Rotating fluids in a cylinder

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Résumé

On étudie dans ce papier divers problèmes de perturbations singulières pour des fluides en rotation dans un cylindre. On considère tout d'abord les équations de Navier–Stokes pour des fluides incompressibles avec viscosité turbulente, dans la limite des faibles nombres de Rossby. On montre la convergence forte des solutions de ces équations, pour des données mal préparées, sous une condition géométrique sur la base du cylindre et une hypothèse de généricité sur l'opérateur singulier.

On discute ensuite le cas de fluides compressibles avec tenseur de viscosité anisotrope, dans la limite des faibles nombres de Mach et des faibles nombres de Rossby. Dans le cas de données bien préparées, on prouve un résultat de convergence des solutions faibles globales en temps avec conditions de Dirichlet vers la solution du modèle bidimensionnel quasi-géostrophique avec un terme supplémentaire dû à la compressibilité. Dans le cas de données mal préparées, on montre seulement que l'on peut espérer la convergence forte sous le même type de conditions que dans le cas incompressible.

Abstract

We study various singularly perturbed models related to rotating flows in a cylinder. At first we consider the three dimensional incompressible Navier–Stokes equations with turbulent viscosity, in the low Rossby limit. We prove a strong convergence result for ill prepared data, under a geometrical assumption on the cylinder section and a genericity condition on the singular operator.

In a second section, we discuss the compressible Navier–Stokes equations with anisotropic viscosity tensor in the combined low Mach and low Rossby number limit. In the case of well prepared initial data, we prove that global weak solutions with Dirichlet boundary conditions converge to the solution of a two–dimensional quasi-geostrophic model taking into account the compressibility. In the case of ill prepared data, we only show that we can hope a strong convergence result under the same kind of assumptions as in the incompressible case.

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Abstract

We study various singularly perturbed models related to rotating flows in a cylinder. At first we consider the three dimensional incompressible Navier–Stokes equations with turbulent viscosity, in the low Rossby limit. We prove a strong convergence result for ill prepared data, under a geometrical assumption on the cylinder section and a genericity condition on the singular operator.

In a second section, we discuss the compressible Navier–Stokes equations with anisotropic viscosity tensor in the combined low Mach and low Rossby number limit. In the case of well prepared initial data, we prove that global weak solutions with Dirichlet boundary conditions converge to the solution of a two–dimensional quasi-geostrophic model taking into account the compressibility. In the case of ill prepared data, we only show that we can hope a strong convergence result under the same kind of assumptions as in the incompressible case.

Keywords: Singular perturbations, geophysical flows, incompressible and compressible Navier–Stokes equations, propagation of waves, rapidly rotating fluids, low Mach number, quasi-geostrophic equations, free surface.

AMS subject classification: 35Q30.

1 Introduction

A variety of applications such as acoustic phenomena in atmosphere dynamics, see [35], or some industrial processes using centrifugation involve rapidly rotating incompressible fluids, or rapidly rotating compressible fluids in the low Mach number regime. The purpose of this work is to study such singularly perturbed fluid models. The three dimensional domain is assumed to be a cylinder $\Omega = S \times (0,1)$, directed along the rotation vector e_3. Homogeneous Dirichlet boundary conditions on the velocity field u are considered on the boundary $\partial\Omega$, which require to introduce boundary layers correctors both on the lateral boundary and on the top and bottom of the domain.

We shall consider, at first, the anisotropic incompressible Navier–Stokes equations under the action of the Coriolis force.

$$\begin{cases} \partial_t u + \dfrac{e_3 \times u}{\mathrm{Ro}} - \operatorname{div} \nabla_\mu u + \dfrac{\nabla p}{\mathrm{Ro}} + u \cdot \nabla u = f, \\ \operatorname{div} u = 0, \end{cases} \tag{1}$$

where $\nabla_\mu = (\mu \nabla_{\mathrm{x}}, \mu_z \partial_z) = (\mu\partial_x, \mu\partial_y, \mu_z\partial_z)$. We complete this system with the following boundary and initial conditions

$$u_{|\partial\Omega} = 0, \qquad u_{|t=0} = u_0, \tag{2}$$

where u_0 is assumed to be divergence free with finite initial energy, that means

$$\int_\Omega |u_0|^2 \, d\mathrm{x} \, dz \le C_0.$$

In what follows, the Rossby number will be replaced by ε and solutions will be denoted $(u^\varepsilon, p^\varepsilon)$. We will assume

$$\mu \approx 1, \qquad \mu_z = r_0^2 \varepsilon.$$

The Coriolis operator being skew-symmetric, the global existence of weak solutions for the rotating incompressible fluid equations is well known since the famous work by Jean Leray [39].

Concerning the asymptotic analysis as $\varepsilon \to 0$, u^ε formally converges to $\overline{u}$ satisfying the modified quasi-geostrophic equations, see Section 2. As we shall see later on, under some geometrical condition, oscillations of the solutions are damped in a boundary layer located near the lateral surface of the cylinder. Mathematically, it allows us to recover a strong convergence result. The proof relies on the construction of precise approximate solutions, taking into account the boundary layers, and the possible influence of "corners" $\partial S \times \{0\}$ and $\partial S \times \{1\}$.

In the second part, we shall consider the barotropic compressible Navier–Stokes equations under the action of the Coriolis force with anisotropic viscosity tensor. Anisotropic viscosities are usually introduced in Geophysics as a crude model of the anisotropy of the small scale part of the flow. More precisely the horizontal viscosity is denoted μ and the vertical one μ_z.

The fluid's density is denoted by ρ and the pressure law is assumed to be of γ–type, where $\gamma > 3/2$. Finally, denoting by Ma the Mach number and Ro the Rossby number of the flow, the rescaled compressible Navier–Stokes equations can be written as follows

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \mathrm{div}\,\sigma + \dfrac{e_3 \times (\rho u)}{\mathrm{Ro}} = 0, \end{cases} \tag{3}$$

in $\mathcal{D}'((0,T)\times\Omega)^d$ with

$$\sigma = \nabla_\mu u + {}^t\nabla_\mu u - (\mathrm{div}\,u)\,\mathrm{diag}(\mu,\mu,\mu_z) + \widetilde{\lambda}(\mathrm{div}\,u)\mathrm{Id} - \frac{\rho^\gamma}{\gamma \mathrm{Ma}^2}\,\mathrm{Id},$$

where $\nabla_\mu = (\mu\partial_x, \mu\partial_y, \mu_z\partial_z)$ and $\widetilde{\lambda}$ denotes a bulk viscosity coefficient such that $\widetilde{\lambda} \geq -\min(\mu,\mu_z)$. The initial conditions

$$\rho_{|t=0} = \rho_0 \geq 0, \quad \rho u_{|t=0} = m_0, \tag{4}$$

are taken in such a way that the initial total energy is finite

$$\int_\Omega \left\{ \pi_{|t=0} + \frac{|m_0|^2}{2\rho_0} \right\} dx \leq C_0, \quad \text{where} \quad \pi = \frac{\rho^\gamma - 1 - \gamma(\rho - 1)}{\mathrm{Ma}^2 \gamma(\gamma-1)}. \tag{5}$$

Here we agree that $m_0(x) = 0$ on $\{x \in \Omega \,/\, \rho_0(x) = 0\}$.

In what follows, the Mach number and the Rossby number are assumed to be of the same order of magnitude ε and the vertical viscosity to be much smaller than the horizontal one (which corresponds to the fact that turbulent eddies are more likely to move horizontally than vertically because of the high rotation of the system):

$$\mathrm{Ma} = \mathrm{Ro} = \varepsilon, \qquad \mu \approx 1, \qquad \mu_z = r_0^2\varepsilon \text{ with } r_0 \approx 1.$$

Notice that $\mathrm{Ro} = \mathrm{O}(\mathrm{Ma})$ corresponds to the effective Rossby number for the larger acoustic scales in atmospheric flows, see [35] page 777.

Since we are interested in the limit of small ε, solutions will be denoted $(\rho^\varepsilon, u^\varepsilon)$. The initial data $(\rho_0^\varepsilon, m_0^\varepsilon)$ are assumed to satisfy (5) for some uniform constant C_0 and

$$(\rho_0^\varepsilon)^{-\frac{1}{2}} m_0^\varepsilon \text{ converges strongly in } L^2(\Omega)^d \text{ to some } u_0.$$

The existence of global weak solutions to (3) for a given Mach number $\varepsilon > 0$ and isotropic viscosity ($\mu_z = \mu$ in (3)) was obtained by P.–L. Lions in [43] for large enough exponents γ and later optimally improved by E. Feireisl in [24]. Assuming $\widetilde{\lambda} + \mu > 0$, they proved that for any fixed $\varepsilon > 0$, there exists a global weak solution $(\rho^\varepsilon, u^\varepsilon)$ to (3) such that $\rho^\varepsilon \in L^\infty(0, \infty; L^\gamma(\Omega))$, $u^\varepsilon \in L^2(0, \infty; H_0^1(\Omega))^d$ satisfying in addition: $\rho^\varepsilon \in C([0, \infty); L^p_{loc}(\Omega))$ if $1 \le p < \gamma$, $\rho^\varepsilon |u^\varepsilon|^2 \in L^\infty(0, \infty; L^1(\Omega))$, $\rho^\varepsilon u^\varepsilon \in C([0, \infty); L^{2\gamma/(\gamma+1)}(\Omega) \text{ weak})^d$. Furthermore, the energy inequality holds for almost all $t \ge 0$

$$\int_\Omega \left\{ \frac{1}{2} \rho^\varepsilon |u^\varepsilon|^2 + \pi^\varepsilon \right\}(t)\, dx + \int_0^t \int_\Omega \left\{ \mu |\nabla u^\varepsilon|^2 + \widetilde{\lambda} |\operatorname{div} u^\varepsilon|^2 \right\} dxds \le C_0. \qquad (6)$$

The reader interested by a survey on the compressible Navier–Stokes equations is referred to [21].

Concerning the asymptotic analysis as $\varepsilon \to 0$, ρ^ε formally converges to 1, since $\pi^\varepsilon \ge 0$ is bounded uniformly in ε. Moreover, from $(3)_1$, we deduce that $\operatorname{div} u^\varepsilon$ converges to 0, and u^ε converges, for "well prepared" data, to some $\overline{u}$ satisfying the modified quasi-geostrophic equations, see Section 3. This heuristics does not take account of waves, which carry out the energy of some part of the velocity, and are likely to propagate with the high speed $1/\varepsilon$ in the domain. As we shall see later on, the analysis is indeed different in the case of "well prepared" data (velocity and density close to the kernel of the singular operator: see Theorem 6) and in the case of "ill prepared" data. In the well prepared case, we perform an asymptotic result taking into account the Ekman boundary layers. The ill prepared case is discussed and we will see that strong convergence can be obtained as soon as it is possible to build correctors which do not affect the damping phenomenon. Such construction seems far from being proved.

1.1 High rotating limit of the incompressible Navier–Stokes equations

The low Rossby number limit of the incompressible rotating Navier–Stokes equations has been extensively studied mathematically over the past decade.

Recent progress are presented in [31], [7], [2], [46]. See also [6] for a review. The case of periodic boundary conditions was treated by using the group generated by the wave operator introduced in [50] [29]. Because of fast propagating Rossby waves, the velocity u^ε converges only weakly to a solution $\overline{u}$ of the 2D Navier–Stokes equations.
The whole space case $\Omega = \mathbb{R}^d$ was considered in [7], using Strichartz' estimates for the linear wave equations. It expresses mathematically the dispersion of Rossby waves at infinity, and allowed to prove in [7] that the convergence to incompressible solutions is strong in $L^2_{loc}(\mathbb{R}^+_t \times \mathbb{R}^d_x)$. The strong convergence in a domain $\Omega = \mathbb{R}^2 \times (0,1)$ has been proved in [8] using again Strichartz' estimates.
In the case of a domain $\Omega = \mathrm{T}^2 \times (0,1)$ with homogeneous Dirichlet boundary conditions, convergence results were obtained in [9], [30] in the well prepared case or for suitable boundary conditions. The weak convergence of u^ε to $\overline{u}$ was proved in [45], [7] for general data. We also mention the recent work [26] with a rotating term of the form $B(x) \times u$ instead of $e_3 \times u$. This paper shows the weak convergence of Leray-type solutions towards a vector field which satisfies the usual 2D Navier-Stokes equation in regions of space where B is constant, with Dirichlet boundary conditions, and a heat-type equation elsewhere. The proof uses weak compactness arguments. This case may have application in Geophysics since the Coriolis force depends on the latitude. Let us also mention the work on qualitative properties related to rotating incompressible flows in [10].

1.2 Low Mach number limit of the compressible Navier–Stokes equations without Coriolis force.

The low Mach number limit of the compressible Navier-Stokes equations has been studied mathematically by many authors. Recent progress in the isotropic viscous case (that means with the standard Laplace operator) are presented in [44], [16], [17].
The case of periodic boundary conditions was treated in [44] by using the group generated by the wave operator introduced in [50] [29]. Because of fast propagating acoustic waves, the velocity u^ε converges only weakly to a solution $\overline{u}$ of the incompressible Navier–Stokes equations. Let us observe at this point that the evolution of the mean velocity $\overline{u}$ is decoupled from the acoustic waves dynamics.

The whole space case $\Omega = \mathbb{R}^d$ was considered in [53] [31] [32] for strong solutions locally in time, and in [44] for weak solutions by using the so called group method [50] [29]. Using Strichartz' estimates for the linear wave equations, which expresses mathematically the dispersion of acoustic waves at infinity, it was proved in [16] that the convergence to incompressible solutions is strong in $L^2_{loc}(\mathbb{R}^+_t \times \mathbb{R}^d_x)$. The zero Mach number limit has been also studied in critical spaces in [11], see also [12].
In the case of a C^2 bounded domain Ω of $\mathbb{R}^d$ convergence results were obtained in [44] in the well prepared case. For homogeneous Dirichlet Boundary conditions, The weak convergence of u^ε to $\overline{u}$ was proved in [17] for general data. Moreover, if Ω satisfies the property

$$\begin{aligned} &\text{If } -\Delta\Psi = \lambda^2\Psi \text{ in } \Omega,\ \lambda \neq 0,\ \partial\Psi/\partial n = 0 \\ &\qquad\qquad\qquad \text{and } \Psi = C^{te} \text{ on } \partial\Omega, \quad \text{then } \Psi \equiv 0, \end{aligned} \tag{7}$$

then u^ε converges strongly to $\overline{u}$ in $(L^2((0,T) \times \Omega))^d$ (see [17]). Indeed, the energy of acoustic waves is dissipated in a boundary layer of size $\sqrt{\varepsilon}$ on each mode, which yields exponential decay of the potential part of u^ε. Notice that condition (7) is closely connected to Schiffer's conjecture. It states that every bounded domain $\Omega \subset\subset \mathbb{R}^d$ which does not satisfy (7) is an Euclidian ball. In that case, some waves may propagate in the domain and the convergence of u^ε to $\overline{u}$ is only weak in $(L^2((0,T) \times \Omega))^d$.

When an additional equation related to the entropy a^ε is added to System (3), that means the non-isentropic case, very few mathematical results are known for the time–dependent problem. For results on time discretized models, we refer to [41] and [42]. For the non-isentropic Euler equations, we refer the reader to [47] where the asymptotic problem is studied in $\mathbb{R}^d$ with a decreasing assumption on a^ε at infinity. Let us mention [48] where the problem in the torus is studied without describing the resonance due to acoustic waves. In [3] a semi-formal derivation is given for the nonstationary problem with ill–prepared initial data. The dynamics governing the mean velocity $\overline{u}$ and the acoustic waves turns out to be strongly coupled to each other. The nonhomogeneity yields an extra non gradient term depending on the waves in the mean momentum equation. The description of the waves dynamics might be of physical and numerical interest.

1.3 Low Froude number limit of high rotating viscous compressible Navier–Stokes equations.

The low Froude number limit of high rotating shallow water equations has been recently studied in [5] in $\Omega = \mathrm{T}^2$. This system is of the same kind of the low Mach number for rotating compressible Navier–Stokes equations but with a degenerate diffusive term of the form $\mathcal{D} = -\mu \mathrm{div}\,(\rho\, D(u))$ where $D(u) = (\nabla u + {}^t\nabla u)/2$. The strong convergence of the solutions of the shallow water system, with well prepared data, to the solution of the quasi-geostrophic equation with one term describing the free surface has been proved. The ill prepared case has been also discussed. The reader interested by the shallow water equations is referred to [4], [28], [43]. The high rotating limit of the Navier-Stokes equations with free surface to the quasi-geostrophic equation with the free surface term is an interesting open question.

1.4 Outline of the paper

The outline of the paper reads as follows: in Section 2, we build an approximate solution of the flow associated with the rotating Navier–Stokes equations. More precisely, this approximate velocity takes into account the well prepared and the ill prepared data for which we have to use a geometric condition on the surface S. We also have to define correctors in the corners. Then we can prove the mathematical strong convergence to the solution of the 2D quasi-geostrophic equations using a Gronwall type Lemma. Section 3 is devoted to compressible flows. At first we explain the main difficulties for proving the existence of global weak solutions. The anisotropy of the stress tensor seems to prevent from adapting P.–L. Lions' existence proof, see [44]. Subsection 3.2 is devoted to rigorous global mathematical stability of the anisotropic compressible Navier–Stokes equations to a modified 2D quasi-geostrophic model. The derivation in the ill prepared case is discussed in Section 3.2.

In the appendix, we give the construction that leads to the geometrical condition on the cylinder section. It is related to the possible damping of waves created by ill prepared data, for the incompressible and compressible cases.

Remark. In all what follows, the time variable will be denoted by t and the spaces variables $(\mathrm{x}, z) = (x, y, z)$. Moreover,

— The letter u will always refer to a 3D vector field,
— The letter v will always refer to a 2D vector field,
— The letter w will always refer to the vertical component of a 3D field.

2 Rotating incompressible fluids in a cylinder

2.1 Statement of the results

Throughout section 2, we focus on the incompressible system (1) under the classical anisotropic scaling $\mu_z = r_0^2\varepsilon$, where r_0 is a constant (see for instance [7]). Equations (1), (2) become

$$\begin{cases} \partial_t u + u\cdot\nabla u + \dfrac{\nabla p}{\varepsilon} + \dfrac{e_3\times u}{\varepsilon} - \mu\Delta_{\mathrm{x}} u - r_0^2\varepsilon\partial_z^2 u = 0 \quad \text{in } \Omega, \\ \operatorname{div} u = 0 \quad \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{8}$$

where $\Omega = S\times(0,1)$, S smooth bounded domain of R^2. We assume general initial data $u_0 \in H(\Omega)$, where for any open set $\mathcal{O}$ of R^d, $H(\mathcal{O})$ is defined by

$$H(\mathcal{O}) = \left\{ u \in \left(L^2(\mathcal{O})\right)^d, \quad \operatorname{div} u = 0, \quad u\cdot n_{|\partial\mathcal{O}} = 0 \right\}.$$

As an element of $H(\Omega)$, it has the following unique decomposition

$$u_0 = \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} \bar v_0 \\ 0 \end{pmatrix} + \sum_{n\geq 1} \begin{pmatrix} \bar v_n \cos n\pi z \\ -(n\pi)^{-1}\operatorname{div}_{\mathrm{x}} \bar v_n \sin n\pi z \end{pmatrix} \tag{9}$$

with $\bar v_n \in H(S)$ for all n.

The goal of this part is to prove the strong convergence of all sequences of weak solutions u^ε of the rotating Navier–Stokes equations (1) to $u^0 = (\bar v, 0)$ where $(\bar v, \bar p) = ((\bar v_1, \bar v_2), \bar p)$ is the unique solution of the 2D damped Navier–Stokes equations

$$\begin{cases} \partial_t \bar v + \bar v\cdot\nabla_{\mathrm{x}}\bar v + \nabla_{\mathrm{x}}\bar p + \sqrt{2} r_0 \bar v - \mu\Delta_{\mathrm{x}}\bar v = 0 \quad \text{in} \quad S, \\ \operatorname{div}_{\mathrm{x}} \bar v = 0 \quad \text{in} \quad S, \\ \bar v_{|\partial S} = 0, \qquad \bar v_{|t=0} = v_0, \end{cases} \tag{10}$$

In what follows we will denote L^ε the differential operator

$$L^\varepsilon : u \mapsto \partial_t u + \frac{e_3 \times u}{\varepsilon} - \operatorname{div} \nabla_\mu u.$$

We will note A_0 the Coriolis operator

$$A_0 : H(\Omega) \to H(\Omega), \quad u \mapsto P(e_3 \times u),$$

where P is the Leray projector on divergence free vector fields. We recall that A^0 is a skew-symmetric operator and that

$$\ker A_0 = \left\{ \begin{pmatrix} \bar v_0 \\ 0 \end{pmatrix}, \quad \bar v_0 = \bar v_0(x,y) \in H(S) \right\}.$$

In particular, if Π is the orthogonal projection on $\ker A^0$, and $u_0 \in H(\Omega)$ admits decomposition (9), then

$$\Pi u_0 = \begin{pmatrix} \bar v_0 \\ 0 \end{pmatrix}, \quad (\mathrm{Id} - \Pi) u_0 = \sum_{n \geq 1} \begin{pmatrix} \bar v_n \cos n\pi z \\ -(n\pi)^{-1} \nabla_{x,y} \cdot \bar v_n \sin n\pi z \end{pmatrix}$$

In order to be able to establish the asymptotic result, we assume the following hypothesis:

$$\text{If } -\Delta\Psi = \lambda^2 \Psi \text{ in } S,\ \lambda \neq 0,\ \partial\Psi/\partial n = 0 \\ \text{and } \Psi = C^{te} \text{ on } \partial\Omega, \quad \text{then } \Psi \equiv 0. \tag{11}$$

$$\text{The Coriolis operator } A_0 \text{ admits an eigenbasis.} \tag{12}$$

Remark. We can assume this eigenbasis to be orthonormal, since A_0 is skew-symmetric, and $\mathcal{C}^\infty$, as an easy consequence of the spectral analysis performed in the appendix.

Remark. Assumption (11) is considered in order to ensure that oscillations are damped in the viscous boundary layer near the lateral side of the cylinder (far from the corners). Such kind of hypothesis has been formerly made in [17] for the study of the low Mach number limit of compressible fluids in bounded domains. Note that in the two dimensional case, it is proved that every bounded, simply connected open set $\Omega \subset \mathrm{R}^2$ whose boundary is Lipschitz but not real analytic satisfies (11) (see [17]).

Remark. Hypothesis (12) is made since, to our knowledge, spectral properties in bounded domains for such an operator seems to be unknown. If we try to write the problem in terms of the pressure term, boundary conditions depending on the potential eigenvalues appear. Remind to the reader that all previous works on rotating fluids in the ill prepared case were considering in infinite domains, periodic, or vorticity dependent conditions with respect to the horizontal coordinates.

We will prove the following convergence result

Theorem 1 *Let* (11) *and* (12) *be satisfied and* u^ε *a family of weak solutions of* (8)*. Then, for all* $T > 0$,

$$u^\varepsilon \to u^0 \ \textit{in} \ (L^2(0,T) \times \Omega)^3 \cap L^\infty_{loc}(0,T;(L^2(\Omega))^3)$$

when $\varepsilon \to 0$ *with* $u^0 = \begin{pmatrix} \bar{v} \\ 0 \end{pmatrix}$*, where* $(\bar{v}, \bar{p})$ *is the unique solution of* (10).

As usual in boundary layer problems, the proof of this theorem is based on the construction of an approximate solution u^ε_{app} of solutions u^ε of (1)–(2). This approximate solution is decomposed into two parts: $u^\varepsilon_{app} = u^{\varepsilon,wp} + u^{\varepsilon,ip}$. The velocity $u^{\varepsilon,wp}$ (resp. $u^{\varepsilon,ip}$) is associated with the well prepared part Πu_0 (resp. the ill-prepared part $(\mathrm{Id}-\Pi)\, u_0$) of initial data u_0. Such decomposition has been used in [8] for a domain $\Omega = \mathbb{R}^2 \times (0,1)$ where the key observation is that dispersive phenomena are not affected by boundary layers. Let us note again that one of the ingredients in this paper is a Strichartz–type estimate using that $S = \mathbb{R}^2$, and that this kind of estimate can not be used for $\Omega = S \times (0,1)$ with S a bounded set.

Therefore, the changes and difficulties come from the lateral side of the cylinder and thus from the presence of corners (*i.e.* $\partial S \times \{0\}$ and $\partial S \times \{1\}$), which require additional correctors.

As a preliminary step of the proof, we approximate the initial data by a smoother one, which involves only a finite number of eigenmodes. Let $\delta > 0$. Using Hypothesis (12) and the attached remark, there exists $\tilde{u}_0$ in $(H^\infty(\Omega))^3$ such that

$$\|\tilde{u}_0 - u_0\|_{L^2(\Omega)} \leq \delta, \quad (\mathrm{Id} - \Pi)\tilde{u}_0 = \sum_{k=1}^{\ell} e^k,$$

where $\ell = \ell(\delta) \in \mathrm{N}$ and e^k is a smooth eigenvector of A_0, *i.e.* $A_0 e^k = \alpha_k e^k$, with α_k non-zero and purely imaginary since A_0 is skew-symmetric.

2.2 The well prepared part

The well prepared approximation $u^{\varepsilon,wp}$ is made of two parts $u_p^{\varepsilon,wp}$ and $u_c^{\varepsilon,wp}$.

Velocity $u_p^{\varepsilon,wp}$ is a classical Ansatz with *profiles* similar to the ones used in [30] for $\Omega = \mathrm{T}^2 \times (0,1)$. It takes account of the mean value of the velocity and of the bottom and top boundary layers.

Velocity $u_c^{\varepsilon,wp}$ is a corrector built in order to get homogeneous boundary conditions at the lateral boundary of the cylinder. With this construction, we will prove the following proposition

Proposition 2 *There exists $u^{\varepsilon,wp} \in (C^\infty(\Omega))^3$ such that*

$$\begin{cases} \partial_t u^{\varepsilon,wp} + u^{\varepsilon,wp} \cdot \nabla u^{\varepsilon,wp} + \dfrac{e_3 \times u^{\varepsilon,wp}}{\varepsilon} \\ \qquad\qquad\qquad\qquad + \dfrac{\nabla p^{\varepsilon,wp}}{\varepsilon} - \operatorname{div} \nabla_\mu u^{\varepsilon,wp} = r^{\varepsilon,wp}, \\ \operatorname{div} u^{\varepsilon,wp} = 0, \\ u^{\varepsilon,wp}_{|\partial\Omega} = 0, \end{cases} \tag{13}$$

where

$$\| u^{\varepsilon,wp}_{|t=0} - \Pi \tilde{u}_0 \|_{(L^2(\Omega))^3} \to 0, \quad as \quad \varepsilon \to 0,$$

and the remainder $r^{\varepsilon,wp}$ satisfies for small enough ε: for all f in $(H_0^1(\Omega))^3$, for all $t \geq 0$,

$$\begin{aligned} &\langle\, r^{\varepsilon,wp}(t,\cdot),\ f \,\rangle_{((H^{-1}(\Omega))^3 \times (H_0^1(\Omega))^3)} \\ &\qquad \leq C_1\, \varepsilon^{1/4} + C_2\, \varepsilon^{1/4} \Big(\|f\|^2_{(L^2(\Omega))^3} + \|\nabla_\mu f\|^2_{(L^2(\Omega))^9} \Big) \end{aligned} \tag{14}$$

where the C_i's are positive constants independent of t.

Ansatz

We look for approximate velocity $u_p^{\varepsilon,wp}$ in the form

$$\begin{aligned} u_p^{\varepsilon,wp} &= \sum_{i=1}^n \varepsilon^i u_p^i\Big(t,x,y,z,\frac{z}{\varepsilon},\frac{1-z}{\varepsilon}\Big) \ +\ r^\varepsilon \\ &= \sum_{i=1}^n \varepsilon^i \Bigg(u^{int,i}(t,x,y,z) + u^{b,i}\Big(t,x,y,\frac{z}{\varepsilon}\Big) \\ &\qquad\qquad\qquad + u^{t,i}\Big(t,x,y,\frac{1-z}{\varepsilon}\Big) \Bigg) \ +\ r^\varepsilon \end{aligned} \tag{15}$$

with $\|r^\varepsilon\|_{L^\infty(0,T;L^2)} \leq\ C\ \varepsilon^{n+1}$. The construction of such approximate solution has been achieved in [30] in the case $\Omega = \mathrm{T}^2 \times (0,1)$. For $\Omega = S \times (0,1)$, we obtain the existence of $u_p^{\varepsilon,wp}$ in the form (15) solution of

$$\begin{cases} \partial_t u_p^{\varepsilon,wp} + u_p^{\varepsilon,wp} \cdot \nabla u_p^{\varepsilon,wp} + \dfrac{e_3 \times u_p^{\varepsilon,wp}}{\varepsilon} + \dfrac{\nabla p^{\varepsilon,wp}}{\varepsilon} - \mathrm{div}\, \nabla_\mu u_p^{\varepsilon,wp} = r_p^{\varepsilon,wp}, \\ \mathrm{div}\, u_p^{\varepsilon,wp} = 0, \\ u_p^{\varepsilon,wp}|_{z=0} = u_p^{\varepsilon,wp}|_{z=1} = 0, \end{cases}$$

where

$$\|r_p^{\varepsilon,wp}\|_{L^\infty(0,T;(L^2(\Omega))^3)} = O(\varepsilon^{n-2}) \quad \|u_p^{\varepsilon,wp}|_{t=0} - \Pi\tilde{u}_0\|_{(L^2(\Omega))^3} \to 0, \quad \varepsilon \to 0.$$

(n can be chosen as large as desired). The detailed construction of $u_p^{\varepsilon,wp}$ will not be given here, since it is similar to [30]. We only recall some properties of profiles u^i which will be useful in what follows to control the corrector part $u_c^{\varepsilon,wp}$.

Remark. In the sequel, we omit the top boundary layer near $z = 1$ since it leads to similar correctors as for the bottom one at $z = 0$.

Main properties of the profiles

Leading order. The term $u^{int,0}$ is two–dimensional, $u^{int,0} = (\bar{v}(t,x,y), 0)$, where $\bar{v}$ is solution of (10), with initial data $\bar{v}_0$ replaced by $\tilde{v}_0$ (the well-prepared part of $\tilde{u}_0$).

The term $u^{b,0}$ satisfies $w_3^{b,0} = 0$, and

$$v_1^{b,0} + iv_2^{b,0} = -e^{-\frac{(1+i)\theta}{\sqrt{2}}} \left(v_1^{int,0}(t,x,y,0) + iv_2^{int,0}(t,x,y,0) \right).$$

Order ε. Following the same procedure as in [27], we see that $v^{int,1}$ has the form

$$v^{int,1} = v^{int,1}(x,y) = -\sqrt{2} r_0 \bar{v}^\perp + v^1(x,y),$$

where v^1 is solution in S of the linearized Navier–Stokes equations, of type

$$\begin{cases} \partial_t v^1 + \bar{v} \cdot \nabla v^1 + v^1 \cdot \nabla \bar{v} + \nabla \bar{p} - \mu \Delta v^1 + \sqrt{2} r_0 v^1 = f, \\ \mathrm{div}_{\,\mathrm{x}} v^1 = 0, \end{cases}$$

with Dirichlet boundary condition $v^1 = 0$ on ∂S. Thus, the trace of $v^{int,1}$ on the lateral side of the cylinder vanishes. The associated vertical components satisfy

$$w^{int,1} = \frac{\partial_x v_2^{int,0} - \partial_y v_1^{int,0}}{\sqrt{2}}\,(1-2z)\,,$$

$$w^{b,1} = -\frac{1}{\sqrt{2}} e^{-\frac{\theta}{\sqrt{2}}} \left(\partial_x v_2^{int,0} - \partial_y v_1^{int,0}\right)\left(\sin\left(\frac{\theta}{\sqrt{2}}\right) + \cos\left(\frac{\theta}{\sqrt{2}}\right)\right).$$

We note that

$$\partial_\theta w^{b,1} = e^{-\frac{\theta}{\sqrt{2}}} \sin\left(\frac{\theta}{\sqrt{2}}\right)\left(\partial_x v_2^{int,0} - \partial_y v_1^{int,0}\right). \tag{16}$$

Higher orders. The higher order terms do not a priori vanish on the lateral side.

Conclusion. Due to the $O(1)$ horizontal viscosity, the leading terms of (15) satisfy homogeneous boundary conditions on all $\partial\Omega$. There is only a "residual" trace at $\partial S \times (0,1)$, namely those of $\varepsilon w^{int,1}$, $\varepsilon w^{b,1}$ and higher order terms. Lifting this boundary data is the purpose of the next part.

Lift of the residual trace

The goal of this part is to get an approximate solution vanishing on $\partial\Omega$. For this, we build a corrector $u_c^{\varepsilon,wp}$ as a sum of three terms,

$$u_c^{\varepsilon,wp} = u_c^{\varepsilon,1} + u_c^{\varepsilon,2} + u_c^{\varepsilon,3},$$

so that
— $u_c^{\varepsilon,1}$ is a correction to $\varepsilon w^{int,1}$ and $\varepsilon w^{b,1}$.
— $u_c^{\varepsilon,2}$ is a correction to higher order terms.
— $u_c^{\varepsilon,2}$ allow to recover the divergence free condition, which is not satisfied by the previous correctors. Moreover, for $i = 1, \cdots, 3$, $R^{\varepsilon,i} = L^\varepsilon u_c^{\varepsilon,i}$ satisfies the inequality: for all f in $(H_0^1(\Omega))^3$, for all $t \geq 0$,

$$\begin{aligned} &\langle\, R^{\varepsilon,i}(t,\cdot),\, f\,\rangle_{((H^{-1}(\Omega))^3\times(H_0^1(\Omega))^3)} \\ &\qquad \leq C_{1,i}\,\varepsilon^{1/4} + C_{2,i}\,\varepsilon^{1/4}\left(\|f\|^2_{(L^2(\Omega))^3} + \|\nabla_\mu f\|^2_{(L^2(\Omega))^9}\right). \end{aligned} \tag{17}$$

Correction of w_c^1 and $w_c^{b,1}$

We define $u_c^{\varepsilon,1} = (v_c^{\varepsilon,1}, w_c^{\varepsilon,1})$ as follows

$$\begin{aligned} v_c^{\varepsilon,1} &= \varepsilon^\gamma\Big(-1 + e^{-d(x,y)/\varepsilon^\gamma}\Big)\frac{\nabla d(x,y)}{|\nabla d(x,y)|^2}\ \ \partial_\theta w^{b,1}(t,x,y,\varepsilon^{-1}z),\\ w_c^{\varepsilon,1} &= -e^{-d(x,y)/\varepsilon^\gamma}\ \Big(\varepsilon\ w^{int,1}(t,x,y,z) + \varepsilon\ w^{b,1}(t,x,y,\varepsilon^{-1}z)\Big), \end{aligned}$$

where $d : \bar{S} \mapsto \mathbb{R}$ is a mollified distance to the boundary S, and $\gamma \in (1/2, 1)$. The corrector $u_c^{\varepsilon,1}$ satisfies the following properties:
— The horizontal component $v_c^{\varepsilon,1}$ vanishes on $\partial\Omega$: it vanishes in particular for $z = 0$ using equation (16).
— The vertical component $w_c^{\varepsilon,1}$ vanishes on $z = 0$ and satisfies:

$$\begin{aligned} &\forall\ (x,y,z) \in \partial S \times (0,1),\\ &\qquad w_c^{\varepsilon,1}(t,x,y,z) = -\varepsilon\left(w^{int,1}(t,x,y,z) + w^{b,1}(t,x,y,\varepsilon^{-1}z)\right). \end{aligned}$$

Thus $u_c^{\varepsilon,1}$ corrects the order ε trace of $u_p^{\varepsilon,wp}$.
Moreover, the corrector $u_c^{\varepsilon,1}$ satisfies the following estimates

$$\begin{aligned} &\|\mathrm{div} u_c^{\varepsilon,1}\|_{W^{1,\infty}(0,T;L^2(\Omega))} = O(\varepsilon^{\gamma+1/2}),\\ &\|u_c^{\varepsilon,1}\|_{W^{1,\infty}(0,T;(L^2(\Omega))^3)} = O(\varepsilon^{\gamma+1/2}), \qquad \|u_c^{\varepsilon,1}\|_{W^{1,\infty}(0,T;(H^1(\Omega))^3)} = O(\varepsilon^{\gamma-1/2}). \end{aligned}$$

Choosing now for instance $\gamma = 3/4$, we deduce from the last two estimates: for all f in $H_0^1(\Omega)^3$, for all $t \geq 0$,

$$\begin{aligned} &\langle\ R^{\varepsilon,1}(t,\cdot),\ f\ \rangle_{\big((H^{-1}(\Omega))^3\times(H_0^1(\Omega))^3\big)}\\ &= \int_\Omega\left(\partial_t u_c^{\varepsilon,1}(t,\cdot) + \frac{e_3\times u_c^{\varepsilon,1}(t,\cdot)}{\varepsilon}\right)\cdot f + \int_\Omega \nabla u_c^{\varepsilon,1}\cdot\nabla_\mu f\\ &\leq \frac{2}{\varepsilon}\ \|u_c^{\varepsilon,1}\|_{W^{1,\infty}((L^2(\Omega))^3)}\|f\|_{(L^2(\Omega))^3} + \|u_c^{\varepsilon,1}\|_{L^\infty((H^1(\Omega))^3)}\|\nabla_\mu f\|_{(L^2(\Omega))^3}\\ &\leq c_1\,\varepsilon^{1/4}\|f\|_{(L^2(\Omega))^3} + c_2\ \varepsilon^{1/4}\|\nabla_\mu f\|_{(L^2(\Omega))^9}\\ &\leq (c_1^2\varepsilon^{1/4} + c_2^2\varepsilon^{1/4}) + \varepsilon^{1/4}\left(\|f\|^2_{(L^2(\Omega))^3} + \|\nabla_\mu f\|^2_{(L^2(\Omega))^9}\right), \end{aligned}$$

so that inequality (17) is satisfied for $i = 1$.

Correction of higher order terms

We want now to correct the trace on $\partial S \times (0,1)$ of the other terms of the Ansatz, *i.e.* of

$$\Phi^\varepsilon = \sum_{i\geq 2} \varepsilon^i \, u_p^i(t,x,y,z,\frac{z}{\varepsilon},\frac{(1-z)}{\varepsilon}).$$

The trace φ^ε of this term on $\partial\Omega$ satisfies:

$$\varphi^\varepsilon \in W^{1,\infty}(0,T;(H^{1/2}(\partial\Omega))^3), \quad \|\varphi^\varepsilon\|_{W^{1,\infty}(0,T;(H^{1/2}(\partial\Omega))^3)} = O(\varepsilon^{3/2}).$$

Then, see [23] and [53], there exists $u_c^{\varepsilon,2} \in W^{1,\infty}(0,T;(H^1(\Omega))^3)$, such that

$$u_c^{\varepsilon,2} = -\varphi^\varepsilon, \quad \|u_c^{\varepsilon,2}\|_{W^{1,\infty}(0,T;(H^1(\Omega))^3)} \leq C\|\varphi^\varepsilon\|_{W^{1,\infty}(0,T;(H^{1/2}(\partial\Omega))^3)} = O(\varepsilon^{3/2}).$$

Proceeding as in the case i=1, we then show that inequality (17) is satisfied for $i=2$.

Finally, it remains to recover the divergence free condition, which is not fulfilled because of the previous correctors. The function $u^\varepsilon_{app,p} = u_p^{\varepsilon,wp} + u_c^{\varepsilon,1} + u_c^{\varepsilon,2}$ has zero trace on $\partial\Omega$, and satisfies

$$\operatorname{div} u^\varepsilon_{app,p} = O(\varepsilon^{3/4}).$$

We know, see [23] and [53], that there exists $u_c^{\varepsilon,3} \in (H^1_0(\Omega))^3$ such that

$$\operatorname{div} u_c^{\varepsilon,3} = -\operatorname{div} u^\varepsilon_{app,p}.$$

and that it satisfies the following estimate:

$$\|u_c^{\varepsilon,3}\|_{W^{1,\infty}(0,T;(H^1(\Omega))^3)} \leq C\|\operatorname{div} u^\varepsilon_{app,p}\|_{W^{1,\infty}(0,T;L^2(\Omega))} = O(\varepsilon^{3/4}).$$

In particular, Inequality (17) is satisfied for $i=3$.

Proof of Proposition 2.

By the previous construction, the divergence free condition, Dirichlet condition on $\partial\Omega$, and the convergence of the initial data are satisfied by $u^{\varepsilon,wp} = u_p^{\varepsilon,wp} + u_c^{\varepsilon,wp}$, where $u_c^{\varepsilon,wp} = u_c^{\varepsilon,1} + u_c^{\varepsilon,2} + u_c^{\varepsilon,3}$. Then $u_p^{\varepsilon,wp}$ is solution of System (13) if and only if

$$r^{\varepsilon,wp} = L^\varepsilon u_c^{\varepsilon,wp} + u_p^{\varepsilon,wp}\cdot\nabla u_c^{\varepsilon,wp} + u_c^{\varepsilon,wp}\cdot\nabla u_c^{\varepsilon,wp} + u_c^{\varepsilon,wp}\cdot\nabla u_p^{\varepsilon,wp}.$$

The linear term $L^\varepsilon u_c^{\varepsilon,wp}$ satisfies an estimate of the type (14), thanks to Inequalities (17). The quadratic terms are treated in a similar manner.

2.3 The ill prepared part

We now turn to the construction of the ill prepared approximation $u^{\varepsilon,ip}$. As mentioned in the introduction, this part carries high frequency waves, which may prevent from getting the strong convergence of solutions u^ε of (8). In our case of study, we will show that a boundary layer develops at the lateral boundary of the cylinder, and that this layer dissipates almost instantaneously the energy of the waves, see the appendix. Similarly to the well-prepared case, $u^{\varepsilon,ip}$ will be constructed in two steps.

The Ansatz $u^{\varepsilon,ip}$ is a combination of profiles taking into account interior and boundary layer terms (Ekman layers and lateral layers). As will be shown below, properties of this Ansatz are directly linked to the spectral properties of the incompressible viscous Coriolis operator.

In the second step, $u_c^{\varepsilon,ip}$ is designed to lift the trace due to the boundary layer terms in the corner of the cylinder. Choosing $u^{\varepsilon,ip} = u_p^{\varepsilon,ip} + u_c^{\varepsilon,ip}$, we can state the following proposition.

Proposition 3 *The ill prepared part $u^{\varepsilon,ip}$ satisfies*

$$\begin{cases} \partial_t u^{\varepsilon,ip} + u^{\varepsilon,ip} \cdot \nabla u^{\varepsilon,ip} + \dfrac{e_3 \times u^{\varepsilon,ip}}{\varepsilon} + \dfrac{\nabla p^{\varepsilon,ip}}{\varepsilon} - \operatorname{div} \nabla_\mu u^{\varepsilon,ip} = r^{\varepsilon,ip}, \\ \operatorname{div} u^{\varepsilon,ip} = 0, \\ u^{\varepsilon,ip}_{|\partial\Omega} = 0, \end{cases} \tag{18}$$

with

$$\|u^{\varepsilon,ip}_{|t=0} - (\mathrm{Id} - \Pi)\tilde{u}_0\|_{(L^2(\Omega))^3} \to 0, \qquad \text{as } \varepsilon \to 0,$$

and $r^{\varepsilon,ip}$ satisfying: for all f in $L^2(0,T;(H_0^1(\Omega))^3)$, for all $0 \le t \le T$,

$$\begin{aligned} \int_0^t \langle\, r^{\varepsilon,ip}(s,\cdot),\, f(s,\cdot)\,\rangle_{((H^{-1}(\Omega))^3 \times (H_0^1(\Omega))^3)}\, ds \\ \le C_1\varepsilon^{1/4} + C_2\varepsilon^{1/4} \int_0^t \|f(s,\cdot)\|^2_{(L^2(\Omega))^3}\, ds \\ + C_3 \int_0^t \frac{\exp(-as/\sqrt{\varepsilon})}{\varepsilon^{1/4}} \|f(s,\cdot)\|^2_{(L^2(\Omega))^3}\, ds \\ + C_4\varepsilon^{1/4} \int_0^t \|\nabla_\mu f(s,\cdot)\|^2_{(L^2(\Omega))^9}\, ds, \end{aligned} \tag{19}$$

where the C_i's and a are positive constants, eventually depending on the δ chosen in section 2.1.

Ansatz

We look for an approximate solution of the **linear** rotating fluids system in the form

$$\begin{aligned} u_p^{\varepsilon,ip} &= \sum_{k=1}^{\ell} e^{\lambda_k(\varepsilon)t} \sum_{j=0}^{n} \varepsilon^{j/2}\, u_p^{j,k}\left(\frac{d(x,y)}{\sqrt{\varepsilon}}, x, y, z, \frac{z}{\varepsilon}, \frac{(1-z)}{\varepsilon}\right) + r^\varepsilon, \\ u_p^{j,k}\left(\zeta, x, y, z, \theta, \lambda\right) &= u_{int}^{j,k}(x,y,z) + u_b^{j,k}(x,y,\theta) \\ &\qquad + u_t^{j,k}\left(x,y,\lambda\right) + u_\ell^{j,k}\left(\zeta,x,y,z\right), \end{aligned} \tag{20}$$

where $\ell = \ell(\delta)$ has been defined in section 2.1, and

$$\|r^\varepsilon\|_{L^\infty(0,T;(L^2(\Omega))^3)} = O(\varepsilon^{n+1/4}), \quad \lambda_k(\varepsilon) = \frac{\lambda_k^0}{\varepsilon} + \frac{\lambda_k^1}{\sqrt{\varepsilon}} + \lambda_k^2 + \cdots$$

If we set $U^{\varepsilon,k} = \sum \varepsilon^{j/2} u_p^{j,k}\left(\varepsilon^{-1/2} d(x,y), x, y, z, \varepsilon^{-1} z, \varepsilon^{-1}(1-z)\right)$, we see that this problem can be reformulated as a search for approximate eigenmodes $(\pm i\lambda_{k,\varepsilon}^{\pm}, U^{\varepsilon,k})$ of the incompressible viscous Coriolis operator

$$A_\varepsilon : u \mapsto P(e_3 \times u) - \varepsilon \operatorname{div} \nabla_\mu u$$

where $\pm i\lambda_{k,\varepsilon}^{\pm} = -\varepsilon\lambda_k(\varepsilon)$. Such a spectral analysis is performed in the appendix. As a result, we get the existence of $u_p^{\varepsilon,ip}$ in the form (20) solution of

$$\begin{cases} \partial_t u_p^{\varepsilon,ip} + \dfrac{e_3 \times u_p^{\varepsilon,ip}}{\varepsilon} + \dfrac{\nabla p^{\varepsilon,ip}}{\varepsilon} - \Delta^\mu u_p^{\varepsilon,ip} = r_p^{\varepsilon,ip}, \\ \operatorname{div} u_p^{\varepsilon,ip} = 0, \end{cases}$$

where $\|r_p^{\varepsilon,ip}\|_{L^\infty(0,T;(L^2(\Omega))^3))} = O(\varepsilon^{n-2})$, where n can be taken as large as we wish. This approximate solution of the linear system is such that:

$$\forall\, k \in \{1, \cdots, \ell\}, \quad \lambda_k^0 = \alpha_k, \quad \operatorname{Re}(\lambda_k^1) < 0, \quad u_p^{0,k} = e^k,$$

where (α_k, e^k) is an eigenmode of the Coriolis operator A_0, that we can choose to be the one defined in section 2.1. Note that

$$\forall\, 1 \le k \le \ell, \quad |\exp(\lambda_k(\varepsilon)t)| \le \exp(-\alpha t/\sqrt{\varepsilon}),$$

where $\alpha > 0$ depends on δ as it depends on the number ℓ of eigenmodes.

This approximate solution does not vanish on the boundary of Ω: the horizontal boundary layer has a trace on the vertical side and conversely.

Therefore, we have to define a corrector $u_c^{\varepsilon,ip}$, which modifies the solution in the vicinity of the corner.

As in the previous part, we omit the horizontal boundary layer near $z = 1$ since the behavior is the same and it could be treated in a similar manner.

Remark. In the study of the incompressible limit of isentropic Navier–Stokes equations, an "approximate spectral analysis" of the type evoked above is performed in [17] on the classical wave operator. As in our case, it is the first step in the proof of a strong convergence result. But the rest of the proof contained in [17] does not adapt to our system. Indeed, it relies in a crucial way on the uniform H^1 bound of velocity field u^ε, which we do not get here because of the anisotropic diffusion. This explains why we use another method, namely the construction of precise approximate solutions.

Corner Layers

The corrector $u_c^{\varepsilon,ip}$ is searched in the form

$$u_c^{\varepsilon,ip} = \sum_{k=1}^{\ell}\sum_{i=1}^{5}\mathcal{U}_c^{\varepsilon,k,i}, \quad \mathcal{U}_c^{\varepsilon,k,i} = e^{\lambda_k(\varepsilon)t}U_c^{\varepsilon,k,i}(x,y,z). \tag{21}$$

We will only build the first mode in time ($k = 1$), the k-th mode, $k \geq 2$, is obtained in a similar way. To shorten notations, we will set

$$\mathcal{U}_c^{\varepsilon,i} := \mathcal{U}_c^{\varepsilon,1,i}, \quad U_c^{\varepsilon,i} := U_c^{\varepsilon,1,i}, \quad u_{int}^j := u_{int}^{j,1}, \quad u_b^j := u_b^{j,1}, \quad u_\ell^j := u_\ell^{j,1}$$

where $\mathcal{U}_c^{\varepsilon,i,1}$, $\cdots$, $u^{j,1}$, $\cdots$ are defined in (21) and (20). Each corrector plays the following role:
— $\mathcal{U}_c^{\varepsilon,1}$ lifts the tangential part of the trace due to the layers (the leading order term).
— $\mathcal{U}_c^{\varepsilon,2}$ and $\mathcal{U}_c^{\varepsilon,3}$ lift the normal part of the trace due to the layers (the leading order term).
— Finally, $\mathcal{U}_c^{\varepsilon,4}$ and $\mathcal{U}_c^{\varepsilon,5}$ handle respectively the higher order term and divergence free condition.

We build $\mathcal{U}_c^{\varepsilon,i}$ in order that for all i, $R^{\varepsilon,i} := L^\varepsilon\mathcal{U}_c^{\varepsilon,i}$ satisfies: for all f in

$L^2(0,T;(H_0^1(\Omega))^3)$, for all $0 \le t \le T$,

$$
\begin{aligned}
&\int_0^t \langle\, R^{\varepsilon,i}(s,\cdot),\ f(s,\cdot)\,\rangle_{((H^{-1}(\Omega))^3\times(H_0^1(\Omega))^3)}\, ds \\
&\quad \le C_{1,i}\varepsilon^{1/4} + C_{2,i}\varepsilon^{1/4}\int_0^t \|f(s,\cdot)\|^2_{(L^2(\Omega))^3} \\
&\qquad + C_{3,i}\int_0^t \frac{\exp(-as/\sqrt{\varepsilon})}{\varepsilon^{1/4}}\|f(s,\cdot)\|^2_{(L^2(\Omega))^3}\, ds \\
&\qquad + C_{4,i}\varepsilon^{1/4}\int_0^t \ \|\nabla_\mu f(s,\cdot)\|^2_{(L^2(\Omega))^9}\, ds,
\end{aligned}
\tag{22}
$$

where the constants are positive, eventually depending on δ. Modulo the rest r^ε in (20), the correctors $U_c^{\varepsilon,i}$ have to satisfy the following boundary conditions

$$
\begin{aligned}
\sum_{i=1}^{5} U_c^{\varepsilon,i} &= -\sum_{j=1}^{n} \varepsilon^{j/2} u_b^j \text{ on } \partial S\times(0,1)\\
\sum_{i=1}^{5} U_c^{\varepsilon,i} &= -\sum_{j=1}^{n} \varepsilon^{j/2} u_\ell^j \text{ on } S\times\{0\}.
\end{aligned}
$$

Correction of the tangential part

We define $U_c^{\varepsilon,1} = (V_c^{\varepsilon,1}, W_c^{\varepsilon,1})$ as follows

$$
V_c^{\varepsilon,1} = -\frac{(\nabla d(x,y))^\perp}{|\nabla d(x,y)|^2}\, f\left(\frac{d(x,y)}{\sqrt{\varepsilon}}, x, y, \frac{z}{\varepsilon}\right), \qquad W_c^{\varepsilon,1} = 0,
$$

where $f(\zeta,x,y,\theta)$ satisfies: $\forall\ (x,y)\in S$,

$$
\begin{aligned}
f(0,x,y,\theta) &= (\nabla d(x,y))^\perp \cdot v_b^0(x,y,\theta),\\
f(\zeta,x,y,0) &= (\nabla d(x,y))^\perp \cdot v_\ell^0(\zeta,x,y,0).
\end{aligned}
$$

The profile $U_c^{\varepsilon,1}$ satisfies the following properties:

$$
U_c^{\varepsilon,1} = -\begin{pmatrix} v_b^0\cdot\tau \\ 0\end{pmatrix} \text{ on } \partial S\times(0,1)
$$

where τ is the unit tangent at ∂S. On the bottom, it satisfies

$$
U_c^{\varepsilon,1} = -\begin{pmatrix} v_\ell^0 \\ 0\end{pmatrix} \text{ on } S\times\{0\},
$$

observing that $v_\ell^0 \cdot \nabla d = 0$.

Moreover, we get the following controls

$$\begin{aligned}
&\|\mathrm{div} U_c^{\varepsilon,1}\|_{L^2(\Omega)} = O(\varepsilon^{3/4}), \quad \|U_c^{\varepsilon,1}\|_{(L^2(\Omega))^3} = O(\varepsilon^{3/4}),\\
&\|\nabla U_c^{\varepsilon,1}\|_{(L^2(\Omega))^9} = O(\varepsilon^{-1/4}), \quad \|\mathrm{div}\, \nabla_\mu U_c^{\varepsilon,1}\|_{(L^2(\Omega))^3} = O(\varepsilon^{-1/4}).
\end{aligned}$$

In particular, we get the estimates

$$\begin{aligned}
\|\mathcal{U}_c^{\varepsilon,1}(s,\cdot)\|_{(L^2(\Omega))^3} =& O(\varepsilon^{3/4} \exp(-\alpha s/\sqrt{\varepsilon})),\\
&\|\mathrm{div}\, \nabla_\mu \mathcal{U}_c^{\varepsilon,1}(s,\cdot)\|_{(L^2(\Omega))^3} = O(\varepsilon^{-1/4} \exp(-\alpha s/\sqrt{\varepsilon})).
\end{aligned}$$

We deduce from these estimates

$$\begin{aligned}
&\langle R^{\varepsilon,1}(s,\cdot),\, f(s,\cdot)\,\rangle_{((H^{-1}(\Omega))^3 \times (H_0^1(\Omega))^3)}\\
&\leq \frac{C_1}{\varepsilon} \|\mathcal{U}_c^{\varepsilon,1}(s,\cdot)\|_{(L^2(\Omega))^3} \|f(s,\cdot)\|_{(L^2(\Omega))^3}\\
&\qquad\qquad + \frac{C_2}{\varepsilon} \|\mathrm{div}\, \nabla_\mu \mathcal{U}_c^{\varepsilon,1}(s,\cdot)\|_{(L^2(\Omega))^3} \|f(s,\cdot)\|_{(L^2(\Omega))^3}\\
&\leq \frac{C}{\varepsilon^{1/4}} \exp(-\alpha s/\sqrt{\varepsilon}) \|f(s,\cdot)\|_{(L^2(\Omega))^3}\\
&\leq \frac{C^2}{2\varepsilon^{1/4}} \exp(-\alpha s/\sqrt{\varepsilon}) + \frac{1}{2\varepsilon^{1/4}} \exp(-\alpha s/\sqrt{\varepsilon}) \|f(s,\cdot)\|_{(L^2(\Omega))^3}.
\end{aligned}$$

Integrating from 0 to t, we obtain inequality (22) for $i = 1$.

Lift of the normal trace

Vertical part. We correct at first the vertical part with $U_c^{\varepsilon,2} = (V_c^{\varepsilon,2}, W_c^{\varepsilon,2})$ where

$$V_c^{\varepsilon,2} = 0, \quad W_c^{\varepsilon,2} = -w_\ell^0 \left(\frac{d(x,y)}{\sqrt{\varepsilon}}, x, y, 0 \right).$$

The profile $U_c^{\varepsilon,2}$ satisfies the following properties: On the lateral side

$$U_c^{\varepsilon,2} = 0 \text{ on } \partial S \times (0,1),$$

since for all $(x,y) \in \partial S$,

$$W_c^{\varepsilon,2} = w_\ell^0(0,x,y,0) = -w_{int}^0(0,x,y,0) = w_b^0(0,x,y,0) = 0.$$

On the bottom:

$$U_c^{\varepsilon,2} = -(0, w_\ell^0) \text{ on } S \times \{0\}.$$

Moreover $U_c^{\varepsilon,2}$ is divergence free and satisfies the following estimates

$$\|U_c^{\varepsilon,2}\|_{(L^2(\Omega))^3} = O(\varepsilon^{1/4}), \quad \|U_c^{\varepsilon,2}\|_{(H^1(\Omega))^3} = O(\varepsilon^{-1/4}). \tag{23}$$

Now, $R^{\varepsilon,2} = L^\varepsilon \mathcal{U}_c^{\varepsilon,2}$ has zero horizontal component, whereas the vertical component satisfies

$$\begin{aligned} R_3^{\varepsilon,2} = &\exp(\lambda_1(\varepsilon)t)\, \Big[\varepsilon^{-1}\lambda_1^0 w_\ell^0\left(\varepsilon^{-1/2}d(x,y),x,y,0\right) \\ &- \varepsilon^{-1}\mu|\nabla d|^2\partial_\zeta^2 w_\ell^0\left(\varepsilon^{-1/2}d(x,y),x,y,0\right)\Big] + \exp(\lambda_1(\varepsilon)t)\ S^\varepsilon(s,\cdot) \end{aligned} \tag{24}$$

where $S^\varepsilon(s,\cdot) = (\varepsilon^{-1/2}\lambda_1^1 + \lambda_1^2 + \ )w_\ell^0 + \dots$ satisfies

$$\|S^\varepsilon(s,\cdot)\|_{(L^2(\Omega))^3} = O(\varepsilon^{-1/4}\exp(-\alpha t/\sqrt{\varepsilon})).$$

The first bracket at the right-hand side of (24) vanishes from the equation satisfied by w_ℓ^0 (see the appendix), so that $R^{\varepsilon,2}$ satisfies Estimate (22).

Horizontal part. The treatment of the normal component of v_b^0 at $\partial S \times (0,1)$ is more delicate for the following reason: it is not difficult to realize oneself that any lift $\mathcal{U}_c^{\varepsilon,3}$ of this boundary data can not lead to an estimate better than

$$\|\mathcal{U}_c^{\varepsilon,3}(s,\cdot)\|_{(H^1(\Omega))^3} = O(\sqrt{\varepsilon}\exp(-\alpha s/\sqrt{\varepsilon})),$$

It is *a priori* not enough to obtain an inequality of type (22). To handle this term, we then must inject another information, namely its rapid decrease in variable $\varepsilon^{-1}z$. This requires to build a lift which shares the same properties. It is the meaning of the following lemma:

Lemma 4 *There exists $U_c^{\varepsilon,3} \in (H^1(\Omega))^3$ such that*

$$U_c^{\varepsilon,3} = -(v_b^0\cdot\nu, \varepsilon w_b^2) \text{ on } \partial S \times (0,1),$$

where ν is the outward unit normal at ∂S, and

$$U_c^{\varepsilon,3} = -(0, \varepsilon w_\ell^2) \text{ on } S \times \{0\}.$$

Such lift can be taken in the form

$$U_c^{\varepsilon,3} = \bar{U}_c^{\varepsilon,3} + (\zeta_{\mathrm{x}}(x,y,\varepsilon^{-1}z), \varepsilon\,\zeta_3(x,y,\varepsilon^{-1}z))$$

where $\bar U_c^{\varepsilon,3}$ satisfies the estimates

$$\|\bar U_c^{\varepsilon,3}\|_{(H^1(\Omega))^3} = O(\varepsilon^{3/4}), \quad \|\mathrm{div}\,\nabla_\mu \bar U_c^{\varepsilon,3}\|_{(L^2(\Omega))^3} = O(\varepsilon^{-1/4}),$$

and $\zeta = (\zeta_{\mathrm x}, \zeta_3) = \zeta(x, y, \theta)$ with $\zeta_{\mathrm x} = (\zeta_1, \zeta_2)$ satisfies

$$\zeta, \quad \theta\zeta \in (L^\infty(S \times \mathbb{R}^+))^3 \cap (H^1(S \times \mathbb{R}^+))^3, \quad \mathrm{div}\,\zeta = 0. \tag{25}$$

Proof of Lemma 4: Let us define

$$\varphi : \partial S \times \mathbb{R}^+ \mapsto \mathbb{R}^2, \quad ((x, y), \theta) \mapsto v_b^0(x, y, \theta)\cdot \nu(x, y),$$

where ν is the outward normal unit at ∂S. It satisfies the "compatibility" condition

$$\int_{\partial S} \varphi(\cdot, \theta) = \int_S \mathrm{div}_{\,\mathrm x} v_b^0 = -\int_S \partial_\theta w_b^2.$$

We know, see [23] and [53], that there exists $\Phi = \Phi(x, y, \theta)$ smooth such that: for all $\theta > 0$, for all k in N,

$$\mathrm{div}_{\,\mathrm x}\Phi(\cdot, \theta) = -\partial_\theta w_b^2(\cdot, \theta), \quad \Phi(\cdot, \theta)|_{\partial S} = \varphi(\cdot, \theta),$$

with

$$\begin{aligned}\|\partial_\theta^k \Phi(\cdot, \theta)\|_{(H^2(S))^3} &\le C_k \left(\|\partial_\theta^{k+1} w_b^2(\cdot, \theta)\|_{(H^1(S))^3} + \|\partial_\theta^k \varphi(\cdot, \theta)\|_{(H^{3/2}(\partial S))^3}\right)\\ &\le C_k' \exp(-c\theta), \quad c > 0.\end{aligned}$$

As a consequence,

$$\Phi, \quad \theta\Phi \in \left(L^\infty\left(S \times \mathbb{R}^+\right)\right)^2 \cap \left(H^1(\left(S \times \mathbb{R}^+\right)\right)^2.$$

Now, set

$$\begin{aligned}\tilde U_c^{\varepsilon,3} &= -\Big(\Phi(x, y, \frac{z}{\varepsilon}), \varepsilon w_b^2(x, y, \frac{z}{\varepsilon}) + \varepsilon w_{int}^2(x, y, 0) + \varepsilon w_\ell^2(\frac{d(x, y)}{\sqrt{\varepsilon}}, x, y, 0)\Big)\\ &= (\tilde\zeta_{\mathrm x}(x, y, \varepsilon^{-1}z), \varepsilon\, \tilde\zeta_3(x, y, \varepsilon^{-1}z)) + \bar U_c^{\varepsilon,3},\end{aligned}$$

with

$$\tilde\zeta = (\Phi(x, y, \theta), w_b^2(x, y, \theta)), \quad \bar U_c^{\varepsilon,3} = (0, \varepsilon w_{int}^2(x, y, 0) + \varepsilon w_\ell^2(\frac{d(x, y)}{\sqrt{\varepsilon}}, x, y, 0))$$

Corrector $\tilde{U}_c^{\varepsilon,3}$ is almost convenient since it satisfies all the properties of Lemma 4 except the value on $S \times \{0\}$ (term $-\Phi(x,y,0)$) which must be modified. Let

$$\mathcal{O} = \{(x,y,\theta) \in S \times (0,1)\},$$

and $\chi = \chi(\theta)$ defined on $(0,1)$ such that

$$\chi \in \mathcal{C}_c^\infty(0,1), \quad \chi(0) = 1.$$

We consider

$$\psi : \partial\mathcal{O} \mapsto \mathbb{R}^3, \quad \psi = (-\Phi(x,y,0),0) \text{ on } S \times \{0\}, \quad \psi = 0 \text{ elsewhere.}$$

One easily checks that ψ is the trace of the function

$$\Psi = \chi(\theta)(v^0, \varepsilon w^2) + U_c^{\varepsilon,1}(x,y,\varepsilon\,\theta) + \bar{U}_c^{\varepsilon,3}(x,y,\varepsilon\,\theta)$$

where

$$v^0 = v_b^0(x,y,\theta) + v_\ell^0(\frac{d(x,y)}{\sqrt{\varepsilon}}, x, y, 0) + v_{int}^0(x,y,0),$$
$$w^2 = w_b^2(x,y,\theta) + w_\ell^2(\frac{d(x,y)}{\sqrt{\varepsilon}}, x, y, 0) + w_{int}^2(x,y,\theta).$$

Since Ψ is in $(W^{1,4}(\mathcal{O}))^3$ then ψ is in $(W^{3/4,4}(\partial\mathcal{O}))^3$. Moreover, using its definition, $\int_{\partial\mathcal{O}} \psi \cdot \vec{n} = 0$. Now, see [23] and [53], we know that there exists ξ in $(W^{1,4}(\mathcal{O}))^3$, such that

$$\partial_x \xi_1 + \partial_y \xi_2 + \partial_\theta \xi_3 = 0, \quad \xi_{|\partial\mathcal{O}} = \psi.$$

We extend ξ by 0 for $\theta > 1$ to get a function

$$\xi \in (W^{1,4}(S \times \mathbb{R}^+))^3 \hookrightarrow (L^\infty(S \times \mathbb{R}^+))^3.$$

As it has compact support with respect to θ, we even have

$$\theta\xi \in (L^\infty(S \times \mathbb{R}^+))^3 \cap (H^1(S \times \mathbb{R}^+))^3.$$

Finally, defining $\zeta = \tilde{\zeta} + \xi$, and

$$U_c^{\varepsilon,3} = \bar{U}_c^{\varepsilon,3} + \Big(\zeta_{\mathrm{x}}(x,y,\frac{z}{\varepsilon}), \varepsilon\zeta_3(x,y,\frac{z}{\varepsilon})\Big),$$

Lemma 4 is proved since $U_c^{\varepsilon,3}$ is convenient.

It remains to show that **the corrector** $R^{\varepsilon,3}$ **satisfies Inequality** (22). The study of $\langle R^{\varepsilon,3}(t,\cdot), f(t,\cdot)\rangle_{((H^{-1}(\Omega))^3\times(H_0^1(\Omega))^3}$ can be decomposed into three parts.

— The term $\bar{U}_c^{\varepsilon,3}$ can be treated as $U_c^{\varepsilon,1}$.

— The viscous part of L^ε leads to integrals of type

$$I^\varepsilon = C\exp(\lambda_1(\varepsilon)s)\int_\Omega (\partial_i\zeta_k)(x,y,\varepsilon^{-1}z)\cdot\partial_j f_k(s,x,y,z)\;dxdydz,$$

where $i\in\{x,y,\theta\}$, $j\in\{x,y,z\}$, $k\in\{1,2,3\}$ and ζ given as in lemma 4. As

$$\begin{aligned}|I^\varepsilon| &\le C\;\sqrt{\varepsilon}\exp(-\lambda s/\sqrt{\varepsilon})\|\zeta\|_{H^1(S\times\mathbb{R}^+)^3}\,\|\nabla f(s,\cdot)\|_{L^2(\Omega)^3}\\ &\le C'\sqrt{\varepsilon}\exp(-\lambda s/\sqrt{\varepsilon})\|\nabla f(s,\cdot)\|_{L^2(\Omega)^3}\qquad\text{(using (25))}\\ &\le \frac{C'^2}{\varepsilon^{1/4}}\exp(-2\lambda s/\sqrt{\varepsilon})+\varepsilon^{5/4}\|\nabla f(s,\cdot)\|_{(L^2(\Omega)^3)}\\ &\le \frac{C''}{\varepsilon^{1/4}}\exp(-2\lambda s/\sqrt{\varepsilon})+\varepsilon^{1/4}\|\nabla_\mu f(s,\cdot)\|_{(L^2(\Omega)^3)}\end{aligned}$$

we obtain the desired control by integration from 0 to t.

— Other terms involve integrals of the type

$$J^\varepsilon = C\,\varepsilon^{-1}\,e^{\lambda_1(\varepsilon)t}\int_\Omega \zeta_i(x,y,\frac{z}{\varepsilon})f_j(t,x,y,z)\;dx\;dy\;dz,\quad i,j\in\{1,2,3\},$$

where ζ is defined by Lemma 4. We treat it as follows:

$$\begin{aligned}I^\varepsilon &= C\,\varepsilon^{-1}\,e^{\lambda_1^\varepsilon t}\int_S\left(\int_0^1\left(\int_0^z\partial_{z'}f_j(t,x,y,z')\,dz'\right)\zeta_i(t,x,y,\frac{z}{\varepsilon})\,dz\right)\,dx\,dy\\ &\le C\,\varepsilon^{-1}\,e^{-\alpha t/\sqrt{\varepsilon}}\int_S\left(\int_0^1\left(\int_0^z|\partial_{z'}f_j(t,x,y,z')|^2\;dz'\right)^{1/2}\sqrt{z}\left|\zeta_i(t,x,y,\frac{z}{\varepsilon})\right|\;dz\right)\;dx\,dy\\ &\le\;C\,\varepsilon^{-1}e^{-\alpha t/\sqrt{\varepsilon}}\int_S\left(\int_0^1|\partial_{z'}f_j(t,x,y,z')|^2\;dz'\right)^{1/2}\left(\int_0^1\sqrt{z}\left|\zeta_i(t,x,y,\frac{z}{\varepsilon})\right|\;dz\right)\;dxdy\\ &\le C\,\varepsilon^{-1}\,e^{-\alpha t/\sqrt{\varepsilon}}\|\partial_z f_j(t,\cdot)\|_{L^2(\Omega)}\quad\left(\int_S\left(\int_0^1\sqrt{z}\left|\zeta_i(t,x,y,\frac{z}{\varepsilon})\right|\;dz\right)^2dx\,dy\right)^{1/2}\\ &\le C\,\varepsilon^{-1}\,e^{-\alpha t/\sqrt{\varepsilon}}\|\partial_z f_j(t,\cdot)\|_{L^2(\Omega)}\quad\left(\int_S\left(\int_0^1 z\left|\zeta_i(t,x,y,\frac{z}{\varepsilon})\right|^2\;dz\right)dx\,dy\right)^{1/2}\end{aligned}$$

by Cauchy-Schwarz' Inequality. The change of variable $\theta = \varepsilon^{-1} z$ gives

$$\begin{aligned} I^\varepsilon &\le C\ \sqrt{\varepsilon} e^{-\alpha t/\sqrt{\varepsilon}} \|\partial_z f_j(t,\cdot)\|_{L^2(\Omega)} \|\theta^{1/2}\zeta_i\|_{L^2(S\times\mathbb{R}^+)} \\ &\le C'\,\sqrt{\varepsilon}\, e^{-\alpha t/\sqrt{\varepsilon}} \|\partial_z f_j(t,\cdot)\|_{L^2(\Omega)} \quad \text{(using (25))} \\ &\le \frac{C'^2}{\varepsilon^{1/4}} \exp(-2\alpha s/\sqrt{\varepsilon}) + \varepsilon^{1/4} \|\nabla_\mu f_j\|^2_{L^2(\Omega)} \end{aligned}$$

By integrating from 0 to t, we thus obtain Inequality (22) for $i = 3$.

Lift of higher order terms

We want to correct the traces of the other terms of (20) (Let us remark that the vertical component of the order ε term has been already corrected). We define $U_c^{\varepsilon,4}$ by

$$\begin{aligned} V_c^{\varepsilon,4} &= \sum_{j\ge 1} \sqrt{\varepsilon}^j \Big(v_b^j(x,y,\frac{z}{\varepsilon}) \\ &\qquad\qquad + \chi(\frac{z}{\varepsilon}) v_{int}^j(x,y,z) + \chi(\frac{z}{\varepsilon}) v_\ell^j(\frac{d(x,y)}{\sqrt{\varepsilon}},x,y,z)\Big) \\ W_c^{\varepsilon,4} &= \sqrt{\varepsilon}\, w_\ell^1(\frac{d(x,y)}{\sqrt{\varepsilon}},x,y,0) \\ &\quad + \sum_{j\ge 3} \quad \sqrt{\varepsilon}^j \,\Big(w_b^j(x,y,\frac{z}{\varepsilon}) + w_{int}^j(x,y,0) + w_\ell^j(\frac{d(x,y)}{\sqrt{\varepsilon}},x,y,0)\Big) \end{aligned}$$

We easily check that $\|\mathrm{div}\, U_c^{\varepsilon,4}\|_{L^2(\Omega)} = O(\varepsilon^{3/4})$. It remains to get a divergence free function. Let

$$U^\varepsilon_{app,p} = u_p^{\varepsilon,ip} + \sum_{1\le i\le 4} u_c^{\varepsilon,i} \in (H^1_0(\Omega))^3.$$

We know, see [23] and [53], that there exists $U_c^{\varepsilon,5} \in (H^1_0(\Omega))^3$, with

$$\mathrm{div}\, U_c^{\varepsilon,5} = -\mathrm{div}\, U^\varepsilon_{app,p}, \quad \|U_c^{\varepsilon,5}\|_{(H^1(\Omega))^3} \le C \|\mathrm{div}\, U^\varepsilon_{app,p}\|_{L^2(\Omega)} \le C'\varepsilon^{3/4}$$

Estimates (22), $i = 4,5$, are then easily obtained.

Proof of proposition 3

By construction, the divergence free condition, Dirichlet boundary condition on $\partial\Omega$, and the convergence of the initial data are satisfied by $u^{\varepsilon,ip} =$

$u_p^{\varepsilon,ip} + u_c^{\varepsilon,ip}$ where $u_c^{\varepsilon,ip} = \sum_{k=1}^{\ell}\sum_{i=1}^{5}\mathcal{U}_c^{\varepsilon,i}$. The first equation of System (18) is satisfied iff

$$r^{\varepsilon,ip} = R^{\varepsilon,ip} + L^\varepsilon u_c^{\varepsilon,ip} + u^{\varepsilon,ip}\cdot\nabla u^{\varepsilon,ip}.$$

Remainders $R^{\varepsilon,ip}$ and $L^\varepsilon u_c^{\varepsilon,ip}$ satisfy an estimate of the type (19) (*cf.* Inequalities (22)).The quadratic term is treated in the same way.

2.4 Final energy estimates

Let $u_{app}^\varepsilon = u^{\varepsilon,wp} + u^{\varepsilon,ip}$, $\chi^\varepsilon = u^\varepsilon - u_{app}^\varepsilon$, $\Pi^\varepsilon = p^\varepsilon - p^{\varepsilon,wp} - p^{\varepsilon,ip}$. The flow $(\chi^\varepsilon, \Pi^\varepsilon)$ is solution of

$$\begin{cases} \partial_t\chi^\varepsilon + \chi^\varepsilon\cdot\nabla u_{app}^\varepsilon + u_{app}^\varepsilon\cdot\nabla\chi^\varepsilon + \chi^\varepsilon\cdot\nabla\chi^\varepsilon \\ \qquad\qquad + \dfrac{e_3\times\chi^\varepsilon}{\varepsilon} + \dfrac{\nabla\Pi^\varepsilon}{\varepsilon} - \Delta^{\mu,\varepsilon}\chi^\varepsilon = r_{app}^\varepsilon, \\ \operatorname{div}\chi^\varepsilon = 0, \\ \chi^\varepsilon_{|\partial\Omega} = 0, \end{cases} \tag{26}$$

where for small enough ε, $\|\chi^\varepsilon(0,\cdot)\|^2_{(L^2(\Omega))^3} \le \delta$, $r_{app}^\varepsilon = r^{\varepsilon,wp} + r^{\varepsilon,ip}$. A direct energy estimate gives: $\forall\ t$,

$$\begin{aligned} &\frac{1}{2}\|\chi^\varepsilon(t,\cdot)\|^2_{(L^2(\Omega))^3} + \int_t^0 \|\nabla_{\mathrm{x}}\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^6}\,ds + \varepsilon\int_0^t \|\partial_z\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^3}\,ds \\ &= \frac{1}{2}\|\chi^\varepsilon(0,\cdot)\|^2_{(L^2(\Omega))^3} - \int_0^t \left(\int_\Omega \left(\chi^\varepsilon(s,\cdot)\cdot\nabla u_{app}^\varepsilon(s,\cdot)\right)\cdot\chi^\varepsilon(s,\cdot)\right)\,ds \\ &\qquad + \int_0^t \langle r_{app}^\varepsilon, \chi^\varepsilon(s,\cdot)\rangle_{((H^{-1}(\Omega))^3,(H_0^1(\Omega))^3)}\,ds \end{aligned}$$

From Equations (14) and (19), we get

$$\begin{aligned} \int_0^t \langle r_{app}^\varepsilon, \chi^\varepsilon(s,\cdot)\rangle_{(H^{-1}(\Omega))^3,(H_0^1(\Omega))^3)}\,ds &\le A(\delta)\,\varepsilon^{1/4} \\ &+ B(\delta)\varepsilon^{1/4}\int_0^t \|\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^3}\,dt \\ &+ C(\delta)\varepsilon^{1/4}\int_0^t \left(\varepsilon\|\partial_z\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^3} + \|\nabla_{\mathrm{x}}\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^6}\right)\,dt \\ &+ D(\delta)\varepsilon^{-1/4}\int_0^t \exp(-as/\sqrt{\varepsilon})\|\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^3}\,ds \end{aligned}$$

where A, B, C, D and a depend on δ. It remains to treat the term

$$\int_0^t\int_\Omega \left(\chi^\varepsilon\cdot\nabla u_{app}^\varepsilon\right)\cdot\chi^\varepsilon = \sum_j \left(\chi^\varepsilon\cdot\nabla u_{app}^{\varepsilon,j}\right)\cdot\chi^\varepsilon, \quad u_{app}^\varepsilon = \sum_j u_{app}^{\varepsilon,j} \tag{27}$$

where $u^{\varepsilon,j}_{app}$ is one of the terms seen in the construction of $u^{\varepsilon,wp}$ and $u^{\varepsilon,ip}$. In decomposition (27), most of the terms are easily estimated. For the sake of simplicity, we just look at the bad terms.

i) *Well prepared part.* We refer to [30] for the treatment of $U^{\varepsilon,wp}$ (the worst term corresponds to the first corrector of the boundary layer, see below for the estimate of a term of same kind). The part $V^{\varepsilon,wp}$ is without any difficulty: we get, for small enough ε,

$$\begin{aligned}\left|\int_0^t\int_\Omega(\chi^\varepsilon\cdot\nabla(u^{\varepsilon,wp})\cdot\chi^\varepsilon\right| &\leq \frac{1}{4}\int_0^t\left(\|\nabla_\mu\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^9}\right)\,ds\\ &\quad + C\int_0^t\|\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^3}\,ds\end{aligned}$$

ii) *Ill prepared part.* The worst terms are again the first "boundary layer" terms. The terms with $\varepsilon^{-1}z$ give estimate of the form

$$I_\varepsilon=\int_0^t e^{\lambda(\varepsilon)t}\int_\Omega(\chi_3^\varepsilon\cdot\partial_z\zeta^\varepsilon)\cdot\chi^\varepsilon,\quad |\lambda(\varepsilon)|\leq-\beta/\sqrt{\varepsilon},\quad \beta=\beta(\delta)>0,$$

and $\zeta^\varepsilon(x,y,z)=\zeta(x,y,\varepsilon^{-1}z)$ satisfies the condition (25). We have

$$\begin{aligned}\int_\Omega(\chi^\varepsilon\cdot\partial_z\zeta^\varepsilon)\cdot\chi^\varepsilon &= -\int_\Omega(\partial_z\chi_3^\varepsilon\chi^\varepsilon+\chi_3^\varepsilon\partial_z\chi^\varepsilon)\cdot\zeta^\varepsilon\\ &= -\int_S\left(\int_0^1\left(\partial_z\chi_3^\varepsilon\frac{\chi^\varepsilon}{z}+\frac{\chi_3^\varepsilon}{z}\partial_z\chi^\varepsilon\right)z\zeta^\varepsilon\,dz\right)\\ &\leq \sup_{x,y,z}\left(z\,|\zeta^\varepsilon(x,y,z)|\right)\int_\Omega\left(|\partial_z\chi_3^\varepsilon|\left|\frac{\chi^\varepsilon}{z}\right|+\left|\frac{\chi_3^\varepsilon}{z}\right||\partial_z\chi^\varepsilon|\right)\\ &\leq \varepsilon\sup_{x,y,\theta}\left(\theta\,|\zeta(x,y,\theta)|\right)\|\nabla\chi_3^\varepsilon\|_{(L^2(\Omega))^3}\|\nabla_{\mathrm{x}}\chi^\varepsilon\|_{(L^2(\Omega))^9}\end{aligned}$$

where we have used Hardy's Inequality. Notice that

$$\begin{aligned}\int_\Omega|\nabla\chi_3^\varepsilon|^2 &= \int_\Omega|\partial_x\chi_3^\varepsilon|^2+\int_\Omega|\partial_y\chi_3^\varepsilon|^2+\int_\Omega|\partial_z\chi_3^\varepsilon|^2\\ &= \int_\Omega|\partial_x\chi_3^\varepsilon|^2+\int_\Omega|\partial_y\chi_3^\varepsilon|^2+\int_\Omega|\partial_x\chi_1^\varepsilon+\partial_y\chi_2^\varepsilon|^2\\ &\leq 3\|\nabla_{\mathrm{x}}\chi^\varepsilon\|^2_{(L^2(\Omega))^9}.\end{aligned}$$

We deduce from the previous estimates that for ε small enough, we have for instance

$$I^\varepsilon\leq\frac{1}{4}\int_0^t\left(\varepsilon\|\partial_z\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^3}+\mu\|\nabla_{\mathrm{x}}\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^6}\right)\,dt$$

The boundary layer terms in $\varepsilon^{-1/2}d(x,y)$ give integral of the form

$$J_\varepsilon = \int_0^t e^{\lambda(\varepsilon)s} \int_\Omega (\chi_i^\varepsilon \cdot \partial_i G^\varepsilon) \cdot \chi^\varepsilon$$

where $i \in \{x, y\}$ and $G^\varepsilon(x,y,z) = G(\varepsilon^{-1/2}d(x,y), x, y, z)$. We deduce

$$\begin{aligned}
&\int_\Omega (\chi_i^\varepsilon \cdot \partial_i G^\varepsilon) \cdot \chi^\varepsilon = -\int_\Omega (\partial_i \chi_i^\varepsilon \chi^\varepsilon + \chi_i^\varepsilon \partial_i \chi^\varepsilon) \cdot G^\varepsilon \\
&\le \sqrt{\varepsilon} \sup_{\zeta,x,y,z} (\zeta\, |G(\zeta,x,y,z)|) \int_0^1 \left(\int_S \left(|\partial_i \chi_i^\varepsilon| \left| \frac{\chi^\varepsilon}{d(x,y)} \right| + \left| \frac{\chi_i^\varepsilon}{d(x,y)} \right| |\partial_i \chi^\varepsilon| \right) dx\, dy \right) dz \\
&\le C\sqrt{\varepsilon}\, \|\nabla_{\mathrm{x}} \chi^\varepsilon\|^2_{(L^2(\Omega))^6}
\end{aligned}$$

where we have used again Hardy's Inequality, this times on S with z fixed. We bound by one the exponential term, hence

$$|J_\varepsilon| \le C\, \varepsilon^{1/2} \int_0^t \|\nabla_{\mathrm{x}} \chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^6}\, ds$$

Finally, all the previous estimates give for small enough ε

$$\begin{aligned}
&\frac{1}{2}\|\chi^\varepsilon(t,\cdot)\|^2_{(L^2(\Omega))^3} + \frac{1}{4}(\mu \int_0^t \|\nabla_\mu \chi^\varepsilon(t,\cdot)\|^2_{(L^2(\Omega))^9}\, dt \\
&\quad \le \frac{1}{2}\|\chi^\varepsilon(0,\cdot)\|^2_{(L^2(\Omega))^3} + A_\delta \varepsilon^{1/4} \qquad (28)\\
&\quad + B_\delta \int_0^t \exp(-a_\delta s/\sqrt{\varepsilon}) \|\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^3}\, ds + C\int_0^t \|\chi^\varepsilon(s,\cdot)\|^2_{(L^2(\Omega))^3}
\end{aligned}$$

where A_δ and B_δ are constants depending on δ, and C is a constant independent of δ, coming from the well-prepared estimates. By Gronwall's Lemma, we get in particular that

$$\begin{aligned}
&\frac{1}{2}\|\chi^\varepsilon(t,\cdot)\|^2_{(L^2(\Omega))^3} \\
&\quad \le \left(\frac{1}{2}\|\chi^\varepsilon(0,\cdot)\|^2_{(L^2(\Omega))^3} + A_\delta \varepsilon^{1/4} \right) \exp\left(\int_0^t \left(B_\delta \varepsilon^{-1/4} e^{-a_\delta s/\sqrt{\varepsilon}} + C \right) ds \right) \\
&\quad \le \left(\frac{1}{2}\|\chi^\varepsilon(0,\cdot)\|^2_{(L^2(\Omega))^3} + A_\delta \varepsilon^{1/4} \right) \exp\left(B_\delta\, a_\delta^{-1} \varepsilon^{1/4} + CT \right),
\end{aligned}$$

for all $t \le T$. Now there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,

$$\|\chi^\varepsilon(t,\cdot)\|^2_{(L^2(\Omega))^3} \le 2\, \exp(2CT)\, \delta.$$

Last lines of the proof are straightforward and left to the reader.

3 Rotating low Mach compressible flows in a cylinder

3.1 Existence of global weak solutions.

In this section, we follow the main steps introduced by P.–L. Lions in [44] to obtain global existence of weak solutions in the isotropic case. As we shall see, the arguments used to control the oscillations of the density do not seem to extend to the anisotropic case. The existence of weak solutions will be derived from a stability property in the next sections, assuming small enough ε.

Let ε be given, System (3) writes

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \Delta_\mu u - \widetilde{\lambda}\nabla \mathrm{div}\, u + \dfrac{\nabla \rho^\gamma}{\gamma \varepsilon^2} + \dfrac{e_3 \times (\rho u)}{\varepsilon} = 0. \end{cases} \tag{29}$$

Multiplying by u the momentum equation and integrating by parts one obtain the energy inequality

$$\int_\Omega \Big\{ \frac{1}{2}\rho |u|^2 + \pi \Big\}(t)\, dx + \int_0^t \int_\Omega \Big(\sum_{i\,j} \mu_i |\partial_i\, u_j|^2 + \widetilde{\lambda} |\mathrm{div}\, u|^2 \Big) dx ds \leq C_0. \tag{30}$$

In the isotropic case, the momentum equation reads

$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div}\, u + \frac{\nabla \rho^\gamma}{\gamma \mathrm{Ma}^2} + \frac{e_3 \times (\rho u)}{\mathrm{Ro}} = 0, \tag{31}$$

where $\lambda + 2\mu > 0$. For large enough exponents γ, the oscillations of the density can be controlled by using the transport equation on $\rho \log \rho$, namely the equation

$$\partial_t(\rho \log \rho) + \mathrm{div}\,(u \rho \log \rho) + \rho \mathrm{div}\, u = 0. \tag{32}$$

Considering sequences of approximate weak solutions $(\rho_n, u_n)_{n \in N}$ and denoting $\overline{f}$ a weak limit of f_n, $\overline{\rho \mathrm{div}\, u}$ may be expressed in terms of $\overline{\rho} \mathrm{div}\, \overline{u}$ applying $\Delta^{-1}\mathrm{div}$ to the momentum equation and analyzing the so–called effective viscous flux $\rho^\gamma / (\gamma \mathrm{Ma}^2) - (\lambda + 2\mu)\mathrm{div}\, u$. We obtain

$$\overline{\rho \mathrm{div}\, u} = \overline{\rho} \mathrm{div}\, \overline{u} + \frac{\overline{\rho^{\gamma+1}} - \overline{\rho}\, \overline{\rho^\gamma}}{\gamma \mathrm{Ma}^2 (\lambda + 2\mu)}.$$

The convexity of $x \mapsto x^{\gamma+1}$ implies that $\overline{\rho \mathrm{div}\, u} \geq \overline{\rho} \mathrm{div}\, \overline{u}$. Using now DiPerna-Lions [18] uniqueness results for the transport equation, we deduce that $\overline{\rho \log \rho} = \overline{\rho} \log \overline{\rho}$, so that strong compactness of the density ρ is obtained. We refer to [44] for details.

Let us turn to System (29) with anisotropic stress tensor. Due to the structure of the momentum equation, the natural operator to obtain the quantity $\rho \mathrm{div}\, u$ is $A_\mu \mathrm{div}$ where $A_\mu = (\Delta_\nu + \widetilde{\lambda}\Delta)^{-1}$. We get the following effective viscous flux

$$\Big(\frac{A_\mu \Delta \rho^\gamma}{\gamma \mathrm{Ma}^2} - \mathrm{div}\, u\Big) = -\partial_t A_\mu \mathrm{div}\,(\rho u) - A_\mu \mathrm{div}\,\mathrm{div}\,(\rho u \otimes u) - A_\mu \mathrm{div}\,\Big(\frac{e_3 \times \rho u}{\mathrm{Ro}}\Big).$$

Using compactness arguments similar to [44], this gives

$$\overline{\rho \mathrm{div}\, u} = \overline{\rho} \mathrm{div}\, \overline{u} + \frac{\overline{\rho\, (A_\mu \Delta)\, \rho^\gamma} - \overline{\rho}\, \overline{(A_\mu \Delta)\, \rho^\gamma}}{\gamma \mathrm{Ma}^2}.$$

Unfortunately, the second term of the right-hand side does not seem to be of positive sign. This is the main reason why P.–L. Lions' proof seems to fail in the above anisotropic case.

3.2 The 3D problem with well prepared data.

Formal limit.

Assume that solutions (ρ, u) satisfy the following Ansatz

$$(u, \rho) = \sum_{k \geq 0} \varepsilon^k (u_k, \rho_k).$$

Using $(29)_2$ at order $1/\varepsilon^2$ and $1/\varepsilon$, we get

$$\nabla \rho_0^\gamma = 0, \qquad v_0^\perp + \nabla_{\mathrm{x}}(\rho_0^{\gamma-1} \rho_1) = 0, \qquad \partial_z \rho_1 = 0, \tag{33}$$

where $f^\perp = (-f_2, f_1)$. Hence ρ_0 is independent of x and z, and ρ^1 is independent of z. Moreover $\mathrm{div}_{\,\mathrm{x}} v_0 = 0$. Equations on u_0 and ρ_1 are called geostrophic wind relations. The mass equation at order ε^0 reads

$$\partial_t \rho_0 + \partial_z(\rho_0 w_0) = 0.$$

Integrating with respect to z from the bottom to the top gives $\partial_t \rho_0 = 0$. Therefore, $\rho_0 = 1$, which leads $w_0 = 0$.

To get an evolution equation on ρ_1, we have to study the next order of the expansion. Taking the two-dimensional curl of $(29)_2$ at order ε^0 and setting $\xi_0 = \operatorname{curl} v_0$, we get

$$\partial_t \xi_0 + v_0 \cdot \nabla_{\mathrm{x}} \xi_0 - \mu \Delta_{\mathrm{x}} \xi_0 + \operatorname{div}_{\mathrm{x}} v_1 = 0. \tag{34}$$

Using (33) and the divergence free relation on v_0, the mass equation at order ε gives

$$\partial_t \rho_1 + \operatorname{div} u_1 = 0. \tag{35}$$

Thus, (34) reads

$$\partial_t \xi_0 + v_0 \cdot \nabla_{\mathrm{x}} \xi_0 - \mu \Delta_{\mathrm{x}} \xi_0 - \partial_t \rho_1 - \partial_z w_1 = 0.$$

The constraint involving ρ_1 and v_0 then yields

$$\partial_t \xi_0 + v_0 \cdot \nabla_{\mathrm{x}} \xi_0 - \mu \Delta_{\mathrm{x}} \xi_0 - \partial_t \, \Delta_{\mathrm{x}}^{-1} \xi_0 - \partial_z w_1 = 0. \tag{36}$$

Since ξ_0, v_0 are independent of z, we can integrate (36) to get

$$\partial_t \xi_0 + v_0 \cdot \nabla_{\mathrm{x}} \xi_0 - \mu \Delta_{\mathrm{x}} \xi_0 - \partial_t \, \Delta_{\mathrm{x}}^{-1} \xi_0 = w_1(t,x,1) - w_1(t,x,0). \tag{37}$$

It remains to evaluate w at $z = 0$ and $z = 1$. For this, we need as usual to analyze the Ekman boundary layers. Remark that the extra term $\partial_t \, \Delta_{\mathrm{x}}^{-1} \xi_0$ appears in the limit equation, unlike the now classical mathematical analysis of incompressible rotating fluid flows with rigid lid assumption given in [9], [30]. By this way, we obtain the usual quasi-geostrophic equation with free surface term, see [49] p. 234.

Ekman layer.

The procedure is classical and the reader is referred to [14], [30], [49] for instance. The boundary layer equations are obtained by looking for solutions on the form

$$u(t,x,z) = u_0(t,x,z) + u^b(t,x,\zeta) + u^t(t,x,\widehat{\zeta})$$

with

$$\zeta = z/\sqrt{2\varepsilon\mu_z}, \qquad \widehat{\zeta} = (1-z)/\sqrt{2\varepsilon\mu_z}.$$

Plugging the Ansatz in the equation and keeping the leading terms, we prove that as usual in fluid boundary layers, the pressure (the density) does not vary in the layer. Recalling that $\mu_z = r_0^2 \varepsilon$ and using the behavior of the correctors at infinity and on the boundary, the standard Ekman layers give

$$w_1(t,x,0) = \frac{r_0}{\sqrt{2}} \xi_0(t,x), \qquad w_1(t,x,1) = -\frac{r_0}{\sqrt{2}} \xi_0(t,x). \tag{38}$$

The quasi-geostrophic equation.

Let us observe that, identifying ρ_1 with a stream function $\overline{\Psi}$, Equation (37) with (38) may be written

$$\begin{cases} \partial_t(\Delta_{\mathrm{x}}\overline{\Psi} - \overline{\Psi}) + \nabla_{\mathrm{x}}^{\perp}\overline{\Psi} \cdot \nabla_{\mathrm{x}}\Delta_{\mathrm{x}}\overline{\Psi} - \mu\Delta_{\mathrm{x}}^2\overline{\Psi} + \sqrt{2}r_0\Delta_{\mathrm{x}}\overline{\Psi} = 0, \\ \overline{\Psi} = \nabla_{\mathrm{x}}\overline{\Psi} \cdot n = 0 \text{ on } \partial S, \qquad \overline{\Psi}_{|t=0} = \overline{\Psi}_0. \end{cases} \tag{39}$$

This gives the usual quasi-geostrophic equations with the extra term $\partial_t\Psi$.

Before getting into the main stability result in the well prepared case, let us rewrite (39) in a semi-velocity form:

$$\begin{cases} \partial_t\overline{v} + \overline{v} \cdot \nabla_{\mathrm{x}}\overline{v} - \mu\Delta_{\mathrm{x}}\overline{v} + \sqrt{2}r_0\overline{v} + \nabla_{\mathrm{x}}\overline{p} - \partial_t\nabla_{\mathrm{x}}^{\perp}\Delta_{\mathrm{x}}^{-1}\overline{\Psi} = 0, \\ \mathrm{div}_{\,\mathrm{x}}\overline{v} = 0, \qquad \overline{v} = \nabla_{\mathrm{x}}^{\perp}\overline{\Psi}, \\ \overline{v} = 0 \text{ on } \partial S, \qquad \overline{v}_{|t=0} = \overline{v}_0. \end{cases} \tag{40}$$

Let us observe that formulation (39) in terms of $\overline{\Psi}$ is more convenient to establish the following result similar to the standard incompressible 2D Navier–Stokes boundary value problem.

Theorem 5 *Let S be an open bounded set of class $\mathcal{C}^3$. Let $\overline{v}_0 \in (H_0^1(S))^2 \cap (H^2(S))^2$ such that $\mathrm{div}_{\,\mathrm{x}}\overline{v}_0 = 0$. There exists a unique global strong solution $\overline{v}$ of (40) in $L^\infty(0,T;(H_0^1(S))^2 \cap (H^2(S))^2) \cap L^2(0,T;(H^3(S))^2)$ such that $\partial_t\overline{v}$ belongs to $L^2(0,T;(H^1(S))^2)$.*

Stability result.

Then we are able to state and prove the following stability property.

Theorem 6 *Let us assume $\gamma = 2$. Let $(u_0^\varepsilon, \rho_0^\varepsilon) = \sum_{k\geq 0}\varepsilon^k(u_0^k,\rho_0^k)$ with $\rho_0^0 = \rho_0 = 1$, $\overline{v}_0$ and ρ_0^1 independent of z, $\overline{w}_0 = 0$, $\overline{v}_0 = \nabla_{\mathrm{x}}^{\perp}\rho_1$ where $u_0^0 = (\overline{v}_0, \overline{w}_0)$. Assuming $\overline{v}_0 \in (H_0^1(S))^2 \cap (H^2(S))^2$ and*

$$(u_0^\varepsilon, \rho_0^\varepsilon) \to ((\overline{v}_0, 0), 1) \text{ in } (L^2(\Omega))^4,$$

then, every global weak solution $(u^\varepsilon, \rho^\varepsilon)$ of (3) satisfies

$$u^\varepsilon \to (\overline{v}, 0) \text{ in } L^\infty(0,T;(L^2(\Omega))^3), \qquad \rho^\varepsilon \to 1 \text{ in } L^\infty(0,T;L^2(\Omega))$$

when $\varepsilon \to 0$, where $\overline{v} = \nabla_{\mathrm{x}}^{\perp}\overline{\Psi}$ is global strong solution of the quasi–geostrophic equation (40) with initial data $\overline{v}_0 = \nabla_{\mathrm{x}}^{\perp}\overline{\Psi}_0$.

Proof. Let us assume that there exists a sequence $(\rho^\varepsilon, u^\varepsilon)$ of global weak solutions to (3) satisfying the energy inequality (30). The proof of Theorem 6 will be divided into three parts. At first we define a suitable energy quantity from which we get Inequality (42) to control the difference between solutions and the limit. In a second part, we prove the convergence and for the reader's convenience, in the last part, we prove Inequality (42).

Introduction of a suitable energy quantity and the corresponding inequality. The strategy of the proof is similar to [44] since it is based upon a weak–strong uniqueness argument. However, as we have seen in 3.2, the limit solution is two dimensional, which requires to introduce boundary layers. Moreover the singular operator has mixed Coriolis/acoustic features and therefore does not have the same kernel space as the standard acoustic or Coriolis operators. In particular, the well prepared assumption on ρ does not write $\rho = 1 + \mathrm{O}(\varepsilon^2)$, see (33). Therefore we have to define a suitable energy quantity to estimate the difference between weak solutions and the solution of the limit system.

The limit solution corrected by boundary layers $\widetilde{u} = \overline{u} + B$, where $\overline{u} = (\overline{v}, 0)$, satisfies

$$\partial_t \widetilde{u} + \widetilde{u} \cdot \nabla \widetilde{u} - \Delta_\mu \widetilde{u} + \sqrt{2} r_0 \widetilde{u} + \nabla_{\mathrm{x}} \overline{p} = \partial_t \nabla_{\mathrm{x}}^\perp \Delta_{\mathrm{x}}^{-1} \overline{\Psi} + F(\overline{u}, B) + \partial_t B - \Delta_\mu B, \quad (41)$$

where F tends to 0 in $L^\infty(0,T;(L^2(\Omega))^3)$. Recall that the corrector B is divergence free and defined in order that $\widetilde{u}$ satisfies the homogeneous Dirichlet condition on $\partial\Omega$, and takes into account the order 1 in powers of ε in the asymptotics. In the present case, this corrector is exactly the same as in Subsection 2.2 to which the reader is referred for details. See also [49] for physical viewpoint on Ekman layers. Equation (41) is the same as in [30], except the term involving $-\partial_t \nabla_{\mathrm{x}}^\perp \Delta_{\mathrm{x}}^{-1} \overline{\Psi}$ which comes from the assumption that Rossby and Mach numbers are of the same order of magnitude. Observe that the energy contribution of this term writes

$$\frac{d}{dt} \int_\Omega \frac{|\overline{\Psi}|^2}{2}.$$

As we shall see, at the end of the proof, using the energy inequality for weak solutions (30), the energy equality for the limit solution corrected by boundary layers (41), the mass and momentum equations of weak solutions (29) tested against $(\overline{\Psi}, \overline{u})$, we are able to derive the following inequality

$$
\begin{aligned}
\int_\Omega \rho^\varepsilon \frac{|u^\varepsilon - \widetilde{u}|^2}{2} + \int_\Omega \Big(\pi^\varepsilon - \frac{|\Psi^\varepsilon|^2}{2}\Big) + \int_\Omega \frac{|\Psi^\varepsilon - \overline{\Psi}|^2}{2} \\
+ \int_0^t \int_\Omega \sum_{i,j} \mu_i |\partial_i (u_j^\varepsilon - \widetilde{u}_j)|^2 + \widetilde{\lambda} |\mathrm{div}\, u^\varepsilon|^2 \le \sum_{i=1}^9 I_i
\end{aligned}
\tag{42}
$$

where

$$
I_1 = \int_\Omega \rho_0^\varepsilon \frac{|u_0^\varepsilon - \widetilde{u}_0|^2}{2} + \int_\Omega \Big(\pi_0^\varepsilon - \frac{|\Psi_0^\varepsilon|^2}{2}\Big) + \int_\Omega \frac{|\Psi_0^\varepsilon - \overline{\Psi}_0|^2}{2} - \int_\Omega (\rho_0^\varepsilon - 1) \frac{|\widetilde{u}_0|^2}{2},
$$

$$
\begin{aligned}
I_2 = \int_\Omega (\rho^\varepsilon - 1) \frac{|\widetilde{u}|^2}{2} - \int_0^t \int_\Omega (\rho^\varepsilon - 1) u^\varepsilon \cdot \partial_t \overline{u} \\
- \int_0^t \int_\Omega (\rho^\varepsilon - 1)((\widetilde{u} \cdot \nabla \widetilde{u}) \cdot u^\varepsilon + (u^\varepsilon \cdot \nabla \widetilde{u}) \cdot \widetilde{u}) + \int_0^t \int_\Omega (\rho^\varepsilon - 1)(\widetilde{u} \cdot \nabla \widetilde{u}) \cdot \widetilde{u},
\end{aligned}
$$

$$
I_3 = - \int_0^t \int_\Omega (\widetilde{u} \cdot \nabla \widetilde{u}) \cdot u^\varepsilon + \int_0^t \int_\Omega \partial_t \nabla_{\mathrm{x}}^\perp \Delta_{\mathrm{x}}^{-1} \overline{\Psi} \cdot B,
$$

$$
I_4 = \int_0^t \int_\Omega \sqrt{2} r_0 u^\varepsilon \cdot \overline{u} - \int_0^t \int_\Omega \sqrt{2} r_0 |\widetilde{u}|^2,
$$

$$
I_5 = \int_0^t \int_\Omega F(\overline{u}, B) \cdot \widetilde{u},
$$

$$
I_6 = - \int_0^t \int_\Omega \rho^\varepsilon ((u^\varepsilon - \widetilde{u}) \cdot \nabla \widetilde{u}) \cdot (u^\varepsilon - \widetilde{u}),
$$

$$
I_7 = - \int_0^t \int_\Omega (\frac{|\overline{u}|^2}{2} + \overline{p}) \mathrm{div}_{\mathrm{x}} v^\varepsilon + \int_0^t \int_\Omega (u^\varepsilon - \nabla^\perp \Psi^\varepsilon) \cdot \partial_t \nabla_{\mathrm{x}}^\perp \Delta_{\mathrm{x}}^{-1} \overline{\Psi} - \int_0^t \int_\Omega \frac{|\widetilde{u}|^2}{2} \mathrm{div}\, u^\varepsilon,
$$

$$
I_8 = \int_0^t \int_\Omega \frac{e_3 \times (\rho^\varepsilon u^\varepsilon)}{\varepsilon} \cdot \overline{u} + \frac{\mathrm{div}\, (\rho^\varepsilon u^\varepsilon)}{\varepsilon} \overline{\Psi},
$$

and

$$
\begin{aligned}
I_9 = \int_0^t \int_\Omega \frac{e_3 \times (\rho^\varepsilon u^\varepsilon)}{\varepsilon} \cdot B + \int_0^t < \Delta_\mu B, u^\varepsilon - \widetilde{u} >_{(H^{-1}(\Omega))^3 \times (H_0^1(\Omega))^3} \\
- \int_0^t \int_\Omega \rho^\varepsilon u^\varepsilon \cdot \partial_t B + \int_0^t \int_\Omega \widetilde{u} \cdot \partial_t B.
\end{aligned}
$$

The convergence. This part is devoted to the proof of the convergence claimed in Theorem 6. More precisely, classical arguments based upon Gronwall's Lemma allow to pass to the limit.

The quantity I_1 tends to 0 when ε goes to 0 thanks to the well preparedness assumptions on the data. For I_2, the convergence to 0 follows from the bounds on u^ε and $\overline{u}$ and from the fact that $\rho^\varepsilon \to 0$ in $\mathcal{C}([0,T];L^p(\Omega))$ for all $p<\gamma$. Recall that $\overline{u}$ is independent of z, $\overline{w}=0$ and $B\partial_z B$ is regular enough. The quantities I_3, I_4 and I_5 tend to 0 using again the bounds on u^ε and $\overline{u}$. We just have to notice that $\overline{u}$ does not depend on z and that the L^2 norms of the horizontal derivatives of B and the L^2 norms of $\varepsilon\partial_z B$ tend to 0 when $\varepsilon\to 0$. The quantity I_6 is controlled by splitting $\widetilde{u}$ into two parts $\widetilde{u}=\overline{u}+B$. For the term involving $\overline{u}$, Gronwall's type arguments can be used since

$$\Big|\int_0^t\int_\Omega \rho^\varepsilon((u^\varepsilon-\widetilde{u})\cdot\nabla\overline{u})\cdot(u^\varepsilon-\widetilde{u})\Big| \le \int_0^t \|\nabla_{\mathrm{x}}\overline{u}\|_{(L^\infty(S))^2}\int_\Omega \rho^\varepsilon|u^\varepsilon-\widetilde{u}|^2.$$

For the term involving B, we have to use Hardy's inequality as for the homogeneous case in [30]. Indeed, denoting d the distance to the top and the bottom, the bad term reads

$$J=\Big|\int_0^t\int_\Omega \rho^\varepsilon((w^\varepsilon-\widetilde{w})\cdot\partial_z B)\cdot(u^\varepsilon-\widetilde{u})\Big|.$$

It is controlled as follows, writing $\rho^\varepsilon = 1+\varepsilon\widetilde{\Psi}+\varepsilon(\Psi^\varepsilon-\widetilde{\Psi})$

$$\begin{aligned}
J &\le \Big(1+\varepsilon\|\widetilde{\Psi}\|_{L^\infty((0,T)\times\Omega)}\Big)\left\|\frac{w^\varepsilon-\widetilde{w}}{d}\right\|_{L^2((0,T)\times\Omega)} \\
&\qquad\qquad \times\left\|\frac{u^\varepsilon-\widetilde{u}}{d}\right\|_{L^2(0,T;L^2(\Omega)^3)}\sup\Big(d^2\,|\partial_z B|\Big) \\
&\quad +\varepsilon\int_0^t\left\|\Psi^\varepsilon-\widetilde{\Psi}\right\|_{L^2(\Omega)}\left\|\left|\frac{w^\varepsilon-\widetilde{w}}{d}\right|^{\frac12}\right\|_{L^4(\Omega)} \\
&\qquad\qquad \times\left\||w^\varepsilon-\widetilde{w}|^{\frac12}\right\|_{L^{12}(\Omega)}\|u^\varepsilon-\widetilde{u}\|_{L^6(\Omega)^3}\sup\Big(d^{\frac12}\partial_z B\Big) \\
&\le C\varepsilon\|\nabla(w^\varepsilon-\widetilde{w})\|_{L^2((0,T)\times\Omega)}\|\nabla(u^\varepsilon-\widetilde{u})\|_{L^2(0,T;L^2(\Omega)^3)} \\
&\quad + C\varepsilon^{\frac12}\int_0^t\left\|\Psi^\varepsilon-\widetilde{\Psi}\right\|_{L^2(\Omega)}\|\nabla(w^\varepsilon-\widetilde{w})\|_{L^2(\Omega)}\|\nabla(u^\varepsilon-\widetilde{u})\|_{L^2(\Omega)^3)}ds \\
&\le \frac{\varepsilon}{4}\|\nabla(u^\varepsilon-\widetilde{u})\|^2_{L^2((0,T)\times\Omega)}+C\varepsilon\|\nabla(w^\varepsilon-\widetilde{w})\|^2_{L^2((0,T)\times\Omega)} \\
&\quad +\frac{\varepsilon}{8}\|\nabla(u^\varepsilon-\widetilde{u})\|^2_{L^2((0,T)\times\Omega)}+C\int_0^t\left\|\Psi^\varepsilon-\widetilde{\Psi}\right\|^2_{L^2(\Omega)}\|\nabla(w^\varepsilon-\widetilde{w})\|^2_{L^2(\Omega)}ds.
\end{aligned}$$

We conclude by using the viscous terms on the left hand side, observing that the energy *a priori* bounds on weak and approximate solutions yield uniform

$L^2(0,T;H^1_0(\Omega))$ bounds on $w^\varepsilon - \widetilde{w}$ (use indeed the uniform bounds on the divergence of the velocity together with the fact that $\partial_z w = \operatorname{div} u - \operatorname{div}_{\mathrm{x}} v$). The final argument is based upon Gronwall's lemma.

The group I_7 converges to 0 since $\int_{-1}^0 \operatorname{div}_{\mathrm{x}} v^\varepsilon$ tends weakly to 0 and $\operatorname{div} u^\varepsilon$ and $u^\varepsilon - \nabla^\perp \Psi^\varepsilon$ tend weakly to 0. The quantity I_8 vanishes integrating by parts. Indeed, since $\overline{v}^\perp = -\nabla_{\mathrm{x}}\Psi$, we get

$$\int_\Omega e_3 \times (\rho^\varepsilon u^\varepsilon)\cdot \overline{u} + \operatorname{div}(\rho^\varepsilon u^\varepsilon)\overline{\Psi} = -\int_\Omega (\rho^\varepsilon v^\varepsilon)\cdot \overline{v}^\perp - \int_\Omega (\rho^\varepsilon v^\varepsilon)\cdot \nabla_{\mathrm{x}}\overline{\Psi} = 0.$$

The last one I_9 tends to 0 using the definition of B. Indeed B has been built in order that

$$\begin{cases} \partial_t B - \Delta_\mu B + \dfrac{e_3 \times B}{\varepsilon} = r_\varepsilon, \quad \operatorname{div} B = 0 \text{ in } \Omega, \\ B = -\overline{u} \text{ on } \partial\Omega, \end{cases}$$

with B and its horizontal derivatives with L^2 norm tending to 0 when $\varepsilon \to 0$ and the rest r_ε converging to 0 as well. The reader is referred to Proposition 2 in Subsection 2.2.

Now we remark that we have to add a term on the left-hand side to make sure that the second term in the left-hand side of (42) is positive. More precisely,

$$\int_\Omega \left(\pi^\varepsilon - \frac{|\Psi^\varepsilon|^2}{2} - \frac{(\gamma-2)}{6}\varepsilon |\Psi^\varepsilon|^2\Psi^\varepsilon\right)$$
$$= \int_\Omega \varepsilon^2 \frac{|\Psi^\varepsilon|^4}{6}(\gamma-2)(\gamma-3)\int_0^1 (1-s)^3(1+s\varepsilon\Psi^\varepsilon)^{\gamma-4} ds.$$

However, this extra cubic term in Ψ^ε does not seem to be easily estimated in terms of the left-hand side, unless $\gamma = 2$, which is the case considered here.

Proof of Estimate (42). Developing the right-hand side of Inequality (42), we obtain

$$\begin{aligned}
&\int_\Omega \rho^\varepsilon \frac{|u^\varepsilon - \widetilde{u}|^2}{2} + \int_\Omega \Big(\pi^\varepsilon - \frac{|\Psi^\varepsilon|^2}{2}\Big) + \int_\Omega \frac{|\Psi^\varepsilon - \overline{\Psi}|^2}{2} \\
&\qquad + \int_0^t \int_\Omega \sum_{i,j} \mu_i |\partial_i (u_j^\varepsilon - \widetilde{u}_j)|^2 + \widetilde{\lambda} |\mathrm{div}\, u^\varepsilon|^2 \\
=& \frac{1}{2} \int_\Omega \rho^\varepsilon |u^\varepsilon|^2 + \int_\Omega \pi_\varepsilon + \sum_{i,j} \int_0^t \int_\Omega \mu_i |\partial_i u_j^\varepsilon|^2 + \widetilde{\lambda} \int_0^t \int_\Omega |\mathrm{div}\, u^\varepsilon|^2 \qquad (43) \\
&+ \frac{1}{2} \int_\Omega \rho^\varepsilon |\widetilde{u}|^2 + \sum_{i,j} \int_0^t \int_\Omega \mu_i |\partial_i \widetilde{u}_j|^2 - \int_\Omega \rho^\varepsilon u^\varepsilon \cdot \widetilde{u} - \int_\Omega \Psi^\varepsilon \overline{\Psi} \\
&+ \int_\Omega \frac{|\overline{\Psi}|^2}{2} - 2 \sum_{i,j} \int_0^t \int_\Omega \mu_i \partial_i u_j^\varepsilon \partial_i \widetilde{u}_j
\end{aligned}$$

Now let us give various inequalities that will be used to bound the right-hand side in order to get (42). The first one is the energy inequality for weak solutions

$$\frac{1}{2} \int_\Omega \rho^\varepsilon |u^\varepsilon|^2 + \int_\Omega \pi^\varepsilon + \sum_{i,j} \int_0^t \int_\Omega \mu_i |\partial_i u_j^\varepsilon|^2 - \frac{1}{2} \int_\Omega \rho_0^\varepsilon |u_0^\varepsilon|^2 - \int_\Omega \pi_0^\varepsilon \leq 0.$$

The second one is otained multiplying Equation (41) by $\widetilde{u}$

$$\begin{aligned}
\frac{1}{2} \int_\Omega |\widetilde{u}|^2 - \frac{1}{2} \int_\Omega |\widetilde{u}_0|^2 + \sum_{i,j} \int_0^t \int_\Omega \mu_i |\partial_i \widetilde{u}_j|^2 + \sqrt{2} r_0 \int_0^t \int_\Omega |\widetilde{u}|^2 \\
= \int_0^t \int_\Omega \partial_t \nabla_{\mathrm{x}}^\perp \Delta_{\mathrm{x}}^{-1} \overline{\Psi} \cdot (\overline{u} + B) + \int_0^t \int_\Omega F(\overline{u}, B) \cdot \widetilde{u} \\
- \int_0^t < \Delta_\mu B, \widetilde{u} >_{(H^{-1}(\Omega))^3 \times (H_0^1(\Omega))^3} + \int_0^t \int_\Omega \partial_t B \cdot \widetilde{u}.
\end{aligned}$$

Since $\overline{u} = \nabla_{\mathrm{x}}^\perp \Psi$, the following identity is also obtained

$$\int_0^t \int_\Omega \partial_t \nabla_{\mathrm{x}}^\perp \Delta_{\mathrm{x}}^{-1} \overline{\Psi} \cdot \overline{u} = -\frac{1}{2} \frac{d}{dt} \int_\Omega |\overline{\Psi}|^2$$

Next, multiplying Equation $(40)_1$ by u^ε, we remark that

$$-\int_\Omega \rho^\varepsilon u^\varepsilon \cdot \widetilde{u} = \int_0^t \int_\Omega (1-\rho^\varepsilon)\partial_t \overline{u} \cdot u^\varepsilon - \int_\Omega \rho_0^\varepsilon u_0^\varepsilon \cdot \widetilde{u}_0$$
$$- \int_0^t \int_\Omega \partial_t(\rho^\varepsilon u^\varepsilon)\cdot \widetilde{u} - \int_0^t \int_\Omega (\frac{|\overline{u}|^2}{2} + p)\mathrm{div}_{\,\mathrm{x}} v^\varepsilon + \int_0^t \int_\Omega \nabla_\mu \widetilde{u} : \nabla u^\varepsilon$$
$$+ \sqrt{2}\int_0^t \int_\Omega r_0 u^\varepsilon \cdot \overline{u} - \int_0^t \int_\Omega \partial_t B \cdot \rho^\varepsilon u^\varepsilon$$
$$- \int_0^t \int_\Omega \partial_t \nabla_{\mathrm{x}}^\perp \Delta_{\mathrm{x}}^{-1} \overline{\Psi} \cdot u^\varepsilon + \int_0^t < \Delta_\mu B, u^\varepsilon >_{(H^{-1}(\Omega))^3 \times (H_0^1(\Omega))^3} .$$

Multiplying Equation $(29)_2$ by $\widetilde{u}$, we get

$$-\int_0^t \int_\Omega \partial_t(\rho^\varepsilon u^\varepsilon)\widetilde{u} = \int_0^t \int_\Omega \nabla_\mu u^\varepsilon : \nabla \widetilde{u}$$
$$+ \int_0^t \int_\Omega \frac{e_3 \times \rho^\varepsilon u^\varepsilon}{\varepsilon} \cdot \widetilde{u} - \int_0^t \int_\Omega (\rho^\varepsilon u^\varepsilon \cdot \nabla \widetilde{u}) \cdot u^\varepsilon.$$

Moreover, multiplying Equation $(29)_1$ by $\overline{\Psi}$ and recalling that $\Psi^\varepsilon = (\rho^\varepsilon - 1)/\varepsilon$, we have

$$\int_\Omega \Psi^\varepsilon \overline{\Psi}(t) - \int_\Omega \Psi_0^\varepsilon \overline{\Psi}_0 - \int_0^t \int_\Omega \nabla_{\mathrm{x}}^\perp \Psi^\varepsilon \cdot \partial_t \nabla_{\mathrm{x}}^\perp \Delta_{\mathrm{x}}^{-1} \overline{\Psi} + \int_0^t \int_\Omega \frac{\mathrm{div}\,(\rho^\varepsilon u^\varepsilon)}{\varepsilon} \overline{\Psi} = 0.$$

Collecting all the previous relations together and using (43), we get Inequality (42). □

Remark. Recall that we assumed the existence of global weak solutions of System (29) for small enough ε. We have proved that solutions of System (3) are *a priori* close to the unique smooth solution of (39). Since the difference between the two solutions is small, it seems possible to prove that this difference is actually globally smooth which gives *a priori* regularity and thus compactness on $(\rho^\varepsilon, u^\varepsilon)$ for small enough ε. No further details will be given here. The interested reader is referred to [25], [2]. □

3.3 The 3D problem with ill prepared data.

To obtain a rigorous mathematical convergence proof, the analysis of boundary layers in the neighborhood of corners $\partial S \times \{0\} \cup \partial S \times \{1\}$ has to be performed, as in Subsection 2.3 for incompressible flows. Nevertheless, due to the increased complexity of the equations, the ideas introduced in this subsection might need some strong technical refinements. We do not go further in that direction. In the following appendix Section 4, we provide formal asymptotics of the eigenvalues and eigenvectors of the wave operator perturbed by the anisotropic viscosity $\varepsilon\Delta_\mu$. The aim is to obtain the expression $\lambda_{k,1}^{\pm}$ of the order $\sqrt{\varepsilon}$ of the eigenvalue expansion in terms of the inviscid eigenvectors. Depending on whether $\mathcal{R}e(\pm i\lambda_{k,1}^{\pm})$ is positive or not, the associated waves seem to be exponentially damped as $\exp(\mathcal{R}e(-\pm i\lambda_{k,1}^{\pm})t/\sqrt{\varepsilon})$.

The strategy of the analysis now roughly follows the lines of [17]. The difference is that we are looking for solutions of the form $\Phi^\varepsilon(t,x,\xi,\eta)$ where $\xi = d(x)/\sqrt{\varepsilon}$ and $\eta = d(x)/\varepsilon$, $x \to d(x)$ denoting the distance to the boundary $\partial\Omega$. As will be shown later on, the two preceding boundary layer scales have to be introduced to take account of the anisotropy of the stress tensor near the boundary.

4 Appendix

We perform in this section the spectral analysis that leads to the geometrical condition (11). As already mentioned, it is related to the possible damping of waves in the case of ill prepared data.

Spectral analysis of the viscous wave operator

The incompressible Navier-Stokes equations.

This section is devoted to the spectral problem associated with the viscous Coriolis operator A_ε, in terms of eigenvalues and eigenvectors of the inviscid Coriolis operator A_0, in the incompressible case.

We only give the definition of the different operators and refer to the compressible part for details. Indeed, minor changes such as neglecting the acoustic part $\pm i\lambda_{k,\varepsilon}^{\pm}\Psi_{k,\varepsilon}^{\pm}$ in (44) allows to deal with the incompressible case. The conclusion remains the same.

The operator A_0 is defined on $(\mathcal{D}'(\Omega))^d$ by

$$A_0(u) = P(e_3 \times u)$$

where P is the Leray projector on divergence free vector fields. As a skew symmetric operator, A_0 has purely imaginary eigenvalues $(\pm i\lambda_k)$ (where $\lambda_k \in$

$\mathbb{R}^+$). The eigenvalues and eigenvectors problem reads

$$\begin{cases} e_3 \times u^{\pm}_{k,0} + \nabla p^{\pm}_{k,0} = \pm i\lambda^{\pm}_{k,0} u^{\pm}_{k,0}, \\ \operatorname{div} u^{\pm}_{k,0} = 0. \end{cases}$$

The goal is then to study the spectral problem associated to the viscous operator $A_\varepsilon : u \mapsto P(e_3 \times u) - \varepsilon \Delta_\mu u$ which can be written

$$\begin{cases} \nabla p^{\pm}_{k,\varepsilon} + e_3 \times u^{\pm}_{k,\varepsilon} - \varepsilon \Delta_\mu u^{\pm}_{k,\varepsilon} = \pm i\lambda^{\pm}_{k,\varepsilon} u^{\pm}_{k,\varepsilon}, \\ \operatorname{div} u^{\pm}_{k,\varepsilon} = 0, \end{cases}$$

in terms of the eigenvalues and eigenvectors of the inviscid wave operator. We obtain the same conclusion as in the compressible case: that means there exists approximate eigenvalues $\pm i\lambda^{\pm}_{k,\varepsilon}$ and eigenvectors for which

$$\pm i\lambda^{\pm}_{k,\varepsilon} = \pm i\lambda^{\pm}_{k,0} \pm i\sqrt{\varepsilon}\lambda^{\pm}_{k,1} + \mathcal{O}(\varepsilon)$$

where

$$\operatorname{Re}(\pm i\lambda^{\pm}_{k,1}) > 0$$

if S satisfies conditions (7) involved in Schiffer's conjecture.

The compressible Navier-Stokes equations.

This section is devoted to the description of the spectral problem associated with the viscous wave operator A_ε with a rotating term in terms of eigenvalues and eigenvectors of the inviscid wave Coriolis operator A_0. In what follows, density fluctuations will be denoted $\Psi^\varepsilon := (\rho^\varepsilon - 1)/\varepsilon$, and $\Phi^\varepsilon = (\Psi^\varepsilon, m^\varepsilon)^t$. The operators A_0 and A_ε are defined on $\mathcal{D}'(\Omega) \times (\mathcal{D}'(\Omega))^d$ by

$$A_0 \begin{pmatrix} \Psi \\ m \end{pmatrix} = \begin{pmatrix} \operatorname{div} m \\ \nabla\Psi + e_3 \times m \end{pmatrix}$$

and

$$A_\varepsilon \begin{pmatrix} \Psi \\ m \end{pmatrix} = A_0 \begin{pmatrix} \Psi \\ m \end{pmatrix} - \varepsilon \begin{pmatrix} 0 \\ \Delta_\mu m + \tilde{\lambda}\nabla \operatorname{div} m \end{pmatrix}.$$

As a skew symmetric operator, A_0 has purely imaginary eigenvalues $(\pm i\lambda_k)_{k\in N}$ (where $\lambda_k \in \mathbb{R}^+$). As will be shown later on (cf "the inviscid spectral problem" part):

$$\text{for all } k, \quad \lambda_k \neq 1.$$

Moreover, the eigenvectors can be written as follows $(\Phi^{\pm}_{k,0})_{k\in N} = (\Psi_{k,0}, m^{\pm}_{k,0} = B^{\pm}\nabla\Psi_{k,0})_{k\in N} \in C^{4N}$, where $B^{\pm} \in M_3(C)$ is given by

$$B^{\pm} = \frac{1}{1-\lambda_k^2}\begin{pmatrix} \pm i\lambda_k & -1 & 0 \\ 1 & \pm i\lambda_k & 0 \\ 0 & 0 & \dfrac{1-\lambda_k^2}{\pm i\lambda_k} \end{pmatrix}.$$

It satisfies $A_0\Phi^{\pm}_{k,0} = \pm i\lambda^{\pm}_{k,0}\Phi^{\pm}_{k,0}$ in Ω (where $\lambda^{\pm}_{k,0} = \lambda_k$), and $m^{\pm}_{k,0}\cdot n = 0$ on the boundary $\partial\Omega$.

Remark. In the sequel, we will suppose that $0 < \lambda_k < 1$, the case $\lambda_k > 1$ being treated similarly.

The perturbed spectral problem reads

$$\begin{cases} \nabla\Psi^{\pm}_{k,\varepsilon} + e_3 \times m^{\pm}_{k,\varepsilon} - \varepsilon\Delta_\mu m^{\pm}_{k,\varepsilon} = \pm i\lambda^{\pm}_{k,\varepsilon} m^{\pm}_{k,\varepsilon}, \\ \operatorname{div} m^{\pm}_{k,\varepsilon} = \pm i\lambda^{\pm}_{k,\varepsilon}\Psi^{\pm}_{k,\varepsilon}. \end{cases} \tag{44}$$

As mentioned before, we are looking for solutions of the form $f^\varepsilon(t,x,\xi,\eta)$ where $\xi = d(x)/\sqrt{\varepsilon}$ and $\eta = d(x)/\varepsilon$, $x \to d(x)$ denoting the distance to the boundary $\partial\Omega$. By straightforward calculations, it can be proved for any function f^ε as above that

$$\nabla(f^\varepsilon) = \nabla f^\varepsilon + \frac{\nabla d}{\sqrt{\varepsilon}} \otimes \partial_\xi f^\varepsilon + \frac{\nabla d}{\varepsilon} \otimes \partial_\eta f^\varepsilon \tag{45}$$

and

$$\begin{aligned} \Delta_\mu(f^\varepsilon) =& \Delta_\mu f^\varepsilon + \frac{2\nabla_\mu d\cdot\nabla\partial_\xi f^\varepsilon}{\sqrt{\varepsilon}} + \frac{2\nabla_\mu d\cdot\nabla\partial_\eta f^\varepsilon}{\varepsilon} + \frac{2|\nabla_\mu d|^2\partial^2_{\eta\xi} f^\varepsilon}{\varepsilon^{3/2}} \\ &+ \frac{|\nabla_\mu d|^2}{\varepsilon}\partial^2_\xi f^\varepsilon + \frac{|\nabla_\mu d|^2}{\varepsilon^2}\partial^2_\eta f^\varepsilon + \frac{\Delta_\mu d}{\sqrt{\varepsilon}}\partial_\xi f^\varepsilon + \frac{\Delta_\mu d}{\varepsilon}\partial_\eta f^\varepsilon. \end{aligned}$$

Using that $\Delta_\mu = \mu_z \partial_z^2 + \mu \Delta_{\mathrm{x}}$ and $\mu_z = \varepsilon r_0^2$, we get

$$
\begin{aligned}
\Delta_\mu(f^\varepsilon) = & \frac{1}{\varepsilon^2} \mu |\nabla_{\mathrm{x}} d|^2 \partial_\eta^2 f^\varepsilon + \frac{2}{\varepsilon^{3/2}} \mu |\nabla_{\mathrm{x}} d|^2 \partial_{\xi\,\eta}^2 f^\varepsilon \\
& + \frac{1}{\varepsilon}\Big(r_0^2 |\partial_z d|^2 \partial_\eta^2 f^\varepsilon + 2\mu \nabla_{\mathrm{x}} d \cdot \nabla_{\mathrm{x}} \partial_\eta f^\varepsilon \\
& \qquad\qquad + \mu |\nabla_{\mathrm{x}} d|^2 \partial_\xi^2 f^\varepsilon + \mu \Delta_{\mathrm{x}} d\, \partial_\eta f^\varepsilon \Big) \\
& + \frac{1}{\sqrt{\varepsilon}} \Big(2 r_0^2 |\partial_z d|^2 \partial_{\xi\,\eta}^2 f^\varepsilon + 2\mu \nabla_{\mathrm{x}} d \cdot \nabla_{\mathrm{x}} \partial_\xi f^\varepsilon + \mu \Delta_{\mathrm{x}} d\, \partial_\xi f^\varepsilon \Big) \\
& + 2 r_0^2 \partial_z d\, \partial_{\eta\, z}^2 f^\varepsilon + r_0^2 \partial_z^2 d\, \partial_\eta f^\varepsilon + r_0^2 |\partial_z d|^2 \partial_\xi^2 f^\varepsilon + \mu \Delta_{\mathrm{x}} f^\varepsilon \\
& + \sqrt{\varepsilon} (2 r_0^2 \partial_z d\, \partial_{z\,\xi}^2 f^\varepsilon + r_0^2 \partial_z^2 d\, \partial_\xi f^\varepsilon), \\
& + \varepsilon r_0^2 \partial_z^2 f^\varepsilon .
\end{aligned} \tag{46}
$$

The idea is to build approximate modes and eigenvalues of A_ε in terms of $\Phi_{k,0}^\pm$ and $\lambda_k = \lambda_{k,0}^\pm$ as in [17]. More precisely, we make for $\Phi_{k,\varepsilon,N}^\pm$ and $\lambda_{k,\varepsilon,n}^\pm$ the following Ansatz

$$
\begin{cases}
\Phi_{k,\varepsilon,n}^\pm(x) = \sum_{i=0}^n \Big(\sqrt{\varepsilon}^i \Phi_{k,i}^{\pm,\mathrm{int}}(x) + \sqrt{\varepsilon}^i \Phi_{k,i}^{\pm,tb}\Big(x, \dfrac{d(z)}{\varepsilon}\Big) + \sqrt{\varepsilon}^i \Phi_{k,i}^{\pm,\ell}\Big(x, \dfrac{d(x)}{\sqrt{\varepsilon}}\Big)\Big) \\
\lambda_{k,\varepsilon,n}^\pm = \sum_{i=0}^n \sqrt{\varepsilon}^i \lambda_{k,i}^\pm ,
\end{cases}
$$

and expect to obtain in suitable functional spaces

$$
A_\varepsilon \Phi_{k,\varepsilon,n}^\pm = \pm i \lambda_{k,\varepsilon,n}^\pm \Phi_{k,\varepsilon,n}^\pm + o\,(\varepsilon^{n/2}).
$$

The profiles $\Phi_{k,i}^{\pm,tb}$ represent the boundary layer part in the neighborhood of the horizontal top and bottom boundaries $S \times \{1\}$ and $S \times \{0\}$, whereas $\Phi_{k,i}^{\pm,\ell}$ stands for the boundary layer flow near the lateral boundary $\partial S \times (0,1)$.

The top and bottom boundary layers.

The aim of this part is to obtain the expression of boundary layer profiles away from the corners. In that case, the distance $d(x)$ depends only on z ($d(x) = \min(z, 1-z)$), which allows to suppress the dependence upon ξ and all the terms containing $\nabla_{\mathrm{x}} d$ in (45) and (46).

System (44) then yields

$$\nabla\Psi_{k,\varepsilon}^{\pm,tb}+\begin{pmatrix}0\\0\\\partial_z d\end{pmatrix}\frac{\partial_\eta\Psi_{k,\varepsilon}^{\pm,tb}}{\varepsilon}+e_3\times m_{k,\varepsilon}^{\pm,tb}-\varepsilon\Big[\frac{1}{\varepsilon}|\partial_z d|^2 r_0^2\partial_\eta^2 m_{k,\varepsilon}^{\pm,tb}+2r_0^2\partial_z d\,\partial_{\eta z}^2 m_{k,\varepsilon}^{\pm,tb}$$
$$+r_0^2\partial_z^2 d\,\partial_\eta m_{k,\varepsilon}^{\pm,tb}+\mu\Delta_{\mathrm{x}} m_{k,\varepsilon}^{\pm,tb}+\varepsilon r_0^2\partial_z^2 m_{k,\varepsilon}^{\pm,tb}\Big]=\pm i\lambda_{k,\varepsilon}^{\pm}m_{k,\varepsilon}^{\pm,tb}.$$

and

$$\operatorname{div} m_{k,\varepsilon}^{\pm,tb}+\frac{\partial_z d}{\varepsilon}\partial_\eta m_{k,\varepsilon}^{\pm,tb,3}=\pm i\lambda_{k,\varepsilon}^{\pm}\Psi_{k,\varepsilon}^{\pm,tb}.$$

– Order $1/\varepsilon$ in the two previous equations gives

$$\partial_z d\,\partial_\eta\Psi_{k,0}^{\pm,tb}=0,\ \text{hence}\ \Psi_{k,0}^{\pm,tb}\equiv 0$$

in view of the exponential decay property, and

$$\partial_z d\,\partial_\eta m_{k,0}^{\pm,tb,3}=0,\ \text{hence}\ m_{k,0}^{\pm,tb,3}\equiv 0.$$

– Order $1/\sqrt{\varepsilon}$ gives

$$\partial_z d\,\partial_\eta\Psi_{k,1}^{\pm,tb}=0\ \text{hence}\ \Psi_{k,1}^{\pm,tb}\equiv 0$$

and

$$\partial_z d\,\partial_\eta m_{k,1}^{\pm,tb,3}=0\ \text{hence}\ m_{k,1}^{\pm,tb,3}\equiv 0.$$

– Order ε^0 yields

$$\nabla\Psi_{k,0}^{\pm,tb}+\partial_z d\,\partial_\eta\Psi_{k,2}^{\pm,tb}e_3+e_3\times m_{k,0}^{\pm,tb}-|\partial_z d|^2 r_0^2\partial_\eta^2 m_{k,0}^{\pm,tb}=\pm i\lambda_{k,0}^{\pm}m_{k,0}^{\pm,tb}$$

and

$$\operatorname{div} m_{k,0}^{\pm,tb}+\partial_z d\,\partial_\eta m_{k,2}^{\pm,tb,3}=\pm i\lambda_{k,0}^{\pm}\Psi_{k,0}^{\pm,tb}. \tag{47}$$

The horizontal components $m_{k,0}^{\pm,tb,*}$ of $m_{k,0}^{\pm,tb}$ therefore solve

$$(m_{k,0}^{\pm,tb,*})^{\perp}-|\partial_z d|^2 r_0^2\partial_\eta^2 m_{k,0}^{\pm,tb,*}=\pm i\lambda_{k,0}^{\pm}m_{k,0}^{\pm,tb,*}$$

where $f^{\perp}=(-f_2,f_1)$ for any $f\in R^2$. This gives

$$m_{k,0}^{\pm,tb,*}=\frac{1}{2}\begin{pmatrix}\exp_-+\exp_+ & i(\exp_--\exp_+)\\ -i(\exp_+-\exp_-) & \exp_++\exp_-\end{pmatrix}m_{k,0}^{\pm,tb,*}|_{\eta=0}$$

where

$$\exp_+ = \exp\Big((-1+i)\sqrt{\frac{1\pm\lambda_{k,0}}{2r_0^2}}\frac{1}{|\partial_z d|}\eta\Big),$$

and

$$\exp_- = \exp\Big((-1-i)\sqrt{\frac{1\pm\lambda_{k,0}}{2r_0^2}}\frac{1}{|\partial_z d|}\eta\Big).$$

The vertical component of the first equation gives

$$\partial_z d\,\partial_\eta \Psi_{k,2}^{\pm,tb} = 0 \text{ hence } \Psi_{k,2}^{\pm,tb} \equiv 0.$$

Since $\Psi_{k,0}^{\pm,tb} \equiv 0$, Equation (47) yields

$$\partial_z d\,\partial_\eta m_{k,2}^{\pm,tb,3} = -\mathrm{div}\, m_{k,0}^{\pm,tb}.$$

from which we get the expression of $m_{k,2}^{\pm,tb,3}$.
– Order $\sqrt{\varepsilon}$ gives

$$\nabla\Psi_{k,1}^{\pm,tb} + \partial_z d\,\partial_\eta \Psi_{k,3}^{\pm,tb} e_3 + e_3\times m_{k,1}^{\pm,tb} - |\partial_z d|^2 r_0^2 \partial_\eta^2 m_{k,1}^{\pm,tb} = \pm i(\lambda_{k,1}^{\pm} m_{k,0}^{\pm,tb} + \lambda_{k,0}^{\pm} m_{k,1}^{\pm,tb})$$

and

$$\mathrm{div}\, m_{k,1}^{\pm,tb} + \partial_z d\,\partial_\eta m_{k,3}^{\pm,tb,3} = \pm i(\lambda_{k,0}^{\pm}\Psi_{k,1}^{\pm} + \lambda_{k,1}^{\pm}\Psi_{k,0}^{\pm})$$

The horizontal component of the first equation reads

$$(m_{k,1}^{\pm,tb,*})^\perp - |\partial_z d|^2 r_0^2 \partial_\eta^2 m_{k,1}^{\pm,tb,*} = \pm i(\lambda_{k,1}^{\pm} m_{k,0}^{\pm,tb,*} + \lambda_{k,0}^{\pm} m_{k,1}^{\pm,tb,*})$$

since $\Psi_{k,1}^{\pm} \equiv 0$. This allows to obtain the expression of $m_{k,1}^{\pm,tb,*}$. The vertical component reads

$$\partial_z d \partial_\eta \Psi_{k,3}^{\pm,tb} - |\partial_z d|^2 r_0^2 \partial_\eta^2 m_{k,1}^{\pm,tb,3} = \pm i(\lambda_{k,1}^{\pm} m_{k,0}^{\pm,tb,3} + \lambda_{k,0}^{\pm} m_{k,1}^{\pm,tb,3}).$$

Therefore

$$\Psi_{k,3}^{\pm,tb} \equiv 0.$$

The second equation gives $m_{k,3}^{\pm,tb,3}$ in terms of $m_{k,1}^{\pm,tb,*}$.
– Order ε gives

$$\begin{aligned}\nabla\Psi_{k,2}^{\pm,tb} + e_3\partial_z d\,\partial_\eta \Psi_{k,4}^{\pm,tb} + e_3\times m_{k,2}^{\pm,tb} - |\partial_z d|^2 r_0^2 \partial_\eta^2 m_{k,2}^{\pm,tb} - 2r_0^2 \partial_z d\,\partial_{\eta\,z}^2 m_{k,0}^{\pm,tb} \\ - r_0^2 \partial_z^2 d\,\partial_\eta m_{k,0}^{\pm,tb} - \mu\Delta_{\mathrm{x}} m_{k,0}^{\pm,tb} = \pm i(\lambda_{k,1}^{\pm} m_{k,1}^{\pm,tb} + \lambda_{k,2}^{\pm} m_{k,0}^{\pm,tb} + \lambda_{k,0}^{\pm} m_{k,2}^{\pm,tb})\end{aligned}$$

and

$$\operatorname{div} m_{k,2}^{\pm,tb} + \partial_z d \partial_\eta m_{k,4}^{\pm,tb,3} = \pm i(\lambda_{k,1}^{\pm}\Psi_{k,1}^{\pm\,tb} + \lambda_{k,2}^{\pm}\Psi_{k,0}^{\pm\,tb} + \lambda_{k,0}^{\pm}\Psi_{k,2}^{\pm\,tb}).$$

The horizontal components gives $m_{k,2}^{\pm,tb,*}$ in terms of $m_{k,0}^{\pm,tb,*}$, $m_{k,1}^{\pm,tb,*}$, $\lambda_{k,0}^{\pm}$, $\lambda_{k,1}^{\pm}$ and $\lambda_{k,2}^{\pm}$. The vertical component gives $\Psi_{k,4}^{\pm,tb}$ in terms of $m_{k,2}^{\pm,tb,3}$. The second equation provides $m_{k,4}^{\pm\,tb,3}$ in terms of $m_{k,2}^{\pm\,tb}$.

We see that this allows us to build the boundary layer corrector at any order.

The lateral boundary layers.

Let us now look at the lateral boundary layers. Away from the corners, we may use the fact that d depends only on x and thus suppress in (45) and (46) the dependence upon η and the terms containing $\partial_z d$.

System (44) then yields

$$\nabla\Psi_{k,\varepsilon}^{\pm,\ell} + \begin{pmatrix} \partial_{\mathrm{x}} d \\ \partial_y d \\ 0 \end{pmatrix} \frac{\partial_\xi \Psi_{k,\varepsilon}^{\pm,\ell}}{\sqrt{\varepsilon}} + e_3 \times m_{k,\varepsilon}^{\pm,\ell} - \varepsilon\Big[\frac{\mu}{\varepsilon}|\nabla_{\mathrm{x}} d|^2 \partial_\xi^2 m_{k,\varepsilon}^{\pm,\ell} + \frac{1}{\sqrt{\varepsilon}}\Big(\mu\Delta_{\mathrm{x}}\, d\partial_\xi m_{k,\varepsilon}^{\pm,\ell}$$
$$+ 2\mu\nabla_{\mathrm{x}} d\cdot\nabla_{\mathrm{x}}\partial_\xi m_{k,\varepsilon}^{\pm,\ell}\Big) + \mu\Delta_{\mathrm{x}} m_{k,\varepsilon}^{\pm,\ell} + \varepsilon\mu\partial_z^2 m_{k,\varepsilon}^{\pm,\ell}\Big] = \pm i\lambda_{k,\varepsilon}^{\pm} m_{k,\varepsilon}^{\pm,\ell}$$

and

$$\operatorname{div} m_{k,\varepsilon}^{\pm,\ell} + \frac{1}{\sqrt{\varepsilon}}\nabla d\cdot\partial_\xi m_{k,\varepsilon}^{\pm,\ell} = \pm i\lambda_{k,\varepsilon}\Psi_{k,\varepsilon}^{\pm,\ell}.$$

– Order $1/\sqrt{\varepsilon}$ gives

$$\nabla_{\mathrm{x}} d\,\partial_\xi\Psi_{k,0}^{\pm,\ell} = 0, \text{ hence } \Psi_{k,0}^{\pm,\ell} \equiv 0 \tag{48}$$

and

$$\nabla_{\mathrm{x}} d\cdot\partial_\xi m_{k,0}^{\pm,*} = 0, \text{ hence } m_{k,0}^{\pm,*}\cdot\nabla_{\mathrm{x}} d \equiv 0 \tag{49}$$

– Order ε^0 gives

$$\begin{aligned}\nabla\Psi_{k,0}^{\pm,\ell} + \begin{pmatrix} \nabla_{\mathrm{x}} d \\ 0 \end{pmatrix}\partial_\xi\Psi_{k,1}^{\pm,\ell} + e_3\times m_{k,0}^{\pm,\ell} - \mu|\nabla_{\mathrm{x}} d|^2\partial_\xi^2 m_{k,0}^{\pm,\ell} \\ = \pm i\lambda_{k,0} m_{k,0}^{\pm,\ell}.\end{aligned} \tag{50}$$

and

$$\operatorname{div} m_{k,0}^{\pm,\ell} + \nabla_{\mathrm{x}} d\cdot\partial_\xi m_{k,1}^{\pm,\ell,*} = \pm i\lambda_{k,0}\Psi_{k,0}^{\pm,\ell}. \tag{51}$$

Multiplying the horizontal components of (50) by $\nabla_{\mathrm{x}}^{\perp}d$ yields

$$-\mu|\nabla_{\mathrm{x}}d|^2\partial_\xi^2(m_{k,0}^{\pm,\ell,*}\cdot\nabla_{\mathrm{x}}^{\perp}d)=\pm i\lambda_{k,0}^{\pm}(m_{k,0}^{\pm,\ell,*}\cdot\nabla_{\mathrm{x}}^{\perp}d).$$

This gives

$$m_{k,0}^{\pm,\ell,*}\cdot\nabla_{\mathrm{x}}^{\perp}d=\exp\Big(-\frac{(1\mp i)}{|\nabla_{\mathrm{x}}d|}\sqrt{\frac{\lambda_{k,0}}{2\mu}}\xi\Big)(m_{k,0}^{\pm,\ell,*}\cdot\nabla^{\perp}d)_{|\xi=0}. \tag{52}$$

Multiplying the horizontal components of (50) by $\nabla_{\mathrm{x}}d$ gives

$$|\nabla_{\mathrm{x}}d|^2\partial_\xi\Psi_{k,1}^{\pm,\ell}-m_{k,0}^{\pm,\ell,*}\cdot\nabla_{\mathrm{x}}^{\perp}d=0.$$

This gives

$$\Psi_{k,1}^{\pm,\ell}=-\frac{1}{(1\mp i)|\nabla_{\mathrm{x}}d|}\sqrt{\frac{2\mu}{\lambda_{k,0}}}\exp\Big(-\frac{(1\mp i)}{|\nabla_{\mathrm{x}}d|}\sqrt{\frac{\lambda_{k,0}}{2\mu}}\xi\Big)(m_{k,0}^{\pm,\ell,*}\cdot\nabla_{\mathrm{x}}^{\perp}d)_{|\xi=0}.$$

The third component of (50) gives

$$-\mu|\nabla_{\mathrm{x}}d|^2\partial_\xi^2 m_{k,0}^{\pm,\ell,3}=\pm i\lambda_{k,0}^{\pm}m_{k,0}^{\pm,\ell,3}$$

which implies

$$m_{k,0}^{\pm,\ell,3}=\exp\Big(-\frac{(1\mp i)}{|\nabla_{\mathrm{x}}d|}\sqrt{\frac{\lambda_{k,0}}{2\mu}}\xi\Big)(m_{k,0}^{\pm,\ell,3})_{|\xi=0}. \tag{53}$$

Using (48)–(53), Equation (51) gives

$$\begin{aligned}\partial_\xi m_{k,1}^{\pm,*}\cdot\nabla_{\mathrm{x}}d&=-\operatorname{div} m_{k,0}^{\pm,\ell}\\&=-\exp\Big(-\frac{(1\mp i)}{|\nabla_{\mathrm{x}}d|}\sqrt{\frac{\lambda_{k,0}}{2\mu}}\xi\Big)\Big(\nabla_{\mathrm{x}}\Big(\frac{(m_{k,0}^{\pm,\ell,*}\cdot\nabla^{\perp}d)_{|\xi=0}}{|\nabla_{\mathrm{x}}d|^2}\Big)\cdot\nabla^{\perp}d\\&\qquad+\partial_z(m_{k,0}^{\pm,\ell,3})|_{\xi=0}\Big).\end{aligned}$$

Thus

$$\begin{aligned}m_{k,1}^{\pm,*}\cdot\nabla_{\mathrm{x}}d=&\frac{|\nabla_{\mathrm{x}}d|}{(1\mp i)}\sqrt{\frac{2\mu}{\lambda_{k,0}}}\exp\Big(-\frac{(1\mp i)}{|\nabla_{\mathrm{x}}d|}\sqrt{\frac{\lambda_{k,0}}{2\mu}}\xi\Big)\\&\times\Big(\nabla_{\mathrm{x}}((m_{k,0}^{\pm,\ell,*}\cdot\nabla^{\perp}d)_{|\xi=0})\cdot\nabla^{\perp}d+\partial_z(m_{k,0}^{\pm,\ell,3})_{|\xi=0}\Big).\end{aligned}$$

This reads, using that $m_{k,0}^{\pm,b,*}\cdot\nabla^\perp d = -m_{k,0}^{\pm,int,*}\cdot\nabla^\perp d$ and the expression of $m_{k,0}^{\pm,int,*}$ in terms of $\Psi_{k,0}^{\pm,int,*}$ (namely $m_{k,0}^{\pm,int,*} = (\pm i(\lambda_{k,0})^{\pm}\nabla_{\rm x}\Psi_{k,0}^{\pm,int} + \nabla_{\rm x}^\perp\Psi_{k,0}^{\pm,int,*})/(1-(\lambda_{k,0}^\pm)^2)$ and $m_{k,0}^{\pm,int,3} = \partial_z\Psi_{k,0}^{\pm,int}/(\pm i\lambda_{k,0}^\pm)$),

$$m_{k,1}^{\pm,\ell,*}\cdot n = \frac{(-1\pm i)}{2}\sqrt{\frac{2\mu}{\lambda_{k,0}^3}}\Big[\Delta_{x,g}\Psi_{k,0}^\pm + \partial_z^2\Psi_{k,0}^\pm\Big], \tag{54}$$

where $\Delta_{x,g}$ denotes the Laplace Beltrami operator on ∂S.
– Order $\sqrt{\varepsilon}$ yields

$$\nabla\Psi_{k,1}^{\pm,\ell} + \nabla_{\rm x}d\,\partial_\xi\Psi_{k,2}^\ell + e_3\times m_{k,1}^\ell + \mu|\nabla_{\rm x}d|^2\partial_\xi^2 m_{k,1}^{\pm,\ell} + 2\mu\nabla_{\rm x}d\cdot\nabla_{\rm x}\partial_\xi m_{k,0}^\ell + \mu\Delta_{\rm x}d\,\partial_\xi m_{k,0}^\ell = \pm i(\lambda_{k,0}^\pm m_{k,1}^{\pm,\ell} + \lambda_{k,1}^\pm m_{k,0}^{\pm,\ell}).$$

and

$$\operatorname{div} m_{k,1}^{\pm,\ell} + \nabla_{\rm x}d\cdot\partial_\xi m_{k,2}^{\pm,\ell,\star} = \pm i(\lambda_{k,0}^\pm\Psi_{k,1}^{\pm,\ell} + \lambda_{k,1}^\pm\Psi_{k,0}^{\pm,\ell}).$$

The interior.

At the order 1, we get the standard inviscid spectral problem

$$\begin{cases}\nabla\Psi_{k,0}^{\pm,int} + e_3\times m_{k,0}^{\pm,int} = \pm i\lambda_{k,0}^\pm m_{k,0}^{\pm,int},\\ \operatorname{div} m_{k,0}^{\pm,int} = \pm i\lambda_{k,0}^\pm\Psi_{k,0}^{\pm,int}.\end{cases} \tag{55}$$

At the order $\sqrt{\varepsilon}$, we obtain the following system

$$\begin{cases}\nabla\Psi_{k,1}^{\pm,int} + e_3\times m_{k,1}^{\pm,int} = \pm i(\lambda_{k,0}^\pm m_{k,1}^{\pm,int} + \lambda_{k,1}^\pm m_{k,0}^{\pm,int}),\\ \operatorname{div} m_{k,1}^{\pm,int} = \pm i(\lambda_{k,1}^\pm\Psi_{k,0}^{\pm,int} + \lambda_{k,0}^\pm\Psi_{k,1}^{\pm,int}).\end{cases}$$

The inviscid spectral problem.

The horizontal components of $(55)_1$ gives

$$\begin{cases}\Big(1-(\lambda_{k,0}^\pm)^2\Big)m_{k,0}^{\pm,int,1} = (\pm i\lambda_{k,0}^\pm\partial_1\Psi_{k,0}^{\pm,int} - \partial_2\Psi_{k,0}^{\pm,int}),\\ \Big(1-(\lambda_{k,0}^\pm)^2\Big)m_{k,0}^{\pm,int,2} = (\partial_1\Psi_{k,0}^{\pm,int} \pm i\lambda_{k,0}^\pm\partial_2\Psi_{k,0}^{\pm,int})\end{cases}$$

The vertical component of $(55)_1$ and $(55)_2$ reads

$$\begin{cases}\partial_z\Psi_{k,0}^{\pm,int} = \pm i\lambda_{k,0}^\pm m_{k,0}^{\pm,int,3},\\ \partial_z m_{k,0}^{\pm,int,3} = -\partial_{\rm x}m_{k,0}^{\pm,int,1} - \partial_y m_{k,0}^{\pm,int,2} \pm i\lambda_{k,0}^\pm\Psi_{k,0}^{\pm,int}.\end{cases}$$

Using that $m_{k,0}^{\pm,int,3} = 0$ on $z = 0$ and 1, we can use decompose it on the basis $\{\sin n\pi z\}_{n\in N}$. This gives

$$\begin{cases} \pm i\lambda_{k,0}^{\pm}\widehat{m}_{k,0}^{\pm,int,3} = -n\pi\widehat{\Psi}_{k,0}^{\pm,int}, \\ n\pi\widehat{m}_{k,0}^{\pm,int,3} = -\partial_{\mathrm{x}}\widehat{m}_{k,0}^{\pm,int,1} - \partial_y\widehat{m}_{k,0}^{\pm,int,2} \pm i\lambda_{k,0}^{\pm}\widehat{\Psi}_{k,0}^{\pm,int}. \end{cases}$$

This gives, using the expression of $m_{k,0}^{\pm,int,*}$,

$$\begin{cases} -\Delta_{\mathrm{x}}\widehat{\Psi}_{k,0}^{\pm,int} = \Big(1-(\lambda_{k,0}^{\pm})^2\Big)\Big(\dfrac{n^2\pi^2}{(\lambda_{k,0}^{\pm})^2}-1\Big)\widehat{\Psi}_{k,0}^{\pm,int} \text{ in } S \\ \pm i\lambda_{k,0}^{\pm}\nabla\widehat{\Psi}_{k,0}^{\pm,int}\cdot n + \nabla\widehat{\Psi}_{k,0}^{\pm,int}\cdot\tau = 0 \text{ on } \partial S. \end{cases} \tag{56}$$

Let

$$\mu_{k,n} = \Big(1-(\lambda_{k,0}^{\pm})^2\Big)\Big(\frac{n^2\pi^2}{(\lambda_{k,0}^{\pm})^2}-1\Big).$$

Suppose that $\lambda_k := \lambda_{k,0}^{\pm} = 1$, *i.e.* $\mu_{k,n} = 0$. Taking the scalar product in (56) with $\overline{\widehat{\Psi}_{k,0}^{\pm,int}}$, we get

$$\int_S |\nabla\widehat{\Psi}_{k,0}^{\pm,int}|^2 = \mp i\int_S \Big(\nabla\widehat{\Psi}_{k,0}^{\pm,int}\Big)^{\perp}\cdot\Big(\nabla\overline{\widehat{\Psi}_{k,0}^{\pm,int}}\Big)$$

Using Cauchy–Schwartz' inequality, we get that

$$\left|\int_S \Big(\nabla\widehat{\Psi}_{k,0}^{\pm,int}\Big)^{\perp}\cdot\Big(\nabla\overline{\widehat{\Psi}_{k,0}^{\pm,int}}\Big)\right| \le \sqrt{\int_S \left|\Big(\nabla\widehat{\Psi}_{k,0}^{\pm,int}\Big)^{\perp}\right| \int_S \left|\nabla\overline{\widehat{\Psi}_{k,0}^{\pm,int}}\right|} = \int_S |\nabla\widehat{\Psi}_{k,0}^{\pm,int}|^2.$$

Combining last two equations, we see that we have equality in the Cauchy-Schwartz formula, so that there exists a complex number C, $\Big(\nabla\widehat{\Psi}_{k,0}^{\pm,int}\Big)^{\perp} = C\,\nabla\overline{\widehat{\Psi}_{k,0}^{\pm,int}}$. Finally, we get

$$\int_S |\nabla\widehat{\Psi}_{k,0}^{\pm,int}|^2 = C\int_S \Big(\nabla\widehat{\Psi}_{k,0}^{\pm,int}\Big)\cdot\Big(\nabla\widehat{\Psi}_{k,0}^{\pm,int}\Big)^{\perp} = 0$$

It easily leads to a contradiction, so that $\lambda_k \neq 1$.

The order $\sqrt{\varepsilon}$.

Let us take the scalar product with $(m_{\ell,0}^{\pm,int}, \Psi_{\ell,0}^{\pm,int})^t$, integrating by parts and using the system satisfied by $(m_{\ell,0}^{\pm,int}, \Psi_{\ell,0}^{\pm,int})^t$, we get

$$\pm i(\lambda_{\ell,0}^{\pm} - \lambda_{k,0}^{\pm})\Big(\int_\Omega m_{k,1}^{\pm,int}\cdot\overline{m_{l,0}^{\pm,int}} + \int_\Omega \Psi_{k,1}^{\pm,int}\overline{\Psi_{l,0}^{\pm,int}}\Big) + \int_{\partial\Omega}(m_{k,1}^{\pm,int}\cdot n)\overline{\Psi_{k,0}^{\pm,int}} = \pm i\lambda_{k,1}^{\pm}\delta_{k\ell}.$$

Thus, if $k = \ell$ and if we assume to have simple eigenvalues, we get

$$\pm i\lambda_{k,1}^{\pm} = \int_{\partial\Omega} (m_{k,1}^{\pm,int} \cdot n)\overline{\Psi_{k,0}^{\pm,int}}$$

We now use the fact that $m_{k,1}^{\pm,int} \cdot n + m_{k,1}^{\pm,tb} \cdot n = 0$ on the bottom and the top and $m_{k,1}^{\pm,int} \cdot n + m_{k,1}^{\pm,\ell} \cdot n = 0$ on the lateral side in order to get $\pm i\lambda_{k,1}^{\pm}$ in terms of $\Psi_{k,0}^{\pm,int}$. We remark that on the top and the bottom

$$m_{k,1}^{\pm,int} \cdot n = m_{k,1}^{\pm,int,3} = m_{k,1}^{\pm,tb,3} \equiv 0.$$

Thus

$$\pm i\lambda_{k,1}^{\pm} = \int_{\partial S\times(0,1)} (m_{k,1}^{\pm,int} \cdot n)\overline{\Psi_{k,0}^{\pm,int}}$$

Now using (54), we get

$$\pm i\lambda_{k,1}^{\pm} = -(-1 \pm i)\sqrt{\frac{\mu}{2(\lambda_{k,0}^{\pm})^3}}\Big(\int_{\partial S\times(0,1)} \Big(|\partial_z \Psi_{k,0}^{\pm,int}|^2 + |\nabla_{\mathrm{x}} \Psi_{k,0}^{\pm,int}|^2\Big)\Big). \quad (57)$$

Conclusion.

In view of the above expression,

$$\mathcal{R}e\,(\pm i\lambda_{k,1}^{\pm}) > 0$$

if and only if $\Psi_{k,0}^{\pm,int} \neq \text{const}$ on $\partial S \times (0,1)$. In this case, waves related to this mode are likely to be exponentially damped.

In the opposite case, we have

$$\mathcal{R}e\,(\pm i\lambda_{k,1}^{\pm}) = 0. \quad (58)$$

It occurs if and only if there exists n such that $\widehat{\Psi}_{k,0}^{\pm,int} \not\equiv 0$ satisfies

$$\begin{cases} -\Delta_{\mathrm{x}} \widehat{\Psi}_{k,0}^{\pm,int} = \mu_{k,n} \widehat{\Psi}_{k,0}^{\pm,int} \text{ in } S \\ \nabla \widehat{\Psi}_{k,0}^{\pm,int} \cdot n = 0, \qquad \widehat{\Psi}_{k,0}^{\pm,int} = \text{const on } \partial S. \end{cases} \quad (59)$$

This means (58) may be obtained if and only if S is a disk in view of the Schiffer's conjecture. In that case, waves may propagate and the convergence to the limit system is only weak.

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