Homogenization and boundary layers

David Gérard-Varet*and Nader Masmoudi†

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This paper deals with the homogenization of elliptic systems with Dirichlet boundary condition, when the coefficients of both the system and the boundary datum are ε-periodic. We show that, as $\varepsilon \to 0$, the solutions converge in L^2 with a power rate in ε, and identify the homogenized limit system. Due to a boundary layer phenomenon, this homogenized system depends in a non trivial way on the boundary. Our analysis answers a longstanding open problem, raised for instance in [7]. It extends substantially previous results obtained for polygonal domains with sides of rational slopes as well as our previous paper [14] where the case of irrational slopes was considered.

1 Introduction

This paper is about the homogenization of elliptic systems in divergence form

$$-\nabla \cdot \left(A\left(\cdot/\varepsilon\right)\nabla u\right)(x) = 0, \quad x \in \Omega, \tag{1.1}$$

set in a bounded domain Ω of $\mathbb{R}^d$, $d \geq 2$, with an oscillating Dirichlet data

$$u(x) = \varphi(x, x/\varepsilon), \quad x \in \partial\Omega. \tag{1.2}$$

As is customary, $\varepsilon > 0$ is a small parameter, and $A = A^{\alpha\beta}(y) \in M_N(\mathbb{R})$ is a family of functions of $y \in \mathbb{R}^d$, indexed by $1 \leq \alpha, \beta \leq d$, with values in the set of $N \times N$ matrices. Also, $u = u(x)$ and $\varphi = \varphi(x, y)$ take their values in $\mathbb{R}^N$. We recall, using Einstein convention for summation, that for each $1 \leq i \leq N$,

$$(\nabla \cdot A\left(\cdot/\varepsilon\right)\nabla u)_i(x) \;:=\; \partial_{x_\alpha}\left[A^{\alpha\beta}_{ij}\left(\cdot/\varepsilon\right)\,\partial_{x_\beta} u_j\right](x).$$

In the sequel, greek letters $\alpha, \beta, \ldots$ will range between 1 and d and latin letters $i, j, k, \ldots$ will range between 1 and N. We make three hypotheses:

i) Ellipticity: For some $\lambda > 0$, for all family of vectors $\xi = \xi^\alpha_i \in \mathbb{R}^{Nd}$

$$\lambda \sum_\alpha \xi^\alpha \cdot \xi^\alpha \;\leq\; \sum_{\alpha,\beta,i,j} A^{\alpha,\beta}_{ij}\,\xi^\beta_j\,\xi^\alpha_i \;\leq\; \lambda^{-1}\sum_\alpha \xi^\alpha \cdot \xi^\alpha.$$

ii) Periodicity: $\forall y \in \mathbb{R}^d$, $\forall h \in \mathbb{Z}^d$, $\forall x \in \partial\Omega$, $A(y+h) \;=\; A(y), \quad \varphi(x,y) \;=\; \varphi(x, y+h).$

*DMA/CNRS, Ecole Normale Supérieure, 45 rue d'Ulm,75005 Paris, FRANCE

†Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, USA. N. M is partially supported by NSF Grant DMS-0703145.

iii) Smoothness: The functions A and φ, as well as the domain Ω are smooth. It is actually enough to assume that ϕ and Ω are in some H^s for s big enough, but we will not try to compute the optimal regularity.

We are interested in the limit $\varepsilon \to 0$, *i.e.* the homogenization of system (1.1)-(1.2).

Systems of type (1.1) are involved in various domains of material physics, notably in linear elasticity and in thermics [21, 7, 2, 20]. In many cases they come with a right hand side f. Our analysis extends easily to that case. In the context of thermics, $d = 2$ or 3, $N = 1$, u is the temperature, and $\sigma = A(\cdot/\varepsilon)\nabla u$ is the heat flux given by Fourier law. The parameter ε models heterogeneity, that is short-length variations of the material conducting properties. The boundary term φ in (1.2) corresponds to a prescribed temperature at the surface of the body. In the context of linear elasticity, $d = 2$ or 3, $N = d$, u is the unkown displacement, f is the external load and A is a fourth order tensor that models Hooke's law.

Note that other boundary conditions can be encountered, such as the Neumann boundary condition

$$n(x) \cdot (A(\cdot/\varepsilon)\nabla u)\,(x) \;=\; \varphi(x, x/\varepsilon), \quad x \in \partial\Omega, \tag{1.3}$$

where $n(x)$ is the normal vector. Still in thermics, it corresponds to a given heat flux at the solid surface. One could also account for heat sources inside the body, by the addition of a source term in (1.1).

Elliptic systems with periodic coefficients are also a classical topic in the mathematical theory of homogenization. We refer to the reknown book [7] for a good overview (see also the more recent books [17, 10, 9, 22]). As regards divergence form systems, two problems have been widely studied and are by now well understood:

1. *the non-oscillating Dirichlet problem*, that is (1.1) and (1.2) with $\varphi = \varphi(x)$.
2. *the oscillating Neumann problem*, that is (1.1) and (1.3) with a standard compatibility condition on φ.

Note that in both problems, the usual energy estimate provides a uniform bound on the solution u^ε in $H^1(\Omega)$.

For the non-oscillating Dirichlet problem, one shows that u^ε weakly converges in $H^1(\Omega)$ to the solution u^0 of the homogenized system

$$\begin{cases} -\nabla \cdot \left(A^0 \nabla u^0\right)(x) = 0, & x \in \Omega, \\ \qquad\qquad u^0(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.4}$$

The so-called homogenized matrix A^0 comes from the averaging of the microstructure. It involves the periodic solution $\chi = \chi^\gamma(y) \in M_N(\mathbb{R}),\ 1 \le \gamma \le d$, of the *cell problem*:

$$-\partial_{y_\alpha}\left[A^{\alpha\beta}(y)\,\partial_{y_\beta}\chi^\gamma(y)\right] \;=\; \partial_{y_\alpha}A^{\alpha\gamma}(y), \quad \int_{[0,1]^d} \chi^\gamma(y)\,dy = 0. \tag{1.5}$$

The homogenized matrix is then given by:

$$A^{0,\alpha\beta} \;=\; \int_{[0,1]^d} A^{\alpha\beta} \;+\; \int_{[0,1]^d} A^{\alpha\gamma}\partial_{y_\gamma}\chi^\beta.$$

One may even go further in the analysis, and obtain a two-scale expansion of u^ε. Denoting

$$u^1(x,y) \;:=\; -\chi^\alpha(y)\partial_{x_\alpha}u^0(x), \tag{1.6}$$

it is proved in [7] that

$$u^\varepsilon(x) \;=\; u^0(x) \;+\; \varepsilon u^1(x, x/\varepsilon) \;+\; O(\sqrt{\varepsilon}), \;\text{ in } H^1(\Omega). \tag{1.7}$$

Actually, an open problem in this area is to compute the next term in the expansion in the presence of a boundary. This will follow from the analysis of this paper (see below and section5).

For the oscillating Neumann problem, two cases must be distinguished. On one hand, if $\partial\Omega$ does not contain flat pieces, or if it contains finitely many flat pieces whose normal vectors do not belong to $\mathbb{R}\mathbb{Z}^n$, then

$$\varphi(\cdot, \cdot/\varepsilon) \to \overline{\varphi} := \int_{[0,1]^d} \varphi \;\text{ weakly in } L^2(\partial\Omega)$$

and u^ε converges weakly to the solution u^0 of

$$\begin{cases} -\nabla \cdot \left(A^0 \nabla u^0\right)(x) = 0, & x \in \Omega, \\ n(x) \cdot \left(A^0 \nabla u^0\right)(x) = \overline{\varphi}(x), & x \in \partial\Omega. \end{cases} \tag{1.8}$$

On the other hand, if $\partial\Omega$ does contain a flat piece whose normal vector belongs to $\mathbb{R}\,\mathbb{Q}^d$, then the family $\varphi(\cdot, \cdot/\varepsilon)$ may have a continuum of accumulation points as $\varepsilon \to 0$. Hence, u^ε may have a continuum of accumulation points in H^1 weak, corresponding to different Neumann boundary data. We refer to [7] for all details.

On the basis of these results, it seems natural to address the homogenization of (1.1)-(1.2) with an oscillating Dirichlet data. At first glance, this case looks similar to the forementioned ones. *However, this homogenization problem turns out to be much different, and much more difficult.* Up to our knowledge, besides restrictive settings to be described later on, it has remained unsolved. There are two main sources of difficulties:

i) One has uniform L^p bounds on the solutions u^ε of (1.1)-(1.2), but no uniform H^1 bound *a priori.* This is due to the fact that

$$\|x \mapsto \varphi(x, x/\varepsilon)\|_{H^{1/2}(\partial\Omega)} = O(\varepsilon^{-1/2}), \;\text{ resp. } \|x \mapsto \varphi(x, x/\varepsilon)\|_{L^p(\partial\Omega)} = O(1),\; p > 1.$$

The usual energy inequality, resp. the estimates in article [5, page 8, Thm 3] yield

$$\|u^\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^{-1/2}), \;\text{ resp. } \|u^\varepsilon\|_{L^p(\Omega)} = O(1),\; p > 1.$$

This indicates that singularities of u^ε are *a priori* stronger than in the usual situations. It will be rigorously established in the core of the paper.

ii) Furthermore, one can not expect these stronger singularities to be periodic oscillations. Indeed, the oscillations of φ are at the boundary, along which they do not have any periodicity property. Hence, it is reasonable that u^ε should exhibit concentration near $\partial\Omega$, with no periodic character, as $\varepsilon \to 0$. This is a so-called *boundary layer phenomenon.* The key point is to describe this boundary layer, and its effect on the possible weak limits of u^ε. This causes strong mathematical difficulties. Quoting [7, page xiii]:

Of particular importance is the analysis of the behavior of solutions near boundaries and, possibly, any associated boundary layers. Relatively little seems to be known about this problem.

We stress that there is also a boundary layer in the non-oscillating Dirichlet problem, although it has in this case a lower amplitude. More precisely, it is responsible for the $O(\sqrt{\varepsilon})$ loss in the error estimate (1.7). If either the L^2 norm, or the H^1 norm in a relatively compact subset $\omega \Subset \Omega$ is considered, one may avoid this loss as strong gradients near the boundary are filtered out. Following Allaire and Amar (see [3, Theorem 2.3]):

$$u^\varepsilon \;=\; u^0(x) \;+\; O(\varepsilon) \;\text{ in }\; L^2(\Omega), \quad u^\varepsilon(x) \;=\; u^0(x) + \varepsilon u^1(x, x/\varepsilon) \;+\; O(\varepsilon) \;\text{ in }\; H^1(\omega). \tag{1.9}$$

Still following [3], another way to put the emphasis on the boundary layer is to introduce the solution $u_{bl}^{1,\varepsilon}(x)$ of

$$\begin{cases} -\nabla \cdot A\left(\dfrac{x}{\varepsilon}\right) \nabla u_{bl}^{1,\varepsilon} = 0, & x \in \Omega \subset \mathbb{R}^d, \\ \quad u_{bl}^{1,\varepsilon} = -u^1(x, x/\varepsilon), & x \in \partial\Omega, \end{cases} \tag{1.10}$$

Then, one can show that

$$u^\varepsilon(x) \;=\; u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon u_{bl}^{1,\varepsilon}(x) \;+\; O(\varepsilon), \;\text{ in }\; H^1(\Omega). \tag{1.11}$$

or

$$u^\varepsilon(x) \;=\; u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon u_{bl}^{1,\varepsilon}(x) \;+\; O(\varepsilon^2), \;\text{ in }\; L^2(\Omega). \tag{1.12}$$

Note that system (1.10) is a special case of (1.1)-(1.2). Thus, the homogenization of the oscillating Dirichlet problem may give a refined description of the non-oscillating one. This is another motivation for its study. We refer to section 5 for the study of this case.

Before stating our main result, let us present former works on this problem. Until recently, *they were all limited to convex polygons with rational normals.* This means that

$$\Omega \;:=\; \cap_{k=1}^K \left\{ x, \quad n^k \cdot x > c^k \right\}$$

is bounded by K hyperplanes, *whose unit normal vectors n^k belong to $\mathbb{R}\,\mathbb{Q}^d$.* Under this stringent assumption, the study of (1.1)-(1.2) can be carried out. In short, the keypoint is the addition of boundary layer correctors to the formal two-scale expansion:

$$u^\varepsilon(x) \;\sim\; u^0(x) \;+\; \varepsilon u^1(x, x/\varepsilon) \;+\; \sum_k v_{bl}^k\left(x, \frac{x}{\varepsilon}\right), \tag{1.13}$$

where $v_{bl}^k = v_{bl}^k(x,y) \in \mathbb{R}^n$ is defined for $x \in \Omega$, and y in the half-space

$$\Omega^{\varepsilon,k} \;=\; \left\{ y, \quad n^k \cdot y > c^k/\varepsilon \right\}.$$

These correctors satisfy

$$\begin{cases} -\nabla_y \cdot A(y) \nabla_y v_{bl}^k = 0, & y \in \Omega^{\varepsilon,k}, \\ \; v_{bl}^k = \varphi(x,y) - u_0(x), & y \in \partial\Omega^{\varepsilon,k}. \end{cases} \tag{1.14}$$

We refer to the papers by Moskow and Vogelius [19], and Allaire and Amar [3] for more details. These papers deal with the special case (1.10), but the results adapt to more general oscillating data. Note that x is only involved as a parameter in (1.14). Note also that the assumption $n^k \in \mathbb{R}\,\mathbb{Z}^d$ yields periodicity of the function $A(y)$ tangentially to the hyperplanes. This periodicity property is used in a crucial way in the forementioned references. First, it yields easily well-posedness of the boundary layer systems (1.14). Second, as was shown by Tartar in [18, Lemma 10.1], the solution $v^k_{bl}(x,y)$ converges exponentially fast to some $v^k_{bl,*}(x) = \varphi^k_*(x) - u^0(x)$, when y goes to infinity transversally to the k-th hyperplane. In order for the boundary layer correctors to vanish at infinity (and to be $o(1)$ in L^2), one must have $v^k_{bl,*} = 0$, which provides the boundary condition for u^0. Hence, u^0 should satisfy a system of the type

$$\begin{cases} -\nabla \cdot \left(A^0 \nabla u^0\right)(x) = 0, & x \in \Omega, \\ u^0(x) = \varphi_*(x), & x \in \partial\Omega. \end{cases} \tag{1.15}$$

where $\varphi_*(x) := \varphi^k_*(x)$ on the k-th side of Ω. Nevertheless, this picture is not completely correct. Indeed, there is still *a priori* a dependence of φ^k_* on ε, through the domain $\Omega^{\varepsilon,k}$. In fact, Moskow and Vogelius exhibit examples for which there is an infinity of accumulation points for the φ^k_*'s, as $\varepsilon \to 0$. Eventually, they show that the accumulation points of u^ε in L^2 are the solutions u^0 of systems like (1.15), in which the φ^k_*'s are replaced by their accumulation points. See [19] for rigorous statements and proofs. We stress that their analysis relies heavily on the special shape of Ω, especially the rationality assumption.

A step towards more generality has been made in our recent paper [14], in which generic convex polygonal domains are considered. Indeed, we assume in [14] that *the normals $n = n^k$ satisfy the diophantine condition:*

$$\text{For all } \xi \in \mathbb{Z}^d \setminus \{0\} \quad |P_{n^\perp}(\xi)| > \kappa\, |\xi|^{-l}, \quad \text{for some } \kappa,\ l > 0, \tag{1.16}$$

where $P_{n^\perp}$ is the projector orthogonally to n. Note that for dimension $d = 2$ this condition amounts to:

$$\text{For all } \xi \in \mathbb{Z}^d \setminus \{0\} \quad |n^\perp \cdot \xi| := |-n_2\xi_1 + n_1\xi_2| > \kappa\, |\xi|^{-l}, \quad \text{for some } \kappa,\ l > 0,$$

whereas for $d = 3$, it is equivalent to:

$$\text{For all } \xi \in \mathbb{Z}^d \setminus \{0\} \quad |n \times \xi| > \kappa\, |\xi|^{-l}, \quad \text{for some } \kappa,\ l > 0.$$

Condition (1.16) is generic in the sense that it holds for almost every $n \in S^{d-1}$, see section 2 for more details.

Under this diophantine assumption, one can perform the homogenization of problem (1.1)-(1.2). *Stricto sensu*, only the case (1.10), $d = 2, 3$ is treated in [14], but our analysis extends straightforwardly to the general setting. Despite a loss of periodicity in the tangential variable, we manage to solve the boundary layer equations, and prove convergence of v^k_{bl} away from the boundary. The main idea is to work with quasi-periodic functions instead of periodic ones. Interestingly, and contrary to the "rational case", the field φ^k_* does not depend on ε. As a result, we establish convergence of the whole sequence u^ε to the single solution u^0 of (1.15). We stress that, even in this polygonal setting, the boundary datum φ_* depends in a non trivial way on the boundary. In particular, it is not simply the average of φ with respect to y, contrary to what happens in the Neumann case.

Pondering on this previous study, we are able to treat in this paper the case of smooth domains. Our main result is the following

Theorem 1 (Homogenization in smooth domains)
Let Ω be a smooth bounded domain of $\mathbb{R}^d$, $d \geq 2$. We assume that it is uniformly convex (all the principal curvatures are bounded from below).
Let u^ε be the solution of system (1.1)-(1.2)*, under the ellipticity, periodicity and smoothness conditions i)-iii).*
There exists a boundary term φ_* (depending on φ, A and Ω), with $\varphi_* \in L^p(\partial\Omega)$ for all finite p, and a solution u^0 of (1.15)*, with $u^0 \in L^p(\Omega)$ for all finite p, such that:*

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \;\leq\; C_\alpha\, \varepsilon^\alpha, \quad \textit{for all } 0 < \alpha < \frac{d-1}{3d+5}. \tag{1.17}$$

Let us make a few remarks on this theorem:

1. We only treat with full details the case where Ω is the disc. The general case of uniformly convex Ω follows from a much similar analysis, and is briefly discussed in section 4.

2. As regards (more) general domains, one can still carry out most of the analysis if there is no flat piece in the boundary which has a normal vector which belongs to $\mathbb{R}\,\mathbb{Q}^d$. In such a case, one can still prove a result similar to theorem 1 with a worse rate of convergence. This will be done in a forthcoming paper. In the case there is a flat part of the boundary with a normal vector which belongs to $\mathbb{R}\,\mathbb{Q}^d$, we expect that the limit problem depends on the choice of a subsequence, as was observed in polygonal domains with rational slopes (see [19]).

3. The value $\frac{d-1}{3d+5}$ in the theorem comes from the optimization of several small parameters involved and hence is not sharp. Finding the sharp rate seems a very interesting open problem.

4. The dependence of φ_* in x only happens through the normal $n(x)$ and through the function $\varphi(x,.)$, where x is fixed. More precisely, $\varphi_*(x) = \mathcal{A}[\varphi(x,.), A(.), n(x)]$ where $\mathcal{A}$ is a functional that will be constructed in the next section.

The outline of the paper is as follows. We investigate in section 2 the case where Ω is a half-space: $\Omega = \{x,\ x\cdot n > c\}$, under condition (1.16). We remind some results obtained in [14], and give some refined ones. In particular, we construct the functional $\mathcal{A}$. In section 3, we prove the theorem in the case where $d = 2$, Ω is the unit disk and φ factors into $\varphi(x,y) = v_0(y)\,\varphi_0(x)$ for some smooth $v_0 \in M_N(\mathbb{R})$ and $\varphi_0 \in \mathbb{R}^N$. Then, we indicate in section 4 how to extend the proof to general smooth, uniformly convex domains Ω and general boundary data φ. Finally, we give an application of our result to the study of the higher-order approximation of (1.4).

2 The half-space problem

We consider here a half-space: $\Omega = \{x,\ x \cdot n > c\}$. We suppose that the unit inward normal n satisfies the small divisor assumption (1.16). This assumption is almost surely satisfied. More precisely, let $(d-1)l > 1$ and let $\mathcal{A}_\kappa$ be the set

$$\mathcal{A}_\kappa = \left\{ n \in \mathbb{S}^{d-1}\ ,\ \forall \xi \in \mathbb{Z}^d \setminus \{0\}\ , |P_{n^\perp}(\xi)| \geq \kappa\, |\xi|^{-l} \right\}. \tag{2.1}$$

We claim that there exists a constant C such that $m(\mathcal{A}_\kappa^c) \leq C\kappa^{d-1}$ where m denotes the Lebesgue measure on the sphere $\mathbb{S}^{d-1}$. Indeed,

$$\mathcal{A}_\kappa = \cap_{\xi \in \mathbb{Z}^d, |\xi| \neq 0} \left\{ n,\ \left| P_{n^\perp}\left(|\xi|^{-1}\xi \right) \right| \geq \kappa\, |\xi|^{-(l+1)} \right\}$$

from which

$$\mathcal{A}_\kappa^c = \cup_{\xi \in \mathbb{Z}^d, |\xi| \neq 0} \left\{ n,\ \left| P_{n^\perp}\left(|\xi|^{-1}\xi \right) \right| < \kappa\, |\xi|^{-(l+1)} \right\}.$$

Completing the unit vector $\xi_1 := |\xi|^{-1}\xi$ into an orthonormal basis $\xi_2, \ldots, \xi_d$, and writing $n = \sum_{i=1}^d n_i\, \xi_i$, one has

$$\left\{ n \in \mathbb{S}^{d-1},\ \left| P_{n^\perp}\left(|\xi|^{-1}\xi \right) \right| < \kappa\, |\xi|^{-(l+1)} \right\} = \left\{ n \in \mathbb{S}^{d-1},\ \left(\sum_{i=2}^d n_i^2 \right)^{1/2} < \kappa\, |\xi|^{-(l+1)} \right\}$$

with a Lebesgue measure which is clearly less than $C\kappa^{d-1}\, |\xi|^{(1-d)(l+1)}$. Hence, we deduce that

$$m(\mathcal{A}_\kappa^c) \leq C\kappa^{d-1} \sum_{\xi \in \mathbb{Z}^d, |\xi| \neq 0} |\xi|^{(1-d)(l+1)}. \tag{2.2}$$

This estimate will be used later on.

2.1 The boundary layer analysis

In the half-space case, we expect the solution u^ε of (1.1)-(1.2) to behave like

$$u^\varepsilon(x) \ \sim\ u^0(x) \ +\ \varepsilon u^1(x, x/\varepsilon) \ +\ v_{bl}(x, x/\varepsilon)$$

where u^1 was given in (1.6) and where $v_{bl} = v_{bl}^\varepsilon$ models the boundary layer. At a formal level, it satisfies

$$\left\{ \begin{array}{ll} -\nabla_y \cdot\ A(y)\nabla_y\, v_{bl}(x,y) = 0, & y.n > c/\varepsilon \\ v_{bl}(x,y) = \varphi(x,y) - u^0(x), & y.n = c/\varepsilon. \end{array} \right. \tag{2.3}$$

and should decay when y goes to infinity transversally to the boundary $y.n = c/\varepsilon$. Remark that x is not involved in the differential operators and that the ε dependence only comes from the domain, namely c/ε. This suggests to have a look at the problem

$$\left\{ \begin{array}{rl} -\nabla_y \cdot\ A(y)\nabla_y\, v(y) = 0, & y.n > a \\ v(y) = v_0(y), & y.n = a. \end{array} \right. \tag{2.4}$$

for a periodic and smooth $v_0 = v_0(y)$. We consider v_0 and v with values in $\mathbb{R}^N$, but of course all results can be extended to $M_{N,p}(\mathbb{R})$, treating the p columns of the matrices separately. Here $M_{N,p}(\mathbb{R})$ denotes the set of matrices with N lignes and p columns. In particular $M_{N,N}(\mathbb{R}) = M_N(\mathbb{R})$.

System (2.4) has been examined in our recent paper [14]. Losely, we have shown:

1. Well-posedness of (2.4), in an appropriate space of *quasiperiodic* functions. Our well-posedness result holds for general normal vector n, with or without the diophantine assumption. Moreover, it is valid for any $N \geq 1$. We stress that in the scalar case $N = 1$, simpler arguments based on the maximum principle would lead to well-posedness in L^∞.

2. Convergence of the solution v to some constant field v_* as y goes to infinity transversally to the boundary. This convergence result uses assumption (1.16).

We shall recall here a few elements of these two aspects of the boundary layer analysis. We shall then refine these elements, focusing on the dependence of v and v_* on a and n.

Well-posedness

Let M be an orthogonal matrix of $O(d)$ that maps the canonical vector $e_d = (0, ..., 0, 1)$ to the normal vector n. The matrix M is not unique: it is only defined modulo an orthogonal matrix of $O(d-1)$. By the change of variable $y = Mz$, system (2.4) becomes

$$\begin{cases} -\nabla_z \cdot B(Mz)\nabla_z \mathbf{v}(z) = 0, & z_d > a, \\ \mathbf{v}(z) = v_0(Mz), & z_d = a, \end{cases} \tag{2.5}$$

where $\mathbf{v}(z) = v(Mz)$. Denoting $A_{ij}^{\alpha\beta}$, resp. $B_{ij}^{\alpha\beta}$, $1 \leq i, j \leq N$, the coefficients of $A^{\alpha\beta}$, resp. $B^{\alpha\beta}$, we get that

$$\forall i, j, \quad B_{ij} = (M^t) A_{ij} M$$

which is a product of matrices in $M_d(\mathbb{R})$. Indeed, from $y = Mz$, we get that $\nabla_z = M^t \nabla_y$ and $\nabla_y = M\nabla_z$. Hence for any vector e, $\mathrm{div}_y(e) = \mathrm{div}_z(M^t e)$. We also denote $z = (z', z_d)$ the tangential and normal component of z.

Let now $N \in M_{d,d-1}(\mathbb{R})$ be defined by

$$Nz' = M(z', 0),$$

which means that N is obtained from M by removing the last column of the square matrix M. The structure of (2.5) suggests to look for a solution of the type:

$$\mathbf{v}(z) = V\big(Nz', z_d\big), \quad V\big(\theta, t\big) \text{ 1-periodic in } \theta \in \mathbb{R}^d. \tag{2.6}$$

This means that we look for a $\mathbf{v}$ which is quasiperiodic in z'. We point out that if n is the multiple of a rational vector, as in former papers [19, 3], one can choose M in such a way that all coefficients of (2.5) are periodic in z' (with an integer period, possibly greater than one). In such a case, one can look for a $\mathbf{v}$ periodic in z', which simplifies greatly the boundary layer analysis. In the case n is not a multiple of a rational vector, we are replacing v which depends on d variables by V which depends on $d+1$ variables and is periodic in d of those variables.

Accordingly to (2.6), we define

$$\mathcal{B}\big(\theta, t\big) = B\big(\theta + t\, n\big), \quad V_0\big(\theta, t\big) = v_0\big(\theta + t\, n\big)$$

This leads to the following system, for $\theta \in \mathbb{T}^d,\ t > a$:

$$\begin{cases} -\begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}(\theta,t) \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} V(\theta,t) = 0, \quad t > a \\ \qquad\qquad\qquad\qquad V(\theta,t) = V_0(\theta,t), \quad t = a. \end{cases} \tag{2.7}$$

The well-posedness of this "degenerate" elliptic system (2.7) is established in Proposition 2 of [14] which states that

Proposition 2 *There exists a unique smooth solution V of* (2.7) *such that*

$$\int_{\mathbb{T}^d} \int_a^{+\infty} \left(|N^t \nabla_\theta \partial_\theta^\gamma V|^2 + |\partial_t^l \partial_\theta^\gamma V|^2 \right) dt\, d\theta \ < \ C$$

for $l \in \mathbb{N}$, $l \geq 1$, and $\gamma \in \mathbb{N}^d$ and where we denote $\partial_\theta^\gamma = \partial_{\theta_1}^{\gamma_1} ... \partial_{\theta_d}^{\gamma_d}$. Here and in all the paper $\mathbb{N}$ denotes the set of integers including 0.

We recall that this proposition is deduced from careful energy estimates. Since the solution V given by the proposition is smooth, $\mathbf{v}(z) := V(Nz', z_d)$ defines a smooth solution of (2.5).

Behaviour at infinity

At this stage, one still needs to understand the asymptotic behaviour of $V(\theta,t)$, as $t \to +\infty$. In the "periodic case", this follows from a lemma of Tartar (see [18, Lemma 10.1]). In the wider quasiperiodic setting, *and together with the diophantine assumption* (1.16), we have

Proposition 3 (see [14]) *There exists a constant vector $v_* \in \mathbb{R}^N$ such that*

$$\lim_{t \to +\infty} V = v_*.$$

Moreover,

$$\left| \ \partial_\theta^\alpha \, \partial_t^k \, (V - v_*) \, (\theta,t) \right| \ \leq C \, (1+t)^{-m} \, ,$$

for all $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$, $k \in \mathbb{N}$, uniformly in θ.

In general and without any diophantine assumption on n, we have

$$\|V(\cdot,t)\|_{H^s(\mathbb{T}^d)} \leq C + Ct^{1/2} \tag{2.8}$$

which can be obtain by writing $V(\cdot,t) = V(\cdot,0) + \int_0^T \partial_t V(\cdot,s)\, ds$ and then using the L^2 bound on $\partial_t V$.

Note that V and v_* depend *a priori* on n and a in (2.7). But, as n satisfies the small divisor assumption, it does not belong to $\mathbb{R}\mathbb{Q}^d$, which implies

Proposition 4 (see [14]) *The limit at infinity v_* does not depend on a.*

As mentioned in the introduction, this is in sharp contrast with the rational case where it is known that the limit depends on a (see [19]).

2.2 Refined estimates

The results described above are not enough to be used within the context of smooth domains. Roughly, our idea to handle a smooth convex domain Ω is to see it as the intersection of the half spaces whose boundaries are the tangent hyperplanes to $\partial\Omega$. Using a good sequence of such half-spaces and the corresponding boundary layer correctors, one may hope to obtain in the limit a homogenized problem in Ω. However, this idea will require some uniform control of the correctors, with respect to the normal vectors n at $\partial\Omega$. This is the purpose of the present paragraph. We start with uniform L^∞ bounds on the correctors and their derivatives:

Proposition 5 *For all $n \in \cup_{\kappa>0} A_\kappa$, the solution v of (2.4) given by $v(Mz) = V(Nz', z_d)$, where V solves (2.7), satisfies:*

$$\sup_y |\partial^\alpha v(y)| \le M_\alpha, \quad \forall \alpha \in \mathbb{N}^d \tag{2.9}$$

The constant M_α depends linearly on the $W^{s,\infty}$ norm of v_0 for some $s = s(\alpha)$ large enough. It depends neither on n nor on a. In particular, $v_* = v_*(n)$ is bounded uniformly in n.

Proof. We set $\Omega_a := \{y,\ y \cdot n > a\}$. We also introduce, for any $r > 0$ and $y \in \Omega_a$,

$$D(y,r) := \{y',\ |y'-y| < r\} \cap \Omega_a, \quad \Gamma(y,r) := \{y',\ |y'-y| < r\} \cap \partial\Omega_a.$$

By Sobolev imbedding and classical local elliptic estimates (see [1, Chapter 4, Section 10.2]), one has for $\alpha \in \mathbb{N}^d$ and $y \in \Omega_a$:

$$\begin{aligned} \|\partial^\alpha v\|_{L^\infty(D(y,1/2))} &\le C_\alpha \|v\|_{H^{|\alpha|+d/2+1}(D(y,1/2))} \\ &\le C'_\alpha \left(\|v\|_{L^2(D(y,1))} + \|v_0\|_{H^{|\alpha|+d/2+1/2}(\Gamma(y,1))} \right) \\ &\le C''_\alpha \left(\|v\|_{L^\infty(\Omega_a)} + \|v_0\|_{W^{s,\infty}(\Omega_a)} \right) \end{aligned} \tag{2.10}$$

where the constant C''_α does not depend on n and a, and where for instance $s = |\alpha| + d/2 + 1$. Using a covering of Ω_a with disks of radius $1/2$, we end up with

$$\|\partial\alpha v\|_{L^\infty(\Omega_a)} \le C''_\alpha \left(\|v\|_{L^\infty(\Omega_a)} + \|v_0\|_{W^{s,\infty}(\Omega_a)} \right)$$

Thus, it is enough to establish (2.9) in the special case $\alpha = 0$ (L^∞ bound).

Let us remark that the L^∞ bound is trivial in the scalar case, thanks to the maximum principle. For the vector case, we will need an integral representation of v, using the Poisson kernel associated to our elliptic system. This representation is not straightforward in our setting, because v has no space decay. Thus, the Poisson kernel must be controlled over large space distances, that is over many periods of the elliptic matrix A. This kind of problem has been adressed by Avellaneda and Lin in their paper [5], by taking advantage of the underlying homogenization process. We shall adapt their arguments to our half-space case in what follows.

We start by considering the Green matrix $G(y, \tilde{y})$, that satisfies

$$\begin{cases} -\nabla_y \cdot A(y) \nabla_y G(y, \tilde{y}) = \delta(y - \tilde{y})\, I_N, & y \in \Omega_a, \\ \qquad\qquad\qquad\quad G(y, \tilde{y}) = 0, & y \in \partial\Omega_a, \end{cases} \tag{2.11}$$

where I_N denotes the identity matrix over $\mathbb{R}^N$. The existence of Green matrices for elliptic systems in a half-space is established in [16, Theorem 5.4] in the case $d \geq 3$, and in [12, Theorem 2.21] in the case $d = 2$. We also refer to [13, 11, 4] for bounded domains or the whole space. Following [16, 12], the Green matrix $G(y, \tilde{y})$ is defined as the unique matrix function satisfying:

a) G is continous over $(\Omega_a \times \Omega_a) \setminus \{y = \tilde{y}\}$.

b) For all $y \in \Omega_a$, $G(y, \cdot)$ is locally integrable in Ω_a.

c) For all $f \in C_c^\infty(\Omega)$, the function

$$u(y) = \int_\Omega G(y, \tilde{y}) f(\tilde{y})\, d\tilde{y}$$

is the variational solution of $-\nabla_y \cdot A(y)\nabla_y u = f$ in Ω_a, $\quad u|_{\partial\Omega_a} = 0$.

Moreover, for all $\tilde{y} \in \Omega_a$, $\nabla G(\cdot, \tilde{y})$ belongs to $L^p_{loc}(\Omega)$ for p small enough (depending on the dimension d),

$$(1 - \eta)\, \tilde{\eta}\, G(\cdot, \tilde{y}) \in H^1_0(\Omega) \tag{2.12}$$

for all $\eta, \tilde{\eta} \in C_c^\infty(\mathbb{R}^d)$ with $\eta = 1$ near $\tilde{y}$. Also, one has for all $\varphi \in C_c^\infty(\Omega)$, for all $\tilde{y} \in \Omega_a$:

$$\int_\Omega A^{\alpha\beta}_{ij} \partial_{y_\beta} G_{jk}(y, \tilde{y}) \partial_\alpha \varphi_i(y)\, dy \;=\; \varphi_k(\tilde{y}). \tag{2.13}$$

Note that equations (2.12) and (2.13) are the weak formulation of system (2.11). Finally, it is shown in [16, 12] that

$$G(\tilde{y}, y) \;=\; (G^t(y, \tilde{y}))^T, \quad \text{that is } G_{ij}(\tilde{y}, y) = G^t_{ji}(y, \tilde{y}), \tag{2.14}$$

where G^t is the Green Matrix corresponding to the transpose of the operator $-\nabla_y \cdot A(y)\nabla_y$, that is the operator $-\nabla_y \cdot A^T(y)\nabla_y$, where $(A^T)^{\alpha\beta}_{ij} \;:=\; A^{\beta\alpha}_{ji}$.

In our case, the coefficients $A^{\alpha\beta}_{ij}$ and the domain Ω_a are smooth, so that for any $y \in \Omega_a$, $G^t(y, \cdot)$ is smooth away from y, up to the boundary of Ω_a. Therefore, we can define the Poisson kernel

$$P(y, \tilde{y}) := -n \cdot \left(A^T(\tilde{y}) \nabla_{\tilde{y}} G^t(y, \tilde{y})\right) \;=\; -n_\alpha\, (A^T)^{\alpha\beta} \partial_{\tilde{y}_\beta} G^t(y, \tilde{y})$$

where $\tilde{y} \in \partial\Omega_a$. The next key lemma collects various estimates on G and P.

Lemma 6 (Bounds on the Green function and the Poisson kernel)

i) *For all $y \neq \tilde{y}$ in Ω_a, one has:*

$$|G(y, \tilde{y})| \;\leq\; \frac{C}{|y - \tilde{y}|^{d-2}} \quad \textit{for } d \geq 3, \tag{2.15}$$

$$|G(y, \tilde{y})| \;\leq\; C\,(|\ln(|y - \tilde{y}|)| + 1) \quad \textit{for } d = 2, \tag{2.16}$$

$$|G(y, \tilde{y})| \;\leq\; \frac{C\,\delta(y)\,\delta(\tilde{y})}{|y - \tilde{y}|^d}, \quad \delta(y) \;:=\; y \cdot n - a, \quad \delta(\tilde{y}) \;:=\; \tilde{y} \cdot n - a, \quad \textit{for all } d. \tag{2.17}$$

ii) *For all $y \in \Omega_a$, $\tilde{y} \in \partial\Omega_a$,*

$$|P(y,\tilde{y})| \le \frac{C\,\delta(y)}{|y-\tilde{y}|^d}. \tag{2.18}$$

iii) *For all $y \neq \tilde{y}$ in Ω_a,*

$$|\nabla_y G(y,\tilde{y})| \le \frac{C}{|y-\tilde{y}|^{d-1}}, \tag{2.19}$$

$$|\nabla_y G(y,\tilde{y})| \le C\left(\frac{\delta(\tilde{y})}{|y-\tilde{y}|^d} + \frac{\delta(y)\,\delta(\tilde{y})}{|y-\tilde{y}|^{d+1}}\right). \tag{2.20}$$

iv) *For all $y \in \Omega_a$, $\tilde{y} \in \partial\Omega_a$,*

$$|\nabla_y P(y,\tilde{y})| \le C\left(\frac{1}{|y-\tilde{y}|^d} + \frac{\delta(y)}{|y-\tilde{y}|^{d+1}}\right). \tag{2.21}$$

The constant C appearing in above inequalities depends neither on n nor on a.

We postpone the proof of the lemma to Appendix A. This proof follows very closely the work of Avellaneda and Lin [5].

Thanks to Lemma 6, the fact that v is bounded uniformly in n and a will follow easily from the integral representation

$$v(y) = -\int_{\partial\Omega_a} P(y,\tilde{y}) v_0(\tilde{y})\, d\tilde{y}. \tag{2.22}$$

Indeed, by (2.18), it will follow that for all $y \in \Omega_a$,

$$|v(y)| \le C\|v_0\|_{L^\infty} \int_{\mathbb{R}^{d-1}} \frac{|y\cdot n - a|}{(|\tilde{y}'| + |y\cdot n - a|)^d} d\tilde{y}' \le C\|v_0\|_{L^\infty}.$$

Hence, it remains to prove (2.22). Let

$$w(y) := -\int_{\partial\Omega_a} P(y,\tilde{y})\, v_0(\tilde{y})\, d\tilde{y}$$

the r.h.s of (2.22). Thanks to the bound (2.18), this vector function is well-defined and uniformly bounded over Ω_a. We will show that it satisfies (2.4) (step 1), and then prove that $v = w$ by a duality argument (step 2).

Step 1. Let $\psi^k \in C_c^\infty(\mathbb{R}^d)$ satisfying $\psi^k = 1$ for $|y| \le k$. We claim that

$$w^k(y) := -\int_{\partial\Omega_a} P(y,\tilde{y})\, \psi^k(\tilde{y}) v_0(\tilde{y})\, d\tilde{y},$$

satisfies

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y\, w^k = 0, & y \in \Omega_a \\ w^k(y) = -\psi^k(y) v_0(y), & y \in \partial\Omega_a. \end{cases} \tag{2.23}$$

Indeed, by property c) of the Green matrix, the function

$$\tilde{w}^k(y) := \int_{\Omega_a} G(y,\tilde{y})\, f^k(\tilde{y})\, d\tilde{y}, \quad f_k(y) := -\nabla_y \cdot A(y)\nabla_y(\psi^k(y)\, v_0(y)) \tag{2.24}$$

is the variational solution of

$$-\nabla_y \cdot A(y)\nabla_y \tilde{w}^k = f^k \quad \text{in } \Omega_a, \quad \tilde{w}^k|_{\partial\Omega_a} = 0$$

(note that property c) is stated above for $f \in C_c^\infty(\Omega_a)$, but extends to $f \in C_c^\infty(\overline{\Omega_a})$ by an easy approximation argument). For a given y in Ω_a, we can then introduce $\phi_y \in C_c^\infty(\Omega_a)$, with $\phi_y = 1$ in a neighborhood of y. We split $\tilde{w}^k(y)$ according to

$$\begin{aligned}\tilde{w}^k(y) = &-\int_{\Omega_a} G(y,\tilde{y})\,\nabla_{\tilde{y}} \cdot A(\tilde{y})\nabla_{\tilde{y}} \Big(\phi_y(\tilde{y})\,\psi^k(\tilde{y})\,v_0(\tilde{y})\Big)\,d\tilde{y} \\ &-\int_{\Omega_a} G(y,\tilde{y})\,\nabla_{\tilde{y}} \cdot A(\tilde{y})\nabla_{\tilde{y}} \Big((1-\phi_y)(\tilde{y})\,\psi^k(\tilde{y})\,v_0(\tilde{y})\Big)\,d\tilde{y} \;:=\; I_1(y) + I_2(y).\end{aligned} \tag{2.25}$$

We can then integrate by parts each term at the r.h.s. On one hand, combining (2.13) and (2.14), we get

$$\begin{aligned} I_1(y) \;&=\; \int_{\Omega_a} A^T(\tilde{y})\nabla_{\tilde{y}} G^T(y,\tilde{y})\,\nabla_{\tilde{y}} \Big(\phi_y(\tilde{y})\,\psi^k(\tilde{y})\,v_0(\tilde{y})\Big)\,d\tilde{y} \\ &=\; \int_{\Omega_a} A^T(\tilde{y})\nabla_{\tilde{y}} G^t(\tilde{y},y)\,\nabla_{\tilde{y}} \Big(\phi_y(\tilde{y})\psi^k(\tilde{y})v_0(\tilde{y})\Big)\,d\tilde{y} \;=\; \phi_y(y)\,\psi^k(\tilde{y})\,v_0(\tilde{y}) \;=\; \psi^k(\tilde{y})\,v_0(\tilde{y}). \end{aligned} \tag{2.26}$$

On the other hand, as $1-\phi$ vanishes for $\tilde{y}$ near y, the integrand in $I_2(y)$ is a smooth function of $\tilde{y}$, and two successive integrations by parts yield:

$$\begin{aligned} I_2(y) \;&=\; \int_{\Omega_a} A^T(\tilde{y})\nabla_{\tilde{y}} G^T(y,\tilde{y})\,\nabla_{\tilde{y}} \Big((1-\phi_y)(\tilde{y})\,\psi^k(\tilde{y})\,v_0(\tilde{y})\Big)\,d\tilde{y} \\ &=\; \int_{\Omega_a} \big(-\nabla_{\tilde{y}} \cdot A^T(\tilde{y})\nabla_{\tilde{y}} G^t(\tilde{y},y)\big)\,\Big((1-\phi_y)(\tilde{y})\,\psi^k(\tilde{y})\,v_0(\tilde{y})\Big)\,d\tilde{y} \\ &\quad + \int_{\partial\Omega_a} n \cdot \big(A^T(\tilde{y})\nabla_{\tilde{y}} G^t(\tilde{y},y)\big)\,\Big((1-\phi_y)(\tilde{y})\psi^k(\tilde{y})v_0(\tilde{y})\Big)\,d\tilde{y} \;=\; 0 + w^k(y). \end{aligned} \tag{2.27}$$

Thus, $w^k = \tilde{w}^k - \psi^k v_0$, which proves that w_k solves the Dirichlet problem (2.23).

Eventually, we let k go to infinity. On one hand, using (2.18) in the integral formula (2.24) for w^k, we get that w^k converges locally uniformly to w over the closed half-plane $\overline{\Omega_a}$. On the other hand, passing to the limit in system (2.23), one obtains that w solves (2.4). Note that, as w is bounded, one can use the elliptic bounds (2.10) with w instead of v, so that w is smooth with bounded derivatives at any order.

Step 2. We now define $u := v - w$. By Proposition 3, v and all its derivatives are bounded (*a priori* not uniformly with respect to n). As we have just seen, w is also a smooth function with all derivatives bounded, and consequently so is u. It satisfies the homogeneous system

$$-\nabla_y \cdot A(y)\nabla_y\, u = 0, \quad y \in \Omega_a, \quad u = 0, \quad y \in \partial\Omega_a.$$

We can prove that $u = 0$ by a duality argument. More precisely, let f be smooth and compactly supported in Ω_a. Since f is arbitrary, it is enough to show that $\int u \cdot f = 0$. To this end, let us introduce U given by

$$U(y) \;=\; \int_{\Omega_a} G^T(\tilde{y},y)\,f(\tilde{y})dy.$$

By (2.14), it satisfies

$$-\nabla_y \cdot A^T(y)\nabla_y U = f, \quad y \in \Omega_a, \quad U = 0, \quad y \in \partial\Omega_a.$$

The idea is to write

$$\int_{\Omega_a} u \cdot f \;=\; -\int_{\Omega_a} u(y) \cdot \big(\nabla_y \cdot A^T(y)\nabla_y U(y)\big) \tag{2.28}$$

$$= +\int_{\Omega_a} A(y)\nabla u(y) \cdot \nabla_y U(y) \;=\; 0, \tag{2.29}$$

where the last two equalities come from successive integration by parts. To make this reasoning rigorous, one must have some decay properties for the integrands. Precisely, it is enough to show that

$$I_1(R) \;:=\; \int_{\{y\cdot n>a,\, |y|=R\}} u \cdot \big(n \cdot (A^T(y)\nabla_y U)\big) \quad \text{and} \quad I_2(R) \;:=\; \int_{\{y\cdot n>a,\, |y|=R\}} A(y)\nabla u \cdot n\, U$$

go to zero as $R \to +\infty$. By the first part of proposition 3, we know that v is bounded. Moreover, by (2.18), w is also bounded, and so is u. Besides, from the dual version of (2.20) (that is with G^t replacing G), we have that for y far enough from the support of f, $|\nabla U(y)| \;\le\; C/|y|^d$. Combining these bounds yields: $I_1(R) \to 0$ as $R \to +\infty$. As regards I_2, we use the second part of proposition 3, which shows that $\delta(y)^m \nabla v(y)$ is bounded for all $m \in \mathbb{N}$. Moreover, using (2.21), we get that $\delta(y)\nabla w(y)$, and so $\delta(y)\nabla u(y)$, is bounded. Finally, by (2.17), we obtain that $|U(y)| \;\le\; C\delta(y)/|y|^d$. Hence, $I_2(R) \to 0$ as $R \to +\infty$. This concludes the proof of the proposition.

Besides this bound, we need some extra decay estimates on $V - v_*$ and their derivatives. For such estimates, the diophantine assumption $n \in \mathcal{A}_\kappa$ plays a role, and the decay deteriorates as κ goes to zero. This is made quantitative in

Proposition 7 *The solution V of (2.7) satisfies: for all $\alpha \in \mathbb{N}^d$, $k \in \mathbb{N}$, $m \in \mathbb{N}$,*

$$\left|\partial_\theta^\alpha \partial_t^k \left(V(\theta, t) - v_*\right)\right| \;\le\; \frac{C_{m,\alpha,k}}{\kappa}\Big(1 + \kappa(t-a)\Big)^{-m}, \quad \textit{uniformly in } \theta. \tag{2.30}$$

The constant $C_{m,\alpha,k}$ depends linearly on the $W^{s,\infty}$ norm of v_0 for some $s = s(m, \alpha, k)$ large enough as well as on the regularity of the matrix A.

Proof. Throughout the sequel, $\kappa \le 1$. Moreover, $C_{m,\alpha,k}$ will denote a constant that depends only on the ellipticity constant λ and the regularity of the matrix A as long as v_0 satisfies the fact that $\|v_0\|_{W^{s,\infty}} \le 1$ for some $s = s(m, \alpha, k)$ large enough. As the map $v_0 \mapsto V$ is linear, this shows that the constant $C_{m,\alpha,k}$ in Proposition 7 can be chosen linear in $\|v_0\|_{W^{s,\infty}}$, s large enough. We will also take $a = 0$ since the general case can be recovered by the change of variable $t' = t - a$ and $\mathcal{B}(\theta, t' + a) = \mathcal{B}(\theta, t)$ in (2.7). Of course it is important here that the constants $C_{m,\alpha,k}$ only depend on the regularity of A.

To prove (2.30), it is enough to prove

$$|\partial_\theta^\alpha \partial_t^k \left(V(\theta, t) - v_*\right)| \;\le\; C_{m,\alpha,k}\, \kappa^{-1}, \qquad 0 \le t \le 1, \tag{2.31}$$

$$|\partial_\theta^\alpha \partial_t^k \left(V(\theta, t) - v_*\right)| \;\le\; C_{m,\alpha,k}\, \kappa^{-1} \left(\kappa t\right)^{-m}, \qquad t \ge 1. \tag{2.32}$$

We recall the Sobolev bounds

$$\|N^t \nabla_\theta V(t)\|^2_{H^s} + \|\partial_t V(t)\|^2_{H^s} \ \leq \ C_s\,, \quad \forall s \tag{2.33}$$

that follow from Proposition 2. As $V = \int_0^t \partial_t V \, + \, V_0$, these bounds yield a uniform bound on V and its derivatives for $t \leq 1$. Combined with the uniform bound on v_* coming from the previous proposition, it implies the first inequality (one can even take κ^0 instead of κ^{-1}).

To obtain the second inequality, that is the decay of $V - v_*$ as $t \to +\infty$, we go along the lines of [14, Proposition 4], but keep track of the dependence on κ. If $n \in \mathcal{A}_\kappa$, then

$$\int_{\mathbb{T}^d} |N^t \, \nabla_\theta \, \tilde{W}|^2 \ \geq \ c\kappa^2 \, \|\tilde{W}\|^2_{H^{-l}(\mathbb{T}^d)} \tag{2.34}$$

for smooth enough $\tilde{W} = \tilde{W}(\theta)$ with zero average. Hence, the previous Sobolev bounds yield

$$\int_a^{+\infty} \kappa^2 \|\tilde{V}\|^2_{H^s(\mathbb{T}^d)} + \|\partial_t^k V\|^2_{H^s(\mathbb{T}^d)} \ \leq \ C(s,k) \ < \ +\infty, \tag{2.35}$$

for all $k \geq 1$ where we decompose

$$V(\theta,t) = \tilde{V}(\theta,t) \ + \ \bar{V}(t), \quad \int_{\mathbb{T}^d} \tilde{V} \, d\theta = 0.$$

Proceeding exactly as in the proof of Proposition 4 in [14], we introduce

$$f(T) \ := \ \int_{\mathbb{T}^d} \int_T^{+\infty} \Big(|N^t \, \nabla_\theta \ V|^2 + |\partial_t \, V|^2 \Big) \, dt \, d\theta,$$

and

$$W \ := \ V - \int_{\mathbb{T}^d} V(\theta,T) d\theta.$$

After multiplication of (2.7) by W, integration from T to infinity and integration by parts in θ, one ends up with (see also [14, Proposition 4])

$$f(T) \ \leq \ C(-f'(T))^{1/2} \left(\int_{\mathbb{T}^d} |\tilde{V}(\theta,T)|^2 \, d\theta \right)^{1/2}. \tag{2.36}$$

To estimate $\int_{\mathbb{T}^d} |\tilde{V}(\theta,T)|^2 \, d\theta$, we use interpolatioin between H^{-l} and $H^{l/(p-1)}$:

$$\left(\int_{\mathbb{T}^d} |\tilde{V}|^2 \, d\theta \right)^{1/2} \leq \ C \left(\|\tilde{V}\|_{H^{-l}(\mathbb{T}^d)} \right)^{1/p} \ \left(\|\tilde{V}\|_{H^{l/(p-1)}(\mathbb{T}^d)} \right)^{1-1/p} \tag{2.37}$$

By (2.34), the first factor at the right-hand side of (2.37) is controlled by $\left(\frac{-f'(T)}{\kappa^2}\right)^{1/2p}$. For the second factor, we use a simple interpolation inequality:

Lemma 8 *If $h \in H^1(\mathbb{R})$, then we have $\|h\|_\infty \leq C \|h\|^{1/2}_{L^2} \|h'\|^{1/2}_{L^2}$.*

Proof of the lemma. We write for each $t \in \mathbb{R}$ and $r > 0$,

$$|h(t)| = |h(t-r) + \int_{t-r}^{t} h'(s)ds| \leq |h(t-r)| + r^{1/2} \left(\int_{t-r}^{t} h'(s)^2 ds \right)^{1/2}.$$

Integrating in r between 0 and $R > 0$, we get $R\,|h(t)| \leq R^{1/2}\,\|h\|_{L^2} + R^{3/2}\,\|h'\|_{L^2}$. The result follows by optimizing in R.

Thanks to this lemma and the uniform Sobolev bounds on $\kappa\tilde{V}$ and $\partial_t V$, the second factor in the right-hand side of (2.37) is controlled by $C/\kappa^{1/2-1/(2p)}$. Finally, (2.36) leads to

$$f(T) \leq C_p \left(-\frac{f'(T)}{\kappa} \right)^{\frac{p+1}{2p}}. \tag{2.38}$$

Notice that this is exactly equation (2.16) of [14] with the precise κ dependence. Hence, we deduce that $f(T) \leq C_m(\kappa T)^{-m}$ for each $m > 1$, where $m = \frac{p+1}{p-1}$.

As regards higher-order derivatives, we argue as in [14] and consider the function

$$f_K(T) := \sum_{|\alpha|+k \leq K} f_{\alpha,k}(T) = \sum_{|\alpha|+k \leq K} \int_{\mathbb{T}^d} \int_T^{+\infty} \left(|N^t\, \nabla_\theta\, \partial_\theta^\alpha\, \partial_t^k\, V|^2 + |\partial_t\, \partial_\theta^\alpha\, \partial_t^k V|^2 \right) dt\, d\theta$$

instead of f. We are going to prove that $f_K(T)$ satisfies the same bound:
$f_K(T) \leq C_{K,m}\,(\kappa T)^{-m}$. This is proved by induction on K. Assume that

$$f_j(T) \leq C_{j,m}\,(\kappa T)^{-m}, \quad \forall j \leq k-1.$$

Let α, k be such that $|\alpha| + k = K$. Applying $\partial_\theta^\alpha\, \partial_t^k$ to (2.7) leads to the equation

$$-\begin{pmatrix} N^t\, \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}(\theta,t) \begin{pmatrix} N^t\, \nabla_\theta \\ \partial_t \end{pmatrix} \partial_\theta^\alpha\, \partial_t^k V = \begin{pmatrix} N^t\, \nabla_\theta \\ \partial_t \end{pmatrix} \cdot G_{\alpha,k}, \tag{2.39}$$

where

$$|G_{\alpha,k}| \leq C_{\alpha,k} \sum_{|\beta|+l \leq K-1} \left| (N^t\, \nabla_\theta, \partial_t) \partial_\theta^\beta\, \partial_t^l V \right|.$$

If we multiply equation (2.39) by

$$W_{\alpha,k} := \partial_\theta^\alpha\, \partial_t^k V - \int_{\mathbb{T}^d} \partial_\theta^\alpha\, \partial_t^k V(\theta, T) d\theta,$$

and integrate by parts, we get

$$\begin{aligned} f_{\alpha,k}(T) &\leq C \left((-f'_{\alpha,k}(T))^{1/2} + \|G_{\alpha,k}(\cdot,T)\|_{L^2(\mathbb{T}^d)} \right) \|\partial_\theta^\alpha\, \partial_t^k \tilde{V}(\cdot,T)\|_{L^2(\mathbb{T}^d)} \\ &\quad + \|G_{\alpha,k}\|_{L^2(\mathbb{T}^d \times \{t>T\})}\, f_{\alpha,k}(T)^{1/2} \\ &\leq C_{\alpha,k} \left((-f'_K(T))^{1/2} \|\partial_\theta^\alpha\, \partial_t^k \tilde{V}(\cdot,T)\|_{L^2(\mathbb{T}^d)} + \|G_{\alpha,k}\|^2_{L^2(\mathbb{T}^d \times \{t>T\})} \right) \\ &\leq C_{\alpha,k} \left((-f'_K(T))^{1/2} \|\partial_\theta^\alpha\, \partial_t^k \tilde{V}(\cdot,T)\|_{L^2(\mathbb{T}^d)} + C_{K-1,m}(\kappa T)^{-m} \right) \end{aligned}$$

using the induction assumption. Summing over α and k such that $|\alpha| + k = K$ and using as above the interpolation argument to control $\|\partial_\theta^\alpha \partial_t^k \tilde{V}(\cdot, T)\|_{L^2(\mathbb{T}^d)}$, we end up with

$$f_K(T) = \sum_{|\alpha|+k\le K} f_{\alpha,k}(T) \;\le\; C_{s,p}\left(-\frac{f_K'(T)}{\kappa}\right)^{\frac{p+1}{2p}} \;+\; C_{K-1,m}\,(\kappa T)^{-m}.$$

which gives the desired bound.

Using these bounds and the Sobolev imbeddings (see also lemma 8) we get that

$$\begin{aligned} \left|\,\partial_\theta^\alpha\, \tilde{V}(\theta,t)\right| &\le C_{\alpha,m}\frac{1}{\sqrt{\kappa}}(\kappa t)^{-m} \\ \left|\,\partial_\theta^\alpha\, \partial_t^{k+1} V(\theta,t)\right| &\le C_{\alpha,k,m}(\kappa t)^{-m} \end{aligned} \tag{2.40}$$

for all $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$, $k \in \mathbb{N}$, uniformly in θ. As regards $\bar{V}(t)$, we use that

$$|\bar{V}(t+h) - \bar{V}(t)| \le \int_t^{t+h} \left|\frac{d}{dt}\bar{V}\right| \;\le\; C_m \int_t^{t+h} (1+\kappa s)^{-m-1}\, ds \le C\frac{1}{\kappa}(\kappa t)^{-m}.$$

This implies

$$|\bar{V}(t) - v_*| \le C\frac{1}{\kappa}(\kappa t)^{-m}. \tag{2.41}$$

The estimates (2.40), (2.41) imply (2.32). This concludes the proof of the proposition.

Thanks to the previous propositions, we have at hand refined estimates on v and $v - v_*$. Such estimates will be crucial in our homogenization proof for smooth domains. Indeed, our proof will rely on the construction of accurate expansions of u^ε, in which correctors like v will appear as leading terms. Still, for the next terms of the expansion, other boundary layer correctors will be needed. They will satisfy the same type of equations as v, but with additional source terms. Therefore, we need to extend the estimates of the previous propositions to this slightly larger setting.

Instead of (2.7), we consider the system

$$\begin{cases} -\begin{pmatrix} N^t\nabla_\theta \\ \partial_t \end{pmatrix} \cdot\, \mathcal{B}(\theta,t) \begin{pmatrix} N^t\nabla_\theta \\ \partial_t \end{pmatrix} U(\theta,t) = F(\theta,t), & t > a \\ \qquad\qquad\qquad\qquad\qquad\qquad U(\theta,t) = 0, & t = a, \end{cases} \tag{2.42}$$

set on $\mathbb{T}^d \times \{t > a\}$. We assume that the source term $F = F(\theta,t)$ is smooth and in the Schwartz class with respect to t. As explained in our previous paper [14] (see the explanation below system (3.11) in [14]), the well-posedness and asymptotic properties of (2.7) extend to the system (2.42). In particular, there is a unique smooth solution $U = U(\theta,t)$, with the Sobolev bounds

$$\begin{aligned} &\|N^t\nabla_\theta U\|^2_{H^s(\mathbb{T}^d\times\{t>a\})} + \|\partial_t U\|^2_{H^s(\mathbb{T}^d\times\{t>a\})} \\ &\qquad\qquad \le\; C\,\left(\|(t-a)\,F\|^2_{H^s(\mathbb{T}^d\times\{t>a\})} \;+\; \|F\|^2_{H^s(\mathbb{T}^d\times\{t>a\})}\right). \end{aligned} \tag{2.43}$$

Moreover, there is a constant u_* such that $U - u_*$ is in the Schwartz class with respect to t.

Like the solution of (2.7) provides a solution to (2.4), the solution of (2.42) provides a solution to

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y u(y) = f(y), & y.n > a \\ u(y) = 0, & y.n = a. \end{cases} \tag{2.44}$$

As before, u and U, resp. f and F are related through

$$u(y = Mz) \;=\; \mathbf{u}(z) \;=\; U(Nz', z_d), \quad f(y = Mz) \;=\; \mathbf{f}(z) \;=\; F(Nz', z_d).$$

We want to derive some bounds on u and $U - u_*$, in terms of f and F. We state

Proposition 9 *Let $\mu \geq 0$, $\nu \geq 1$, $m_0 \geq 4$. Assume that for all $m \geq m_0$, $\alpha \in \mathbb{N}^d$ and for all $k \in \mathbb{N}$,*

$$\begin{aligned} \left|\partial_y^\alpha f(y)\right| \;&\leq\; C^f_\alpha\, \kappa^{-\mu} && \textit{uniformly in } y, \\ \left|\partial_\theta^\alpha \partial_t^k F(\theta,t)\right| \;&\leq\; C^F_{m,\alpha,k} \left(\kappa^\nu(1+t-a)\right)^{-m} && \textit{uniformly in } \theta. \end{aligned} \tag{2.45}$$

Then, for all $\delta > 0$, there exists $m_1 = m_1(m_0,\mu,\nu,\delta)$ such that: for all $m \geq m_1$, $\alpha \in \mathbb{N}^d$ and for all $k \in \mathbb{N}$

$$\begin{aligned} \left|\partial_y^\alpha u(y)\right| \;&\leq\; C^u_{\alpha,\delta}\, \kappa^{-\mu}\, \kappa^{-2\nu} && \textit{uniformly in } y, \\ \left|\partial_\theta^\alpha \partial_t^k (U-u_*)(\theta,t)\right| \;&\leq\; C^U_{m,\alpha,k,\delta} \left(\kappa^{\nu+\delta}\,(1+t-a)\right)^{-m} && \textit{uniformly in } \theta. \end{aligned} \tag{2.46}$$

Proof. Before we start the proof, let us notice that here we are combining bounds in the physical space y with bounds in the periodic variable θ. Indeed, we will take advantage of both formulations. We will assume that $a = 0$ since we can recover the general case by making the change of variable $t' = t - a$ and replacing $\mathcal{B}(\theta,t)$ by $\mathcal{B}(\theta,t'+a)$. Also, we can restrict to the case $\mu = 0$ as well. Indeed, suppose that the result holds in such a case, and take $\mu > 0$. If (2.45) is satisfied, it implies trivially that

$$\left|\partial_y^\alpha f(y)\right| \;\leq\; C^f_\alpha\, \kappa^{-\mu}, \quad \left|\partial_\theta^\alpha \partial_t^k F(\theta,t)\right| \;\leq\; C^F_{m,\alpha,k}\, \kappa^{-\mu} \left(\kappa^\nu(1+t)\right)^{-m}.$$

By linearity of the equations, and using the result with $\mu = 0$, and $\delta/2$ instead of δ, we get (for $m \geq m_1(m_0, 0, \nu, \delta/2)$)

$$\left|\partial_y^\alpha u(y)\right| \;\leq\; C^u_{\alpha,\delta/2}\, \kappa^{-\mu}\, \kappa^{-2\nu}, \quad \left|\partial_\theta^\alpha \partial_t^k (U-u_*)(\theta,t)\right| \;\leq\; C^U_{m,\alpha,k,\delta/2}\, \kappa^{-\mu} \left(\kappa^{\nu+\delta/2}\,(1+t)\right)^{-m}.$$

The last inequality reads

$$\left|\partial_\theta^\alpha \partial_t^k (U-u_*)(\theta,t)\right| \;\leq\; C^U_{m,\alpha,k,\delta/2} \left(\kappa^{\nu+\delta/2+\mu/m}\,(1+t)\right)^{-m} \;\leq C^U_{m,\alpha,k,\delta/2} \left(\kappa^{\nu+\delta}\,(1+t)\right)^{-m}$$

for $m \geq \max(m_1, 2\mu/\delta)$, which proves our claim. From now on, $\mu = 0$.

We start with the inequality on u. As u satisfies an elliptic system, it is enough to treat the case $\alpha = 0$: regularity arguments similar to those used in the proof of Proposition 5 provide the bound for higher-order derivatives (see (2.10) and [1, 15]). By a combination of the two inequalities in (2.45), we have

$$\left|f(y)\right| \;\leq\; \frac{C_m}{1+(\kappa^\nu t)^m}, \;\text{ uniformly in } y',$$

for $m \geq m_0$. We use here the notation $y = y' + t\,n$, with $y' \cdot n = 0$ and $t \geq 0$. We rescale system (2.44), introducing $\tilde{y} := \kappa^\nu y$, $\tilde{u}(\tilde{y}) := u(y)$, $\tilde{f}(\tilde{y}) := f(y)$ and so on. *Dropping the tildes*, we get

$$\begin{cases} -\nabla_y \cdot A(\cdot/\kappa^\nu) \nabla_y u(y) = \dfrac{1}{\kappa^{2\nu}} f(y), \quad y.n > 0 \\ \qquad\qquad\qquad\qquad\quad u(y) = 0, \quad y.n = 0. \end{cases} \tag{2.47}$$

where the source f satisfies in particular (for some C depending on m_0)

$$|f(y)| \leq \frac{C}{1+t^4}. \tag{2.48}$$

We must show that $\kappa^{2\nu} u$ is uniformly bounded. We use temporarily the notation ε instead of κ^ν. Let $G^\varepsilon = G^\varepsilon(y_1, y_2)$ the Green function associated to the operator $-\nabla_y \cdot A(\cdot/\varepsilon)\nabla_y$ in the domain $\{y \cdot n > 0\}$. Then,

$$\varepsilon^2 u(y_1) = \int_{\{y_2 \cdot n > 0\}} G^\varepsilon(y_1, y_2)\, f(y_2)\, dy_2.$$

This representation formula can be established similarly to what we did for (2.22). Indeed, let w^k be the solution of system (2.47) with the right-hand side $f^k = f\psi^k$ where $\psi^k \in C_c^\infty(\mathbb{R}^d)$ satisfies $\psi^k = 1$ for $|y| \leq k$. Hence, w^k has the following representation

$$\varepsilon^2 w^k(y_1) := \int_{\{y_2 \cdot n > 0\}} G^\varepsilon(y_1, y_2)\, f^k(y_2)\, dy_2. \tag{2.49}$$

As in proposition 5, we rely on the estimates of Lemma 6 to prove uniform bounds on w^k, in particular estimates (2.15) or (2.16) and (2.17). To apply these estimates to the Green formula, we decompose the integral into

$$\int_{y_2 \cdot n > 0} G^\varepsilon(y_1, y_2)\, f^k(y_2)\, dy_2 = \int_{\substack{|y_1 - y_2| < 1, \\ y_2 \cdot n > 0}} G^\varepsilon(y_1, y_2)\, f^k(y_2)\, dy_2 + \int_{\substack{|y_1 - y_2| > 1, \\ y_2 . n > 0}} G^\varepsilon(y_1, y_2)\, f^k(y_2)\, dy_2.$$

Combining (2.48) and (2.15) or (2.16) yields a uniform (in ε, k and y_1) bound for the first term. As regards the second term, we use (2.17): we denote $t_1 := y_1 \cdot n$, $t_2 := y_2 \cdot n$, and write

$$\begin{aligned} \left| \int_{\substack{|y_1 - y_2| > 1, \\ y_2 \cdot n > 0}} G^\varepsilon(y_1, y_2)\, f^k(y_2)\, dy_2 \right| &\leq C \int_{y_2 \cdot n > 0} \frac{t_1\, t_2}{|y_1 - y_2|^d} \frac{1}{1+t_2^4} dy_2 \\ &\leq C' \int_{\mathbb{R}_+} \frac{t_1\, t_2}{|t_1 - t_2| + 1} \frac{1}{1+t_2^4} dt_2. \end{aligned} \tag{2.50}$$

The last integral comes from integration with respect to the tangential variable. Hence, it is bounded by a constant that is independent of ε, k and y_1. It is then clear that w^k converges locally uniformly to w which is given by

$$\varepsilon^2 w(y_1) := \int_{\{y_2 \cdot n > 0\}} G^\varepsilon(y_1, y_2)\, f(y_2)\, dy_2 \tag{2.51}$$

and which solves the same equation as u. Hence, to conclude it is enough to prove that $u = w$. This follows from the same uniqueness argument used in the proof of Proposition 5. The only difference is that the fact that w is bounded follows now from the Green's representation instead of the Poisson integral. Also, at this stage, we already know that u is bounded (but without an exact dependence in ε). This concludes the proof of the first inequality in (2.46).

The estimate of $U - u_*$ is established much like the estimate of $V - v_*$ in proposition 7, and we will only sketch the proof. As for V, the estimate for $t \le 1$ comes from the global Sobolev estimate (2.43) and the uniform bound on u (so on u_*). Note that, using the bound for F in (2.43), we obtain

$$\|N^t \nabla_\theta U\|_{H^s}^2 + \|\partial_t U\|_{H^s}^2 \;<\; C_s\, \kappa^{-2m_0\nu} \quad \forall s \tag{2.52}$$

which corresponds to a fixed loss in κ.

For $T \ge 1$ and $K \in \mathbb{N}$, we introduce again the functions

$$f_K(T) \;:=\; \sum_{|\alpha|+k \le K} f_{\alpha,k}(T) \;:=\; \sum_{|\alpha|+k\le K} \int_{\mathbb{T}^d} \int_T^{+\infty} \Big(|N^t\, \nabla_\theta\, \partial_\theta^\alpha\, \partial_t^k\, U|^2 + |\partial_t\, \partial_\theta^\alpha\, \partial_t^k U|^2 \Big)\, dt\, d\theta.$$

We obtain, along the same lines as in the proof of Proposition 7: for all $p > 1$,

$$\begin{aligned} f_0(T) \;\le\; C_p \left(\|N^t \nabla_\theta U\|_{H^{l/(p-1)}} + \|\partial_t U\|_{H^{l/(p-1)}} \right)^{1-\frac{1}{p}} \left(-\frac{f_0'(T)}{\kappa} \right)^{\frac{p+1}{2p}} \\ + \left| \int_{\mathbb{T}^d} \int_T^{+\infty} F(\theta,t) \left(U(\theta,t) - \int_{\mathbb{T}^d} U(\cdot,T) \right) d\theta dt \right|. \end{aligned} \tag{2.53}$$

An integration by parts provides

$$\begin{aligned} \int_{\mathbb{T}^d} \int_T^{+\infty} F(\theta,t) \left(U(\theta,t) - \int_{\mathbb{T}^d} U(\cdot,T) \right) d\theta dt = - \int_{\mathbb{T}^d} \int_T^{+\infty} \mathcal{F}(\theta,t) \partial_t U(\theta,t) d\theta dt \\ - \int_{\mathbb{T}^d} \mathcal{F}(\theta,T) \left(U(\theta,T) - \int_{\mathbb{T}^d} U(\cdot,T) \right) d\theta \end{aligned}$$

where $\mathcal{F}(\theta,t) = - \int_t^{+\infty} F(\theta,s) ds$. It follows that

$$\begin{aligned} &\left| \int_{\mathbb{T}^d} \int_T^{+\infty} F(\theta,t) \left(U(\theta,t) - \int_{\mathbb{T}^d} U(\cdot,T) \right) d\theta dt \right| \\ &\qquad \le \; \|\mathcal{F}\|_{L^2(\mathbb{T}^d \times \{t>T\})} \|\partial_t U\|_{L^2(\mathbb{T}^d \times \{t > T\})} \;+\; \|\mathcal{F}(\cdot,T)\|_{H^l(\mathbb{T}^d)}\, \kappa^{-1} \|N^t \nabla_\theta U(\cdot,T)\|_{L^2(\mathbb{T}^d)} \end{aligned}$$

using (2.34) for the last term. From there, combining (2.45), (2.52) and (2.53), we obtain easily that

$$f_0(T) \;\le\; C_{m,p}\, \kappa^{-M} \left(\left(-\frac{f_0'(T)}{\kappa} \right)^{\frac{p+1}{2p}} \;+\; (\kappa^\nu\, T)^{-m} \right)$$

for all $m \ge m_0$ and some fixed M depending only on m_0. Now, given some $\delta > 0$, if $m \ge \max(m_0, M/\delta)$ and $\frac{p+1}{p-1} \ge \max(m_0, M/\delta)$, then

$$f_0(T) \;\le\; C_{m,p} \left(\left(-\frac{f_0'(T)}{\kappa^{\nu+\delta}} \right)^{\frac{p+1}{2p}} \;+\; \left(\kappa^{\nu+\delta}\, T \right)^{-m} \right).$$

Hence $f_0(T) \leq C_m(\kappa^{\nu+\delta}T)^{-m}$ for m large enough. Similar bounds hold for f_K, $K \in \mathbb{N}$ which are obtained, as before, recursively. We have just to differentiate the equation using $\partial_\theta^\alpha \partial_t^k$ first and perform the energy estimate as above. This concludes the proof of the proposition.

We note that, by the linearity of the map $(F, f) \mapsto (U, u)$, one can be more specific about the constants $C^u_{\alpha,\delta}$ and $C^U_{m,\alpha,k,\delta}$ in (2.46): one has

$$C^u_{\alpha,\delta} + C^U_{m,\alpha,k,\delta} \;\leq\; \mathcal{C}_{m,\alpha,k,\delta} \sum_{(m',\alpha',k')\in I_{m,\alpha,k}} \left(C^f_{\alpha'} \;+\; C^F_{m',\alpha',k'} \right) \tag{2.54}$$

where $\mathcal{C}_{m,\alpha,k} > 0$ does not depend on f, F, and $I_{m,\alpha,k}$ is a finite subset of indices also independent of f, F.

Corollary 1 *The function $n \mapsto v_*(n)$ is Lipschitz on $\mathcal{A}_\kappa$, with a Lipschitz constant which is $O(\kappa^{-2})$ as κ goes to zero.*

Proof of the corollary. Let n_1 and n_2 be in $\mathcal{A}_\kappa$. We wish to show that

$$|v_*(n_1) - v_*(n_2)| \;\leq\; \frac{C}{\kappa^2}\,|n_1 - n_2|.$$

For $i = 1, 2$, we introduce the solution V_{n_i} of (see previous section for notations):

$$\begin{cases} -\begin{pmatrix} N_i^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\, B(\theta + tn_i) \begin{pmatrix} N_i^t \nabla_\theta \\ \partial_t \end{pmatrix} V_{n_i} = 0, \quad t > 0, \\ \qquad\qquad\qquad\qquad V_{n_i}(\theta, t = 0) = \chi^\gamma(\theta), \quad t = 0. \end{cases} \tag{2.55}$$

We set $V := V_{n_1} - V_{n_2}$, $N := N_1 - N_2$, and so on. We have

$$\begin{cases} -\begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\, B(\theta + tn_1) \begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} V = F, \quad t > 0, \\ \qquad\qquad\qquad\qquad V(\theta, t = 0) = 0, \quad t = 0. \end{cases} \tag{2.56}$$

where

$$\begin{aligned} F \;&=\; \left(-\begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\, B(\theta + tn_1) \begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} \;+\; \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\, B(\theta + tn_2) \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} \right) V_{n_2} \\ &=\; -\begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\, B(\theta + tn_1) \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} V_{n_2} \\ &\quad + \left(\begin{pmatrix} -N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\, B(\theta + tn_1) \;+\; \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\, (B(\theta + tn_2) - B(\theta + tn_1)) \right) \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} V_{n_2}. \end{aligned}$$

We also introduce the corresponding

$$v_{n_i}(y = M_1 z) \;=\; \mathbf{v}_{\mathbf{n_i}}(z) \;=\; V_{n_i}(N_1 z', z_d), \quad i = 1, 2,$$

$v(y)$, $f(y)$. By the estimates of propositions 5 and 7, one has the following bounds:

$$\begin{aligned} \left|\partial_y^\alpha f(y)\right| \;&\leq\; C_{m,\alpha}|n_1 - n_2| && \text{uniformly in } y, \\ \left|\partial_\theta^\alpha \partial_t^k F(\theta, t)\right| \;&\leq\; C_{m,\alpha,k,\delta}|n_1 - n_2| \left(\kappa^{1+\delta}(1 + t - a)\right)^{-m} && \text{uniformly in } \theta \end{aligned}$$

for all $\delta > 0$ and m such that $\delta m > 1$. Applying our last proposition (see also (2.54)), we get that

$$|v(y)| \;=\; |v_{n_1}(y) - v_{n_2}(y)| \;\le\; \frac{C_\delta}{\kappa^{2+2\delta}}\,|n_1 - n_2|$$

uniformly in y, for all $\delta > 0$. Actually, one can improve a little this inequality and take $\delta = 0$. Indeed, the source term F can be split into

$$F \;=\; F' \;+\; F'' \;:=\; -\begin{pmatrix} N_1^t\nabla_\theta \\ \partial_t \end{pmatrix} G \;+\; L(n_1 - n_2, \theta, t, \partial_\theta, \partial_t)\begin{pmatrix} N_2^t\nabla_\theta \\ \partial_t \end{pmatrix} V_{n_2}$$

where G satisfies:

$$|\partial_\theta^\alpha \partial_t^k G| \;\le\; C_{m,\alpha,k,\delta}\,|n_1 - n_2|\,\left(\kappa^{1+\delta} t\right)^{-m}, \quad \forall \delta > 0,\ \forall t > a = 0,$$

where $L(n, \theta, t, \partial_\theta, \partial_t)$ is a first-order smooth matricial operator, whereas

$$\left|\partial_\theta^\alpha \partial_t^k \begin{pmatrix} N_2^t\nabla_\theta \\ \partial_t \end{pmatrix} V_{n_2}\right| \;\le\; C_{m,\alpha,k}\;(\kappa t)^{-m}\,.$$

We insist that this last inequality involves only κ: one evaluates $(N_2^t\nabla_\theta, \partial_t)V_{n_2}$, so that additional estimates of type (2.40)-(2.41) (responsible for an additional loss in κ) are not needed.

This special form of the source term F allows to refine the estimate on $v = v_{n_1} - v_{n_2}$. One can write $v = v' + v''$, with

$$\begin{cases} -\nabla_y\cdot\, A(\cdot)\nabla_y\, v'(y) = f'(y) = \nabla\cdot g(y), & y.n > 0 \\ v'(y) = 0, & y.n = 0. \end{cases}$$

and

$$\begin{cases} -\nabla_y\cdot\, A(\cdot)\nabla_y\, v''(y) = f''(y), & y.n > 0 \\ v''(y) = 0, & y.n = 0. \end{cases}$$

We then proceed very much like in the proof of Proposition 9: we have the representation formula:

$$\kappa^{1+\delta} v'(\kappa^{1+\delta} y_1) \;=\; -\int_{y_2\cdot n>0} \nabla_{y_2} G^{\kappa^{1+\delta}}(y_1, y_2)\, g(\frac{y_2}{\kappa^{1+\delta}}) dy_2 \tag{2.57}$$

$$\kappa^2 v''(\kappa y_1) \;=\; -\int_{y_2\cdot n>0} G^\kappa(y_1, y_2)\, f''(\frac{y_2}{\kappa}) dy_2,$$

where $G^\varepsilon(y_1, y_2)$ is as before the Green function associated to the operator $-\nabla_y\cdot A(\cdot/\varepsilon)\nabla_y$ in the domain $y\cdot n > 0$. Moreover, the source terms satisfy

$$|g(\frac{y_2}{\kappa^{1+\delta}})| \;+\; |f''(\frac{y_2}{\kappa})| \;\le\; \frac{C_{\delta,m}|n_1 - n_2|}{t_2^m}.$$

Proceeding exactly as in the proof of Proposition 9, one has

$$|v''(y)| \;\le\; \frac{C\,|n_1 - n_2|}{\kappa^2}.$$

For v', we use the bounds on the gradient of the Green function, namely (2.19) and (2.20) (more precisely their symmetric version, obtained by considering G^t instead of G). We decompose the integral in (2.57) into two parts. One for which $|y_1 - y_2| \le 1$ and one for which $|y_1 - y_2| \ge 1$. The first one is controlled using (2.19). For the second one, we argue as in (2.50)

$$
\begin{aligned}
&\left| \int_{\substack{|y_1-y_2|>1,\\ y_2\cdot n>0}} \nabla_{y_2} G^{\kappa^{1+\delta}}(y_1, y_2)\, g(\kappa^{1+\delta} y_2)\, dy_2 \right| \\
&\le C \int_{y_2\cdot n>0} \left(\frac{t_1\, t_2}{|y_1 - y_2|^{d+1}} + \frac{t_1}{|y_1 - y_2|^{d}} \right) \frac{C|n_1 - n_2|}{1 + t_2^4} dy_2 \\
&\le C' \int_{\mathbb{R}_+} \frac{t_1\,(1 + t_2)}{|t_1 - t_2| + 1} \frac{C|n_1 - n_2|}{1 + t_2^4} dt_2 \le C|n_1 - n_2|.
\end{aligned}
$$

The result follows letting y go to infinity transversally to the boundary. This concludes the proof of the corollary.

3 The disk

We turn in this section to the core of the paper, that is the homogenization of system (1.1)-(1.2) for smooth domains Ω. To get rid of confusing technicalities, we will first consider the case of a unit disk:

$$d = 2, \quad \Omega \;=\; \{x,\ |x| < 1\},$$

with boundary datum φ that factors into

$$\varphi(x, y) \;=\; v_0(y)\, \varphi_0(x)$$

for some smooth v_0 on $\mathbb{T}^d$ with values in $M_N(\mathbb{R})$ and some smooth φ_0 on $\partial\Omega$ with values in $\mathbb{R}^N$. The extension to the general framework of theorem 1 will be discussed in section 4. Let us stress that this extension, althoug a bit heavy to write down, contains no mathematical difficulties. Thus, all ideas are already contained in the simplified configuration studied here.

For all $x \in \mathbb{S}^1$, we denote by $n(x) = -x$ the unit inward normal vector. If $x \in \cup_{\kappa>0}\mathcal{A}_\kappa$, $n(x)$ satisfies the small divisor assumption (1.16). Thus, we can use the results of section 2: the boundary layer system (2.4) with $n = n(x)$ and with boundary data $v_0 \in M_N(\mathbb{R})$ has a solution $v = v(y) \in M_N(\mathbb{R})$ that converges (transversally to the boundary) to some $v_* = v_*(n) \in M_N(\mathbb{R})$. We set:

$$\varphi_*(x) \;:=\; v_*(n(x))\, \varphi_0(x).$$

From the beginning of section 2, we know that $\cup_{\kappa>0}\mathcal{A}_\kappa$ has full measure, so that φ_* is defined almost everywhere on the circle. Moreover, $\varphi_* \in L^\infty(\mathbb{S}^1;\, \mathbb{R}^N)$: corollary 1 implies its measurability and proposition 5 yields a uniform bound.

Theorem 1 is a direct consequence of

Proposition 10 *Let u^0 be the solution of system* (1.15), *with the boundary data φ_* defined above. Then,*

$$\|u^\varepsilon - u^0\|_{L^2} \;=\; O(\varepsilon^\alpha),$$

as ε goes to zero, for all $\alpha < 1/11$.

We note that, because φ_* has L^∞ regularity, the limit field u^0 is in $L^p(\Omega)$ for all $1 \le p \le \infty$.

We can also prove (using the next interpolation argument) that u^0 belongs to $W^{s_p,p}(\Omega)$ for some $s_p > 0$ for all $1 < p < \infty$. But this will not be used in the convergence proof.

The rest of the section is devoted to the proof of this proposition. We first split the problem in two: we write $u^\varepsilon = u^\varepsilon_{reg} + u^\varepsilon_{bl}$, with

$$\begin{cases} -\nabla \cdot A\left(\dfrac{x}{\varepsilon}\right) \nabla u^\varepsilon_{reg} = 0, & x \in \Omega \subset \mathbb{R}^d, \\ u^\varepsilon_{reg} = \varphi_*, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\nabla \cdot A\left(\dfrac{x}{\varepsilon}\right) \nabla u^\varepsilon_{bl} = 0, & x \in \Omega \subset \mathbb{R}^d, \\ u^\varepsilon_{bl} = \varphi(x, x/\varepsilon) - \varphi_*, & x \in \partial\Omega. \end{cases}$$

We will bound $\|u^\varepsilon_{reg} - u^0\|_{L^2(\Omega)}$ and $\|u^\varepsilon_{bl}\|_{L^2(\Omega)}$ separately. We stress that the difficult part is the bound on u^ε_{bl}. It is where the boundary layer analysis is involved, notably the sets $\mathcal{A}_\kappa$. The treatment of u^ε_{reg} enters the classical framework discussed in the introduction, and is essentially contained in previous studies.

Nevertheless, there is a little technical difficulty for this problem, namely the lack of regularity of φ_*. Indeed, the classical estimates on $u^\varepsilon_{reg} - u^0$ rely on expansions that require differentiating u^0. As u^0 is only in L^∞, we will need some regularizing sequences, indexed by another parameter δ. The choice of these sequences will be specified in the next subsection. Remark that we have now three small parameters: ε, κ, and δ. Special attention will be paid to the way our estimates depend on them. The rate $\varepsilon^{1/11}$ will follow from optimizing in κ, δ and α which will be defined later.

3.1 Classical approximation

We derive here estimates on $u^\varepsilon_{reg} - u^0$. We take care of the smoothness problem as follows. By corollary 1, φ_* is Lipschitz over A_κ, with Lipschitz constant less than C/κ^2. By standard results (see [8]), $\varphi_*|_{\mathcal{A}_\kappa}$ admits a Lipschitz extension, uniformly bounded in κ (because φ_* is), and with the same Lipschitz constant.

Let us call this extension φ^κ_*. With obvious notations, we associate to this boundary data the fields $u^{\varepsilon,\kappa}_{reg}$ and $u^{0,\kappa}$.

Now, we notice that

$$\varphi^\kappa_* - \varphi_* \;=\; \varphi^\kappa_* \mathbf{1}_{\mathcal{A}^c_\kappa} \;-\; \varphi_* \mathbf{1}_{\mathcal{A}^c_\kappa}.$$

So, from estimate (2.2), we have

$$\|\varphi^\kappa_* - \varphi_*\|_{L^2(\partial\Omega)} \;\le\; C\,\kappa^{1/2}.$$

Thus, using the results of [5, Theorem 3(ii)], we get

$$\|u^{\varepsilon,\kappa}_{reg} - u^\varepsilon_{reg}\|_{L^2(\Omega)} \;\le\; C\,\kappa^{1/2}, \quad \|u^{0,\kappa} - u^0\|_{L^2(\Omega)} \;\le\; C\,\kappa^{1/2}.$$

It remains to estimate $u^{\varepsilon,\kappa}_{reg} - u^{0,\kappa}$. We introduce a sequence of smooth fields $\varphi^{\kappa,\rho}_*$ such that $\varphi^{\kappa,\rho}_* \;\to\; \varphi^\kappa_*$ in $L^2(\partial\Omega)$, as $\rho \to 0$. More precisely, we chose it in such a way that

$$\|\varphi^{\kappa,\rho}_* - \varphi^\kappa_*\|_{L^2(\partial\Omega)} \;\le\; C\,\|\nabla\varphi^\kappa_*\|_{L^\infty}\,\rho \;\le\; C'\,\frac{\rho}{\kappa^2}, \quad \|\varphi^{\kappa,\rho}_*\|_{H^s} \;\le\; C_s\,\rho^{1-s}\|\nabla\varphi^\kappa_*\|_{L^\infty} \;\le\; C'_s\,\frac{\rho^{1-s}}{\kappa^2},$$

for all $s \geq 0$. For instance, one can use a partition of unity to come down to local charts, and in each chart, use a convolution by an approximation of unity with support in $(-\rho, \rho)$.

Since $\varphi_*^{\kappa,\rho}$ is smooth, we claim that the following bound holds:

$$\|u_{reg}^{\varepsilon,\kappa,\rho} - u^{0,\kappa,\rho}\|_{L^2(\Omega)} \;\leq\; C\,\|u^{0,\kappa,\rho}\|_{H^2(\Omega)}\,\varepsilon. \tag{3.1}$$

Let us explain where this bound comes from. Following the notations of the introduction, we define the first order corrector

$$u^{1,\kappa,\rho}(x,y) \;:=\; -\chi^\alpha(y)\partial_{x_\alpha} u^{0,\kappa,\rho}$$

where χ solves the cell problem (1.5), as well as the boundary layer corrector $u_{bl}^{1,\varepsilon,\kappa,\rho}$, satisfying system (1.10) with $u^{1,\kappa,\rho}$ instead of u^1. Then, one can show the following bound:

$$\|u_{reg}^{\varepsilon,\kappa,\rho} - u^{0,\kappa,\rho} - \varepsilon u^{1,\kappa,\rho}(\cdot,\cdot/\varepsilon) - \varepsilon u_{bl}^{1,\varepsilon,\kappa,\rho}\|_{H^1(\Omega)} \;\leq\; C\,\|u^{0,\kappa,\rho}\|_{H^2(\Omega)}\,\varepsilon.$$

We refer to [19, section 2] for a proof, or [14, Section 3.2, "global error estimate"] for the proof of a similar bound. Moreover, one has clearly

$$\|\varepsilon u^{1,\kappa,\rho}(\cdot,\cdot/\varepsilon)\|_{L^2(\Omega)} \;\leq\; C\,\|u^{0,\kappa,\rho}\|_{H^1(\Omega)}\,\varepsilon,$$

and using again [5, Theorem 3(ii)],

$$\|\varepsilon u_{bl}^{1,\varepsilon,\kappa,\rho}\|_{L^2(\Omega)} \;\leq\; C\,\|\nabla u^{0,\kappa,\rho}\|_{L^2(\partial\Omega)}\,\varepsilon \;\leq\; C'\,\|u^{0,\kappa,\rho}\|_{H^2(\Omega)}\,\varepsilon.$$

Combining the last three inequalities yields (3.1).

Hence, we have

$$\begin{aligned}\|u_{reg}^{\varepsilon,\kappa,\rho} - u^{0,\kappa,\rho}\|_{L^2(\Omega)} \;&\leq\; C\,\|u^{0,\kappa,\rho}\|_{H^2(\Omega)}\,\varepsilon \\ &\leq\; C'\,\|\varphi_*^{\kappa,\rho}\|_{H^{3/2}(\partial\Omega)}\,\varepsilon \;\leq\; C''\,\frac{\varepsilon}{\rho^{1/2}\kappa^2}.\end{aligned}$$

Moreover, using again the results of Avellaneda and Lin:

$$\|u_{reg}^{\varepsilon,\kappa,\rho} - u_{reg}^{\varepsilon,\kappa}\|_{L^2(\Omega)} \;\leq\; C\,\frac{\rho}{\kappa^2}, \quad \|u^{0,\kappa,\rho} - u^{0,\kappa}\|_{L^2(\Omega)} \;\leq\; C\,\frac{\rho}{\kappa^2}$$

Gathering all previous bounds, we end up with

$$\|u_{reg}^{\varepsilon} - u^0\|_{L^2(\Omega)} \;\leq\; C\,\left(\kappa^{1/2} \;+\; \frac{\varepsilon}{\rho^{1/2}\kappa^2} \;+\; \frac{\rho}{\kappa^2}\right) \tag{3.2}$$

3.2 Boundary layer approximation

We shall construct in this paragraph an approximation of u_{bl}^ε, of boundary layer type. To construct the boundary layer, we will divide the circle into small arcs, each of length ε^α, with $1 > \alpha > 0$ to be determined, and we will approximate each arc by a segment so as to use the half-space analysis.

We first parametrize the boundary of $\partial\Omega$ by $\theta \to e^{i\theta}$ with $\theta \in [0, 2\pi]$. We divide $[0, 2\pi]$ into $Q = [\frac{1}{\varepsilon^\alpha}]$ small intervals, namely

$$[0, 2\pi] = \bigcup_{q=1}^{Q} I_q, \quad I_q = [2\pi\frac{q-1}{Q}, 2\pi\frac{q}{Q}].$$

We also denote $\tilde{I}_q$ the interval which has the same center as I_q and half the size: $\tilde{I}_q = [2\pi\frac{q-3/4}{Q}, 2\pi\frac{q-1/4}{Q}]$. We also denote θ_q the center of I_q, namely $\theta_q = 2\pi\frac{q-1/2}{Q}$.

Let $\Psi = \Psi(\xi)$ be a smooth function with compact support satisfying:

i) $\Psi = 1$ for $|\xi| < \pi/2$.

ii) $\Psi = 0$ for $|\xi| > 2\pi$.

iii) $\sum_{i=1}^{Q} \Psi(Q(\theta - \theta_q)) = 1$.

It induces a partition of unity in the vicinity of the circle: for $x = (r\cos\theta, r\sin\theta)$ in an ε^α neighborhood of the circle,

$$1 \;=\; \sum_{i=1}^{Q} \phi_q(x) \;:=\; \sum_{i=1}^{Q} \Psi\left(Q(\theta - \theta_q)\right)\, \Psi(Q(r-1)).$$

Clearly, we can write $\phi_q(x) = \psi\left(\frac{x - x_q}{\varepsilon^\alpha}\right)$ where $x_q = e^{i\theta_q}$ and *all derivatives of* ψ *are uniformly bounded.* We now divide the set $\{1, \ldots, Q\}$ into two sets

$$\mathbb{Q}^g = \{q,\, 1 \le q \le Q,\, \tilde{I}_q \cap \mathcal{A}_\kappa \neq \emptyset\}. \tag{3.3}$$

$$\mathbb{Q}^b = \{q,\, 1 \le q \le Q,\, \tilde{I}_q \cap \mathcal{A}_\kappa = \emptyset\}. \tag{3.4}$$

It is clear that the cardinality of $\mathbb{Q}^b$ is bounded by $\frac{C\kappa}{\varepsilon^\alpha}$ for some constant C. We write

$$u^\varepsilon_{bl} \;=\; u^{\varepsilon,g} \;+\; u^{\varepsilon,b} \;:=\; \sum_{q\in\mathbb{Q}^g} u^\varepsilon_q \;+\; \sum_{q\in\mathbb{Q}^b} u^\varepsilon_q$$

where u^ε_q satisfies

$$\begin{cases} -\nabla \cdot A\left(\dfrac{x}{\varepsilon}\right) \nabla u^\varepsilon_q = 0, \quad x \in \Omega, \\ u^\varepsilon_q = \left(\varphi(x, x/\varepsilon) - \varphi_*(x)\right) \phi_q(x), \quad x \in \partial\Omega. \end{cases} \tag{3.5}$$

The boundary datum for u^ε_q is localized in a small arc around x_q.

For $u^{\varepsilon,b}$, we use [5] and the bound on the cardinality of $\mathbb{Q}^b$ to get

$$\|u^{\varepsilon,b}\|_{L^2(\Omega)} \le \|u^{\varepsilon,b}\|_{L^2(\partial\Omega)} \le C\kappa^{1/2}. \tag{3.6}$$

It remains to handle $u^{\varepsilon,g}$, that is u^ε_q for $q \in \mathbb{Q}^g$. First, we pick for such q some n_q with $-n_q \in \tilde{I}_q$. Then, we give the following ansatz:

$$u^{\varepsilon,app}_q = \sum_{\substack{k,l\ge 0 \\ k(1-\alpha)+l\le K_0}} \varepsilon^{k(1-\alpha)+l} v^{k,l}_q\left(\frac{x}{\varepsilon}, \frac{x - x_q}{\varepsilon^\alpha}, x\right). \tag{3.7}$$

For each k, l, the boundary layer corrector $v_q^{k,l}$ will be a function of (y, Y, x), with compact support in Y, and decaying fast to zero as y goes to infinity along n_q. The constant K_0 will be fixed in due course. Actually, to be more precise the boundary profile $v_q^{k,l}$ also depends on ε through the boundary condition (see for instance that the boundary datum is taken at the hyperplane $y.n_q = -1/\varepsilon$ in (3.8)). However, the bounds will be uniform in ε and we chose not to keep an ε in the notation $v_q^{k,l}$.

Let us detail the construction of the first correctors, that is for $k+l \leq 1$. The higher-order terms are handled similarly. Remember that

$$\varphi_*(x) \; := \; v_*(n(x))\, \varphi_0(x) \qquad a.e.$$

We take $v_q^{0,0}$ to satisfy

$$\begin{cases} -\nabla_y \cdot \, A(y)\nabla_y \, v_q^{0,0}(y, Y, x) = 0, \quad y.n_q > -1/\varepsilon \\ v_q^{0,0}(y, Y, x) = \Big(\varphi(x, y) - v_*(n_q)\, \varphi_0(x)\Big)\, \psi(Y), \quad y.n_q = -1/\varepsilon. \end{cases} \tag{3.8}$$

Of course, the idea is that $v_q^{0,0}(x/\varepsilon, x/\varepsilon^\alpha, x)$ should cancel the trace of u_q^ε at the boundary. This is still not exactly so: First, to be able to construct the corrector, we replace the circle $|\varepsilon y| = |x| = 1$ by the flat line $y.n_q = -\frac{1}{\varepsilon}$ (recall that n_q points inward). Second, we replace $v_*(n(x))$ by $v_*(n_q)$. However, we will show in the next subsection that these approximations result in small errors, and do not affect the homogenization.

Note that $v_q^{0,0}$ has separate variables, in the sense that it reads

$$v_q^{0,0}(y, Y, x) \; = \; w_q^{0,0}(y)\, \varphi_0(x)\, \psi(Y) \tag{3.9}$$

where $w_q^{0,0} \in M_N(\mathbb{R})$ satisfies (2.4) with $n = n_q$, $a = -1/\varepsilon$, and boundary datum $v_0 - v_*(n_q)$. By definition of v_*, it goes to zero as y goes to infinity along n_q.

The $v_q^{1,0}$ term is chosen as a solution of (we drop the lowerscript q for easier reading):

$$\begin{cases} -\nabla_y \cdot \, A(y)\nabla_y \, v^{1,0}(y, Y, x) = \nabla_y \cdot \, A(y)\nabla_Y \, v^{0,0}(y, Y, x) \\ \qquad\qquad + \nabla_Y \cdot \, A(y)\nabla_y \, v^{0,0}(y, Y, x), \quad y.n_q > -\dfrac{1}{\varepsilon}, \\ v^{1,0}(y, Y, x) = v_{bd}^{1,0}(Y, x), \qquad y.n_q = -\dfrac{1}{\varepsilon}, \end{cases} \tag{3.10}$$

for some good boundary datum $v_{bd}^{1,0}(Y, x)$ (independent of y). Roughly, this corrector takes care of the source terms of amplitude $O(\varepsilon^{-\alpha-1})$ generated by $v^{0,0}$, while the boundary datum $v_{bd}^{1,0}$ ensures that it decays at infinity. As before, we can factorize these fields, through

$$\begin{aligned} v^{1,0}(y, Y, x) \; &= \; \sum_{\alpha'=1}^{d} w_{\alpha'}^{1,0}(y)\, \varphi_0(x)\, \partial_{\alpha'}\psi(Y) \\ v_{bd}^{1,0}(Y, x) \; &= \; \sum_{\alpha'=1}^{d} w_{bd,\alpha'}^{1,0}\, \varphi_0(x)\, \partial_{\alpha'}\psi(Y), \end{aligned}$$

where $w^{1,0}_{\alpha'}$ solves

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y w^{1,0}_{\alpha'} = \nabla_{y_{\beta'}} \cdot (A^{\beta'\alpha'}(y)w^{0,0}) + A_{\alpha'\beta'}(y)\nabla_{y_{\beta'}} w^{0,0} & y.n_q > -\dfrac{1}{\varepsilon} \\ w^{1,0}_{\alpha'}(y) = w^{1,0}_{bd,\alpha'}, & y.n_q = -\dfrac{1}{\varepsilon}. \end{cases} \tag{3.11}$$

Note that, up to considering a lift of the boundary datum, this system is of the type (2.44). Note also that the source term decays fast as $y \cdot n_q$ goes to infinity. As we have already discussed, for any constant boundary datum $w^{1,0}_{bd,\alpha'}$, this problem admits a solution that converges to a constant field as $y \cdot n_q$ goes to infinity. We chose precisely $w^{1,0}_{bd,\alpha'}$ so that this constant at infinity is zero. Of course, this gives rise to another error term, to be controlled in the next subsection.

The construction of $v^{0,1}$ follows the same lines. Thus, $v^{0,1}$ satisfies

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y v^{0,1}(y,Y,x) = \nabla_y \cdot A(y)\nabla_x v^{0,0}(y,Y,x) & \\ \qquad\qquad + \nabla_x \cdot A(y)\nabla_y v^{0,0}, & y.n_q > -\dfrac{1}{\varepsilon}, \\ v^{0,1} = v^{0,1}_{bd}(Y,x), & y.n_q = -\dfrac{1}{\varepsilon}. \end{cases} \tag{3.12}$$

so as to cancel the $O(\varepsilon^{-1})$ remainder terms due to $v^{0,0}$. Again, one can separate variables:

$$v^{0,1}(y,Y,x) \;=\; \sum_{\alpha'=1}^{d} w^{0,1}_{\alpha'}(y)\partial_{x_{\alpha'}}\varphi_0(x)\psi(Y)$$

where $w^{0,1}_{\alpha'} \;=\; w^{1,0}_{\alpha'}(y)$ solves the classical boundary layer system, with a rapidly decaying source term. The higher-order profiles are built recursively, following this scheme. They satisfy the same type of equations, with source terms coming from the lower order profiles. More precisely, $v^{k,l}$ solves

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y v^{k,l}(y,Y,x) = \nabla_y \cdot A(y)\nabla_x v^{k,l-1}(y,Y,x) & \\ \quad + \nabla_x \cdot A(y)\nabla_y v^{k,l-1} + \nabla_y \cdot A(y)\nabla_Y v^{k-1,l}(y,Y,x)+ & \\ \quad + \nabla_Y \cdot A(y)\nabla_y v^{k-1,l}(y,Y,x) & \\ \quad + \nabla_Y \cdot A(y)\nabla_Y v^{k-2,l}(y,Y,x) + \nabla_x \cdot A(y)\nabla_x v^{k,l-2}(y,Y,x) & \\ \quad + \nabla_x \cdot A(y)\nabla_Y v^{k-1,l-1}(y,Y,x) + \nabla_Y \cdot A(y)\nabla_x v^{k-1,l-1}(y,Y,x) & y.n_q > -\dfrac{1}{\varepsilon}, \\ v^{k,l} = v^{k,l}_{bd}(Y,x), & y.n_q = -\dfrac{1}{\varepsilon}. \end{cases} \tag{3.13}$$

Note that the bounds on the $v^{k,l}$ and $v^{k,l}_{bd}$ for $k+l \geq 1$ depend on κ. More precisely, at each step of the construction, a little more than a power κ^2 is lost: uniformly in $q \in \mathbb{Q}^g$,

$$\forall \delta > 0,\ \forall k,l,\ \forall s, \quad \|\nabla^s_{y,Y,x} v^{k,l}\|_{L^\infty} \;+\; \|\nabla^s_{Y,x} v^{k,l}_{bd}\|_{L^\infty} \;\leq\; \frac{C_{\delta,k,l,s}}{\kappa^{(2+\delta)(k+l)}} \tag{3.14}$$

These inequalities are a simple consequence of propositions 5, 7 and 9. For $k+l=0$, it follows straightforwardly from Proposition 5. For $k+l=1$, we notice that $v^{k,l}(y,x,Y) -$

$v_{bd}^{k,l}(x,Y)$ satisfies the equations in (2.44), with a zero boundary datum and a source term $f^{k,l}$ that depends on $v^{0,0}$. More precisely, this system is derived from an enlarged system of type (2.42) with a source term $F^{k,l}$ depending on $V^{0,0}$. From the estimates of propositions 5 and 7, we obtain

$$\begin{aligned}
\left|\partial_y^\alpha f^{k,l}(y)\right| &\leq C_\alpha, && \text{uniformly in } y,\\
\left|\partial_\theta^\alpha \partial_t^\beta F^{k,l}(\theta,t)\right| &\leq C_{m,\alpha,\beta}\,\kappa^{-1}\Big(\kappa(1+t-a)\Big)^{-m} &&\\
&\leq C_{m,\alpha,\beta}\,\Big(\kappa^{1+\delta}(1+t-a)\Big)^{-m} && \text{uniformly in } \theta,
\end{aligned}$$

for any given $\delta > 0$, as soon as $m\delta \geq 1$. Then, Proposition 9 with $\mu = 0$ and $\nu = 1+\delta$ yields the good L^∞ bounds on $v^{k,l}$, for $k+l = 1$ as well as good decay estimates for $V^{k,l}(\theta,t,Y,x)$. Applying recursively Proposition 9, one obtain (3.14) for all k,l. Notice that at each step, we lose a factor $\kappa^{2+\delta}$.

3.3 Last error estimates and conclusion

To conclude the homogenization proof, we still need: *i)* to estimate in L^2 the approximate boundary layer

$$u^{\varepsilon,g,app} = \sum_{q\in\mathbb{Q}^g} u_q^{\varepsilon,app},$$

where $u_q^{\varepsilon,app}$ has the expansion (3.7) *ii)* to compare it in L^2 to

$$u^{\varepsilon,g} := \sum_{q\in\mathbb{Q}^g} u_q^\varepsilon$$

where u_q^ε satisfies (3.5).

i) Note that for all q, the support of $u_q^{\varepsilon,app}$ has size $O(\varepsilon^\alpha)$ along the boundary and $O(1)$ transversally to the boundary. Moreover, when $|q-q'| \geq 2$, the supports of $u_q^{\varepsilon,app}$ and $u_{q'}^{\varepsilon,app}$ are disjoint. From there, we infer

$$\|u^{\varepsilon,g,app}\|^2_{L^2(\Omega)} \leq 2\sum_{q\in\mathbb{Q}^g} \|u_q^{\varepsilon,app}\|^2_{L^2(\Omega)}$$

and

$$\|u_q^{\varepsilon,app}\|_{L^2(\Omega)} \leq \left\|v_q^{0,0}\left(\cdot,\frac{\cdot}{\varepsilon^\alpha},\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} + C\,\varepsilon^{\alpha/2} \sum_{\substack{k+l\geq 1\\ k(1-\alpha)+l\leq K_0}} \varepsilon^{k(1-\alpha)+l}\|v_q^{k,l}\|_{L^\infty(\Omega)}.$$

Combining this last inequality with (3.14), we get

$$\|u_q^{\varepsilon,app}\|_{L^2(\Omega)} \leq \left\|v_q^{0,0}\left(\cdot,\frac{\cdot}{\varepsilon^\alpha},\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} + C(k_0,\delta)\,\varepsilon^{\alpha/2} \sum_{\substack{k+l\geq 1\\ k(1-\alpha)+l\leq K_0}} \frac{\varepsilon^{k(1-\alpha)+l}}{\kappa^{(2+\delta)(k+l)}}.$$

By construction, $v_q^{0,0}$ goes fast to zero as $t = y\cdot n_q \to +\infty$. Using the notation of (3.9)

$$\left\|v_q^{0,0}\left(\cdot,\frac{\cdot}{\varepsilon^\alpha},\frac{\cdot}{\varepsilon}\right)\right\|^2_{L^2(\Omega)} \leq C\varepsilon^{\alpha+1}\int_{\mathbb{R}_+} \left|w_q^{0,0}(y'+(t+1/\varepsilon)\right|^2 dt \leq C\frac{\varepsilon^{\alpha+1}}{\kappa^3},$$

where the last inequality follows from Proposition 7. By summing over q, we get

$$\|u^{\varepsilon,g,app}\|_{L^2(\Omega)} \;\le\; C(k_0,\delta)\left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \frac{\varepsilon^{1/2}}{\kappa^{3/2}}\right) \;\le\; C'(k_0,\delta)\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}}$$

as soon as $\alpha > 1/2$ and δ small enough, a condition that will be satisfied eventually.

ii) The difference $e^\varepsilon = u^{\varepsilon,g} - u^{\varepsilon,g,app}$ solves

$$\begin{cases} -\nabla\cdot A\left(\dfrac{x}{\varepsilon}\right)\nabla e^\varepsilon = r^\varepsilon, & x\in\Omega,\\ \qquad\qquad\qquad e^\varepsilon = \phi^\varepsilon, & x\in\partial\Omega. \end{cases} \tag{3.15}$$

We now comment on the errors r^ε and ϕ^ε.

The source term r^ε comes from the fact that $u_q^{\varepsilon,app}$ does not satisfy exactly the first equation of (3.5). Indeed, the expansion (1.7) has been cut at $k(1-\alpha)+l = K_0$. Crudely, we get $\|r^\varepsilon\|_{L^2} = O(\varepsilon^{K_0-2})$. Furthermore, estimate (3.14) allows to specify the dependence with respect to κ. Introducing k_0 such that $K_0 = k_0(1-\alpha)$, we get

$$\|r^\varepsilon\|_{L^2(\Omega)} \;\le\; C(\delta,K_0)\left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}}\right)^{k_0}\varepsilon^{-2}, \quad \forall\delta>0. \tag{3.16}$$

For this inequality, we use that $\varepsilon^{1-\alpha}/\kappa^{2+\delta} < 1$, a condition that will be ensured by our choice of parameters.

The boundary term ϕ^ε comes from several approximations:

1. In the boundary datum for $v_q^{0,0}$, we have written $v_*(n_q)$ instead of $v_*(n(x))$. In other words, we have replaced u_q^ε by the solution $\tilde u_q^\varepsilon$ of

$$\begin{cases} \qquad\qquad -\nabla\cdot A\left(\dfrac{x}{\varepsilon}\right)\nabla\tilde u_q^\varepsilon = 0, & x\in\Omega,\\ \tilde u_q^\varepsilon = \left(\varphi(x,x/\varepsilon) - v_*(n_q)\varphi_0(x)\right)\phi_q(x), & x\in\partial\Omega. \end{cases}$$

   Note that the boundary datum for both u_q^ε and $\tilde u_q^\varepsilon$ are non-zero only for θ in a vicinity of I_q. Due to the Lipschitz character of v_*, *cf* corollary 1, we deduce

$$\|\sum_{q=1}^{Q}\left(u_q^\varepsilon - \tilde u_q^\varepsilon\right)\|_{L^2(\partial\Omega)} \;\le\; \frac{C}{\kappa^2}\varepsilon^\alpha.$$

2. To be able to solve the boundary layer systems for $v_q^{k,l}$, $q\in\mathbb{Q}^g$, we have considered the flat line $y.n_q = -1/\varepsilon$, instead of the original circle $|y| = 1/\varepsilon$. Moreover, to force the decay to zero, we have added the inhomogeneous Dirichlet data $v_{q,bd}^{k,l}$, $k+l\ge 1$. All of this results in non zero boundary terms at the circle. Note that the q-th term is supported in a $O(\varepsilon^\alpha)$ neighborhood of x_q, which is at distance at most $O(\varepsilon^{2\alpha})$ from the

flat line. Its amplitude is therefore bounded by

$$\sum_{\substack{k+l\geq 1\\ k(1-\alpha)+l\leq K_0}} \varepsilon^{k(1-\alpha)+l}\, |v^{k,l}_{q,bd}| \;+\; \varepsilon^{2\alpha-1} \sum_{k(1-\alpha)+l\leq K_0} \varepsilon^{k(1-\alpha)+l}\, \|\nabla v^{k,l}_q(y)\|_{L^\infty}$$

$$\leq\; C(\delta, K_0) \Big(\sum_{\substack{k+l\geq 1\\ k(1-\alpha)+l\leq K_0}} \frac{\varepsilon^{k(1-\alpha)+l}}{\kappa^{(2+\delta)(k+l)}} \;+\; \varepsilon^{2\alpha-1} \sum_{k(1-\alpha)+l\leq K_0} \frac{\varepsilon^{k(1-\alpha)+l}}{\kappa^{(2+\delta)(k+l)}} \Big), \quad \forall \delta > 0,$$

$$\leq\; C(\delta, K_0) \Big(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1} \Big), \quad \forall \delta > 0.$$

For the last inequality, we use that $\varepsilon^{1-\alpha}/\kappa^2 < 1$, a condition that will be ensured by our choice of parameters.

Gathering these bounds, we end up with

$$\|\phi^\varepsilon\|_{L^2(\partial\Omega)} \;\leq\; C(\delta, K_0) \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1} \right), \quad \forall \delta > 0. \tag{3.17}$$

Using estimates (3.16), (3.17), and applying Theorem 3 of Avellaneda and Lin [5], we end up with

$$\|e^\varepsilon\|_{L^2(\Omega)} \;\leq\; C(\delta, K_0) \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1} \;+\; \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \right)^{k_0} \varepsilon^{-2} \right), \quad \forall \delta > 0. \tag{3.18}$$

Eventually, we have the following inequalities:

$$\left\{ \begin{aligned} \|u^\varepsilon_{reg} - u^0\|_{L^2(\Omega)} \;&\leq\; C \left(\kappa^{1/2} \;+\; \frac{\varepsilon}{\rho^{1/2}\kappa^2} \;+\; \frac{\rho}{\kappa^2} \right), \\ \|u^\varepsilon_{bl}\|_{L^2(\Omega)} \;&\leq\; C\kappa^{1/2} \;+\; C(\delta, K_0) \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1} \;+\; \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \right)^{k_0} \varepsilon^{-2} \right) \end{aligned} \right. \tag{3.19}$$

for arbitrary $\delta > 0$ and $K_0 \in \mathbb{N}$. To obtain the appropriate rate of convergence, it remains to optimize these inequalities with respect to the parameters κ, α and ρ.

First, for any given values of ε and κ, the r.h.s. of the upper inequality is minimized when $\varepsilon/(\rho^{1/2}\kappa^2) \;\sim\; \rho/\kappa^2$. This yields $\rho \sim \varepsilon^{2/3}$. With this choice,

$$\|u^\varepsilon - u^0\|_{L^2} \;\leq\; C(\delta, K_0) \left(\kappa^{1/2} + \frac{\varepsilon^{2/3}}{\kappa^2} + \frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1} \;+\; \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \right)^{k_0} \varepsilon^{-2} \right).$$

Note that the r.h.s must vanish when $\varepsilon \to 0$, which implies that $2\alpha - 1 > 0$. In turn, this implies that the second term in the sum can be neglected compared to the third one. Now, for any given value of ε, the quantity $\kappa^{1/2} + \frac{\varepsilon^{1-\alpha}}{\kappa^2} + \varepsilon^{2\alpha-1}$ is minimized when all three terms are of the same size. This yields $\alpha = 6/11$, and $\kappa \sim \varepsilon^{2/11}$.

With this scaling, we get

$$\|u^\varepsilon - u^0\|_{L^2} \;\leq\; C(\delta, K_0) \left(\varepsilon^{(1-2\delta)/11} \;+\; \varepsilon^{k_0(1-2\delta)/11-2} \right),$$

for all δ and K_0. Then, for any $\delta \in (0, 1/2)$, we take K_0 large enough so that $(k_0 - 1)(1 - 2\delta)/11 > 2$. Hence,

$$\|u^\varepsilon - u^0\|_{L^2} \;\leq\; C(\delta)\, \varepsilon^{1/11-2\delta},$$

which concludes the proof.

4 Extension to the general setting

We still need to explain how to extend our result to more general Ω, and to the case where ϕ is not factored. We shall follow the analysis and notations of section 3, and point out the arguments that need to be modified.

4.1 Uniformly convex domains

We assume that Ω is a smooth bounded open subset of $\mathbb{R}^d$, uniformly convex. We denote by m the measure on $\partial\Omega$. By our assumptions, the mapping

$$n : \partial\Omega \mapsto \mathbb{S}^{d-1}, \quad x \mapsto n(x)$$

is a diffeomorphism. This implies that for all $\kappa > 0$ the set

$$\mathcal{B}_\kappa \,:=\, \{x \in \partial\Omega, \quad n(x) \in \mathcal{A}_\kappa\}$$

satisfies

$$m(\mathcal{B}_\kappa^c) \leq C\kappa^{d-1}. \tag{4.1}$$

In particular, the set $\cup_{\kappa>0}\mathcal{B}_\kappa$ has full measure in $\partial\Omega$. For x in this set, we can define

$$\varphi_*(x) \,:=\, v_*(n(x))\,\varphi_0(x)$$

which belongs to $L^\infty(\partial\Omega)$. As in section 3, we then introduce u^ε_{reg}, u_0 and u^ε_{bl}. In order to prove Theorem 1, we need to control: *i)* $\|u^\varepsilon_{reg} - u^0\|_{L^2(\Omega)}$ and *ii)* $\|u^\varepsilon_{bl}\|_{L^2(\partial\Omega)}$.

i) The analysis carried for the disk still works for our domains Ω, replacing $\mathcal{A}_\kappa$ by $\mathcal{B}_\kappa$. The only change is the κ^{d-1} in the measure estimate (4.1). Therefore, we end up with

$$\|u^\varepsilon_{reg} - u^0\|_{L^2(\Omega)} \,\leq\, C\,\left(\kappa^{(d-1)/2} \,+\, \frac{\varepsilon}{\rho^{1/2}\kappa^2} \,+\, \frac{\rho}{\kappa^2}\right) \tag{4.2}$$

ii) The analysis is again almost unchanged. Let $\alpha > 0$. As $\partial\Omega$ is diffeomorphic to the sphere $\mathbb{S}^{d-1}$, it is easy to build a partition of unity $(\varphi_q)_{q\in\mathbb{Q}}$ in a vicinity of $\partial\Omega$, with cardinality $O(\varepsilon^{(1-d)\alpha})$, such that $\varphi_q|_{\partial\Omega}$ is supported in a set of measure $O(\varepsilon^{(d-1)\alpha})$. One can again distinguish between a bad set of indices $\mathbb{Q}^b$ and a good set $\mathbb{Q}^g$, and split u^ε_{bl} accordingly. All estimates remain the same, except for (3.6), in which the $\kappa^{1/2}$ term is replaced by $\kappa^{(d-1)/2}$, because of (4.1). Eventually, one obtains

$$\|u^\varepsilon_{bl}\|_{L^2(\Omega)} \,\leq\, C\kappa^{(d-1)/2} \,+\, C(\delta, k_0)\left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \,+\, \varepsilon^{2\alpha-1} \,+\, \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}}\right)^{k_0}\varepsilon^{-2}\right) \tag{4.3}$$

Putting together (4.2)-(4.3) and optimizing yields the theorem.

4.2 General boundary data

So far, we have considered factored data, meaning

$$\varphi(x, y) \,=\, v(y)\,\varphi_0(x)$$

for some smooth periodic $v \in M_N(\mathbb{R})$ and some smooth $\varphi_0 \in \mathbb{R}^N$. We have established in such a case that

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \le C_{\alpha,\varphi}\, \varepsilon^\alpha, \quad \forall \alpha < \frac{d-1}{3d+5}.$$

Actually, the constant $C_{\alpha,\varphi}$ can be further specified. Indeed, since the problem is linear in ϕ and since the only used a finite number of derivatives on the data, we get the bound

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \le C_\alpha\, \|\varphi\|_{H^s(\partial\Omega\times\mathbb{T}^d)}\, \varepsilon^\alpha, \quad \forall \alpha < \frac{d-1}{3d+5}, \tag{4.4}$$

for some s large enough. More precisely, $s = s(\alpha)$ depends on $\frac{d-1}{3d+5} - \alpha$.

This refined estimate (4.4) allows to go from factored to non-factored data. Indeed, let $\varphi = \varphi(x,y) \in C^\infty(\partial\Omega \times \mathbb{T}^d)$. By expanding φ as a Fourier sum, we can write

$$\varphi(x,y) \;=\; \sum_{k\in\mathbb{Z}^d} \varphi^k(x,y) \;=\; \sum_{k\in\mathbb{Z}^d} e^{2\pi i\, k\cdot y}\varphi_0^k(x).$$

For each $k \in \mathbb{Z}^d$, the data φ^k is factored, so that we can apply the analysis of section 3. In particular, we can define a homogenized boundary datum φ_*^k. We can then consider the solution $u^{\varepsilon,k}$ of (1.1)-(1.2) with boundary datum φ^k, *resp.* the solution $u^{0,k}$ of (1.15) with boundary datum φ_*^k. By estimate (4.4):

$$\|u^{\varepsilon,k} - u^{0,k}\|_{L^2(\Omega)} \le C_\alpha\, \|\varphi^k\|_{H^s(\Omega\times\mathbb{T}^d)}\, \varepsilon^\alpha, \quad \forall \alpha < \frac{d-1}{3d+5},$$

for s large enough (independent of k). As φ is smooth and periodic with respect to y, the k-th Fourier coefficient φ_0^k decays in $H^s(\partial\Omega)$ faster than any negative power of k. This leads to

$$\|\varphi^k\|_{H^s(\partial\Omega\times\mathbb{T}^d)} \le C_{s,N}\, |k|^{-N}, \qquad \forall\, k, n.$$

Combining the last two bounds yields the convergence of $u^\varepsilon = \sum_k u^{\varepsilon,k}$ to the solution $u^0 = \sum_k u^{0,k}$ of (1.15) with boundary datum $\varphi_* = \sum_k \varphi_*^k$.

5 Next order approximation

As a byproduct of our main Theorem 1, we can tackle another related homogenization problem. Namely, we can build high order expansions for the non-oscillating Dirichlet problem

$$\begin{cases} -\nabla\cdot\left(A\left(\cdot/\varepsilon\right)\nabla u\right)(x) = 0, & x\in\Omega, \\ \qquad\qquad\qquad u(x) = \varphi(x), & x\in\partial\Omega. \end{cases} \tag{5.1}$$

where φ depends only on x. We have already mentioned this problem in the introduction: one has

$$u^\varepsilon(x) \;=\; u^0(x) + \varepsilon\chi(x/\varepsilon)\nabla u^0(x) + \varepsilon u_{bl}^{1,\varepsilon}(x) \;+\; r^\varepsilon(x),$$

where $r^\varepsilon = O(\varepsilon)$ in $H^1(\Omega)$, $r^\varepsilon = O(\varepsilon^2)$ in $L^2(\Omega)$. The fields u^0 and χ are defined through (1.4) and (1.5), whereas the boundary layer corrector $u_{bl}^{1,\varepsilon}$ satisfies (1.10). This is a special case of system (1.1)-(1.2), where the boundary datum φ is factored into

$$\varphi(x,y) \;:=\; -\chi(y)\nabla u^0(x).$$

We can associate to φ the homogenized boundary datum φ_* and by Theorem 1, we get:

$$\|u_{bl}^{1,\varepsilon} - \overline{u}\|_{L^2(\Omega)} = O(\varepsilon^\alpha), \quad \forall\, \alpha < \frac{d-1}{3d+5}.$$

where $\overline{u}$ is the solution of (1.15). If we set:

$$u^1(x,y) \;:=\; \chi(y)\nabla u^0(x) \;+\; \overline{u}(x).$$

we obtain

Theorem 11 *The solution u^ε of* (5.1) *admits the asymptotic expansion*

$$u^\varepsilon \;=\; u^0 \;+\; \varepsilon\, u^1(x, x/\varepsilon) \;+\; O(\varepsilon^{1+\alpha}) \quad \textit{in } L^2(\Omega), \quad \forall\, \alpha < \frac{d-1}{3d+5}.$$

Thus, we improve the first estimate in (1.9). From this improved L^2 estimate, one can have some improved H^1 estimate in any relatively compact subset $\omega \Subset \Omega$. Namely, one can introduce the family of 1-periodic matrices

$$\Upsilon^{\alpha\beta} = \Upsilon^{\alpha\beta}(y) \in M_n(\mathbb{R}),\; \alpha,\beta = 1,...,d,$$

satisfying

$$-\nabla_y \cdot A\,\nabla_y \Upsilon^{\alpha\beta} \;=\; B^{\alpha\beta} - \int_y B^{\alpha\beta}, \quad \int_y \Upsilon^{\alpha\beta} = 0, \tag{5.2}$$

where

$$B^{\alpha\beta} \;:=\; A^{\alpha\beta} - A^{\alpha\gamma}\frac{\partial \chi^\beta}{\partial y_\gamma} - \frac{\partial}{\partial y_\gamma}\left(A^{\gamma\alpha}\chi^\beta\right).$$

Then, one can define

$$u^2(x,y) \;:=\; \Upsilon^{\alpha,\beta}\frac{\partial^2 u^0}{\partial x_\alpha \partial x_\beta} \;-\; \chi^\alpha \partial_\alpha \bar{u}. \tag{5.3}$$

Proceeding exactly as in [3], one is led to the following asymptotic expansion:

$$u^\varepsilon(x) \;=\; u^0(x) \;+\; \varepsilon u^1(x,x/\varepsilon) \;+\; \varepsilon^2 u^2(x,x/\varepsilon) \;+\; O(\varepsilon^{1+\alpha}) \quad \text{in } H^1(\omega), \quad \forall\, \alpha < \frac{d-1}{3d+5}.$$

A Green function estimates

This section is devoted to the proof of Lemma 6. The proof follows closely the analysis performed by Avellaneda and Lin in [5]. In this paper, they consider elliptic systems of size $n \geq 1$, of the type:

$$-\nabla \cdot \tilde{A}(\cdot/\varepsilon)\nabla u^\varepsilon = \nabla \cdot f, \quad x \in \Omega, \quad u^\varepsilon = g, \quad x \in \partial\Omega, \tag{A.1}$$

set in a $C^{1,\alpha}$ bounded domain Ω of $\mathbb{R}^d$, for some $0 < \alpha \leq 1$, $d \geq 1$. The function

$$\tilde{A} = \left(\tilde{A}_{ij}^{\alpha,\beta}(y)\right)_{1\leq\alpha,\beta\leq d, 1\leq i,j\leq n}$$

shares the same assumptions as ours: it is elliptic and periodic, *cf* conditions i) and ii) in our introduction, and has $C^{0,\gamma}$ regularity, for some $0 < \gamma \leq 1$. The article [5] yields local and global estimates on these systems, uniformly in ε. Notably, it provides some local Hölder and Lipschitz estimates, that are crucial to prove Lemma 6. First, we recall the local interior estimates. We denote $B(0,r) := \{y \in \mathbb{R}^d, |y| < r\}$. We state:

Theorem 12 *(Interior estimates, Lemma 9 page 812 and Lemma 16 page 827)*

i) *Let $\varepsilon > 0$, $\delta > 0$. Let u^ε be a smooth function over $B(0,1)$, satisfying*

$$-\nabla \cdot \tilde{A}(\cdot/\varepsilon)\nabla u^\varepsilon = \nabla \cdot f \quad \text{in } \; B(0,1).$$

Then, there is a constant C depending only on d, n, δ, on the $C^{0,\gamma}$ norm of $\tilde{A}$, and on the ellipticity constant λ, such that:

$$\|u^\varepsilon\|_{C^{0,\mu}(B(0,1/2))} \, \leq \, C \left(\|u^\varepsilon\|_{L^2(B(0,1))} \, + \, \|f\|_{L^{n+\delta}(B(0,1))} \right)$$

with $\mu = 1 - d/(d+\delta)$.

ii) *Moreover, there is a constant C depending only on d, n, δ, γ, on the $C^{0,\gamma}$ norm of $\tilde{A}$, and on the ellipticity constant λ, such that:*

$$\|\nabla u^\varepsilon\|_{L^\infty(B(0,1/2))} \, \leq \, C \left(\|u^\varepsilon\|_{L^\infty(B(0,1))} \, + \, \|f\|_{L^{n+\delta}(B(0,1))} \right)$$

with $\mu = 1 - d/(d+\delta)$.

We then recall the local boundary estimates. More precisely, let $\phi : \mathbb{R}^{d-1} \mapsto \mathbb{R}$ be some $C^{1,\alpha}$ function satisfying $\phi(0) = |\nabla\phi(0)| = 0$. Let $(x'_1, \ldots, x'_d)$ be some orthonormal coordinate system in $\mathbb{R}^d$ (*a priori* distinct from the canonical coordinate system $(x_1, \ldots, x_d)$). Denoting

$$\begin{aligned} D(0,r) \, &:= \, \{x \in \mathbb{R}^d, \quad x'_d > \phi(x'_1, \ldots, x'_{d-1})\} \, \cap \, B(0,r), \\ \Gamma(0,r) \, &:= \, \{x \in \mathbb{R}^d, \quad x'_d = \phi(x'_1, \ldots, x'_{d-1})\} \, \cap \, B(0,r), \end{aligned}$$

we can state

Theorem 13 *(Boundary estimates, Lemma 12 page 217, and Lemma 20 page 835)*

i) *Let $\varepsilon > 0$, $\delta > 0$. Let u^ε a smooth function over $\overline{D(0,1)}$ satisfying*

$$-\nabla \cdot \tilde{A}(\cdot/\varepsilon)\nabla u^\varepsilon = \nabla \cdot f \quad \text{in } \; D(0,1), \quad u^\varepsilon = g \quad \text{in } \; \Gamma(0,1).$$

Then, there is a constant C depending only on d, n, γ, α, δ, on the $C^{0,\alpha}$ norm of $\tilde{A}$, the $C^{1,\alpha}$ norm of ϕ, and on the ellipticity constant λ, such that:

$$\|u^\varepsilon\|_{C^{0,\mu}(D(0,1/2))} \, \leq \, C \left(\|u^\varepsilon\|_{L^2(D(0,1))} \, + \, \|g\|_{C^{0,1}(\Gamma(0,1))} \, + \, \|f\|_{L^{n+\delta}(D(0,1))} \right)$$

where $\mu = 1 - d/(d+\delta)$.

ii) *One has furthermore, for any $\nu > 0$,*

$$\|\nabla u^\varepsilon\|_{L^\infty(D(0,1/2))} \, \leq \, C \left(\|u^\varepsilon\|_{L^\infty(D(0,1))} \, + \, \|g\|_{C^{1,\nu}(\Gamma(0,1))} \, + \, \|f\|_{L^{n+\delta}(D(0,1))} \right)$$

where the constant C depends only on ν and on the parameters mentioned above.

We will deduce the estimates of Lemma 6 from Theorems 12 and 13. To do so, we will mimic the work of Avellaneda and Lin, who derive in [5] similar estimates for the Green matrix G^ε and the Poisson kernel P^ε of the operator $-\nabla\cdot A(\cdot/\varepsilon)\nabla$ in Ω: see pages 819 to 821, and pages 838 to 840.

Let $G = G(y,\tilde y)$ be the Green matrix defined in (2.11). We prove first (2.15).

Let $y \neq \tilde y \in \Omega_a$. Let $r := |y-\tilde y|$, and $f \in C^\infty_c(B(\tilde y, r/3))$. The solution u of

$$-\nabla_y\cdot(A(\cdot)\nabla_y u) = f \quad x\in\Omega_a, \quad u|_{\partial\Omega_a} = 0 \tag{A.2}$$

satisfies

$$u(y) \;=\; \int_{\Omega_a} G(y,z)\,f(z)dz = \int_{\Omega_a\cap B(\tilde y, r/3)} G(y,z)\,f(z)dz. \tag{A.3}$$

As already mentioned in section 2.2, this formula follows from property c) of the Green matrix, that extends to any $f\in C^\infty_c(\overline{\Omega_a})$ by a simple approximation argument. Since f vanishes on $B(y,r/3)$, u solves the system

$$-\nabla\cdot A(\cdot)\nabla u = 0 \quad \text{ in } \Omega_a\cap B(y,r/3), \quad u=0 \quad \text{in } \partial\Omega_a\cap B(y,r/3).$$

From a rescaled version of the Hölder bounds in Theorem 12 i) and in Theorem 13 i), it follows easily that

$$|u(y)| \;\le\; C\left(\fint_{\Omega_a\cap B(y,r/3)} |u|^2\right)^{1/2},$$

where $\fint_A g := \frac{1}{|A|}\int_A g$. We combine this last inequality with (A.3) and the Sobolev embedding theorem, to deduce

$$\begin{aligned}
\left|\int_{\Omega_a\cap B(\tilde y, r/3)} G(y,z)\,f(z)dz\right| &\le\; C\left(\fint_{\Omega_a\cap B(y,r/3)} |u|^2\right)^{1/2} \\
&\le\; C\left(\fint_{\Omega_a\cap B(y,r/3)} |u|^{2d/d-2}\right)^{(d-2)/2d} \\
&\le\; \frac{C}{r^{d/2-1}}\left(\int_{\Omega_a} |u|^{2d/d-2}\right)^{(d-2)/2d} \\
&\le\; \frac{C'}{r^{d/2-1}}\left(\int_{\Omega_a} |\nabla u|^2\right)^{1/2} \;\le\; \frac{C''}{r^{d/2-1}} r\|f\|_{L^2(\Omega_a)}
\end{aligned}$$

We stress that the last inequality comes from a simple energy estimate on system (A.2) and from the Sobolev imbedding theorem:

$$\begin{aligned}
\int_{\Omega_a} |\nabla u|^2 \;\le\; \left|\int_{\Omega_a} f\,u\right| &\le \|f\|_{L^{2d/(d+2)}(B(\tilde y,r/3)\cap\Omega_a)}\|u\|_{L^{2d/(d-2)}(\Omega_a)} \\
&\le C\,r\,\|f\|_{L^2(B(\tilde y,r/3)\cap\Omega_a)}\,\|\nabla u\|_{L^2(\Omega_a)}
\end{aligned}$$

We end up with

$$\left|\fint_{B(\tilde y,r/3)\cap\Omega_a} G(y,z)f(z)dz\right| \;\le\; \frac{C}{r^{d-2}}\left(\fint_{B(\tilde y,r/3)\cap\Omega_a} |f|^2\right)^{1/2}$$

and as f is arbitrary, we get

$$\left(\fint_{B(\tilde{y},r/3)\cap\Omega_a} |G(y,z)|^2 dz\right)^{1/2} \;\le\; \frac{C}{r^{d-2}}.$$

Finally, using that $G(y,\cdot)^T = G^t(\cdot,y)$ satisfies

$$-\nabla\cdot A^T(\cdot)\nabla G^t(\cdot,y) = 0 \quad \text{in } \; B(\tilde{y},r/3)\cap\Omega_a, \quad G^t(\cdot,y) = 0 \quad \text{in } \; B(\tilde{y},r/3)\cap\partial\Omega_a$$

we can again rely on the rescaled versions of the Hölder bounds in Theorem 12 i) and in Theorem 13 i): we conclude that

$$|G(y,\tilde{y})| \le C' \left(\fint_{B(\tilde{y},r/3)\cap\Omega_a} |G(y,z)|^2 dz\right)^{1/2} \;\le\; \frac{C"}{r^{d-2}}$$

which is exactly (2.15).

As regards (2.16), it can be deduced from (2.15), along the exact lines of [5, page 821] . The idea is to introduce the elliptic operator

$$\mathcal{L} \;:=\; -\nabla_y \cdot A(\cdot)\nabla_y \;+\; \frac{\partial^2}{\partial\theta^2} \tag{A.4}$$

defined for y in $\Omega_a \subset \mathbb{R}^2$ and $\theta \in \mathbb{T}$. As $\mathcal{L}$ is three-dimensional, the Green function associated to $\mathcal{L}$, say $\tilde{G}(y,\theta,\tilde{y},\tilde{\theta})$ can be applied the reasoning above (with minor modifications to go from a domain in $\mathbb{R}^3$ to a domain in $\mathbb{R}^2\times\mathbb{T}$). This yields

$$\left|\tilde{G}(y,\theta,\tilde{y},\tilde{\theta})\right| \;\le\; \frac{C}{\left(|y-\tilde{y}|^2 + (\theta-\tilde{\theta})^2\right)^{1/2}}$$

One can then notice that, by separation of variables,

$$G(y,\tilde{y}) = \int_0^1 \tilde{G}(y,0,\tilde{y},\tilde{\theta})d\tilde{\theta}$$

and so

$$|G(y,\tilde{y})| \le \int_0^1 \frac{C}{\left(|y-\tilde{y}|^2 + \tilde{\theta}^2\right)^{1/2}} d\tilde{\theta} \le C'(|\ln|x-y|| + 1).$$

We refer to [5, page 821] for more.

We now turn to the other estimates of Lemma 6, following page 838 to 840 in [5]. The first step towards (2.17) and (2.18) is to prove that

$$|G(y,\tilde{y})| \le \frac{C\delta(\tilde{y})}{|y-\tilde{y}|^{d-1}}. \tag{A.5}$$

Let us start with the case $d \ge 3$. If $\delta(\tilde{y}) \ge \dfrac{|y-\tilde{y}|}{3}$, it follows from (2.15). If $\delta(\tilde{y}) < \frac{|y-\tilde{y}|}{3}$, we introduce $\bar{y} \in \partial\Omega_a$ such that $\delta(\tilde{y}) = |\tilde{y}-\bar{y}|$. Then, $G(y,\cdot)^T = G^t(\cdot,y)$ satisfies

$$-\nabla\cdot A^T(\cdot)\nabla G^t(\cdot,y) = 0 \quad \text{in } \; B(\bar{y},2r/3)\cap\Omega_a, \quad G^t(\cdot,y) = 0 \quad \text{in } \; B(\bar{y},2r/3)\cap\partial\Omega_a$$

and so, using a rescaled version of Lipschitz estimate ii) in Theorem 13, we get: for all $z \in B(\bar{y}, r/3) \cap \Omega_a$,

$$
\begin{aligned}
|G(y,z)| \;&=\; |G(y,z) - G(y,\bar{y})| \;\le\; \frac{r}{3} \|\nabla G(y,\cdot)\|_{L^\infty(B(\bar{y},r/3)\cap\Omega_a)} \;\le\; C\, \|G(y,\cdot)\|_{L^\infty(B(\bar{y},2r/3)\cap\Omega_a)} \\
&\le \sup_{z' \in B(\bar{y},2r/3)\cap\Omega_a} \frac{C'}{|z'-y|^{d-2}} \;\le\; \frac{C''\delta(z)}{r^{d-1}}.
\end{aligned}
$$

Taking $z = \tilde{y}$ yields (A.5) (in the case $d \ge 3$). As regards the case $d = 2$, it follows again from the three-dimensional case, through the operator $\mathcal{L}$ in (A.4). We refer to [5, pages 839-840] for more.

Note that (A.5) implies trivially (2.17) when $\delta(y) \ge \frac{|y-\tilde{y}|}{3}$. When $\delta(y) < \frac{|y-\tilde{y}|}{3}$, we obtain (2.17) from (A.5) in the same way as we obtained (A.5) from (2.15), using the fact that $G(\cdot, \tilde{y})$ is a solution. By (2.14), the same inequality as (2.17) holds replacing G^t by G. From there, if we divide by $\delta(\tilde{y})$ and let $\delta(\tilde{y})$ go to zero, we obtain (2.18).

Thus, it only remains to establish the gradient estimates on the Green matrix and Poisson kernel. To this end, we return to estimate (2.17). We set $r := \min(\delta(y), |y-\tilde{y}|)$, and notice that $G(\cdot, \tilde{y})$ satisfies the equation $\nabla \cdot A(\cdot)\nabla G(\cdot, \tilde{y}) = 0$ in $B(y, r/2)$. Applying a proper rescaling of the interior estimate in Theorem 12 ii), we get

$$
|\nabla_y G(y,\tilde{y})| \;\le\; \frac{C}{r} \sup_{|y'-y|<r/2} |G(y',\tilde{y})| \tag{A.6}
$$

To prove (2.19), we distinguish between two cases. If $r = \delta(y)$, we use (A.5) (more precisely its symmetric version, obtained by considering G^t instead of G). Injecting in inequality (A.6) yields:

$$
|\nabla_y G(y,\tilde{y})| \;\le\; \frac{C}{\delta(y)} \sup_{|y'-y|<r/2} \frac{C'\delta(y')}{|y'-\tilde{y}|^{d-1}} \;\le\; \frac{C''}{|y-\tilde{y}|^{d-1}}.
$$

If $r = |y-\tilde{y}|$, we use (2.15) in (A.6): thus, for $d \ge 3$, we obtain

$$
|\nabla_y G(y,\tilde{y})| \;\le\; \frac{C}{|y-\tilde{y}|} \sup_{|y'-y|<r/2} \frac{C'}{|y'-\tilde{y}|^{d-2}} \;\le\; \frac{C''}{|y-\tilde{y}|^{d-1}}
$$

Finally, the same inequality is shown to be true when $d = 2$, using as before the operator $\mathcal{L}$ in (A.4).

It remains to prove (2.20) and (2.21). We use again (A.6), but this time together with (2.17). We deduce

$$
|\nabla_y G(y,\tilde{y})| \;\le\; \frac{C'}{r} \sup_{|y'-y|<r/2} \frac{\delta(y')\delta(\tilde{y})}{|y'-\tilde{y}|^{d}} \;\le\; \frac{C''}{r} \frac{\delta(y)\,\delta(\tilde{y})}{|y-\tilde{y}|^{d}}
$$

This last inequality clearly implies (2.20). Of course, by considering A^T instead of A, one can obtain the same inequality as (2.20) with G^t instead of G. From there, one may divide by $\delta(\tilde{y})$ and let $\delta(\tilde{y})$ go to zero, to obtain (2.21). This concludes the proof of the Lemma.

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