

The technical constraints (A0)-(A1) being set aside, this shows that for generic polygonal domains, there exists an ε^2 two-scale approximation of u^ε. Note that the higher order correction in (1.8) is independent of the subsequence in ε. In that respect, the case of rational slopes is peculiar. In this case, as can be deduced from [1, 9] in the periodic case, the higher order correction may depend on the sequence.

The main part of the proof of theorem 1 is the treatment of the boundary layer. In previous studies, the rational slopes allowed to get periodicity in the tangential variable. In the case of general irrational slopes, only a quasiperiodicity property is available, making the construction of boundary layer correctors more intricate. Such construction is performed in section 2. The derivation of u^1, u^2, and the proof of estimate (1.8) follows in section 3. As we will see from the proof, we have a more precise version of theorem 1 (see Corollary 1).

2 Homogenization of the boundary layer

2.1 Formal expansion

As emphasized in the introduction, the search for high order approximations resumes to the understanding of the Dirichlet problem (1.12). Formally, one expects $u_{bl}^{1,\varepsilon}$ to be localized in the vicinity of the hyperplanes of Ω:

$$u_{bl}^{1,\varepsilon}(x) = \sum_{k=1}^N u_{bl}^{1,\varepsilon,k}(x),$$

where $u_{bl}^{1,\varepsilon,k}(x)$ describes a boundary layer near K^k. Note that by convexity, Ω lies on one side of K^k, for all $1 \le k \le N$. Hence,

$$\Omega \subset \left\{x, \quad \left(n^k \cdot x - c^k\right) > 0\right\}.$$

We look for an approximation of the type:

$$u_{bl}^{1,\varepsilon,k} \approx v_{bl}^k\left(x, \frac{x}{\varepsilon}\right),$$

where $v_{bl}^k = v_{bl}^k(x,y) \in \mathbb{R}^n$ is defined for $x \in \Omega$, and y in the half-space

$$\Omega^{\varepsilon,k} = \left\{y, \quad n^k \cdot y - c^k/\varepsilon > 0\right\}$$

Plugging this approximation in (1.12) yields

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y v_{bl}^k = 0, & y \in \Omega^{\varepsilon,k}, \\ \qquad v_{bl}^k = -u_1(x,y), & y \in \partial\Omega^{\varepsilon,k}. \end{cases} \tag{2.1}$$

Note that the variable x is only a parameter in this system. Let M^k be an orthogonal matrix that maps the canonical vector $e_d = (0, ..., 0, 1)$ to the normal vector n^k. By the change of variable $y = M^k z$, system (2.1) becomes

$$\begin{cases} -\nabla_z \cdot B^k(M^k z)\nabla_z v^k = 0, & z_d > \dfrac{c^k}{\varepsilon}, \\ \qquad v^k = -u_1(x, M^k z), & z_d = \dfrac{c^k}{\varepsilon}, \end{cases} \tag{2.2}$$

with unknown $v^k(x,z) = v_{bl}^k(x, M^k z)$. Denoting $A_{ij}^{\alpha\beta}$, resp. $B_{ij}^{k,\alpha\beta}$, $1 \le i,j \le n$, the coefficients of $A^{\alpha\beta}$, resp. $B^{k,\alpha\beta}$, we remind the relation

$$\forall i,j, \quad B_{ij}^k = M^k A_{ij} (M^k)^t$$

which is a product of matrices in $M_d(\mathbb{R})$. We also denote $z = (z', z_d)$ the tangential and normal component of z. *We stress that v_{bl}^k and v^k still depend on ε, through the c^k/ε term.* As will be clear from the developments below, this dependence is harmless, so that we omit it in the notations.

The proof of theorem 1 relies mostly on the analysis of system (2.2). In the case of polygons with sides of rational slopes, for which n_k belongs to $\mathbb{R}\mathbb{Q}^d$, one can choose a matrix M^k with columns that are also in $\mathbb{R}\mathbb{Q}^d$, so that system (2.2) has coefficients that are still periodic in z'. Working in spaces of functions periodic in z', one has easily existence and uniqueness of a variational solution. Moreover, using a lemma from Tartar, one can show the convergence towards a constant of this solution, as z_d goes to infinity, exponentially fast. We refer to [1] for all details. The basic ingredient used in the study of this *rational case* is the Poincaré inequality

$$\int_{T^{d-1}} |\tilde{\varphi}|^2 dz' \le C \int_{T^{d-1}} |\nabla \tilde{\varphi}|^2 dz'$$

for $L-$periodic functions $\tilde{\varphi}$ with zero average.

These properties fail to be true for general polygons: the coefficients are not anymore periodic, but quasiperiodic. We refer to [7] for a description of quasiperiodic and almost periodic functions. Quasiperiodicity does not allow to restrict the tangential variable to a bounded domain, and Poincaré's inequality is not anymore valid. As detailed in the next paragraph, we will still be able to deal with system (2.2), under the generic diophantine assumption **(A)**.

2.2 Boundary layer system

Directly inspired by (2.2), we introduce the following system:

$$\begin{cases} -\nabla_z \cdot \; B(Mz)\nabla_z v = 0, & z_d > a \\ \qquad\quad v(z) = v_0(Mz), & z_d = a. \end{cases} \tag{2.3}$$

where B shares the same properties as the original matrix A, v_0 is a smooth $1-$periodic function and M is a $d \times d$ orthogonal matrix. We wish to show the well-posedness of this system. Moreover, as in the case of rational slopes, we expect the solution to converge towards a constant vector as z_d goes to infinity. Let $N \in M_{d,d-1}(\mathbb{R})$ be defined by

$$Nz' \;=\; M(z', 0).$$

The structure of (2.3) suggests to look for a solution of the type:

$$v(z) \;=\; V\big(Nz', z_d\big), \quad V\big(\theta, t\big) \text{ 1-periodic in } \theta \in \mathbb{R}^d. \tag{2.4}$$

Accordingly, we define

$$\mathcal{B}\big(\theta, t\big) = B\big(\theta + M(0,t)\big), \quad V_0\big(\theta, t\big) = v_0\big(\theta + M(0,t)\big)$$

This leads to the following system, for $\theta \in \mathbb{T}^d, \; t > a$:

$$\begin{cases} -\begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \; \mathcal{B}(\theta, t) \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} V = 0, & t > a \\ \qquad\qquad V(\theta, t = a) = V_0(\theta, t = a), & t = a. \end{cases} \tag{2.5}$$

As this new formulation reveals, the solvability of (2.5) is unclear. The problem is the lack of coerciveness of the new operator with respect to θ. For instance, we *do not have* in general

$$\int_{T^d} |N^t \, \nabla_\theta \, \phi|^2 \, d\theta \;\ge\; c \int_{T^d} |\nabla_\theta \phi|^2 \, d\theta \tag{2.6}$$

This can be understood easily in the two-dimensional case: if M is a rotation matrix

$$M = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad N = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

and inequality (2.6) would give (using Plancherel identity): for all $\xi_1, \xi_2 \in \mathbb{Z}^2$

$$(\xi_1 \cos\alpha + \xi_2 \sin\alpha)^2 \;\ge\; c(|\xi_1|^2 + |\xi_2|^2)$$

which is never satisfied uniformly for large ξ_1, ξ_2. The well-posedness issue is considered in the next paragraph.

Another issue to be considered after well-posedness is the asymptotic behavior of V as $t \to +\infty$. Arguments in [1] for the periodic setting do not adapt to our quasiperiodic setting. To overcome this difficulty, we will make a crucial use of the small divisor assumption **(A)**. Note that a straightforward reformulation is

(A) There exists $c, l > 0$, such that for all $\xi \in \mathbb{Z}^d - \{0\}$, $|N^t \xi| \ \geq \ c\, |\xi|^{-l}$

It will be used in this form to show convergence to a constant field at infinity.

2.3 Well-posedness

We have the following well-posedness result for system (2.5):

Proposition 2 *There exists a unique smooth solution V of* (2.5) *such that*

$$\int_{\mathbb{T}^d} \int_a^{+\infty} \left(|N^t \, \nabla_\theta \partial_\theta^\gamma V|^2 + |\partial_t^l \partial_\theta^\gamma V|^2 \right) dt \, d\theta \ < \ +\infty$$

for $l \geq 1$, and $\gamma \in \mathbb{N}^d$ and where we denote $\partial_\theta^\gamma = \partial_{\theta_1}^{\gamma_1} ... \partial_{\theta_d}^{\gamma_d}$. As a consequence, $v(z) = V(Nz', z_d)$ is a smooth solution of (2.3).

The proof of the proposition relies on the following simple estimate.

Lemma 3 *If $Y(\theta, t)$ is a smooth function solving*

$$\begin{cases} -\begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}(\theta, t) \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} Y = H + \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot G, \quad t > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Y = 0, \quad t = 0. \end{cases} \tag{2.7}$$

where tH, $G \in L^2\left(\mathbb{T}^d \times \mathbb{R}_+\right)$, then

$$\int_{\mathbb{T}^d} \int_0^{+\infty} \left(|N^t \, \nabla_\theta Y|^2 + |\partial_t Y|^2 \right) dt \, d\theta \ \leq \ C \int_{\mathbb{T}^d} \int_0^{+\infty} |tH|^2 + |G|^2 \, dt \, d\theta. \tag{2.8}$$

Proof of the lemma.
Multiplying by Y and integrating over $\mathbb{T}^d \times \mathbb{R}^+$, we obtain

$$\int_{\mathbb{T}^d} \int_0^{+\infty} \left(|N^t \, \nabla_\theta Y|^2 + |\partial_t Y|^2 \right) dt \, d\theta \ \leq \ \int_{\mathbb{T}^d} \int_0^{+\infty} (tH) \cdot \frac{Y}{t} \, dt \, d\theta + \int_{\mathbb{T}^d} \int_0^{+\infty} G \cdot \begin{pmatrix} N^t \\ \partial_\theta \end{pmatrix} Y \, dt \, d\theta$$

By Hardy's inequality,

$$\| \frac{Y}{t} \|_{L^2(\mathbb{T}^d \times \mathbb{R}^+)} \ \leq \ C \, \|\partial_t Y\|_{L^2(\mathbb{T}^d \times \mathbb{R}^+)} \ \leq \ C \, \| \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} Y \|_{L^2(\mathbb{T}^d \times \mathbb{R}^+)}$$

Using this bound and Cauchy-Schwartz inequality in the previous inequality yields the result.

Proof of the proposition.
Without loss of generality, one can assume $a = 0$. Let $\delta(t)$ be a smooth truncation function satisfying $\delta = 1$ on $[0, 1/2]$ and $\delta = 0$ outside $[0, 1]$. Introducing

$$Y \ = \ V \ - \ \delta(t) \, V_0,$$

the problem reduces to the well-posedness of

$$\begin{cases} -\begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}(\theta, t) \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} Y = F, \quad t > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad Y = 0, \quad t = 0. \end{cases} \tag{2.9}$$

where F is smooth, periodic in θ, and has support in $t \leq 1$.

A priori estimates

Suppose Y is a smooth solution of system (2.9). Using (2.8) with $H = F$ and $G = 0$ yields the L^2 estimate

$$\int_{\mathbb{T}^d}\int_0^{+\infty}\left(|N^t\,\nabla_\theta Y|^2 + |\partial_t Y|^2\right) dt\,d\theta \;\le\; C\int_{\mathbb{T}^d}\int_0^{+\infty}|F|^2\,dt\,d\theta \qquad (2.10)$$

The same type of estimates extends easily to tangential derivatives. Namely, for $|\alpha| \ge 0$

$$\int\int\left(|N^t\,\nabla_\theta\,\partial_\theta^\alpha\,Y|^2 + |\partial_t\,\partial_\theta^\alpha\,Y|^2\right) \;\le\; C(\alpha)\sum_{|\beta|\le|\alpha|}\int\int|\partial_\theta^\beta\,F|^2\,dt\,d\theta. \qquad (2.11)$$

Indeed, for $|\alpha| = 1$, we differentiate (2.9) with respect to ∂_{θ_α} for some $1 \le \alpha \le d$ and then apply lemma 3 with $H = \partial_{\theta_\alpha}F$ and $G = \partial_{\theta_\alpha}\mathcal{B}(\theta,t)\begin{pmatrix} N^t\nabla_\theta \\ \partial_t\end{pmatrix}Y$. The general case is obtained by induction on the number of derivatives.

Then, standard elliptic arguments provide additional regularity with respect to t. We first notice that equation (2.9) can be written

$$\mathcal{B}_{d,d}\,\partial_t^2 Y = G, \qquad \text{with } G \in L^2_t(H^s(\mathbb{T}^d)), \quad \forall s \in \mathbb{N}, \qquad (2.12)$$

where we used (2.11) to estimate G and where $\mathcal{B}^{ij}_{d,d} = M_{d\alpha}A^{ij}_{\alpha\beta}\,{}^tM_{\beta d}$ satisfies the coercivity condition $|\mathcal{B}_{d,d}\xi.\xi| \ge \lambda|\xi|$ for $\xi \in \mathbb{R}^n$.

Inverting $\mathcal{B}_{d,d}$, we deduce the same regularity for $\partial_t^2 Y$, which implies that $\partial_t Y|_{t=0}$ belongs to $H^s(\mathbb{T}^d)$ for all s. From there, we may differentiate the equation in t, recover a homogeneous Dirichlet condition by a change of unknown, and apply the previous arguments. Reasoning recursively, we obtain easily: for all $\alpha \in \mathbb{N}^d$, for all $k \ge 1$,

$$\int_{\mathbb{R}^+}\|N^t\,\nabla_\theta\partial_\theta^\alpha Y\|^2_{H^s(\mathbb{T}^d)} + \|\partial_t^k Y\|^2_{H^s(\mathbb{T}^d)} \;\le\; C(F,s,k) < +\infty. \qquad (2.13)$$

We point out here that we lack an estimate for Y itself, that is without any derivative.

Well-posedness

The existence of solutions that satisfy the previous energy estimate can be obtained from standard elliptic regularization of the system. On can for instance consider the approximate problems

$$\begin{cases} -\delta\Delta_\theta V - \begin{pmatrix} N^t\nabla_\theta \\ \partial_t\end{pmatrix}\cdot\,\mathcal{B}(\theta,t)\begin{pmatrix} N^t\nabla_\theta \\ \partial_t\end{pmatrix}V = 0, & t > a \\ V(\theta,t=a) = V_0(\theta,t=a), & t = a. \end{cases}$$

for a small parameter $\delta > 0$. As the system is strongly elliptic for each δ, one can show easily existence and uniqueness of a smooth solution V_δ, that satisfy all previous estimates uniformly with respect to δ. As $\delta \to 0$, one gets easily a smooth solution V of (2.5). Uniqueness follows from the basic estimate (2.10).

2.4 Behavior at infinity

The next step in the study of the boundary layer is to understand the behaviour of V as t goes to infinity. In this subsection, we will use the assumption **(A)** to prove the existence of a limit when t goes to infinity for V. First, assumption **(A)** ensures the following inequality:

$$\int_{\mathbb{T}^d} |N^t \nabla_\theta \tilde{\varphi}|^2 \ \geq \ c \, \|\tilde{\varphi}\|^2_{H^{-l}(\mathbb{T}^d)} \tag{2.14}$$

for smooth enough $\tilde{\varphi} = \tilde{\varphi}(\theta)$ with zero average. Combining (2.14) with (2.11), we deduce that for any $s \in \mathbb{N}$,

$$\int_a^{+\infty} \|\tilde{V}\|^2_{H^s(\mathbb{T}^d)} + \|\partial_t^k V\|^2_{H^s(\mathbb{T}^d)} \ \leq \ C(F, s, k) \ < \ +\infty. \tag{2.15}$$

where we decompose

$$V(\theta, t) = \tilde{V}(\theta, t) \ + \ \bar{V}(t), \qquad \int_{\mathbb{T}^d} \tilde{V} \, d\theta = 0.$$

This implies that for all $\alpha \in \mathbb{N}^d$, $k \in \mathbb{N}$, we have, uniformly in θ:

$$\partial_\theta^\alpha \, \partial_t^k \, \tilde{V} \to 0, \quad \partial_\theta^\alpha \, \partial_t^{k+1} \, V \to 0, \quad t \to +\infty$$

However, the behaviour of the average $\bar{V}$ and the speed of convergence are not specified. This is the purpose of the next proposition

Proposition 4 *There exists a constant vector $v^a \in \mathbb{R}^n$ such that*

$$\lim_{t \to +\infty} V = v^a.$$

More precisely,

$$\lim_{t \to +\infty} \left| t^m \, \partial_\theta^\alpha \, \partial_t^k \, (V - v^a) \right| \ = \ 0,$$

for all $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$, $k \in \mathbb{N}$, uniformly in θ.

Note that the solution V of (2.5) depends on a (and also on B, M, V_0), a fact that we have omitted so far in our notations. Here, we only keep track of this dependence in the limit v^a, as it will be of interest to us later on.

Proof.
To prove proposition 4, we establish an integro-differential inequality on

$$f(T) \ := \ \int_{\mathbb{T}^d} \int_T^{+\infty} \left(|N^t \, \nabla_\theta \ V|^2 + |\partial_t \, V|^2 \right) dt \, d\theta.$$

Let $T > a$, and for $t \geq T$, we define

$$W := V - \int_{\mathbb{T}^d} V(\theta, T) d\theta$$

For $t \geq T$, W satisfies

$$- \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \ \mathcal{B}(\theta, t) \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} W = 0.$$

Multiplying by W and integrating for $\theta \in \mathbb{T}^d$, $t \geq T$, we get

$$\int_{\mathbb{T}^d}\int_T^{+\infty}\Big(|N^t\,\nabla_\theta\,W|^2+|\partial_t\,W|^2\Big) \;=\; -\int_{\mathbb{T}^d}\Big[\big(\begin{smallmatrix}0_{d-1}\\1\end{smallmatrix}\big)\cdot\mathcal{B}\Big(\begin{smallmatrix}N^t\nabla_\theta\\\partial_t\end{smallmatrix}\Big)\,W\Big]\,W(\theta,T)\,d\theta$$
$$\leq\; C\left(\int_{\mathbb{T}^d}\Big(|N^t\,\nabla_\theta\,W|^2+|\partial_t\,W|^2\Big)(\theta,T)\,d\theta\right)^{1/2}\left(\int_{\mathbb{T}^d}|W(\theta,T)|^2\,d\theta\right)^{1/2}$$

As $\nabla_\theta W = \nabla_\theta V,\ \partial_t W = \partial_t V,$ and $W(\theta,T)=\tilde V(\theta,T)$, this last inequality reads

$$f(T)\;\leq\;C(-f'(T))^{1/2}\left(\int_{\mathbb{T}^d}|\tilde V(\theta,T)|^2\,d\theta\right)^{1/2}.$$

Now, by assumption **(A)**, for all $1<p<+\infty$, for all smooth enough $\tilde\varphi$ with zero average, we have:

$$\int_{\mathbb{T}^d}|\tilde\varphi|^2\,d\theta\;\leq\;C\left(\int_{\mathbb{T}^d}|N^t\,\nabla_\theta\,\tilde\varphi|^2\right)^{1/p}\left(\|\tilde\varphi\|_{H^{l/(p-1)}(\mathbb{T}^d)}\right)^{2-2/p},$$

where the index l is the same as in **(A)**. Such an inequality is a straightforward consequence of Plancherel formula and Hölder inequality (together with the small divisor assumption). Applying this to $\tilde V(\theta,T)$, we obtain

$$\int_{\mathbb{T}^d}|\tilde V(\theta,T)|^2\,d\theta\;\leq\;(-f'(T))^{1/p}\left(\|\tilde V(\cdot,T)\|_{H^{l/(p-1)}(\mathbb{T}^d)}\right)^{2-2/p}$$
$$\leq\;C\,(-f'(T))^{1/p}$$

bounding the last term thanks to (2.15). This yields the integro-differential inequality

$$f(T)\;\leq\;C(p)\,(-f'(T))^{\frac{p+1}{2p}} \tag{2.16}$$

for any $1<p<+\infty$. This leads in turn to

$$f(T)\;\leq\;C'(p)\,T^{\frac{p+1}{1-p}}.$$

It shows that $f(T)$ decays faster than any power of T as T goes to infinity.

By differentiation of (2.5a) and similar estimates, one shows by induction on $|\alpha|+k$ that

$$f_{\alpha,k}(T)\;:=\;\int_{\mathbb{T}^d}\int_T^{+\infty}\Big(|N^t\,\nabla_\theta\,\partial_\theta^\alpha\,\partial_t^k\,V|^2+|\partial_t\,\partial_\theta^\alpha\,\partial_t^kV|^2\Big)\,dt\,d\theta$$

decays faster than any power of T, for any α, k. More precisely, assuming that such decay holds for all $f_{\beta,l}$ with $|\beta|+l<s$, the energy estimate (2.16) is easily replaced by

$$f_{\alpha,k}(T)\;\leq\;C(p,n)\left(\big(-f'_{\alpha,k}(T)\big)^{\frac{p+1}{2p}}\;+\;T^{-n}\right),\quad \forall n,p>1,\quad \forall\,\alpha,k \text{ with } |\alpha|+k=s.$$

From there, one gets

$$f_{\alpha,k}(T)\;+\;T^{\frac{p+1}{1-p}}\;\leq\;C\left((-f'_{\alpha,k}(T))^{\frac{p+1}{2p}}\;+\;T^{\frac{p+1}{1-p}}\right)\;\leq\;C'\left(-f'_{\alpha,k}(T)+\;T^{\frac{2p}{1-p}}\right)^{\frac{p+1}{2p}}$$

that is

$$g_{\alpha,k}(T) \;\le\; C'' \left(-g'_{\alpha,k}(T)\right)^{\frac{p+1}{2p}}, \quad g_{\alpha,k}(T) \;:=\; f_{\alpha,k}(T) \;+\; T^{\frac{p+1}{1-p}},$$

and one can conclude as above.

Using again (2.14) and Sobolev imbedding, we deduce that

$$\lim_{t\to+\infty} \left| t^m\, \partial_\theta^\alpha\, \partial_t^k \tilde V \right| \;=\; 0, \quad \lim_{t\to+\infty} \left| t^m\, \partial_\theta^\alpha\, \partial_t^{k+1} V \right| \;=\; 0,$$

for all $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$, $k \in \mathbb{N}$, uniformly in θ.

It remains to show the convergence of the average $\bar V = \bar V(t)$. We write

$$|\bar V(t+h) - \bar V(t)| \le \int_t^{t+h} \left| \frac{d}{dt} \bar V \right| \;\le\; C(p) \int_t^{t+h} (1+s)^{-p}\, ds$$

for all p. This shows that $\bar V(t)$ is a Cauchy function, hence convergent to a constant vector v^a as t goes to infinity. Moreover, the rate of convergence is faster than any power function of t.

Back to the original system (2.3), previous results provide a unique smooth solution $v = v(z)$ that converges to a constant v^a as $z_d \to +\infty$. Looking closer at Proposition 4 and its proof, we have: for all $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^{d-1}$, $k \in \mathbb{N}$,

$$\lim_{(z_d-a)\to+\infty} \left| (z_d - a)^m\, \partial_{z'}^\alpha\, \partial_{z_d}^k\, (v - v^a) \right| = 0, \tag{2.17}$$

locally uniformly in z', uniformly in a. We end this section with a crucial property of the constant vector v^a.

Proposition 5 *Let M be the matrix given in* (2.3)*, and $e_d = (0, ..., 0, 1)$ the d-th canonical vector. If $Me_d \notin \mathbb{R}\mathbb{Q}^d$, then v^a is independent of a.*

Note that in the case $M = M^k$, *cf.* (2.2), $Me_d = n^k$ is a normal vector at $\partial\Omega \cap K^k$.

Proof.
We start by the following lemma:

Lemma 6 *v^a depends continuously on a.*

Proof of the lemma.
Let a and a' be two real values, and V, V' the corresponding solutions of (2.5). We denote $\delta = a' - a$. We introduce

$$V'_\delta(\theta, t) \;=\; V'(\theta, t+\delta), \quad \theta \in \mathbb{T}^d,\ t > a.$$

We have:

$$|V'_\delta(\theta,t) - V'(\theta,t)| \;\le\; |\delta|\, \|\partial_t V'\|_{L^\infty} \;\le\; C\,|\delta|. \tag{2.18}$$

Now, V and V'_δ are defined on the same domain, and $W = V - V'_\delta$ satisfies

$$\begin{cases} -\begin{pmatrix} N^t\nabla_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}(\theta,t) \begin{pmatrix} N^t\nabla_\theta \\ \partial_t \end{pmatrix} W = F, & t > a \\ W = W_0, & t = a, \end{cases} \tag{2.19}$$

where

$$F := \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot (\mathcal{B}(\theta,t) - \mathcal{B}(\theta,t+\delta)) \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} V'_\delta, \quad W_0 := V_0(\theta,a) - V_0(\theta,a+\delta).$$

Note that these source terms satisfy

$$|\partial_\theta^\alpha \partial_t^k F| + |\partial_\theta^\alpha \partial_t^k W_0| \le C_{\alpha,k}\,|\delta|, \quad \forall \alpha, k.$$

Moreover, by proposition 4, F and its derivatives converge to zero uniformly in θ, faster than any power of t. With this decay property, it is straightforward to adapt the energy estimates performed in the proof of propositions 2 and 4. As a consequence, using again the assumption **(A)** , we deduce that W satisfies $\|W\|_{L^\infty} \le C|\delta|$ which reads

$$|V(\theta,t) - V'_\delta(\theta,t)| \le C\,|\delta|. \tag{2.20}$$

uniformly in θ, t. Combining (2.18), (2.20) we deduce as t goes to infinity:

$$|v^a - v^{a'}| \le C|a-a'|$$

which proves the lemma.

We can now end the proof of proposition 5. Let $\xi \in \mathbb{Z}^d$. If v satisfies system (2.3), then $v_\xi(z) = v(z + (M)^t\,\xi)$ satisfies

$$\begin{cases} -\nabla_z \cdot\, B(Mz)\nabla_z v_\xi = 0, & z_d > a - \xi\cdot Me_d \\ \qquad\quad v_\xi(z) = v_0(Mz), & z_d = a - \xi\cdot Me_d. \end{cases}$$

It is deduced easily from the periodicity of B and v_0 and the property $(M)^t = M^{-1}$. Hence, the constant at infinity satisfies

$$v^a = v^{a-\xi\cdot Me_d}.$$

If $Me_d \notin \alpha\mathbb{Q}^d$, for any $\alpha \in \mathbb{R}$, then the set $\{\xi\cdot Me_d,\, k \in \mathbb{Z}^d\}$ is dense in $\mathbb{R}$, and by continuity of v^a with respect to a, the result follows.

3 High order approximation

Thanks to the boundary layer analysis of the previous section, we shall prove Theorem 1. From now on, we consider a convex polygonal domain $\Omega = \cap_{k=1}^N \{x,\, n^k\cdot x > c^k\}$ with inward normal vector $n = n^k$ satisfying **(A)** for all k.

3.1 Choice of u^1 and u^2. Discussion of the assumptions (A0) and (A1).

The first step of the proof is to derive the fields u^1 and u^2 for which (1.8) should hold. As described in the introduction, the starting point of this derivation is a formal two-scale expansion of the solution

$$u^\varepsilon \approx u^0(x) + \varepsilon u^1(x,x/\varepsilon) + \varepsilon^2 u^2(x,x/\varepsilon) + \dots$$

whose formal computation is detailed in [1]. The leading term u^0 satisfies the homogenized problem (1.3). The next order term u^1 satisfies (1.5)-(1.11). Finally, the second order term u^2 is given by (1.10).

Of course, system (1.11) is not enough to determine $\bar{u}^1$, as boundary conditions must be prescribed at $\partial\Omega$. These conditions should account for boundary layer phenomena. More precisely, we expect an asymptotic of the type

$$u^\varepsilon \approx u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon u_{bl}^{1,\varepsilon}(x) + \dots$$

where $u_{bl}^{1,\varepsilon}$ satisfies the Dirichlet problem (1.12). Following the formal considerations of section 2, we want to approximate this last term by

$$u_{bl}^{1,\varepsilon}(x) \approx \sum_{k=1}^{N} v_{bl}^k(x, x/\varepsilon)$$

where the boundary layer correctors v_{bl}^k satisfy systems (2.1).

Broadly, the results of the previous section show that there exists some $v^{k,\infty}(x)$ such that

$$v_{bl}^k(x,y) \to v^{k,\infty}(x), \quad \text{as } (y \cdot n^k - c^k/\varepsilon) \to +\infty,$$

uniformly with respect to x and ε. Moreover, the rate of convergence is faster than any negative power of $|y \cdot n^k - c^k/\varepsilon|$. See (2.17). The idea is to chose $\bar{u}^1$ at $\partial\Omega$ so that $v^{k,\infty} = 0$ for all k. In this way, the boundary layer term should be neglectible in all compact subset of Ω, allowing for an estimate like (1.8). To be more specific, let v be the solution of (2.3) provided by Proposition 2, under assumption **(A)**. From Propositions 4 and 5,

$$v(z) \to v^\infty = v^\infty[B, M, v_0], \quad \text{as } z_d \to +\infty, \quad \text{uniformly in } z'.$$

Back to systems (2.1)-(2.2), we introduce for all $1 \le \alpha \le d$, and all $1 \le k \le N$ the matrix $G^{k,\alpha} \in M^n(\mathbb{R})$ whose j-th column is defined by

$$\left(G_{ij}^{k,\alpha}\right)_{1\le i\le n} := -v^\infty \left[B^k, M^k, \left(\chi_{ij}^\alpha\right)_{1\le i\le n}\right], \quad \forall 1 \le j \le n.$$

Finally, we set

$$\bar{u}^1 = G^{k,\alpha} \frac{\partial u^0}{\partial x_\alpha}(x), \quad x \in \partial\Omega \cap K^k, \quad 1 \le k \le N. \tag{3.1}$$

As u^0 is zero at the boundary $\partial\Omega$, this boundary condition is the same as the Robin type condition (1.13), setting

$$\Gamma_{ij}^k := \sum_{\alpha=1}^{d} G_{ij}^{k,\alpha} n_\alpha^k, \quad n^k = \left(n_\alpha^k\right)_{1\le\alpha\le d}, \quad \forall i,j = 1,...,n. \tag{3.2}$$

System (1.11)-(1.13) is well-posed if u^0 is regular enough:

Proposition 7 *If $u^0 \in W^{2,\infty}(\Omega)$, there exists a unique solution $\bar{u}^1 \in H^1(\Omega)$ of* (1.11)-(1.13).

Proof. The main point is to show that the boundary data belongs to $H^{1/2}(\partial\Omega)$, *i.e.* that there exists a $U^1 \in H^1(\Omega)$ such that

$$U^1 = \Gamma^k \, \partial_n u^0, \quad x \in \partial\Omega \cap K^k, \quad 1 \le k \le N.$$

Afterwards, introducing $v^1 = u^1 - U^1$, one obtains an elliptic problem with a homomogeneous boundary condition and a $H^{-1}(\Omega)$ source term. It has a unique variational solution, and yields well-posedness for (1.11)-(1.13).

The difficulty is the lack of regularity near the edges and vertices of Ω. When $d = 2$, the situation is easier. Let O be a vertex. We can assume up to reindexing the hyperplanes,that O belongs to H^1 and H^2. Then, one can even find a constant matrix $G = (G^1, G^2) \in M_n(\mathbb{R}) \times M_n(\mathbb{R})$, such that in the vicinity of 0

$$U^1 \; := \; G^\alpha \partial_\alpha u^0 \; = \; \Gamma^k \, \partial_n u^0, \quad x \in \partial\Omega \cap K^k, \quad k = 1, 2. \tag{3.3}$$

Indeed, condition (3.3) reads

$$G^\alpha \, n^k_\alpha \; = \; \Gamma^k, \quad k = 1, 2.$$

Thus, to prove the existence of G, it is enough to show that the linear mapping

$$M_n(\mathbb{R}) \times M_n(\mathbb{R}) \mapsto M_n(\mathbb{R}) \times M_n(\mathbb{R}), \quad G \mapsto (G^\alpha \, n^1_\alpha, \, G^\alpha n^2_\alpha)$$

is surjective. This follows from its straightforward injectivity. Note that in this case, only H^2 regularity of u^0 is needed.

Note also that the previous reasoning extends directly to the case of an edge (that is the intersection of two hyperplanes) in dimension $d = 3$. Let finally O be a vertex of $\Omega \subset \mathbb{R}^3$, belonging to M sides supported by H^1, ..., H^M. Let us consider a plane H near 0, transverse to the M sides. It intersects Ω along a two-dimensional polygon $\tilde{\Omega}$. Locally near 0, we can describe Ω by spherical type coordinates, that is

$$\Omega \; = \; \{r \, s, \quad 0 < r < \delta, \quad s \in \tilde{\Omega}\}.$$

Applying the results of the case $d = 2$, we can find a smooth function

$$G = (G^1, G^2) : \; \tilde{\Omega} \; \mapsto \; M_n(\mathbb{R}) \times M_n(\mathbb{R})$$

satisfying

$$\sum_{\alpha=1}^{2} G^\alpha(x) \cdot n^k_\alpha = \Gamma_k, \quad x \in K^k \cap \partial\tilde{\Omega}, \quad 1 \le k \le d',$$

Note that G is constant near each vertex of $\tilde{\Omega}$. Back to the domain Ω, we define the lift of the boundary data as

$$U^1(x) \; = \; U^1(t \, s) := \sum_{\alpha=1}^{2} G^\alpha(s) \partial_{x_\alpha} u^0(t \, s).$$

Using the fact that $u^0 \in W^{2,\infty}$ and $\nabla u^0|_{t=0} = 0$, one has easily that $U^1 \in H^1(\Omega)$. This ends the proof of the proposition 7.

The corrections u^1 and u^2 at hand, we will be able to prove the energy estimate (1.8), under the assumptions (A0) and (A1). Let us discuss a little these regularity requirements. Again, the main point is the irregularity of Ω, that limits the smoothing effect of the elliptic operator $\nabla \cdot A^0 \nabla$. Elliptic theory for such polygonal domains has been the matter of many papers. We refer to textbooks [6, 5].

Broadly, for an arbitrary smooth f in (1.3), one can not expect H^s regularity for u^0 when $s > 2$. For the assumption (A0) to hold, f must satisfy some compatibility conditions. These compatibility conditions do not take a simple form, even for a scalar equation ($n = 1$) in dimension 2. For instance, except in the case where the angles of the polygon are of the type $\omega = \pi/n$, $n \in \mathbb{N}$, these conditions are not local near the vertices. We refer to [6] for details. From this point of view, assumption (A0) is restrictive.

We stress however that, if u^0 is regular enough, assumption (A1) is quite natural. For instance, if $n = 1$, $d = 2$, and $u^0 \in H^4(\Omega)$, then $U^1 \in H^3(\Omega) \cap C^1(\overline{\Omega})$ where U^1 is the lift of the boundary data built in the previous proposition. As a result, $v^1 = u^1 - U^1$ satisfies an elliptic equation with constant coefficients, homogenenous boundary condition and source term in H^1. The $H^2 \cap C^1$ regularity of v^1 then follows from standard theory for the Laplace equation in polygonal domains.

Let us stress again, that by the same theory, we do not expect u^1 to be in $H^s(\Omega)$ with $s > 2$. In other words, we do not know if the compatibility conditions imposed on f should be satisfied by the source term in the equation for v^1. We pay attention to this in the next section, where we try to use as little regularity on u^0 and u^1 as possible. From now on, we assume (A0) and (A1).

3.2 Outline of the Proof

For $i = 1, 2$, let $u^i = u^i(x, y)$ be as in the previous paragraph, and let $u_{bl}^{i,\varepsilon}$ be the solutions of

$$\begin{cases} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u_{bl}^{i,\varepsilon} = 0, & x \in \Omega, \\ u_{bl}^{i,\varepsilon} = -u^i(x, x/\varepsilon), & x \in \partial\Omega. \end{cases} \tag{3.4}$$

We shall prove the following error estimates in the next paragraphs:

1. "Global error estimate":

$$\|e^\varepsilon\|_{H^1(\Omega)} = 0(\varepsilon^2), \quad e^\varepsilon := u^\varepsilon - u^0(x) - \varepsilon u^1(x, x/\varepsilon) - \varepsilon^2 u^2(x, x/\varepsilon) - \varepsilon u_{bl}^{1,\varepsilon}(x) - \varepsilon^2 u_{bl}^{2,\varepsilon}(x).$$

2. "Boundary error estimate":

$$\|e_{bl}^\varepsilon\|_{L^2(\Omega)} = 0(\varepsilon), \quad e_{bl}^\varepsilon := u_{bl}^{1,\varepsilon} - \sum_{k=1}^N v_{bl}^k(x, x/\varepsilon) + \varepsilon u_{bl}^{2,\varepsilon},$$

where $v_{bl}^k(x, y)$ is the solution of (2.1) built in the previous section.

Before we establish these bounds, *let us show how they imply Theorem 1.* Let $\omega \Subset \Omega$. By the "global error estimate", we get that

$$\|u^\varepsilon - u^0(x) - \varepsilon u^1(x, x/\varepsilon) - \varepsilon^2 u^2(x, x/\varepsilon)\|_{H^1(\omega)} \le C\,\varepsilon^2 + \|\sum_{k=1}^N v_{bl}^k(x, x/\varepsilon)\|_{H^1(\omega)} + \varepsilon\|e_{bl}^\varepsilon\|_{H^1(\omega)}.$$

By our choice of $\bar{u}^1$, the boundary layer terms $v_{bl}^k(x, y)$ are fastly decreasing to zero as the normal coordinate $(y \cdot n^k - c^k/\varepsilon) \to +\infty$, uniformly in x and ε. The same holds for their derivatives, *c.f.* Proposition 5. Precisely,

$$\| \sum_{k=1}^{N} v_{bl}^k(x, x/\varepsilon) \|_{H^s(\omega')} = O(\varepsilon^m), \quad \forall s \le 2, m, \quad \forall \omega' \Subset \Omega. \tag{3.5}$$

Then, e_{bl}^ε satifies

$$\nabla \cdot A(x/\varepsilon)\nabla e_{bl}^\varepsilon \; = \; r_{bl}^\varepsilon, \quad r_{bl}^\varepsilon \; := \; -\nabla \cdot A(x/\varepsilon)\nabla \sum_{k=1}^{N} v_{bl}^k(x, x/\varepsilon), \quad x \in \Omega.$$

Let $0 \le \varphi(x) \le 1$ compactly supported in Ω, with $\varphi = 1$ in ω. A standard energy estimate yields

$$\int_\Omega \varphi^2 \, A(x/\varepsilon)\nabla e_{bl}^\varepsilon \cdot \nabla \, e_{bl}^\varepsilon dx \; = \; -2 \int_\Omega \varphi \, e_{bl}^\varepsilon \cdot \left(A(x/\varepsilon)\nabla e_{bl}^\varepsilon \cdot \nabla\varphi\right) \, dx + \int_\Omega r_{bl}^\varepsilon \cdot \varphi^2 e_{bl}^\varepsilon$$

Using the decay properties (3.5), the remainder term satisfies $\|r_{bl}^\varepsilon\|_{L^2(\omega')} = O(\varepsilon^m)$, for all s, m, and for any $\omega' \Subset \Omega$ containing the support of φ. Thus, the above inequality implies

$$\|e_{bl}^\varepsilon\|_{H^1(\omega)} \; \le \; C \, \|e_{bl}^\varepsilon\|_{L^2(\Omega)} \; + \; C_m \, \varepsilon^m, \quad \forall m.$$

Thus, we get:

$$\|u^\varepsilon - u^0(x) - \varepsilon u^1(x, x/\varepsilon) - \varepsilon^2 u^2(x, x/\varepsilon)\|_{H^1(\omega)} \; \le \; C\left(\varepsilon^2 \; + \; \varepsilon \, \|e_{bl}^\varepsilon\|_{L^2(\Omega)}\right).$$

Combining this bound with the "boundary error estimate", we obtain (1.8), which ends the proof of theorem 1. Actually, we have the following improved estimate which sees the homogenized boundary layer :

Corollary 1 *Under the assumptions of theorem 1, we also have the following global estimate:*

$$\|u^\varepsilon - u^0(x) - \varepsilon u^1(x, x/\varepsilon) - \varepsilon \sum_{k=1}^{N} v_{bl}^k(x, x/\varepsilon)\|_{L^2(\Omega)} \; \le \; C \, \varepsilon^2 \tag{3.6}$$

We point out that the difference between Theorem 1 and Corollary 1 is that Theorem 1 justifies the term $u^2(x, y)$ in the expansion since it gives a H^1 estimate whereas Corollary 1 justifies the boundary layer behavior since it holds up to the boundary. Of course, it only holds in L^2.

3.3 Global energy estimate

This paragraph is devoted to the proof of a $O(\varepsilon^2)$ estimate for e^ε in $H^1(\Omega)$. It satisfies

$$-\nabla \cdot A\left(\frac{\cdot}{\varepsilon}\right) \nabla e^\varepsilon \; = \; r^\varepsilon, \quad x \in \Omega, \quad e^\varepsilon|_{\partial\Omega} = 0, \tag{3.7}$$

where the remainder term r^ε is given by

$$\begin{aligned} r^\varepsilon(x) \; := \; & \varepsilon \nabla_x \cdot \left(A\nabla_x \widetilde{u^1 + A}\nabla_y u^2\right)\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_y \cdot \left(A\nabla_x u^2\right)\left(x, \frac{x}{\varepsilon}\right) \\ & + \; \varepsilon^2 \nabla_x \cdot \left(A\nabla_x u^2\right)\left(x, \frac{x}{\varepsilon}\right), \end{aligned} \tag{3.8}$$

with the tilde denoting the oscillating part (with zero average with respect to y). As the source term is *a priori* of order ε, one can not obtain a $O(\varepsilon^2)$ bound straightforwardly. To gain extra powers of ε, a standard trick is then to introduce a field $W = W(x,y)$ such that

$$\nabla_y \cdot W \;=\; \nabla_x \cdot \left(\widetilde{A\nabla_x u^1 + A\nabla_y u^2} \right) \tag{3.9}$$

Note that, if W satisfies this relation, setting

$$V(x,y) = W(x,y) + A(y)\nabla_x u^2(x,y),$$

we can write

$$\begin{aligned} r^\varepsilon(x) &= \varepsilon\, \nabla_y \cdot W\left(x, \frac{x}{\varepsilon}\right) + \,+ \varepsilon\, \nabla_y \cdot \left(A\nabla_x u^2\right)\left(x, \frac{x}{\varepsilon}\right) \,+\, \varepsilon^2\, \nabla_x \cdot \left(A\nabla_x u^2\right)\left(x, \frac{x}{\varepsilon}\right) \\ &= \varepsilon\, \nabla_y \cdot V\left(x, \frac{x}{\varepsilon}\right) \,+\, \varepsilon^2 \nabla_x \cdot \left(A\nabla_x u^2\right)\left(x, \frac{x}{\varepsilon}\right) \\ &= \varepsilon^2\, \nabla \cdot \left[V\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right](x) \,-\, \varepsilon^2 \nabla_x \cdot W\left(x, \frac{x}{\varepsilon}\right) \end{aligned} \tag{3.10}$$

This last expression is formally enough to derive a $O(\varepsilon^2)$ bound. But there is a regularity issue. The r.h.s. in (3.9) involves *a priori* three derivatives of u^0 and two derivatives of $\bar{u}^1$. By (A0)-(A1), if we do not choose the solution W of (3.9) carefully, it will only be L^2 with respect to x. It will not be enough to control last term in the above expression for F^ε.

Inspired by ideas of Bensoussan, Lions and Papanicolaou [3], we notice that, as

$$\nabla_y \cdot \left(A\nabla_x u^0 \;+\; A\nabla_y u^1\right) = 0,$$

we can write

$$\widetilde{A\nabla_x u^0 \;+\; A\nabla_y u^1} \;=\; \mathrm{curl}_y \psi,$$

for some $\psi = \psi(x,y)$ with zero average with respect to y. By assumptions on u^0, u^1, the field ψ is smooth with respect to y, has H^2 regularity with respect to x. Then, by construction of u^2,

$$\nabla_y \cdot \left(A\nabla_x u^1 + A\nabla_y u^2\right) = -\nabla_x \cdot \left(\widetilde{A\nabla_x u^0 + A\nabla_y u^1}\right) = -\nabla_x \cdot \mathrm{curl}_y \psi \;=\; \nabla_y \cdot \mathrm{curl}_x \psi.$$

Again, this implies that there exists $\phi = \phi(x,y)$ with zero average in y, such that

$$\widetilde{A\nabla_x u^1 + A\nabla_y u^2} \;-\; \mathrm{curl}_x \psi = \mathrm{curl}_y \phi.$$

The field ϕ is smooth with respect to y and has H^1 regularity with respect to x. Finally, we get that

$$\nabla_x \cdot \left(\widetilde{A\nabla_x u^1 + A\nabla_y u^2}\right) \;=\; \nabla_x \cdot \left(\mathrm{curl}_x \psi \;+\; \mathrm{curl}_y \phi\right) \;=\; \nabla_x \cdot \mathrm{curl}_y \phi \;=\; -\nabla_y \cdot \mathrm{curl}_x \phi.$$

Thus, we can set

$$W(x,y) \;=\; -\mathrm{curl}_x \phi(x,y)$$

which is smooth with respect to y, and has L^2 regularity with respect to x. The keypoint is that $\nabla_x \cdot W = 0$, so that there is no lack of regularity.

From these considerations, it follows easily that

$$\|r^\varepsilon\|_{H^{-1}} \;\leq\; \alpha\varepsilon^2$$

with a constant α depending only on the H^2 norm of u^1, and the H^3 norm of u^0. Back to (3.7), a simple energy estimate gives the $O(\varepsilon^2)$ bound.

3.4 Boundary layer estimate

This paragraph is devoted to the homogenization of the system

$$\begin{cases} -\nabla \cdot A\left(\frac{\cdot}{\varepsilon}\right)\nabla u^\varepsilon_{bl} = 0, \quad x \in \partial\Omega, \\ u^\varepsilon_{bl} = -u^1(x, x/\varepsilon) - \varepsilon u^2(x, x/\varepsilon), \quad x \in \partial\Omega. \end{cases}$$

which is satisfied by $u^\varepsilon_{bl} := u^{1,\varepsilon}_{bl} + \varepsilon u^{2,\varepsilon}_{bl}$. We expect u^ε_{bl} to have an expansion of the type

$$u^\varepsilon_{bl} \approx \sum_{k=1}^{N} \left(v^k_{bl}(x, x/\varepsilon) + \varepsilon w^k_{bl}(x, x/\varepsilon) \right)$$

where $v^k_{bl} = v^k_{bl}(x, y)$, $w^k_{bl} = w^k_{bl}(x, y)$ are defined on the half-space $\Omega^{\varepsilon,k}$, *cf.* section 2.

Plugging the expansion in the system satisfied by u^ε_{bl}, one finds that v^k_{bl} satisfies the system (2.1). The well-posedness and qualitative properties of this system have already been discussed. By our choice of $\bar{u}^1$, the solution v^k_{bl} converges to 0 as $(y \cdot n_k - c^k/\varepsilon) \to +\infty$, with a decay rate better than any power of $|y \cdot n_k - c^k/\varepsilon|$.

The next order term w^k_{bl} satisfies formally:

$$\begin{cases} -\nabla_y \cdot A \nabla_y w^k_{bl} = f^k, \quad y \in \Omega^{\varepsilon,k}, \\ \quad w^k_{bl} = -u_2(x, y), \quad y \in \partial\Omega^{\varepsilon,k}, \end{cases} \tag{3.11}$$

$$f^k := \nabla_x \cdot A \nabla_y v^k_{bl} + \nabla_y \cdot A \nabla_x v^k_{bl}.$$

Remark that, by decay properties of v^k_{bl}, f^k goes rapidly to zero as $(y \cdot n_k - c^k/\varepsilon) \to +\infty$. System (3.11) is of course very similar to system (2.1), and can be solved in a similar manner, taking advantage of a quasiperiodic setting. Proceeding exactly as in section 2, it amounts to solving a problem of the form (2.7), with an H which is not anymore with compact support, but satisfies $t^m \partial^\alpha_\theta \partial^k_t H \in L^2$ for all m, α, k. The arguments for well-posedness and convergence far from the boundary extend easily to this setting. In particular, the conclusions of Propositions 4 and 5 are still valid. Hence, one can find $w^k_{bl} = w^k_{bl}(x, y)$ solving (3.11), that converges fast to some $w^{k,\infty}(x)$ as $(y \cdot n^k - c^k/\varepsilon) \to +\infty$. Note that w^k_{bl} involves linearly second order derivatives of u^0, and first order derivatives of $\bar{u}^1$, so that it has $H^1 \cap C^0$ regularity with respect to x.

Our goal is to derive a $O(\varepsilon)$ bound in $L^2(\Omega)$ for $\tilde{e}^\varepsilon_{bl} := e^\varepsilon_{bl} - \varepsilon \sum w^k_{bl}(x, x/\varepsilon)$. As

$$\|w^k_{bl}(x, x/\varepsilon)\|_{L^2(\Omega)} = O(1).$$

the "boundary error estimate" will follow, concluding the proof of Theorem 1. The field $\tilde{e}^\varepsilon_{bl}$ satisfies

$$-\nabla \cdot A\left(\frac{\cdot}{\varepsilon}\right)\nabla \tilde{e}^\varepsilon_{bl} = r^\varepsilon_{bl}, \quad x \in \Omega, \quad \tilde{e}^\varepsilon_{bl}|_{\partial\Omega} = \varphi^\varepsilon_{bl},$$

where

$$r^\varepsilon_{bl} := \sum \nabla_x \cdot \left(A\nabla_x v^k_{bl} + A\nabla_y w^k_{bl} \right)\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \sum \nabla \cdot \left(A\nabla_x w^k_{bl}\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right)(x), \tag{3.12}$$

$$\varphi^\varepsilon_{bl} := -u^1(x, x/\varepsilon) - \varepsilon u^2(x, x/\varepsilon) - \sum \left(v^k_{bl}(x, x/\varepsilon) + \varepsilon w^k_{bl}(x, x/\varepsilon) \right)|_{\partial\Omega}. \tag{3.13}$$

Control of the source term

The source term r^ε_{bl} is made of two terms.

The second term in the r.h.s. of (3.12) is of the type $\varepsilon\,\nabla\cdot R^\varepsilon$ where $\|R^\varepsilon\|_{L^2(\Omega)} \le C$.

The first term in the r.h.s. of (3.12) reads $\sum r^k(x, x/\varepsilon)$ for some $r^k = r^k(x,y)$ built after $\nabla_x v^k_{bl}$ and $\nabla_y w^k_{bl}$. By properties of these boundary layer profiles, the field r^k has L^2 regularity in x, is smooth in y, and goes to zero as $(y\cdot n^k - c^k/\varepsilon) \to +\infty$, faster than any power of $(y\cdot n^k - c^k/\varepsilon)$ uniformly in x and ε. For any $e \in H^1_0(\Omega)$, we have

$$
\begin{aligned}
&\left|\int_\Omega r^k\left(x, \frac{x}{\varepsilon}\right)\cdot e(x)dx\right| \;\le\; \int_\Omega |r_k|\left(x,\frac{x}{\varepsilon}\right)\, d(x,\partial\Omega)\,\frac{|e|}{d(x,\partial\Omega)}\,dx \\
&\le\; \int_\Omega |r_k|\left(x,\frac{x}{\varepsilon}\right)\, d(x,K^k)\,\frac{|e|}{d(x,\partial\Omega)}\,dx \\
&\le\; \int_\Omega |r_k|\left(x,\frac{x}{\varepsilon}\right)\, |x\cdot n^k - c^k|\,\frac{|e|}{d(x,\partial\Omega)}\,dx \;\le\; \varepsilon\int \sup_y |r^k(x,y)|\,|y\cdot n^k - c^k/\varepsilon|\,\frac{|e|}{d(x,\partial\Omega)}dx \\
&\le\; C\,\varepsilon\,\|\nabla e\|_{L^2(\Omega)}
\end{aligned}
$$

where the last inequality stems from Cauchy-Schwartz and Hardy's inequalities.

Gathering these bounds gives

$$\|r^\varepsilon_{bl}\|_{H^{-1}(\Omega)} \;\le\; C\,\varepsilon. \tag{3.14}$$

Control of the boundary term

We will prove that

$$\|\varphi^\varepsilon_{bl}\|_{W^{1-1/p,p}(\partial\Omega)} \;\le\; C(p)\,\varepsilon, \quad \forall p < 2. \tag{3.15}$$

Before that, let us show how it implies the bound we want on $\tilde e^\varepsilon_{bl}$. First, it allows to introduce a field ϕ^ε satisfying for all $p < 2$:

$$\phi^\varepsilon \in W^{1,p}(\Omega), \quad \phi^\varepsilon|_{\partial\Omega} = \varphi^\varepsilon_{bl}, \quad \|\phi^\varepsilon\|_{W^{1,p}(\Omega)} = O(\varepsilon).$$

The remaining term $e^\varepsilon = \tilde e^\varepsilon_{bl} - \phi^\varepsilon$ satisfies

$$\nabla\cdot(A(x/\varepsilon)\nabla e^\varepsilon) \;=\; F^\varepsilon \text{ in } \Omega, \quad e^\varepsilon|_{\partial\Omega} = 0, \quad F^\varepsilon = r^\varepsilon_{bl} - \nabla\cdot(A(x/\varepsilon)\nabla\phi^\varepsilon) \in W^{-1,p}(\Omega).$$

We can now apply general results of Meyers [8] on elliptic equations in divergence form with bounded coefficients. These results extend straightforwardly to elliptic systems (*i.e.* when $n > 1$). As a result, there exists $p_m < 2$, such that for all $p_m < p < 2$, e^ε satisfies

$$\|e^\varepsilon\|_{W^{1,p}(\Omega)} \;\le\; C(p)\,\|F^\varepsilon\|_{W^{-1,p}(\Omega)} \;\le\; C(p)\,\varepsilon,$$

combining (3.14) and the estimate on ϕ^ε. The L^2 estimate on e^ε, and then on $\tilde e^\varepsilon_{bl}$ follows from Sobolev imbedding.

Hence, the last step is to obtain (3.15). We first focus on one part of φ^ε_{bl}, that is

$$\varphi_v : x \mapsto -u^1(x, x/\varepsilon) \;-\; \sum_k v^k_{bl}(x, x/\varepsilon).$$

We shall prove that

$$\|\varphi_v\|_{W^{1-1/p,p}(\partial\Omega)} \;=\; O(\varepsilon), \quad \forall\, p < 2.$$

By construction of the v_{bl}^k's, one can decompose

$$\varphi_v(x) \;=\; V(x, x/\varepsilon)\nabla u^0(x) \;:=\; \left(-\chi(x/\varepsilon) + \sum_k V^k(x/\varepsilon)\right)\nabla u^0(x),$$

where

$$V = V^\alpha(y), \quad \chi = \chi^\alpha(y), \quad V^k = V^{k,\alpha}(y) \in M_n(\mathbb{R}), \quad \alpha = 1...d, \quad k = 1...N$$

denote as usual families of matrix fields. Note that χ is the solution of the cell problem (1.4). By construction of the boundary layer profiles, v_{bl}^k and its derivatives go to zero uniformly as $y \cdot n^k - c^k/\varepsilon \to +\infty$, faster than any negative power of $y \cdot n^k - c^k/\varepsilon$. Moreover, for any k,

$$\varphi_v|_{\partial\Omega \cap K^k} \;=\; -\sum_{j \neq k} V^j(x/\varepsilon)\, \nabla u^0(x).$$

Let ψ be a smooth function on $\partial\Omega$, compactly supported outside a neighborhood of the edges and vertices of Ω. Above remarks lead to: for all $p < 2$,

$$\|\psi\, \varphi_v\|_{W^{1-1/p,p}(\partial\Omega)} \;\le\; \|\psi\, \varphi_v\|_{H^{1/2}(\partial\Omega)} \;=\; C_{s,m}\, \varepsilon^m\, \|\nabla u^0\|_{H^{1/2}(\partial\Omega)} \;\le\; C'_{s,m}\, \varepsilon^m \quad \forall m, s.$$

Hence, the main problem in establishing the $O(\varepsilon)$ bound comes from the edges and vertices of the polygon. In particular, we will need to use cancellation properties of ∇u^0 there.

Let us first consider the case $n = 2$. Let O be a vertex of Ω. We introduce polar coordinates $r = r(x), \theta = \theta(x)$, centered at O. Let ψ be a smooth function supported this time in a vicinity of O in $\partial\Omega$. We remind the standard estimate: for all $f, g \in L^\infty(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$,

$$\|f\, g\|_{W^{1-1/p,p}(\partial\Omega)} \;\le\; C\left(\|f\|_{L^\infty(\partial\Omega)}\, \|g\|_{W^{1-1/p,p}(\partial\Omega)} \;+\; \|g\|_{L^\infty(\partial\Omega)}\, \|f\|_{W^{1-1/p,p}(\partial\Omega)}\right). \quad (3.16)$$

From there, we deduce

$$\begin{aligned}\|\psi^2\, \varphi_v\|_{W^{1-1/p,p}(\partial\Omega)} \;\le\; & \|\psi\, \frac{\nabla u^0}{r}\|_{W^{1-1/p,p}(\partial\Omega)}\, \|\psi\, r\, V(\cdot/\varepsilon)\|_{L^\infty(\partial\Omega)} \\ & +\; \|\psi\, \frac{\nabla u^0}{r}\|_{L^\infty(\partial\Omega)}\, \|\psi\, r\, V(\cdot/\varepsilon)\|_{W^{1-1/p,p}(\partial\Omega)}.\end{aligned}$$

We emphasize that

$$\psi\, \frac{\nabla u^0}{r} \in L^\infty(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega), \quad \forall p < 2.$$

Indeed, as u^0 satisfies a Dirichlet condition at $\partial\Omega$, ∇u^0 cancels at the vertex O, and Taylor's formula gives

$$\frac{\nabla u^0(x)}{r} \;=\; \frac{x}{|x|} \cdot \int_0^1 \nabla\nabla u^0(tx)\, dt.$$

By assumption (A0), it clearly belongs to $L^\infty(\partial\Omega)$ and to $W^{1,p}(\Omega)$, hence to $W^{1-1/p,p}(\partial\Omega)$. Note however that it does not belong *a priori* to $H^{1/2}(\partial\Omega)$. This is a reason why we consider L^p spaces for $p < 2$ and use Meyers theorem.

It remains to control the function $r\,V(\cdot/\varepsilon)$ in a vicinity of O in $\partial\Omega$. This vertex belongs to two sides, say $K^1\cap\partial\Omega$ and $K^2\cap\partial\Omega$. We can always assume that $\theta=0$ corresponds to K^1 and $\theta=\omega$ corresponds to K^2. Note that by convexity, $0<\omega<\pi$. For $j\neq 1$, by properties of the boundary larer profiles,

$$\|\psi\, r\, V^j(\cdot/\varepsilon)\|_{H^s(\partial\Omega\cap K^2)} \;=\; O(\varepsilon^m), \quad \forall m, s.$$

We can therefore neglect such terms. Then:

$$\begin{aligned}\|\psi\, r\, V(\cdot/\varepsilon)\|_{L^\infty(\partial\Omega\cap K^2)} \;&\le\; C \sup_{r>0}\, r\, \left|V^1(r\cos\omega/\varepsilon, r\sin\omega/\varepsilon)\right| \\ &\le\; c\,\frac{\varepsilon}{\sin\omega}\sup_y |y_2|\,\left|V^1(y_1,y_2)\right| \;\le\; C\,\varepsilon.\end{aligned}$$

Similarly,

$$\|\psi\, r\, V(\cdot/\varepsilon)\|_{L^p(\partial\Omega\cap K^2)} \;\le\; C\,\frac{\varepsilon^{1+1/p}}{(\sin\omega)^{1/p}}\sup_{y_1}\left(\int_0^{+\infty}\left|V^1(y_1,y_2)\right|^p dy_2\right)^{1/p} \;\le\; C'\,\varepsilon^{1+1/p}.$$

Applying the same reasoning to the tangential derivatives, we get

$$\|\,(r\,V(\cdot/\varepsilon))\,\|_{H^1(\partial\Omega\cap K^2)} \;\le\; C'\,\varepsilon^{1/p}.$$

We can of course proceed in a similar way with the other hyperplane K^1, and we end up with

$$\|\psi\, r\, V(\cdot/\varepsilon)\|_{L^\infty(\partial\Omega)} = O(\varepsilon),$$

$$\|\psi\, r\, V(\cdot/\varepsilon)\|_{L^1(\partial\Omega)} = O(\varepsilon^{1+1/p}), \quad \|\psi\, r\, V(\cdot/\varepsilon)\|_{W^{1,p}(\partial\Omega)} = O(\varepsilon^{1/p}).$$

By interpolation of the last two inequalities, we get

$$\|\psi\, r\, V(\cdot/\varepsilon)\|_{W^{1-1/p,p}(\partial\Omega)} = O(\varepsilon^{2/p})$$

which gives the bound we want for the case $d=2$.

When $d=3$, the computations are almost the same. We have to distinguish between the case of an edge and the case of a vertex.

- In the neighborhood of an edge, but far from a vertex, one can use locally cylindrical coordinates (r,θ,z), where $r=0$ corresponds to the edge, z is the variable along the edge, and θ is the angular variable. Again, the edge is the intersection of two hyperplanes K^1 and K^2, with $\theta=0$ corresponding to K^1, whereas $\theta=\omega$ corresponds to K^2. The computation is exactly the same as for $d=2$, and we leave the details to the reader.

- In the neighborhood of a vertex O, one can use spherical type coordinates. Precisely, we consider a plane H near O, transverse to the sides that contain O. Its intersection with Ω is a two-dimensional polygon $\tilde{\Omega}$. We describe $\partial\Omega$ near O by the coordinates $x=r\,s$, $r>0$, $s\in\partial\tilde{\Omega}$. We use this time the decomposition

$$\psi^2\,\varphi_v \;=\; (\psi\,|r\,s|\,V(x/\varepsilon))\left(\psi\,\frac{\nabla u^0(x)}{|rs|}\right)$$

for $\psi = \psi(x)$ a function compactly supported near O. Thanks to (3.16), we must again evaluate $(\psi\,|rs|V)(x/\varepsilon)$. For instance,

$$\|\psi|rs|\,V(x/\varepsilon)\|_{L^\infty(\partial\Omega)} \;\le\; \varepsilon \sup_{\sigma \in \frac{r}{\varepsilon}\partial\tilde\Omega} |\sigma|\,V(\sigma) \;\le\; C\,\varepsilon$$

The treatment of the L^p norm and $W^{1,p}$ norms are similar. We end up with

$$\|\psi|rs|\,V(x/\varepsilon)\|_{W^{1-1/p,p}(\partial\Omega)} \;\le\; C\,\varepsilon^{2/p},$$

which concludes the study of φ_v.

To establish inequality (3.15), it remains to handle the other part of φ^ε_{bl}. Namely,

$$\varphi_w \;:=\; -u^2(x, x/\varepsilon) \;-\; \sum_k w^k_{bl}(x, x/\varepsilon)$$

should satisfy

$$\|\varphi_w\|_{W^{1-1/p,p}(\partial\Omega)} \;=\; O(1), \quad \forall\, p < 2.$$

Again, by properties of the w^k_{bl}'s, we can write

$$\begin{aligned}\varphi_w \;&= \mathcal{W}(x/\varepsilon)\,\nabla^2 u^0(x) \;+\; W(x/\varepsilon)\,\nabla \bar u^1(x) \\ &:= \left(-\Upsilon(x/\varepsilon) \;+\; \sum_k \mathcal{W}^k(x/\varepsilon)\right)\,\nabla^2 u^0(x) \;+\; \left(-\chi(x/\varepsilon) \;+\; \sum_k W^k(x/\varepsilon)\right)\nabla \bar u^1(x),\end{aligned}$$

where

$$\mathcal{W} = \mathcal{W}^{\alpha\beta}(y), \quad \Upsilon = \Upsilon^{\alpha\beta}(y), \quad \mathcal{W}^k = \mathcal{W}^{k,\alpha\beta}(y) \,\in M_n(\mathbb{R}), \quad \alpha,\beta = 1...d, \quad k = 1...N.$$

and

$$W = W^\alpha(y), \quad \chi = \chi^\alpha(y), \quad W^k = W^{k,\alpha}(y) \,\in M_n(\mathbb{R}), \quad \alpha = 1...d, \quad k = 1...N.$$

Note that Υ and χ are the same families as in (1.9)-(1.4).

Contrary to the previous fields V^k, the fields $\mathcal{W}^k$, *resp.* W^k converge to non-zero constant fields $\mathcal{W}^{k,\infty}$, *resp.* $W^{k,\infty}$. To overcome this difficulty, we introduce the field φ^∞_w, defined by

$$\varphi^\infty_w \;:=\; \sum_{j\neq k} \mathcal{W}^{j,\infty}\,\nabla^2 u^0(x) \;+\; \sum_{j \neq k} W^{j,\infty}\,\nabla \bar u^1(x), \quad x \in \partial\Omega \cap K^k$$

Note that φ^∞_w can be decomposed in products of the type fg, where:

- f involves either second derivatives of u^0 or first derivatives of u^1. By (A0) and (A1), it belongs to $H^1(\Omega) \cap C^0(\overline{\Omega})$.
- $g = g^k$ is constant on each hyperplane K^k. Direct verifications show that

$$g \in W^{1-1/p,p}(\partial\Omega), \qquad \text{for all } p < 2.$$

For instance, when $d = 2$, the only regularity problem lies at the vertices of Ω. Let O be such a vertex, belonging for instance to $K^1 \cap K^2$. It is then enough to find $G \in W^{1,p}(\Omega$ such that $G|_{\partial\Omega} = g$ in a vincinity of O. As before, we consider polar coordinates r, θ centered at 0. The angle $\theta = 0$ corresponds to K^1, and $\theta = \omega$ corresponds to H^l.We take

$$G = (1 - \sin\left(\frac{\pi\theta}{2\omega}\right))g_1 \; + \; g_2 \sin\left(\frac{\pi\theta}{2\omega}\right) \in W^{1,p}(\Omega), \quad \forall p < 2$$

We stress that such a field G is not in $H^1(\Omega)$, so that considering $p < 2$ is again needed. When $d = 3$, the treatment is similar and left to the reader.

We deduce: $\|\varphi_w^\infty\|_{W^{1-1/p/p}(\partial\Omega)} < +\infty$. Now, one has for any $k = 1...N$, for all $x \in \partial\Omega \cap K^k$,

$$\begin{aligned} \varphi_w(x) \; &= \; \varphi_w^\infty(x) \\ &+ \sum_{j \neq k} \left(\mathcal{W}^j(x/\varepsilon) - \mathcal{W}^{j,\infty}\right) \nabla^2 u^0(x) \; + \; \left(W^j(x/\varepsilon) - W^{j,\infty}\right) \nabla \bar{u}^1(x). \end{aligned}$$

Hence, up to replacing φ_w by $\varphi_w - \varphi_w^\infty$, we can always assume that $\mathcal{W}^{k,\infty} = 0, \; W^{k,\infty} = 0$.

At this point, the estimate on φ_w can be obtained along the same lines as the estimate on φ_v. As we only need a $O(1)$ bound, the situation is simpler: we do not need extra terms like r (for $d = 2$) or $r\,s$ (for $d = 3$) in front of the boundary layer terms $\mathcal{W}(x, x/\varepsilon)$ and $W(x/\varepsilon)$. In other words, we do not need any cancellation property for $\nabla^2 u^0$ or $\nabla \bar{u}^1$. This concludes the proof of Theorem 1.

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